WEIGHTED HOMOGENEOUS POLYNOMIALS AND BLOW-ANALYTIC EQUIVALENCE

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Abstract- Based on the T. Fukui invariant and the recent motivic invariants proposed by S. Koike and A. Parusiński we give a simple classification of two variable quasihomogeneous polynomials by the blow-analytic equivalence.

1. INTRODUCTION

Unlike the topological triviality of real algebraic germs, the C^1-equisingularity admits continuous moduli. For instance, the Whitney family $W_t(x, y) = xy(x-y)(x-ty)$, $t > 1$, has an infinite number of different C^1-types. Nevertheless, as was noticed by Tzee-Char Kuo, this family is blow-analytically trivial, that is, after composing with the blowing-up $\beta\colon M^2 \to \mathbb{R}^2$, $W_t \circ \beta$ becomes analytically trivial. T.-C. Kuo proposed new notions of blow-analytic equisingularity and the blow-analytic function (see [6, 3] for survey). Let $f\colon U \to \mathbb{R}$, U open in $\mathbb{R}^n$, be a continuous function. We say that f is blow-analytic, if there exists a sequence of blowing-up β such that the composition $f \circ \beta$ is analytic (for instance $f(x, y) = \frac{x^2y}{x^2+y^2}$ is blow-analytic but not C^1). A local homeomorphism $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is called blow-analytic if so are all coordinate functions of h and h^{-1}. Two function germs f_1, $f_2\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent if there is a blow-analytic homeomorphism h such that $f_1 = f_2 \circ h$.

Observation. Let f, $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be weighted homogeneous polynomials with isolated singularities. It is known, for $n = 2$, 3, that if $(\mathbb{C}^n, f^{-1}(0))$ and $(\mathbb{C}^n, g^{-1}(0))$ are homeomophic as germs at $0 \in \mathbb{C}^n$, then, their systems of weights coincide.

We will consider real singularities. We can easily see that the notion of topological equivalence is too weak to consider the same problem for real analytic singularities. For example, consider $f(x, y) = x^3+xy^6$ and $g(x, y) = x^3+y^8$, they are topologically equivalent by Kuiper-Kuo Theorem (see [7, 8]). However, f and g have different weights. We replace the topological equivalence by the blow-analytic equvalence, and we will consider the following problem suggested by T. Fukui.

Problem 1 (T. Fukui, [2], Conjecture 9.2)**.** *Let f, $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be weighted homogeneous polynomials with isolated singularities. Suppose that f and g are blow-analytically equivalent. Then, do their systems of weights coincide?*

The purpose of this paper is to establish this conjecture for two variables. Namely, we will prove the following :

Theorem 1. *Let $f_i\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ be non-degenerate quasihomogeneous polynomials of type $(1; r_{i1}, r_{i2})$ such that $0 < r_{i2} \leq r_{i1}$. If f_1 and f_2 are blow-analytically equivalent, then either both f_1 and f_2 are nonsingular, or both are analytically equivalent to xy, or $(r_{11}, r_{12}) = (r_{21}, r_{22})$.*

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We call a polynomial $f(x_1,\dots,x_n)$ quasihomogeneous of type $(d;\, w_1,\dots,w_n)\in\mathbb{Q}^{n+1}$ if $i_1w_1+\cdots+i_nw_n=d$ for any monomial $\alpha x_1^{i_1}\dots x_n^{i_n}$ of f. We say that a polynomial $f(x)$ is non-degenerate if $\{\frac{\partial f}{\partial x_1}(x)=\cdots=\frac{\partial f}{\partial x_n}(x)=0\}=\{0\}$ as germs at the origin of $\mathbb{R}^n$.

We will next recall some important results on blow-analytic equivalence.

Theorem 2 (T. Fukui - L. Paunescu [4])**.** *Given a system of weights $w=(w_1,\dots,w_n)$, let $f_t\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be an analytic function for $t\in I=[0,1]$. Suppose that for each $t\in I$, the weighted initial form of f_t with respect to w is the same weighted degree and has an isolated singularity at $0\in\mathbb{R}^n$. Then $\{f_t\}_{t\in I}$ is blow-analytically trivial over I.*

T. Fukui ([2]) gave some invariants for blow-analytic equivalence. One of them is defined as follows:

For an analytic function $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$, set

$$A(f)=\{O(f\circ\lambda)\mid \lambda\colon(\mathbb{R},0)\to(\mathbb{R}^n,0)\ C^w arc\}.$$

Then we have

Theorem 3 (Fukui's invariant)**.** *Suppose that analytic functions $f,\, g\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ are blow-analytically equivalent, then $A(f)=A(g)$.*

Recently in [5], S. Koike and A. Parusiński have defined motivic zeta functions (inspired by the work of Denef and Loser [1]) which are invariant for blow-analytic equivalence. We will briefly recall their definition of the zeta functions.

Denote by $\mathcal{L}$ the space of analytic arcs at the origin $0\in\mathbb{R}^n$:

$$\mathcal{L}=\{\gamma\colon(\mathbb{R},0)\to(\mathbb{R}^n,0)\mid \gamma \text{ is analytic }\}$$

and by $\mathcal{L}_k$ the space of truncated arcs:

$$\mathcal{L}_k=\{\gamma\in\mathcal{L}\mid \gamma(t)=v_1t+\cdots+v_kt^k,\ v_i\in\mathbb{R}^n\}.$$

Given an analytic function $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$. For $k\geq 1$ we denote

$$A_k(f)=\{\gamma\in\mathcal{L}_k\mid f\circ\gamma(t)=ct^k+\cdots,\ c\neq 0\}.$$

We define the zeta function of f by

$$Z_f(T)=\sum_{k\geq1}(-1)^{-kn}\chi^c(A_k(f))T^k$$

where χ^c denotes the Euler characteristic with compact support.

Then we have

Theorem 4 (S. Koike - A. Parusiński [5])**.** *Suppose that analytic functions $f,\, g\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ are blow-analytically equivalent, then $Z_f=Z_g$.*

Before starting the proof of Theorem 1, we will make one more remark, as follows.

Remark 5. *Let $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a non-degenerate quasihomogeneous polynomial of type $(d;\, w_1,\dots,w_n)$. Taking a new representative of the blow-analytic class of f if necessary we can suppose that, for each $\alpha\in\mathbb{N}^n$ such that $\langle\alpha\,,\,w\rangle=\alpha_1w_1+\cdots+\alpha_nw_n=d$, the coefficient term $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is not zero in $f(x)$.*

Our remark is a simple consequence of Theorem 2 (we omit the details).

2. PROOF OF THEOREM 1

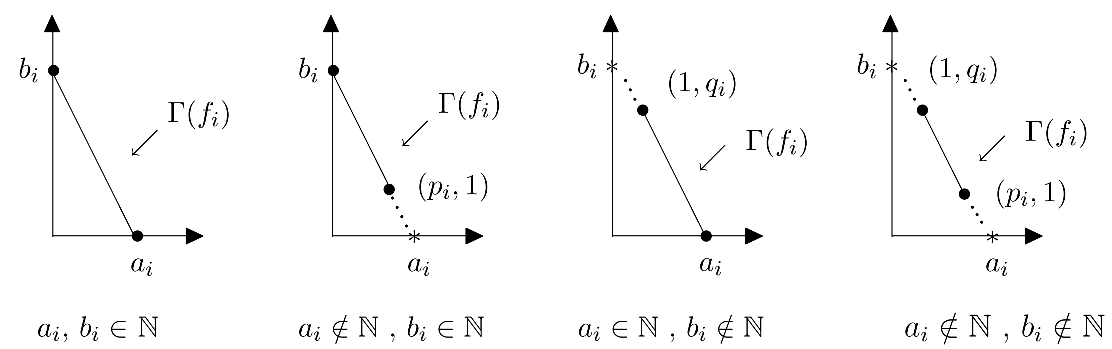
Let $f_i\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ be non-degenerate quasihomogeneous polynomials of type $(1;\, r_{i1}, r_{i2})$. Setting

$$a_i = \frac{1}{r_{i1}} \text{ and } b_i = \frac{1}{r_{i2}} \quad \text{for } i = 1, 2.$$

Modulo a permutation coordinate of $\mathbb{R}^2$, we may assume that $a_i \le b_i$. Moreover, if $a_i < 2$, then f_i is analytically equivalent to $g(x, y) = x$ or xy by the Implicit Function Theorem. But $0 \in \mathbb{R}^2$ is a regular point of x and the polynomial xy is a weighted homogeneous of type $(1;\ \frac{1}{2}, \frac{1}{2})$. Given this, we can assume that

$$2 \le a_i \le b_i \quad \text{for } i = 1, 2. \tag{2.1}$$

Since f_i are non-degenerate quasihomogeneous polynomials, we have the following cases for Newton boundary $\Gamma(f_i)$ as in the following figure:



These figures suggest that the proof of Theorem 1 should be divided into several steps, according to the possible cases for a_i and b_i:

Case 1. In this case, we suppose a_i, $b_i \in \mathbb{N}$ (i.e., f_i nearly convenient). Here $\mathbb{N}$ denotes the set of positive integers and let for any $a \in \mathbb{N}$, $\mathbb{N}_{\ge a} = \{k \in \mathbb{N} \mid k \ge a\}$. We first remark that the Fukui invariant of f_i can be computed easily as follows:

Assertion 6.

$$A(f_i) = \begin{cases} a_i\mathbb{N} \cup b_i\mathbb{N} \cup \{\infty\} & \textit{if } f_i^{-1}(0) = \{0\}, \\ a_i\mathbb{N} \cup b_i\mathbb{N} \cup \mathbb{N}_{\ge [a_i, b_i]} \cup \{\infty\} & \textit{otherwise}. \end{cases} \tag{2.2}$$

Where $[a_i, b_i] = LCM(a_i, b_i)$.

Proof. Let $\lambda\colon (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be an analytic arc. Then $\lambda(t) = (X(t), Y(t))$ can be expressed in the following way:

$$X(t) = \alpha_u t^u + \alpha_{u+1} t^{u+1} + \cdots, \quad Y(t) = c_v t^v + c_{v+1} t^{v+1} + \cdots,$$

where α_u, $c_v \neq 0$ and u, $v \ge 1$. By the above Remark 5, we may assume that there exist the terms X^{a_i} and Y^{b_i} with non-zero coefficients in $f_i(X, Y)$.

We will first consider the case whereby $f_i^{-1}(0) = \{0\}$. If $u\, a_i \neq v\, b_i$, we have

$$f_i(X(t), Y(t)) = d_i\, t^{\min\{u\, a_i\, ,\, v\, b_i\}} + \cdots,\ d_i \neq 0$$

then $O(f_i \circ \lambda) = \min\{u\, a_i\, ,\, v\, b_i\} \in a_i\mathbb{N} \cup b_i\mathbb{N} \cup \{\infty\}$. Thus it remains for us to consider the case $u\, a_i = v\, b_i$. In this case, we have

$$f_i(X(t), Y(t)) = f_i(\alpha_u\, ,\, c_v)\, t^{u\, a_i} + \cdots ,$$

since $f_i(\alpha_u\, ,\, c_v) \neq 0$. Therefore $A(f_i) \subseteq a_i\mathbb{N} \cup b_i\mathbb{N} \cup \{\infty\}$. Any integer $s \in a_i\mathbb{N} \cup b_i\mathbb{N}$, for instance $s = k\, a_i$, is attained by the arc $\gamma(t) = (t^k, 0)$. Hence we have

$$A(f_i) = a_i\mathbb{N} \cup b_i\mathbb{N} \cup \{\infty\}.$$

We will next consider the case whereby $f_i^{-1}(0) \neq \{0\}$. Similarly we have

$$a_i\mathbb{N} \cup b_i\mathbb{N} \cup \{\infty\} \subseteq A(f_i) \subseteq a_i\mathbb{N} \cup b_i\mathbb{N} \cup \mathbb{N}_{\geq [a_i, b_i]} \cup \{\infty\}.$$

Obviously we only have to prove that $\mathbb{N}_{\geq [a_i, b_i]} \subseteq A(f_i)$. Suppose that $k \in \mathbb{N}_{\geq [a_i, b_i]}$. Then there exists an arc γ through $0 \in \mathbb{R}^2$ such that $O(f \circ \gamma) = k$. Setting $[a_i, b_i] = n_i\, a_i = m_i\, b_i$, since f_i is non-degenerate and $f_i^{-1}(0) \neq \{0\}$, there exists a $(\alpha, c) \in f_i^{-1}(0)$ such that $(\frac{\partial f_i}{\partial X}(\alpha, c), \frac{\partial f_i}{\partial Y}(\alpha, c)) \neq (0, 0)$, we may assume that $\frac{\partial f_i}{\partial X}(\alpha, c) \neq 0$. Then it is easy to see that for any positive integers $[a_i, b_i] + s \in A(f)$, $s \in \mathbb{N}$, is attained by an arc $\gamma(t) = (\alpha t^{n_i} + t^{s+n_i}, c t^{m_i})$.

Evidently, this completes the proof of the Assertion. □

From Theorem 3, $A(f_1) = A(f_2)$. Thus, by the above Assertion, we have the following result:

$$\begin{aligned} &a_1 = a_2 \ \text{ same multiplicity for } f_i, \\ &b_1 = b_2 \ \text{ if } b_1 \notin a_1\mathbb{N} \text{ or } b_2 \notin a_2\mathbb{N}, \\ &b_1 = b_2 \ \text{ if } f_i^{-1}(0) \neq \{0\}. \end{aligned}$$

Manifestly, the Fukui invariant determines the weights except in the following case:

$$b_1 = k_1 a,\ b_2 = k_2 a \text{ and } f_i^{-1}(0) = \{0\},$$

where $a = a_1 = a_2$ is the smallest number in $A(f_i)$, and there remains to prove $k_1 = k_2$. In fact, assume that $k_1 \neq k_2$, for example $k_2 > k_1$. We will show that this gives rise to a contraduction by comparing the coefficients of the zeta functions. If $k_2 > k_1$ then we may write

$$\begin{aligned} A_{b_1}(f_2) &= \{\gamma(t) = (c_{k_1} t^{k_1} + \cdots + c_{b_1} t^{b_1}\, ,\, d_1 t^1 + \cdots d_{b_1} t^{b_1}) \mid c_{k_1} \neq 0\} \\ &\simeq \mathbb{R}^* \times \mathbb{R}^{b_1 - k_1} \times \mathbb{R}^{b_1}. \end{aligned}$$

That is

$$\chi^c(A_{b_1}(f_2)) = (-2)\chi^c(\mathbb{R}^{b_1 - k_1 + b_1}) = (-2)(-1)^{2b_1 - k_1}. \tag{2.3}$$

Also, since $f_1^{-1}(0) = \{0\}$, we obtain

$$\begin{aligned} A_{b_1}(f_1) &= \{\gamma(t) = (u_{k_1} t^{k_1} + \cdots + u_{b_1} t^{b_1}\, ,\, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid u_{k_1} \text{ or } v_1 \neq 0\} \\ &\simeq (\mathbb{R}^2 - \{0\}) \times \mathbb{R}^{b - k_1} \times \mathbb{R}^{b_1 - 1} \end{aligned}$$

which means

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\mathbb{R}^2 - \{0\})\, \chi^c(\mathbb{R}^{2b_1 - k_1 - 1}).$$

Since $\chi^c(\mathbb{R}^2 - \{0\}) = 0$ we get by (2.3) that $\chi^c(A_{b_1}(f_1)) \neq \chi^c(A_{b_1}(f_2))$. Therefore $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This ends the proof of Theorem 1 in the first case.

Case 2. In this case, we suppose $a_i \notin \mathbb{N}$, $b_i \in \mathbb{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term $x^{p_i} y$ for some integers $p_i \geq 1$ with non-zero coefficients in

$f_i(x,y)$. By Theorem 2 and (2.1), it is easy to see that for any integers $s \geq 1$, $f_i(x,y)+x^{p_i+s}$ is blow-analytically equivalent to $f_i(x,y)$. Then the Fukui invariant of f_i is determined by

$$A(f_i) = \{p_i+1,\, p_i+2,\, p_i+3, \cdots\} \cup \{\infty\}. \tag{2.4}$$

Moreover $A(f_1) = A(f_2)$, and it follows that $p_1 = p_2$. Consequently it is sufficient to prove that $b_1 = b_2$. Indeed, suppose that $b_1 < b_2$. Then, we let

$$p = p_1 = p_2, \quad \Re_n = \{(r,s) \in (\mathbb{N} - \{0\})^2 \mid rp + s = n\}$$

and

$$\begin{aligned} C^n_{r,s} &= \{\gamma(t) = (u_r t^r + \cdots + u_n t^n \,,\, v_s t^s + \cdots + v_n t^n) \mid u_r, v_s \neq 0\} \\ &\simeq (\mathbb{R}^*)^2 \times \mathbb{R}^{2n-r-s}. \end{aligned}$$

Let us first compute $\chi^c(A_{b_1}(f_i))$. It is easy to see that for any positive integers $n < b_i$, we have that $A_n(f_i) = \bigcup_{(r,s)\in\Re_n} C^n_{r,s}$ (This is immediate from the definitions of zeta functions). Thus, by the additivity of χ^c, we have

$$\chi^c(A_{b_1}(f_2)) = \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s}. \tag{2.5}$$

Similarly if $b_1 - 1 \notin p\,\mathbb{N}$, we obtain

$$\chi^c(A_{b_1}(f_1)) = (-2)(-1)^{2b_1-d} + \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s} \tag{2.6}$$

where d is the smallest number in $\{1,\ldots,b_1\}$ such that $d\,p+1 > b_1$. It follows from (2.5) and (2.6) that $\chi^c(A_{b_1}(f_2)) \neq \chi^c(A_{b_1}(f_1))$. But this implies a contradiction, by comparing the coefficients of the zeta functions. Hence we have $b_1 - 1 \in p\,\mathbb{N}$. Now assume $b_1 = k\,p+1$. Then by elementary computation, we have

$$A_{b_1}(f_1) = C_{f_1} \bigcup_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} C^{b_1}_{r,s},$$

where

$$\begin{aligned} C_{f_1} &= \{\gamma(t) = (u_k t^k + \cdots + u_{b_1} t^{b_1} \,,\, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid f_1(u_k, v_1) \neq 0\} \\ &\simeq \{f_1 \neq 0\} \times \mathbb{R}^{2b_1-k-1}, \end{aligned}$$

Also, by the additivity of the Euler characteristic with compact support, we obtain

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\{f_1 \neq 0\})(-1)^{2b_1-k-1} + \sum_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} (-2)^2(-1)^{2b_1-r-s}.$$

Together with (2.5), it follows that

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.7}$$

We will next compute the $\chi^c(A_{b_1+1}(f_i))$. Let $m = kp+2 = b_1+1$. Then, by the above, $m-1 \notin p\,\mathbb{N}$ and $m \leq b_2$, we can easily see the following

$$\chi^c(A_m(f_2)) = \begin{cases} \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} & \text{if } m < b_2, \\ (-2)(-1)^{2m-k-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} & \text{if } m = b_2 \end{cases} \tag{2.8}$$

Now we compute $\chi^c(A_m(f_1))$. Let $\lambda(t) = (X(t), Y(t))$ be an analytic arc defined by

$$\begin{aligned} X(t) &= u_k t^k + \cdots + u_m t^m, \\ Y(t) &= v_1 t + \cdots + v_m t^m. \end{aligned}$$

We can write

$$f_1(X(t),Y(t)) = f_1(u_k,v_1)t^{m-1} + \langle \nabla f_1(u_k,v_1)\,;\,(u_{k+1},v_2)\rangle \; t^m + \cdots ,$$

where

$$\langle \nabla f_1(u_k,v_1)\,;\,(u_{k+1},v_2)\rangle = \frac{\partial f_1}{\partial x}(u_k,v_1)\,u_{k+1} + \frac{\partial f_1}{\partial y}(u_k,v_1)\,v_2.$$

Moreover, if $f_1(u_k,v_1)=0$ and $\langle \nabla f_1(u_k,v_1)\,;\,(u_{k+1},v_2)\rangle \neq 0$, then we have $O(f_1\circ\lambda) = m$. Let us put

$$\begin{aligned}
B_1 &= \{(u,v,w,z) \in (f_1^{-1}(0)-\{0\})\times\mathbb{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle \neq 0\},\\
B_2 &= \{(u,v,w,z) \in (f_1^{-1}(0)-\{0\})\times\mathbb{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle = 0\},\\
C_{\nabla f_1} &= \{\gamma(t) = (u_k t^k + \cdots + u_m t^m, v_1 t^1 + \cdots + u_m t^m) \mid (u_k,u_{k+1},v_1,v_2)\in B_1\}\\
&\simeq B_1 \times \mathbb{R}^{2m-k-3},
\end{aligned}$$

Then, by the above, the $A_m(f_1)$ given by $A_m(f_1) = C_{\nabla f_1}\bigcup_{(r,s)\in\Re_m} C^m_{r,s}$. Thus the Euler characteristic with support compact of $A_{b_m}(f_1)$ equals

$$\chi^c(A_m(f_1)) = \chi^c(B_1)(-1)^{2m-k-3} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \tag{2.9}$$

By identification of the m-coefficients of both zeta functions of f_i for $i=1,2$, it follows from (2.8) and (2.9) that $\chi^c(B_1)=0$ or -2. On the other hand, $(f_1^{-1}(0)-\{0\})\times\mathbb{R}^2 = B_1\cup B_2$. Therefore

$$\chi^c(f_1^{-1}(0)-\{0\}) = \chi^c(B_1)+\chi^c(B_2),$$

but $B_2 \simeq (f_1^{-1}(0)-\{0\})\times\mathbb{R}$. This is clear because f_1 is non-degenerate, then we have

$$\chi^c(f_1^{-1}(0)-\{0\}) = \chi^c(f_1^{-1}(0)-\{0\})(-1)+\chi^c(B_1).$$

Since $\chi^c(B_1)=0$ or -2, this yields

$$\chi^c(f_1^{-1}(0)) = 1 \text{ or } 0,$$

which contradicts (2.7). This ends the proof of Theorem 1 in the second case.

Remark 7. *If we drop the assumption that b_2 is an integer, then the above proof still holds.*

Case 3. In this case, we suppose $a_i\in\mathbb{N}$, $b_i\notin\mathbb{N}$ for $i=1,2$. Since f_i is non-degenerate, then there exists the term xy^{q_i} for some integers $q_i\geq 1$ with non-zero coefficients in $f_i(x,y)$. For any real α we denote by $e(\alpha)$ the minimum positive integer n such that $n\geq\alpha$. By an argument similar to that of Assertion 6 and (2.4), we can compute the Fukui invariant of f_i as follows:

$$A(f_i) = a_i\mathbb{N}\cup\{e(b_i),\, e(b_i)+1,\cdots\}\cup\{\infty\}.$$

By Theorem 3, $A(f_1)=A(f_2)$. Then we have the following result:

$$a_1 = a_2 \text{ and } e(b_1)=e(b_2) \tag{2.10}$$

Suppose now $b_1\neq b_2$. Then $q_1\neq q_2$, but $\mid b_1-b_2\mid\geq\mid q_1-q_2\mid\geq 1$. It follows that $e(b_1)\neq e(b_2)$, which contradicts (2.10). This complete the proof of Theorem 1 in the third case.

Case 4. In this case, we suppose $a_i,\, b_i \notin \mathbb{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exist the terms $x^{p_i}y$ and xy^{q_i} for some integers $p_i \geq 1$ and $q_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. Thus, the Fukui invariant of f_i can be written as

$$A(f_i) = \{p_i + 1,\, p_i + 2,\, p_i + 3, \cdots\} \cup \{\infty\},$$

which implies $p_1 = p_2$. Thus we only have to prove that $b_1 = b_2$. Indeed, let us assume that $b_1 < b_2$. Then we have $q_1 < q_2$ which implies $b_1 < e(b_1) < b_2$. Let us put

$$p = p_1 = p_2,\ m = e(b_1) \ \text{ and } \ \Re_m = \{(r, s) \in (\mathbb{N} - \{0\})^2 \mid rp + s = m\}.$$

We first observe that $m - 1 \notin p\,\mathbb{N}$. Otherwise, if $m - 1 = r\,p$, then we have:

$$b_1 < q_1 + r < r\,p + 1 < r\,a_1. \tag{2.11}$$

This is a consequence of $b_1 < m = r\,p+1$ and also $(1\,,\, q_1)$ and $(p\,,\, 1)$ are vertices of $\Gamma(f_1)$. But $m = \min\{n \in \mathbb{N} \mid n > b_1\}$, which contradicts (2.11). Hence we have $m - 1 \notin p\,\mathbb{N}$. Using this observation and by elementary computation we obtain the following result:

$$\begin{aligned} \chi^c(A_m(f_2)) &= \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}, \\ \chi^c(A_m(f_1)) &= (-2)^2(-1)^{m+q_1-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \end{aligned} \tag{2.12}$$

This means that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This complete the proof of Theorem 1 in the fourth case.

In order to finish the proof of Theorem 1, it suffices to show the following lemmas.

Lemma 8. $a_1 \in \mathbb{N}$ *if and only if* $a_2 \in \mathbb{N}$.

Proof. Suppose that this is not the case. Namely, $a_1 \in \mathbb{N}$ and $a_2 \notin \mathbb{N}$. Since f_2 is non-degenerate, then there exists the term $x^{p_2}y$ for some integers $p_2 \geq 1$ with non-zero coefficients in $f_2(x, y)$. Again using the same argument in (2.4) one gets

$$A(f_2) = \{p_2 + 1,\, p_2 + 2,\, p_2 + 3, \cdots, \infty\},$$

Since $A(f_1) = A(f_2)$, then we have $a_1 = b_1 = p_2 + 1$, set $m = p_2 + 1$. We shall compute the $\chi^c(A_m(f_i))$ for $i = 1, 2$, that is

$$\begin{aligned} A_m(f_2) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m\,,\, v_1t + \cdots + v_mt^m) \mid u_1, v_1 \neq 0\} \\ &\simeq (\mathbb{R}^*)^2 \times \mathbb{R}^{2m-2}, \end{aligned}$$

so

$$\begin{aligned} A_m(f_1) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m\,,\, v_1t + \cdots + v_mt^m) \mid f_1(u_1, v_1) \neq 0\} \\ &\simeq \{f_1 \neq 0\} \times \mathbb{R}^{2m-2}, \end{aligned}$$

and hence to

$$\chi^c(A_m(f_i)) = \begin{cases} (-2)^2(-1)^{2m-2} & \text{if } i = 2, \\ \chi^c(\{f_1 \neq 0\})(-1)^{2m-2} & \text{if } i = 1. \end{cases} \tag{2.13}$$

Since $\chi^c(A_m(f_1)) = \chi^c(A_m(f_2))$, then we have

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.14}$$

Using the same argument as Case 2, we can compute the $(m+1)$-coefficients of Z_{f_i} for $i=1,2$ as follows:

$$\chi^c(A_{m+1}(f_1)) = \chi^c(B_1) \;\text{ and }\; \chi^c(A_{m+1}(f_2)) = \begin{cases} -4 & \text{if } m \neq b_2, \\ -6 & \text{if } m = b_2. \end{cases}$$

We recall that:

$$\begin{aligned} B_1 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbb{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle \neq 0\,\}, \\ B_2 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbb{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle = 0\,\}. \end{aligned}$$

Finally, by comparing the $(m+1)$-coefficients of both zeta functions Z_{f_i}, it is evident that $\chi^c(B_1) = -4$ or -6, but $(f_1^{-1}(0) - \{0\}) \times \mathbb{R}^2 = B_1 \cup B_2$. It follows from the additivity of the Euler characteristic that $\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(B_1) + \chi^c(B_2)$. On the other hand, by $B_2 \simeq (f_1^{-1}(0) - \{0\}) \times \mathbb{R}$ (because f_1 is non-degenerate), then we have

$$\chi^c(f_1^{-1}(0)) = -1 \text{ or } -2,$$

which contradicts (2.14). This proves the lemma. □

Lemma 9. $b_1 \in \mathbb{N}$ *if and only if* $b_2 \in \mathbb{N}$.

Proof. Suppose now that $b_1 \in \mathbb{N}$ and $b_2 \notin \mathbb{N}$. Since f_2 is non-degenerate, then there exists the term xy^{q_2} for some integers $q_2 \geq 1$ with non-zero coefficients in $f_2(x,y)$.

We first consider $a_i \in \mathbb{N}$ for $i=1,2$. Then, by the same reason as above, we can compute the Fukui invariant of f_i as follows:

$$\begin{aligned} A(f_1) &= a_1\mathbb{N} \cup b_1\mathbb{N} \cup \mathbb{N}_{\geq [a_1,b_1]} \cup \{\infty\}, \\ A(f_2) &= a_2\mathbb{N} \cup \mathbb{N}_{\geq e(b_2)} \cup \{\infty\}. \end{aligned}$$

Since $A(f_1) = A(f_2)$, then we have the following result:

$$a_1 = a_2, \;\; b_1 = k\,a_1, \;\text{ and }\; e(b_2) = b_1 \text{ or } b_1 + 1. \tag{2.15}$$

Since $b_1 = k\,a_1$, then there exists the term $xy^{k(a_1-1)}$ with non-zero coefficients in $f_1(x,y)$. But $|b_2 - b_1| \geq |q_2 - k(a-1)| \geq 1$, which implies $b_2 \geq b_1 + 1$ or $b_1 \geq b_2 + 1$. It follows that $e(b_2) > b_1 + 1$ or $e(b_2) < b_1$, which contradicts (2.15), and ends the first part of the lemma.

Now we consider the case where $a_i \notin \mathbb{N}$ for $i=1,2$. Since f_i is non-degenerate, then there exists the term $x^{p_i}y$ for some integers $p_i \geq 1$ with non-zero coefficients in $f_i(x,y)$. It easy to see that

$$A(f_i) = \{p_i+1,\, p_i+2,\, p_i+3, \cdots\} \cup \{\infty\},$$

Moreover $A(f_1) = A(f_2)$, and we get $p_1 = p_2$. Set

$$p = p_1 = p_2, \;\; m = e(b_2) \;\text{ and }\; \Re_m = \{(r,s) \in (\mathbb{N} - \{0\})^2 \mid rp + s = m\}.$$

As stated in Remark 7, we can exclude the case where $b_1 < b_2$ (because this is proved in exactly the same way as Case 2). Thus it remains to consider the case $b_2 < b_1$.

We next compute the m-coefficients of both zeta functions Z_{f_i} for $i=1,2$. For this, we can assert that $m-1 \notin p\mathbb{N}$. Indeed, suppose that $m-1 = \alpha\,p$ for some positive integer α. Since $b_2 < m = \alpha\,p + 1$ which implies $b_2 < q_2 + \alpha < \alpha p + 1$. This is clear because $(1\,,\,q_2) \in \Gamma(f_2)$. But $m = e(b_2)$ is equal to the smallest integer greater than b_2, which is a contradiction. Therefore we obtain that $m-1 \notin p\,\mathbb{N}$, and so on by elementary computation, we have the following result :

$$\chi^c(A_m(f_2)) = (-2)^2(-1)^{m+q_2-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \tag{2.16}$$

And

$$\begin{aligned} \chi^c(A_m(f_1)) &= \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} && \text{if } m < b_1, \\ \chi^c(A_m(f_1)) &= (-2)(-1)^{m+q_2} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} && \text{if } m = b_1. \end{aligned} \tag{2.17}$$

Now it suffices to note by the above equalities that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This completes the proof. □

Theorem 1 is therefore proved.

Example 10. *Let k be an arbitrary integer greater than or equal to 4. We consider quasihomogeneous polynomial functions f_k, $g_k\colon (\mathbb{R}^2,0) \to (\mathbb{R},0)$ defined by*

$$f_k(x,y) = x^5 + x\,y^{2\,k}, \qquad g_k(x,y) = x^5 - y^{2\,k+2}.$$

Note that the weights of f_k and g_k are $(\frac{1}{5}, \frac{2}{5\,k})$ and $(\frac{1}{5}, \frac{1}{2\,k+2})$ respectively. Since f_k and g_k have different weights for $k > 4$, they are not blow-analytically equivalent by Theorem 1. However, f_k and g_k are topologically equivalent. In fact, the above $f_k(x,y) = x^5 + x\,y^{2\,k} \in J^{2\,k+1}_{\mathbb{R}}(2,1)$ is C^0-sufficient by the Kuiper-Kuo Theorem (see [7, 8]). Therefore, f_k is topologically equivalent to $f_k - y^{2\,k+2}$. On the other hand, g_k and $g_k + x\,y^{2\,k}$ are blow-analytically equivalent by Theorem 2. Besides $f_k - y^{2\,k+2} = g_k + x\,y^{2\,k}$, hence the conclusion holds. Consequently, $f_k \in J^{2\,k+1}_{\mathbb{R}}(2,1)$ is not blow-analytically sufficient for $k > 4$.

In the case $k = 4$, the weights of f_4 and g_4 are equal to $(\frac{1}{5}, \frac{1}{10})$. Furthermore, f_4 is blow-analytically equivalent to g_4. Indeed, consider the family $H_t\colon (\mathbb{R}^2,0) \to (\mathbb{R},0) \quad (t \in [0,1])$ defined by $H_t(x,y) = (1-t)f_4(x,y) + t\,g_4(x,y)$. It is easy to see that for each $t \in [0,1]$, H_t has an isolated singularity at $0 \in \mathbb{R}^2$. Therefore, it follows from Theorem 2 that $\{H_t\}_{0\le t\le 1}$ is blow-analytically trivial over $[0,1]$. In particular, $H_0 = f_4$ is blow-analytically equivalent to $H_1 = g_4$.

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