ON THE ŁOJASIEWICZ EXPONENT AND NEWTON POLYHEDRON

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Abstract- The object of this paper is to give an estimation of the Łojasiewicz exponent of the gradient of a holomorphic function under Kouchnirenko's nondegeneracy condition, using information from the Newton polyhedron.

Let $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function. The Łojasiewicz exponent of gradient of f, $L(f)$ is by definition

$$L(f) := inf\{\, \lambda > 0 \ : \ |\ \mathrm{grad} f| \geq \text{const. } |x|^{\lambda} \text{ near zero } \}.$$

It is well-known that $L(f) < \infty$ if and only if f has an isolated singularity at the origin. Chang and Lu [1] proved that for any integer r greater than $L(f)$, f is a C^0-sufficient, r-jet in holomorphic functions, i.e., adding to the function f monomials of order greater than $L(f)$ does note change its topological type. Originally this was proved by Kuo and Kuiper in the real case (see [4, 5]). Teissier [9] showed that C^0-sufficiency degree of f (i.e., the minimal integer r such that f is C^0-sufficient, r-jet) is equal to $[L(f)] + 1$, where $[L(f)]$ denotes the integral part of $L(f)$. We were motived by the work of Lichtin [7] and Fukui [2] who used the Newton polyhedron of f to give an estimation of $L(f)$, where f is non-degenerate in the sense of Kouchnirenko. In this note, following this procedure, we estimate the Łojasiewicz exponent of gradient $L(f)$ (Theorem 1 below). However, our estimations are based on other ideas, more precisely, we use the Kouchnirenko's theorem [3] on the Newton number and the geometric characterization of μ-constancy in [6, 9].

1. Newton polyhedron, main results

Now we recall some basic notions about the Newton polyhedron (see [3, 8] for details) and state the main result. Let $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be an analytic function defined by a convergent power series $\sum_{\nu} c_{\nu} x^{\nu}$. Also, let $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n, \quad x_i \geq 0, i = 1, \dots, n\}$ and $\mathbb{Z}^n_+ = \mathbb{Z}^n \cap \mathbb{R}^n_+$. A Newton polyhedron $\Gamma_+(f) \subset \mathbb{R}^n$ is defined by the convex hull of $\{\nu + \mathbb{R}^n_+ \,|\, c_{\nu} \neq 0\}$, and $\Gamma(f)$ be the union of the compact faces of $\Gamma_+(f)$. Define f_{γ} by $\sum_{\nu \in \gamma} c_{\nu} x^{\nu}$ for γ face of $\Gamma(f)$. We say that f is non-degenerate in Kouchnirenko's sense if, for any γ face of $\Gamma(f)$, the equations $\frac{\partial f_{\gamma}}{\partial x_1} = \cdots = \frac{\partial f_{\gamma}}{\partial x_n} = 0$ have no common solution on $x_1 \cdots x_n \neq 0$. The power series f is said to be convenient if $\Gamma_+(f)$ meets each of the coordinate axes. We let $\Gamma_-(f)$ denote the compact polyhedron which is the cone over $\Gamma(f)$ with the origin as a vertex. When f is convenient, the Newton number $\nu(f)$ is defined as $\nu(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n$, where V_n is the n-dimensional volumes of $\Gamma_-(f)$ and for $1 \leq k \leq n-1$, V_k is the sum of the k-dimensional volumes of the intersection of $\Gamma_-(f)$ with the coordinate planes of dimension k. The Newton number

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may also be defined for non-convenient analytic function (see [3]). Finally, we let

$$\begin{aligned} a_j &= 1 + f(e_j) \text{ for } j = 1, \ldots, n, \\ r_j(f) &= min\{m \in \mathbb{Z}_+ - \{0\} \mid \nu(f) = \nu(f + a_j x_j^m)\}, \text{ and} \\ r(f) &= \max\{r_j(f) \mid j = 1, \ldots, n\}, \end{aligned} \tag{1.1}$$

where e_j denotes the j-th unit row vector $(0, \ldots, 0, 1, 0, \ldots, 0)$.

Now we can state the main result.

Theorem 1. *Let $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be an analytic function having an isolated singularity at the origin. Suppose that f is non-degenerate in the sense of Kouchnirenko. Then $r(f) - 2 < L(f) \leq r(f) - 1$.*

2. **Proof of the theorem**

First we show that $L(f) > r(f) - 2$. So suppose now that $L(f) \leq r(f) - 2$, and modulo a permutation of coordinates in $\mathbf{C}^n$ we may assume $r(f) = r_1$. Because the non-degenerate condition of Kouchnirenko is an open condition (see [3, 8] for details), we can find an analytic family $F(x,t) = f(x) + \gamma(t)\, x_1^{r_1 - 1}$ such that $F(x, 0) = f(x)$ and $F_t(x) = F(x,t)$ is non-degenerate in Kouchnirenko's sense for each t. Since $L(f) \leq r_1 - 2$, then there exists a positive c such that $|\operatorname{grad} f| \geq c\,|x|^{r_1-2}$ in a neighbourhood U of 0. Also, for t sufficiently small so that $|\gamma(t)| \leq \frac{c}{2}$, we have

$$|\operatorname{grad} F(x,t)| \geq |\operatorname{grad} f| - |\gamma(t)\, x_1^{r_1-2}| \geq \frac{c}{2}|x|^{r_1-2} \quad \text{as } x \in U.$$

Then, we get

$$|\frac{\partial F}{\partial t}(x,t)| = |\frac{\partial \gamma}{\partial t}(t)\, x_1^{r_1-1}| \ll |x|^{r_1-2} \lesssim |\operatorname{grad} F(x,t)| \text{ as } (x,t) \to (0,0). \tag{2.1}$$

It follows from the geometric characterization of Lê and Saito [6] that F_t is μ-constant, where μ denotes the Milnor number. This fact, together with the Kouchnirenko's theorem [3], (i.e., the nondegeneracy condition implies $\mu(F_t) = \nu(F_t)$), gives $\nu(f(x)) = \nu(f(x) + \gamma(t)\, x_1^{r_1-1})$, which contradicts the definition of r_1 in (1.1).

In order to complete the proof of the theorem we need the following lemma.

Lemma 2. *For any subset $J \subset \{1, \ldots, n\}$, we have*

$$\nu(f) = \nu(f + \sum_{j \in J} a_j x_j^{r_j}).$$

Proof. First note that if $\#J = 1$, one finds this lemma by definition of r_j in (1.1). We will prove this lemma only for $\#J = 2$, the general case can be proved in a similar way. Let $J = \{j_1, j_2\}$, then it easy to see that $\Gamma_-(f) = \Gamma_-(f + a_{j_1} x_{j_1}^{r_{j_1}}) \cup \Gamma_-(f + a_{j_2} x_{j_2}^{r_{j_2}})$ is a polyhedral decomposition of $\Gamma_-(f)$, and $\Gamma_-(f + a_{j_1} x_{j_1}^{r_{j_1}} + a_{j_2} x_{j_2}^{r_{j_2}}) = \Gamma_-(f + a_{j_1} x_{j_1}^{r_{j_1}}) \cap \Gamma_-(f + a_{j_2} x_{j_2}^{r_{j_2}})$. Then, we have

$$\begin{aligned} \nu(\Gamma_-(f)) &= \nu(\Gamma_-(f + a_{j_1} x_{j_1}^{r_{j_1}})) + \nu(\Gamma_-(f + a_{j_2} x_{j_2}^{r_{j_2}})) - \nu(\Gamma_-(f + a_{j_1} x_{j_1}^{r_{j_1}} + a_{j_2} x_{j_2}^{r_{j_2}})) \\ \nu(f) &= \nu(f + a_{j_1} x_{j_1}^{r_{j_1}}) + \nu(f + a_{j_2} x_{j_2}^{r_{j_2}}) - \nu(f + a_{j_1} x_{j_1}^{r_{j_1}} + a_{j_2} x_{j_2}^{r_{j_2}}). \end{aligned}$$

Thus, the assumption that $\nu(f) = \nu(f + a_j x_j^{r_j})$ implies $\nu(f) = \nu(f + a_{j_1} x_{j_1}^{r_{j_1}} + a_{j_2} x_{j_2}^{r_{j_2}})$. $\square$

Now we are ready to prove that $r(f)-1$ is an upper bound for the Łojasiewicz exponent $L(f)$. Define an analytic family $F(x,t)=f(x)+\sum_{j=1}^{n}\gamma_j(t)\,x_j^{r_j}$ such that $F(x,0)=f(x)$ and $F_t(x)=F(x,t)$ is non-degenerate in Kouchnirenko's sense for each t. This is again possible because of the nondegenracy condition of Kouchnirenko is an open condition (see [3, 8]). Recall that $\nu(F_t)=\mu(F_t)$ by Kouchnerinko [3], it follows from the above lemma that F_t is μ-constant. According to Teissier, ([9] Remarque 5 and [10], Chap. II), the μ-constancy of F_t implies that

$$L(f)=L(F_0)\le L(F_t). \tag{2.2}$$

On the other hand, fix $t\in\mathbf{C}$. So we can find from Yoshinaga's theorem ([11], Theorem 1.7) that F_t is non-degenerate in Kouchnirenko's sense, if and only if there exists a positive ϵ such that

$$\sum_{i=1}^{n}|x_i\frac{\partial F_t}{\partial x_i}|\ge\epsilon\sum_{\alpha\in ver(F_t)}|x^\alpha|\quad\text{ as } x \text{ near } 0, \tag{2.3}$$

where $ver(F_t)=\{\,\alpha\ :\ \alpha \text{ is a vertex of } \Gamma(F_t)\,\}$. But, $r_je_j\in ver(F_t)$ for $t\ne 0$, the axial vertices of $\Gamma_+(F_t)$ (recall that e_j denotes the j-th unit row vector), which implies that $L(F_t)\le r(f)-1$ for $t\ne 0$. Together with (2.2), this completes the proof of theorem.

Remark 3. *The above inequality (2.3) can be proved directly by an argument, based on the curve selection lemma.*

We conclude with several examples.

Example 4. *Consider the map germ* $f\colon(\mathbf{C}^2,0)\to(\mathbf{C},0)$ *given by* $f(x,y)=xy^8+x^2y^3+yx^7+x^p+y^q$, *where* $p,q\ge 14$. *It is not hard to see that* $r_1(f)=11$ *and* $r_2(f)=13$. *It follows from Theorem 1 that* $11<L(f)\le 12$ *and so* f *is a* C^0*-sufficient, 13-jet. For the comparison, we note that from the Lichtin and Fukui results we have* $L(f)\le\max\{p,q\}-1$. *Their estimation, depens on the choice of the axial vertices* $(p,0)$ *and* $(q,0)$.

Example 5. *Let* $f\colon(\mathbf{C}^3,0)\to(\mathbf{C},0)$ *given by* $f(x,y,z)=x^2(x+y)^2+x(x+y)^4+x^5+(x+y+z)^5$. *This function is degenerate in Kouchnirenko's sense. However, by a linear transformation* $X=x$, $Y=x+y$ *and* $Z=x+y+z$ *we obtain a non-degenerate in Kouchnirenko's sense* $\tilde f(X,Y,Z)=X^2Y^2+XY^4+X^5+Z^5$ *with the same value of the Łojasiewicz exponent of the gradient. Moreover, by the formula of Newton number, it is not difficult to compute that*

$$\begin{aligned}\mu(\tilde f)&=\nu(\tilde f)=\nu(\tilde f+Y^6)=48,\\ \nu(\tilde f+X^4)&=\nu(\tilde f+Y^5)=44 \text{ and}\\ \nu(\tilde f+Z^4)&=36.\end{aligned}$$

Hence, we get $r_1(\tilde f)=5$, $r_2(\tilde f)=6$ *and* $r_3(\tilde f)=5$. *Thus, from the above Theorem 1 we have* $4<L(f)=L(\tilde f)\le 5$. *In this case the Fukui result gives* $L(\tilde f)\le 5$.

References

[1] S. S. Chang and Y.C. Lu, *On C^0-sufficiency of complex jets*, Canad. J. Math. **25** (1973), 874-880.
[2] T. Fukui, *Łojasiewicz type inequalities and Newton diagrams*, Proc.Amer. Math **112** (1991), 1169-1183.
[3] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. math. **32** (1976), 1-31.

[4] T.-C. Kuo, *On C^0-sufficiency of jets of potential functions,* Topology **8** (1969), 167–171.

[5] N. Kuiper, *C^1-equivalence of function near isolated critical points*, Symposium on Infinite Dimensional Topology (Baton Rouge 1967), Ann. of Math. Stud., no.69, Princeton University Press, Princeton, NJ, **98** (1972), 199–218.

[6] D.T. Lê and K. Saito, *La constance du nombre de Milnor donne des bonnes stratification*, C.R. Acad. Sci. Paris Sér. A **272** (1973), 793-795.

[7] B. Lichtin, *Estimation of Łojasiewicz exponent and Newton polygons*, Invt. math **64** (1981), 417-429.

[8] M. Oka, *On the bifurcation on the multiplicity and topology of the Newton boundary*, J. Math. Soc. Japan **31** (1979), 435-450.

[9] B. Teissier, *Variétés polaires*, Invent. Math. **40** (1977), 267-292.

[10] B. Teissier, *Cycles évanescent, section planes, et conditions de Whitney*, Singularités à Cargèse 1972, Astérisque No **7-8**, Soc. Math. Fr. (1973)

[11] E. Yoshinaga, *Topologically principal part of analytic functions*, Trans. Amer. Math. Soc. **314** (1989), 803-814

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