Arc spaces, motivic measure and Lipschitz property of real algebraic varieties

Draft

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Abstract

In this paper, we investigate the motivic measure of the arc spaces of real algebraic varieties with respect to a homeomorphism with reasonable properties concerning arc-analycity and jacobian. We show an improvement of the "change of variables formula" (Theorem 2.2 and its consequences), which is originally suggested by Kontsevich ([9]), a version of inverse mapping theorem (Theorem 2.13) and Lipschitz version of inverse mapping theorem (Theorem 3.2).

A semi-algebraic homeomorphism $f : X \to Y$ between two real algebraic varieties may not preserve analytic arcs. For example, a homeomorphism $h : \mathbb{R} \to \mathbb{R}$, $x \to x^3$, sends analytic arcs to analytic arcs. But h^{-1} is not. So to investigate analytic arcs on real algebraic varieties, it is natural to impose that f is arc-analytic, that is, $f \circ \gamma$ is analytic for any analytic map $\gamma : (-1, 1) \to X$. This condition is much weaker than analyticity when $\dim X \geq 2$. An arc-analytic semi-algebraic map $f : X \to Y$ on a nonsingular algebraic manifold X is blow-Nash, that is, there is a finite composition $h : X \to Y$ of blow-ups whose centers are nonsingular Nash sets such that $f \circ h$ is Nash. But when X is singular, f being blow-Nash is equivalent to that f is generically arc-analytic, that is, there is an algebraic subset S in X so that $f \circ \gamma$ is analytic for any analytic map $\gamma : (-1, 1) \to X$ which is not entirely in S. Thus we are going to investigate arc spaces under the homeomorphism with respect to generic arc-analyticity.

We also introduce the notion of jacobian for semi-algebraic map $f : X \to Y$ in §1.5 to control the behavior of analytic arcs. We show an improved version of the "change of variables formula" (Theorem 2.2 and its consequences), which is originally suggested by Kontsevich ([9]) and compare the motivic measures of arc spaces of M and X via a generically one-to-one map $f : M \to X$ assuming that M is nonsingular. Here we say a map $f : M \to X$ is **generically one-to-one**, if there is nowhere dense subset S of X so that $f^{-1}(y)$ is a point for all $y \in X \setminus S$. This allows us to show that the arc spaces of two varieties germs $(X, 0)$ and $(Y, 0)$ have the same motivic measure if there is a semi-algebraic homeomorphism germ $f : (X, 0) \to (Y, 0)$ so that f and f^{-1} are generically arc-analytic and the Jacoban $\mathcal{J}_f$ is bounded from below and above (Theorem 2.15). This discussion allows us that a version of inverse mapping theorem (Theorem 2.13). As an application, we show that a similar inverse mapping theorem (Theorem 3.2) concerning Lipschitz property.

In last section, we make a short remark on complex case concerning about similar version on our inverse mapping theorem.

Contents

1 Preliminary

In this section, we recall several definitions and properties.

1.1 $\mathcal{AS}$-sets

A subset of $\mathbb{R}^n$ is **semialgebraic** if it is a finite union of the sets of the form

$$\{x \in \mathbb{R}^n : P_1(x) = \cdots = P_k(x) = 0, Q_1(x) > 0, \ldots, Q_l(x) > 0\},$$

where $P_i(x), Q_j(x) \in \mathbb{R}[x_1, \ldots, x_n]$. A subset of the real projective space $P^n(\mathbb{R})$ is semialgebraic if so are its intersections with the affine charts.

We say a semialgebraic subset X of $P^n(\mathbb{R})$ is an $\mathcal{AS}$**-set**, if for every analytic arc $\gamma : (-1, 1) \to P^n(\mathbb{R})$ we have

$$f(-1, 0) \subset X \Longrightarrow \exists \varepsilon > 0 \ \text{ such that } \ f(0, \varepsilon) \subset X.$$

We say a semi-algebraic subset $X \subset \mathbb{R}^n$ is an $\mathcal{AS}$**-set**, if it is so via the natural embedding $X \subset \mathbb{R}^n \subset P^n(\mathbb{R})$.

The notion of $\mathcal{AS}$-sets were introduced in [16] as a version of the arc-symmetric sets of [10]. The $\mathcal{AS}$-sets are more rigid than arbitrary semialgebraic sets and more flexible than the algebraic sets. In particular they satisfy the following properties:

- If X and Y are $\mathcal{AS}$-sets in $\mathbb{R}^n$, then $X \cup Y$, $X \cap Y$ and $X \setminus Y$ are $\mathcal{AS}$-sets.
- The image of an $\mathcal{AS}$-set by an injective regular map is again an $\mathcal{AS}$-set.

For more on the properties of the $\mathcal{AS}$-sets, see [11].

An important example of $\mathcal{AS}$-sets are Nash sets. We say a subset X of $\mathbb{R}^n$ is **Nash**, if X is analytic and semialgebraic. Similarly a map between two real algebraic varieties is called Nash if it is analytic and semialgebraic.

Let X and Y be $\mathcal{C}$-sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ with $\mathcal{C}$ = semi-algebraic or $\mathcal{AS}$, respectively. We say a map $f : X \to Y$ is a **$\mathcal{C}$-map**, if the graph of f is a $\mathcal{C}$-set of $\mathbb{R}^n \times \mathbb{R}^m$.

1.2 Arc-analyticity and arc lifting property

Let X and Y be closed $\mathcal{AS}$-sets. We say that a map $f : X \to Y$ is **arc-analytic** if $f \circ \gamma$ is analytic for every analytic arc $\gamma : (-1, 1) \to X$.

We say that a semialgebraic map $f : X \to Y$ is **generically arc-analytic** if there is an algebraic set S in X with $\dim S < \dim X$ so that $f \circ \gamma$ is analytic for every analytic map $\gamma : (-1, 1) \to X$ which is not entirely in S.

Lemma 1.1. *If X is nonsingular so that each connected component of X has the same dimension and $f : X \to Y$ is a generically arc-analytic map, then f is arc-analytic and continuous.*

Proof. See [4, Lemma 2.23]. Continuity is a consequence of Lemma 6.8 in [3]. □

We say that a map $f : X \to Y$ is **blow-Nash**, if there is a map $h : M \to X$, which is a finite composition of blow-ups whose centers are nonsingular Nash sets, of dimension smaller than $\dim X$, such that $f \circ h$ is Nash.

Lemma 1.2 ([4, Lemma 2.27])**.** *Let X be an algebraic set of dimension n and let $f : X \to Y$ be a semi-algebraic map. Then $f : X \to Y$ is blow-Nash if and only if f is generically arc-analytic.*

We say that $f : X \to Y$ has **the arc-lifting property**, if for any analytic $\beta : (-1, 1) \to Y$ there is analytic $\alpha : (-1, 1) \to X$ so that $f \circ \alpha = \beta$.

We say that $f : X \to Y$ has **the generic arc-lifting property**, if there is an algebraic subset S of Y, $\dim S < \dim Y$, such that for for analytic $\beta : (-1, 1) \to Y$, which is not entirely in S, there is an analytic $\alpha : (-1, 1) \to X$ such that $f \circ \alpha = \beta$.

1.3 Virtual Poincaré polynomial

To an $\mathcal{AS}$-set X we associate the **virtual Poincaré polynomial** $\beta(X) \in \mathbb{Z}[u]$ defined in [13, 14], [7]. The virtual Poincaré polynomial $\beta(X)$ satisfies the following properties:

- $\beta(X) = \beta(Y)$ if there is an $\mathcal{AS}$ bijection (not necessarily continuous) between X and Y.
- $\beta(X) = \beta(X \setminus Y) + \beta(Y)$ if $Y \subset X$.
- $\beta(X \times Y) = \beta(X)\beta(Y)$.
- If X is compact and nonsingular algebraic variety then $\beta(X) = \sum_i \beta_i(X) u^i$, where $\beta_i(X) = \dim H_i(X; \mathbb{Z}_2)$.

1.4 Piecewise trivialization

We say that an $\mathcal{AS}$ map $f : X \to Y$ of $\mathcal{AS}$ sets is a **$\mathcal{AS}$-piecewise trivial fibration** with fiber Z if there are a map $g : X \to Z$ and finite partitions $X = \sqcup_i X_i$, $Y = \sqcup_i Y_i$ into $\mathcal{AS}$-sets such that each $(f, g)|_{X_i} : X_i \to Y_i \times Z$ is an $\mathcal{AS}$-homeomorphism.

Remark that this implies $\beta(X) = \beta(Y)\beta(Z)$ where $\beta(X)$ denote the virtual Poincaré polynomial of X.

Remark 1.3. If a regular map $f : X \to Y$ is injective, then $f : X \to f(X)$ is a piecewise $\mathcal{AS}$-trivial fibration, since $f(X)$ is an $\mathcal{AS}$-set (by Theorem 4.4 in [16], see Theorem 3.9 in [11] also) and this is the case that Z is a point in the context above. We also remark that

$$\beta(Y) = \beta(f(X)) + \beta(Y \smallsetminus f(X)) = \beta(X) + \beta(Y \smallsetminus f(X))$$

and $\beta(Y \smallsetminus f(X)) = \beta(Y) - \beta(X)$. If $\beta(X) = \beta(Y)$, then f must be surjective.

Example 1.4. The natural map $S^1 \to P^1$ cannot be a piecewise $\mathcal{AS}$-trivial fibration. If it is so, then we have $\beta(S^1) = 2\beta(P^1)$, which contradicts with $\beta(S^1) = u + 1 = \beta(P^1)$.

1.5 Jacobian $\mathcal{J}_f$

Let X be an affine real n-dimensional algebraic variety in $\mathbb{R}^N$ defined as the zero locus of $g_1(x), \ldots, g_m(x)$, $g_i \in \mathbb{R}[x_1, \ldots, x_n]$. Let Ω^1_X denote the sheaf of Kähler differentials of X. This is generated by $dx_1, \ldots, dx_N$ over $A = \mathbb{R}[x_1, \ldots, x_N]/\langle g_1, \ldots, g_m \rangle$ with relations $dg_j = 0$, $j = 1, \ldots, m$, where $dg_j = \sum_{i=1}^N \frac{\partial g_j}{\partial x_i} dx_i$. The exterior product $\Omega^n_X = \bigwedge^n \Omega^1_X$ is generated by $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_n}$, $I = \{i_1, \ldots, i_n\}$ $(1 \le i_1 < \cdots < i_n \le N)$ with the relations generated by $dg_j = 0$, $j = 1, \ldots, m$.

Example 1.5. When X is a complete intersection variety, i.e., $\mathscr{I}_X$ is generated by a regular sequence $g_1, \ldots, g_k$, then

$$\omega = (-1)^{s(I)} \frac{dx_{I'}}{\det(dg_I)}, \quad s(I) = \sum_{i \in I} i, \quad dg_I = \left(\frac{\partial g_j}{\partial x_i}\right)_{i \in I,\, j \in \{1, \ldots, k\}}, \ |I| = k,$$

is independent of the choice of I and defines the canonical form on X. Since $dx_{I'} = \det(dg_I)\omega$, Ω^n_X is generated by the forms $\det(dg_I)\omega$, $|I| = k$. Here I' denote the complement of I in $\{1, \ldots, N\}$.

Example 1.6. Consider the curve X defined by $x^p = y^q$ in $\mathbb{R}^2$, where p and q are coprime numbers with $p < q$. Then Ω^1_X is generated by dx and dy. By a map $h : \mathbb{R} \to X$, $t \mapsto (x, y) = (t^q, t^p)$, we have

$$h^*\Omega^1_X = h^*\langle dx,\, dy \rangle = \langle t^{p-1}dt,\, t^{q-1}dt \rangle = \langle t^{p-1}dt \rangle = \langle t^{(p-1)q}\omega \rangle.$$

where $\omega = \frac{dx}{qy^{q-1}} = \frac{dy}{px^{p-1}} = \frac{dt}{t^{(p-1)(q-1)}}$.

Let $f : M \to X$ be a Nash map of a Nash nonsingular manifold M to X such that $f^*\Omega^n_X = \mathcal{J}_f \Omega^n_M$ for some ideal sheaf $\mathcal{J}_f$. We have $f^*dx_I = h_I\omega$ where ω is a local generator of Ω^n_M and $h_I \in \mathcal{O}_M$. Hence $\langle h_I\rangle = \mathcal{J}_f$. Let $(z_1,\dots,z_n)$ be a local coordinate of M and $\omega = dz_1\wedge\cdots\wedge dz_n$. Since

$$f^*dx_I = \frac{\partial(x_{i_1}\circ f,\dots,x_{i_n}\circ f)}{\partial(z_1,\dots,z_n)}\omega,$$

we have

$$\mathcal{J}_f = \Big\langle \frac{\partial(x_{i_1}\circ f,\dots,x_{i_n}\circ f)}{\partial(z_1,\dots,z_n)} : 1\le i_1<\cdots<i_n\le N\Big\rangle.$$

We may assume that $\mathcal{J}_f$ is invertible and normal crossing after composing f with blow-ups if necessary.

Let $f: X\to Y$ be a continuous $\mathcal{AS}$-map of a closed $\mathcal{AS}$-set $X\subset\mathbb{R}^N$ to a closed $\mathcal{AS}$-set $Y\subset\mathbb{R}^{N'}$. We associate to such a map a resolution diagram of f

$$\begin{tikzcd} & M \arrow[d, "\sigma"] & \\ & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd},$$

where Γ is the graph of f and $\sigma: M\to\Gamma$ is a resolution of Γ. The latter is constructed as follows. If $\hat\sigma:\widehat{M}\to\overline{\Gamma}^Z$ is the resolution of the Zariski closure of X then $\hat\sigma^{-1}(\Gamma)$ is the union of connected components of $\widehat{M}$. We then set $M := \hat\sigma^{-1}(\Gamma)$ and σ the restriction of $\hat\sigma$ to M. Clearly such a resolution diagram of f is not unique.

Definition 1.7. Let $\sigma: M\to\Gamma$, $p:\Gamma\to X$, $q:\Gamma\to Y$ be a resolution diagram of f such that $\mathcal{J}_{p\circ\sigma}$ and $\mathcal{J}_{q\circ\sigma}$ are invertible (this can be always assumed after a composition with further blow-ups if necessary). We define the **jacobian sheaf** $\mathcal{J}_f$ of $f: X\to Y$ by

$$\mathcal{J}_f = (p\circ\sigma)_*(\mathcal{J}_{q\circ\sigma}\mathcal{J}^{-1}_{p\circ\sigma}).$$

Remark 1.8. The definition of $\mathcal{J}_f$ does not depend on the choice of M. Indeed, one can show that $(p\circ\sigma_1)_*(\mathcal{J}_{q\circ\sigma_1}\mathcal{J}^{-1}_{p\circ\sigma_1}) = (p\circ\sigma_2)_*(\mathcal{J}_{q\circ\sigma_2}\mathcal{J}^{-1}_{p\circ\sigma_2})$ for two such resolutions $\sigma_i: M_i\to\Gamma$ $(i=1,2)$. Let $\sigma: M\to\Gamma$ be the fiber product of σ_1 and σ_2. It can be singular so we consider a resolution $\tau:\widehat{M}\to M$, and let $\tau_i:\widehat{M}_i\to M_i$ be the blow-ups of the ideal sheaves $\mathcal{J}_\sigma$, $\mathcal{J}_{\sigma_i}$, respectively.

Then we have

$$\begin{aligned}(p\circ\sigma)_*(\mathcal{J}_{q\circ\sigma}\mathcal{J}^{-1}_{p\circ\sigma}) =&(p\circ\sigma\circ\tau)_*(\tau^*\mathcal{J}_{q\circ\sigma}\mathcal{J}_\tau\mathcal{J}^{-1}_\tau\mathcal{J}^{-1}_{p\circ\sigma})\\ =&(p\circ\sigma\circ\tau)_*(\tau^*\mathcal{J}_{q\circ\sigma\circ\tau}\mathcal{J}^{-1}_{p\circ\sigma\circ\tau})\\ =&(\sigma_i\circ\tau_i\circ\hat\pi_i)_*(\tau^*\mathcal{J}_{q\circ\sigma_i\circ\tau_i\circ\hat\pi_i}\mathcal{J}^{-1}_{p\circ\sigma_i\circ\tau_i\circ\hat\pi_i})\\ =&(p\circ\sigma_i\circ\tau_i\circ\hat\pi_i)(\tau^*\mathcal{J}_{q\circ\sigma_i}\mathcal{J}_{\tau_i\circ\hat\pi_i}\mathcal{J}^{-1}_{\tau_i\circ\hat\pi_i}\mathcal{J}^{-1}_{p\circ\sigma_i})\\ =&(p\circ\sigma_i)_*(\mathcal{J}_{q\circ\sigma_i}\mathcal{J}^{-1}_{p\circ\sigma_i})\end{aligned}$$

where $\hat\pi_i:\widehat{M}\to\widehat{M}_i$ are the natural maps.

Example 1.9. Consider the map $f : X \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$, where $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^{2k}\}$, k is odd positive integer. We have

$$\omega = \frac{dy \wedge dz}{2x} = -\frac{dx \wedge dz}{2y} = -\frac{dx \wedge dy}{2kz^{2k-1}} \quad \text{on } X,$$

and Ω_X^2 is generated by $x\omega$, $y\omega$, $z^{2k-1}\omega$. Set $M = \mathbb{R} \times S^1$ and define a map $\sigma : M \to X$ by $(r, \theta) \mapsto (r^k \cos\theta, r^k \sin\theta, r)$. Since $\sigma^*\omega = -\frac{d(r^k\cos\theta)\wedge d(r^k \sin\theta)}{2kr^{2k-1}} = -\frac{1}{2}dr \wedge d\theta$

$$\sigma^*\Omega_X^2 = \langle r^k \cos\theta\omega, r^k \sin\theta\omega, r^{2k-1}\omega\rangle = \langle r^k\omega\rangle, \quad (f \circ \sigma)^*\Omega_{\mathbb{R}^2}^2 = \langle r^{2k-1}\omega\rangle,$$

and $\mathcal{J}_f = \sigma_*\langle r^{k-1}\rangle = \langle z^{k-1}\rangle$.

Let $\mathcal{F}$ be a subsheaf of the sheaf of rational functions on X generated by $g_1, \dots, g_m$ as $\mathcal{O}_X$-module. We say $\mathcal{F}$ is **bounded (from above)** if $\min_i\{\operatorname{ord}_\gamma g_i\} \geq 0$ for any analytic arc $\gamma : (\mathbb{R}, 0) \to X$. We say $\mathcal{F}$ is **bounded from below** if $\max_i\{\operatorname{ord}_\gamma g_i\} \leq 0$ for any analytic arc $\gamma : (\mathbb{R}, 0) \to X$.

2 Arc spaces

2.1 Arc spaces and "change of variables formula"

Let X be a closed $\mathcal{AS}$ set and Y a closed $\mathcal{AS}$ subset of X. Set

$$\mathcal{L}(X) = \{\gamma : (\mathbb{R}, 0) \to X : \text{analytic}\}, \quad \text{and} \qquad \mathcal{L}(X, Y) = \{\gamma \in \mathcal{L}(X) : \gamma(0) \in Y\}.$$

We denote by π_k the projection of $\mathcal{L}(X)$ to the k-jet space $J^k(\mathbb{R}, X)$. Set $L_k(X, 0)$ denote the set of k-jets of $\gamma \in \mathcal{L}(X, 0)$, that is,

$$L_k(X, 0) = \{[\gamma(t)] \in \mathbb{R}\{t\}/\langle t^{k+1}\rangle : g_j(\gamma(t)) \equiv 0 \mod t^{k+1}\}$$

where $\mathscr{I}_X = \langle g_j \rangle$.

Proposition 2.1 ([4, Proposition 2.33])**.** *Let $X \subset \mathbb{R}^N$ be an algebraic subset of dimension n. Then*

- $\dim \pi_k(\mathcal{L}(X)) = (k + 1)n$.
- *The fibers of the natural map $\pi_m(\mathcal{L}(X)) \to \pi_k(\mathcal{L}(X))$, $m \geq k$, are of dimension smaller than or equal to $(m - k)n$.*

Let X be an n-dimensional algebraic subset in $\mathbb{R}^N$ defined as the zero locus of the ideal $\mathscr{I}_X = \langle g_1, \dots, g_s\rangle$. Set

$$\mathcal{L}^{(m)}(X) = \{\alpha \in \mathcal{L}(X) : \operatorname{ord}_\alpha I_c(dg) \leq m\}$$

where $I_c(dg)$ denote the ideal generated by $c \times c$ minors of the jacobi matrix

$$dg = \Big(\frac{\partial g_j}{\partial x_i}\Big)_{i=1,\dots,N; j=1,\dots s}$$

and $c = N - n$. We have $\mathcal{L}(X) \setminus \mathcal{L}(X_{\text{sing}}) = \bigcup_m \mathcal{L}^{(m)}(X)$.

By convention, we set, for $J \subset \{1, \dots, s\}$, $I \subset \{1, \dots, N\}$

$$g^J = (g_j)_{j \in J},\ dg^J = \left(\frac{\partial g_j}{\partial x_i}\right)_{i=1,\dots,N; j \in J} \text{ and } dg_I^J = \left(\frac{\partial g_j}{\partial x_i}\right)_{i \in I; j \in J}.$$

If $\operatorname{ord}_\gamma \det dg_I^J = \operatorname{ord}_\gamma I_c(dg)$, $|I| = |J| = c$, then γ is in the zero locus X_J of $\langle g_j \rangle_{j \in J}$, and $\gamma(t)$, $t \neq 0$, is in the regular locus of X_J.

By Taylor's theorem, we have

$$g^J(\gamma(t) + t^{k+1} v) = g^J(\gamma(t)) + t^{k+1} dg^J(\gamma(t)) v + t^{2(k+1)} S(\gamma(t), v)$$

and therefore, if $\gamma(t) + t^{k+1} v \in X$ then

$$dg_I^J(\gamma(t)) v_I + dg_{I'}^J(\gamma(t)) v_{I'} + t^{k+1} S(\gamma(t), v) = 0.$$

Multiplying by the cofactor matrix $(dg_I^J)^*$ of dg_I^J, we obtain

$$\det(dg_I^J(\gamma(t))) v_I + (dg_I^J(\gamma(t)))^* dg_{I'}^J(\gamma(t)) v_{I'} + t^{k+1} (dg_I^J(\gamma(t)))^* S(\gamma(t), v) = 0.$$

If $\operatorname{ord}_\gamma \det dg_I^J = \operatorname{ord}_\gamma I_c(dg^J)$ and $\operatorname{ord}_\gamma \det dg_I^J \leq k$ then, by implicit function theorem, v_I is a function of t and $v_{I'}$ and thus v_I is determined by t and $v_{I'}$ uniquely.

Theorem 2.2. *Let X be a closed $\mathcal{AS}$ set. Let M be a nonsingular $\mathcal{AS}$-set so that each connected component of X has the same dimension and let $h : M \to X$ be a proper C^∞-map with $\mathcal{AS}$ graph. We assume that the set of points at which h is a local isomorphism is dense in M. Set $\mathcal{B}_e^{(m)} = \{\gamma \in \mathcal{L}(M) : h \circ \gamma \in \mathcal{L}^{(m)}(X),\ \operatorname{ord}_\gamma \mathcal{J}_h = e\}$, and assume that $k \geq \max\{2e, m\}$.*

- *Let $\alpha \in \mathcal{L}(X)$. If there is $\gamma \in \mathcal{B}_e^{(m)}$ with $j^k \alpha(0) = j^k(h \circ \gamma)(0)$, then there is $\tilde{\alpha} \in \mathcal{L}(M)$ such that $\alpha(t) = h \circ \tilde{\alpha}(t)$. Moreover, if there is $\beta \in \mathcal{L}(M)$ with $h \circ \beta = \alpha$ and $j^{k-e+1} \tilde{\alpha}(0) = j^{k-e+1} \beta(0)$, then $\tilde{\alpha} = \beta$.*
- *If h is generically one-to-one, then $h_{*,k}^{-1}(j^k(h \circ \gamma(t)))$ is homeomorphic with $\mathcal{AS}$-graph to $\mathbb{R}^e$, where $h_{*,k} : \pi_k \mathcal{L}(M) \to \pi_k \mathcal{L}(X)$ denotes the induced map. In particular, we have $\pi_k^{-1}(\pi_k h_* \mathcal{B}_e^{(m)}) = h_* \mathcal{B}_e^{(m)}$.*

Proof. Take a point y of M and a local coordinate system $(y_1, \dots, y_n)$ at y where $n = \dim_y M$ and we consider the arc germs at y. Set $\omega = dy_1 \wedge \dots \wedge dy_n$. Then $h^* dx_{I'} = (\det dh_{I'}) \omega$. For J with $|J| = c$, we set $\omega^J = (-1)^{s(I)} \frac{dx_{I'}}{(\det dg_I^J)}$.
We have

$$h^* dx_{I'} = (-1)^{s(I)} h^* [(\det dg_I^J) \omega^J]$$

and then

$$(-1)^{s(I)} h^* (\det dg_I^J) \hat{h}^J = \det(dh_{I'}), \quad \text{where } \hat{h}^J \text{ is a rational function with } h^* \omega^J = \hat{h}^J \omega.$$

This shows that for an analytic arc γ

$$\operatorname{ord}_{h \circ \gamma} \det(dg_I^J) = \operatorname{ord}_{h \circ \gamma} I_c(dg^J) \iff \operatorname{ord}_\gamma \det(dh_{I'}) = \operatorname{ord}_\gamma I_n(dh). \tag{1}$$

For such γ, we have

$$\operatorname{ord}_\gamma h^* \det(dg_I^J) + \operatorname{ord}_\gamma \hat{h}^J = \operatorname{ord}_\gamma h^* I_c(dg^J) + \operatorname{ord}_\gamma \hat{h}^J = \operatorname{ord}_\gamma (dh_{I'}) = \operatorname{ord}_\gamma I_n(dh).$$

We show that for any $v \in \mathbb{R}\{t\}^N$ with $h(\gamma(t)) + t^{k+1}v \in \mathcal{L}(X)$ there is a unique $u \in \mathbb{R}\{t\}^n$ such that

$$h(\gamma(t) + t^{k+1-e}u) - h(\gamma(t)) = t^{k+1}v.$$

By Taylor's theorem, we have

$$h(\gamma(t) + t^{k+1-e}u) - h(\gamma(t)) = t^{k+1-e}dh(\gamma(t))u + t^{2(k+1-e)}R(\gamma(t), u).$$

Assume that $k \geq \max\{2e, m\}$. We thus obtain

$$v = t^{-e}dh(\gamma(t))u + t^{k+1-2e}R(\gamma(t), u).$$

or, equivalently,

$$\begin{aligned} v_I =& t^{-e}dh_I(\gamma(t))u + t^{k+1-2e}R_I(\gamma(t), u), \\ v_{I'} =& t^{-e}dh_{I'}(\gamma(t))u + t^{k+1-2e}R_{I'}(\gamma(t), u). \end{aligned}$$

By multiplying the latter equation by the cofactor matrix $(dh_{I'})^*$, we obtain

$$(dh_{I'})^*(\gamma(t))v_{I'} = t^{-e}\det dh_{I'}(\gamma(t))u + t^{k+1-2e}(dh_{I'})^*(\gamma(t))R_{I'}(\gamma(t), u).$$

Since $t^{-e}\det dh_{I'}(\gamma(t))$ is a unit, we may use the implicit function theorem to show that this equation determines u.

Denote $A = dh_{I'}(\gamma(t))$. Let $\lambda_i = \sigma_i^2$ denote the eigenvalues of ${}^tA\,A$ and u_i the corresponding eigenvectors. We can assume that ${}^tu_iu_j = \delta_{i,j}$. Then $v_i = Au_i/\sigma_i$ is the unit eigenvector of $A\,{}^tA$ with eigenvalue λ_i. Since ${}^tv_iAu_j = {}^t(Au_i/\sigma_i)Au_j = \sigma_i\delta_{i,j}$, we have ${}^tVAU = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, setting $U = (u_i)$, $V = (v_j)$. Remark that $U \in \mathrm{SO}(n)$, $V \in \mathrm{SO}(n)$. We thus have the singular value decomposition $A = V\Sigma{}^tU$ where $\Sigma = \operatorname{diag}(\sigma_1(t), \ldots, \sigma_n(t))$, $\sigma_i(t) = t^{e_i}\bar{\sigma}_i(t)$, $\bar{\sigma}_i(0) \neq 0$. Therefore

$$t^{k+1}\bar{v} = t^{k+1-e}\Sigma\bar{u} + t^{2(k+1-e)}\bar{R}, \quad \bar{v} = {}^tVv_{I'}, \quad \bar{u} = {}^tUu, \quad \bar{R} = {}^tVR_{I'}.$$

In other words,

$$t^{e-e_j}\bar{v}_j = \bar{\sigma}_j(t)\bar{u}_j(t) + t^{k-e-e_j+1}\bar{R}_j, \quad j = 1, \ldots, n.$$

Setting $\bar{v}_j = \sum_{i=0}^\infty \bar{v}_{j,i}t^i$, $\bar{\sigma}_j = \sum_{a=0}^\infty \bar{\sigma}_{j,a}t^a$, $\bar{u}_j = \sum_{b=0}^\infty \bar{u}_{j,b}t^b$, we have

$$\sum_{a+b=s} \bar{\sigma}_{j,a}\bar{u}_{j,b} = \begin{cases} 0, & s = 0, 1, 2, \ldots, e - e_j - 1 \\ \bar{v}_{j,i}, & s = e - e_j + i \ (i = 0, 1, \ldots, k - 2e) \end{cases}$$

and thus $\bar{u}_{j,0} = \cdots = \bar{u}_{j,e-e_j-1} = 0$. So the image of the map $\mathbb{R}^e \to \pi_k\mathcal{L}(M)$ defined by

$$(\bar{u}_{1,e-e_1}, \ldots, \bar{u}_{1,e-1}, \ldots, \bar{u}_{n,e-e_n}, \ldots, \bar{u}_{n,e-1}) \mapsto j^k(\gamma(t) + t^{k-e+1}U\bar{u}) \tag{2}$$

is in the fiber of $j^k(h\circ\gamma(t))$ by h, when we show that $v_{I'}$ determine v_I. By Taylor's theorem, we have

$$g^J(h\circ\gamma(t)+t^{k+1}v)=g^J(h\circ\gamma(t))+t^{k+1}dg^J(h\circ\gamma(t))v+t^{2(k+1)}S^J(\gamma(t),v).$$

Since $h\circ\gamma(t)\in\mathcal{L}(X)$ and $h\circ\gamma(t)+t^{k+1}v\in\mathcal{L}(X)$, we have

$$0=dg^J(h\circ\gamma(t))v+t^{k+1}S^J(\gamma(t),v).$$

Setting $v=\binom{v_I}{v_{I'}}$, we have

$$dg^J_I(h\circ\gamma(t))v_I=-dg^J_{I'}(h\circ\gamma(t))v_{I'}-t^{k+1}S^J(h\circ\gamma(t),v).$$

Multiplying by $(dg^J_I)^*(h\circ\gamma(t))$ from the left, we obtain

$$v_I+\frac{(dg^J_I)^*dg^J_{I'}(h\circ\gamma(t))v_{I'}+t^{k+1}(dg^J_I)^*(h\circ\gamma(t))S^J(h\circ\gamma(t),v)}{\det(dg^J_I)(h\circ\gamma(t))}=0.$$

By (1), we have $\mathrm{ord}_{h\circ\gamma}(\det dg^J_I)\le\mathrm{ord}_{h\circ\gamma}(dg^J_I)^*dg^J_{I'}$, and the second term of the left-hand side is analytic if $k\ge m$. We thus conclude the assertion by the implicit function theorem. $\square$

Lemma 2.3. *We continue the notation above. If there is another analytic (or formal) solution $\hat{u}=\hat{u}_0+\hat{u}_1t+\cdots$, $\hat{u}_0\neq 0$ of*

$$h(\gamma(t)+t^d\hat{u})=h(\gamma(t))+t^{k+1}v,\qquad d\le k-e,$$

then $d\le e_{\max}$ where $e_{\max}=\max\{e_1,\dots,e_n\}$.

Proof. Remark that

$$\begin{aligned}t^{k+1}\bar{v}=&{}^tV[h(\gamma(t)+t^d\hat{u})-h(\gamma(t))]=t^d\Sigma\bar{\hat{u}}+t^{2d}\bar{R}(t,\hat{u})\\ =&t^d\Sigma(\bar{\hat{u}}_0+\bar{\hat{u}}_1t+\cdots)+t^{2d}\bar{R}(t,\hat{u}).\end{aligned}$$

Setting $d_j=\min\{i:\bar{\hat{u}}_{i,j}\neq 0\}$, $j=1,\dots,n$, we have that

$$t^{k+1}\bar{v}_j=t^{d+d_j+e_j}\bar{\sigma}_j(t)(\bar{\hat{u}}_{d_j,j}+\bar{\hat{u}}_{d_j+1,j}t+\cdots)+t^{2d}\bar{R}_j(t,\hat{u}),\qquad j=1,\dots,n.$$

We obtain that

$$2d\le 2d+\mathrm{ord}_t\,\bar{R}_j(t,\hat{u})=d+d_j+e_j\quad\text{whenever}\quad d+d_j+e_j\le k.$$

Take j_0 with $d_{j_0}=0$, then we have $d+e_{j_0}\le k-e+e_{j_0}\le k$ and $d\le e_{j_0}$. $\square$

Corollary 2.4. *Let X be an n-dimensional algebraic set. Let M be an n-dimensional nonsingular $\mathcal{AS}$-set and let $h:M\to X$ be a C^∞-map and generically one-to-one with $\mathcal{AS}$-graph. We assume that there is a normal crossing divisor $E=\cup_jE_j$ in M so that*

$$(\det(dh))_0=\sum_j\nu_jE_j,\quad (h^*I_c(dg^J))_0=\sum_j\lambda^J_jE_j.$$

Then $h_{*,k}(\pi_k\mathcal{B}_e^{(m)})$ is an $\mathcal{AS}$-set, and the map $h_{*,k}:\pi_k\mathcal{B}_e^{(m)}\to h_{*,k}(\pi_k\mathcal{B}_e^{(m)})$ is a piecewise $\mathcal{AS}$-trivial fibration with fiber $\mathbb{R}^e$. So we have a decomposition of the jet space as follows:

$$\pi_{\boldsymbol{k}}(\mathcal{L}^{(m)}(X)\cap h_*\mathcal{L}(M))=Z_{\boldsymbol{k}}^{(m)}\sqcup\bigcup_{\boldsymbol{j}\in A_k^{(m)}}\pi_k h_*\mathcal{B}_{\boldsymbol{j}},$$

where $A_k^{(m)}=\{\boldsymbol{j}\in A^{(m)}:2\langle\boldsymbol{\nu},\boldsymbol{j}\rangle\le k\}$, $A^{(m)}=\{\boldsymbol{j}:\exists J\subset\{1,\dots,s\}\ \langle\boldsymbol{\lambda}^J,\boldsymbol{j}\rangle\le m\}$, $\mathcal{B}_{\boldsymbol{j}}=\{\gamma\in\mathcal{L}(M):\mathrm{ord}_{E_i}\gamma=j_i\}$, $Z_k^{(m)}$ is a semi-algebraic subset with

$$\dim Z_k^{(m)}<n(k+1)-\frac{k}{\max\{2\nu_{\max},\,\lambda_{\max}\}},\quad \begin{array}{l}\nu_{\max}=\max\{\nu_j\},\\ \lambda_{\max}=\max\{\lambda_j^J:J\subset\{1,\dots,s\}\},\end{array}$$

and $\dim\pi_k h_*\mathcal{B}_{\boldsymbol{j}}=n(k+1)-s_{\boldsymbol{j}}-\langle\nu,\boldsymbol{j}\rangle$, $s_{\boldsymbol{j}}=\sum_i j_i$.

Proof. We have $\mathcal{B}_e^{(m)}=\bigsqcup_{\boldsymbol{j}\in A^{(m)}:\langle\boldsymbol{\nu},\boldsymbol{j}\rangle=e}\mathcal{B}_{\boldsymbol{j}}$. Take $\gamma\in\mathcal{B}_{\boldsymbol{j}}$, $\boldsymbol{j}\in A^{(m)}$. Then there are I and J so that $\mathrm{ord}_{h\circ\gamma}\det dg_I^J=\mathrm{ord}_{h\circ\gamma}I_c(dg)\le m\le k$. Since $j_i\le s_{\boldsymbol{j}}\le\langle\boldsymbol{\nu},\boldsymbol{j}\rangle=e<k-e+1$, we have $\gamma(t)+t^{k-e+1}u\in\mathcal{B}_{\boldsymbol{j}}\subset\mathcal{B}_e^{(m)}$ in the expression (2), and the natural map $\pi_k\mathcal{B}_e^{(m)}\to\pi_k h_*\mathcal{B}_e^{(m)}$ has the fiber $\mathbb{R}^e$.

Since the fiber of the natural map $\pi_k\mathcal{B}_e^{(m)}\to\pi_k h_*\mathcal{B}_e^{(m)}$ has odd Euler characteristic, $h_{*,k}(\pi_k\mathcal{B}_e^{(m)})$ is $\mathcal{AS}$-constructible (see Theorem 3.9 in [11]). By the expression (2), we conclude that $h_{*,k}:\pi_k\mathcal{B}_e^{(m)}\to h_{*,k}(\pi_k\mathcal{B}_e^{(m)})$ is an $\mathcal{AS}$-piecewise trivial fibration with fiber $\mathbb{R}^e$.

So the dimension of $\pi_k(\mathcal{B}_{\boldsymbol{j}})=n(k+1)-s_{\boldsymbol{j}}$. When $\boldsymbol{j}\in A_k^{(m)}$, we have $\mathrm{ord}_\gamma I_n(dh)=\langle\boldsymbol{\nu},\boldsymbol{j}\rangle=e$, $\mathrm{ord}_\gamma I_c(dg^J)=\langle\boldsymbol{\lambda}^J,\boldsymbol{j}\rangle\le m\le k$,

$$\dim h_{*,k}\pi_k(\mathcal{B}_{\boldsymbol{j}})=n(k+1)-s_{\boldsymbol{j}}-\langle\boldsymbol{\nu},\boldsymbol{j}\rangle.$$

If $\boldsymbol{j}\notin A_k^{(m)}$, then $k\le\max\{2\langle\boldsymbol{\nu},\boldsymbol{j}\rangle,\ \langle\boldsymbol{\lambda}^J,\,\boldsymbol{j}\rangle\}\le\max\{2\nu_{\max}s_{\boldsymbol{j}},\,\lambda_{\max}s_{\boldsymbol{j}}\}$. So

$$s_{\boldsymbol{j}}\ge\frac{k}{\max\{2\nu_{\max},\lambda_{\max}\}}$$

and we obtain that

$$\dim h_{*,k}\pi_k\mathcal{B}_{\boldsymbol{j}}\le\dim\pi_k\mathcal{B}_{\boldsymbol{j}}\le n(k+1)-s_{\boldsymbol{j}}\le n(k+1)-\frac{k}{\max\{2\nu_{\max},\lambda_{\max}\}}.$$

Since $Z_k^{(m)}$ is the union of $h_{*,k}\pi_k\mathcal{B}_{\boldsymbol{j}}$, $\boldsymbol{j}\notin A_k^{(m)}$, we are done. $\square$

Remark 2.5. Similar versions of Theorem 2.2 and Corollary 2.4 were proved by J. Denef and F. Loeser ([5, Lemma 3.4]) using a slightly different $\mathcal{L}^{(m)}(X,0)$ (see (2.6) in [5]). J.-B. Campesato also gave this theorem ([4, Lemma 4.5]) using another version of $\mathcal{L}^{(m)}(X,0)$ (see Definition 4.2 in [4]).

2.2 Motivic measure

Let $K_0(\mathcal{AS})$ denote the Grothendieck ring of $\mathcal{AS}$-sets, i.e., the ring generated by the symbols $[X]$ for $\mathcal{AS}$-sets X, with the relations

- $[X] = [X']$ if X and X' are $\mathcal{AS}$-isomorphic,
- $[X] = [X \setminus Y] + [Y]$ if Y is closed in X,
- $[X] \cdot [Y] = [X \times Y]$.

Taking the virtial Poincaré polynomial, we have a natural map

$$\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u], \quad [X] \mapsto \beta(X).$$

By Fichou [8, Theorem 1.16], it is an isomorphism.

Denote $\mathcal{M} = K_0(\mathcal{AS})[\mathbb{L}^{-1}]$, the localization of $K_0(\mathcal{AS})$ with respect to the multiplicative set generated by $\mathbb{L}$ where $\mathbb{L} = [\mathbb{R}]$. The isomorphism $\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$ extends to an isomorphism $\mathcal{M} \to \mathbb{Z}[u, u^{-1}]$, which we also denote by β.

Set $\widehat{\mathcal{M}} = \varprojlim_q \mathcal{M}/F^q\mathcal{M}$, the completion of $\mathcal{M}$ with respect to the filtration $F^q\mathcal{M}$ where $F^q\mathcal{M}$ is the subgroup generated by $\{[X]\mathbb{L}^{-i} : \dim X - i \leq -q\}$.

Example 2.6. For a positive integer d, $\lim_{i\to\infty} \mathbb{L}^{-id} = 0$ in $\widehat{\mathcal{M}}$.

Lemma 2.7. $\widehat{\mathcal{M}} \simeq \mathbb{Z}[\mathbb{L}][[\mathbb{L}^{-1}]] \simeq \mathbb{Z}[u][[u^{-1}]]$.

Proof. We see that the natural map

$$\mathbb{Z}[\mathbb{L}][[\mathbb{L}^{-1}]] \to \mathcal{M}/F^q\mathcal{M}, \qquad \sum_{i=-\infty}^{p} a_i\mathbb{L}^i \mapsto \sum_{i=-q}^{p} a_i\mathbb{L}^i \bmod F^q\mathcal{M}$$

induces an isomorphism $\rho : \mathbb{Z}[\mathbb{L}][[\mathbb{L}^{-1}]] \simeq \widehat{\mathcal{M}}$. In fact, for $(m_q)_{q\in\mathbb{Z}}$, $m_q \in \mathcal{M}/F^q\mathcal{M}$, so that $m_{q'} \bmod F^q\mathcal{M} = m_q$ $(q \leq q')$, the map

$$(m_q) \mapsto \sum_{q\in\mathbb{Z}} \mathbb{L}^{1-q}(\lim_{\mathbb{L}\to 0} m_q\mathbb{L}^{q-1})$$

is the inverse of ρ. The isomorphism $\beta : \mathcal{M} \to \mathbb{Z}[u, u^{-1}]$ extend to the isomorphism $\widehat{\mathcal{M}} \to \mathbb{Z}[u][[u^{-1}]]$, and we are done. $\square$

Remark 2.8. Usually these rings have been considered in the context of algebraic varieties, namely the rings $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are constructed similarly from the Grothendieck group $K_0(\mathrm{Var}_\mathbb{R})$ of real algebraic varieties. But these rings are more complicated as shown below:

- $K_0(\mathrm{Var}_\mathbb{R})$ is not a domain ([17]).
- The map $K_0(\mathrm{Var}_\mathbb{R}) \to \mathcal{M} = K_0(\mathrm{Var}_\mathbb{R})$ is not injective ([1]).
- It is not known whether $\mathcal{M} \to \widehat{\mathcal{M}} = \varprojlim_q \mathcal{M}/F^q\mathcal{M}$ is injective or not, where $F^q\mathcal{M}$ is the subgroup generated by $\{[X]\mathbb{L}^{-i} : \dim X - i \leq -q\}$.

In the paper, we prefer to state the results in $\mathcal{AS}$ category, even though many results make sense in real algebraic category.

A subset $\mathcal{A}$ of $\mathcal{L}(X)$ is called to be **sound**[1] if, for $m \gg 1$, $\pi_m(\mathcal{A})$ is a constructible subset, $\mathcal{A} = \pi_m^{-1}(\pi_m(\mathcal{A}))$, and

$$\pi_m^{m+1} : \pi_{m+1}(\mathcal{A}) \to \pi_m(\mathcal{A})$$

is a piecewise trivial fibration with fiber $\mathbb{R}^d$ where $d \le n = \dim X$. For such a set we define

$$\mu_X(\mathcal{A}) = \lim_{m\to\infty} [\pi_m(\mathcal{A})]\mathbb{L}^{-n(m+1)}.$$

Let $\mathcal{A} = \pi_{k_0}^{-1}(C)$ for a constructible subset C of $\pi_{k_0}(\mathcal{L}(X))$. Then $\dim \pi_k(\mathcal{A}) - n(k+1)$ does not depend on $k \ge k_0$. We call this integer **the virtual dimension of** $\mathcal{A}$ and denote it by $\dim \mathcal{A}$. A subset $\mathcal{B}$ of $\mathcal{L}(X)$ is said to be **measurable** if for all $q \ge 1$ there are sound sets $\mathcal{C}_q$ and $\mathcal{C}_{q,i}$ $(i \in \mathbb{N})$ such that

$$(\mathcal{B} \setminus \mathcal{C}_q) \cup (\mathcal{C}_q \setminus \mathcal{B}) \subset \bigcup_{i\in\mathbb{N}} \mathcal{C}_{q,i}$$

and $\dim \mathcal{C}_{q,i} \le -q$ for all i. For a measurable subset $\mathcal{B}$, define $\mu_X(\mathcal{B})$ by

$$\mu_X(\mathcal{B}) = \lim_{q\to\infty} \mu_X(\mathcal{C}_q).$$

Example 2.9. Let M be nonsingular real algebraic variety of dimension n, and let $E = \sum_i E_i$ be a simple normal crossing divisor of M. For $\boldsymbol{j} = (j_i)$ set

$$E_{\boldsymbol{j}}^\circ = \left(\bigcap_{i:j_i>0} E_i\right) \setminus \bigcup_{k:j_k=0} E_k, \quad s_{\boldsymbol{j}} = \sum_i j_i, \quad |\boldsymbol{j}| = \#\{i : j_k \ne 0\},$$

and $\mathcal{B}_{\boldsymbol{j}} = \{\gamma \in \mathcal{L}(M,E) : \mathrm{ord}_{E_i} \alpha = j_i \text{ for } i \text{ with } j_i > 0\}$. If $k \ge s_{\boldsymbol{j}} + 1$, then

$$\dim \pi_k \mathcal{B}_{\boldsymbol{j}} = \dim E_{\boldsymbol{j}}^\circ + nk - s_{\boldsymbol{j}}, \qquad [\pi_k \mathcal{B}_{\boldsymbol{j}}] = [E_{\boldsymbol{j}}^\circ](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{nk-s_{\boldsymbol{j}}},$$

and we have

$$\dim \mathcal{B}_{\boldsymbol{j}} = -|\boldsymbol{j}| - s_{\boldsymbol{j}}, \qquad \mu_M(\mathcal{B}_{\boldsymbol{j}}) = [E_{\boldsymbol{j}}^\circ](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{-n-s_{\boldsymbol{j}}}.$$

Example 2.10. Let $h : M \to X$ be as in the previous subsection. If $k \ge 2\langle \boldsymbol{\nu}, \boldsymbol{j}\rangle$ and $k \ge \langle \boldsymbol{\lambda}^J, \boldsymbol{j}\rangle$, we have

$$[h_{*,k}\pi_k \mathcal{B}_{\boldsymbol{j}}] = [\pi_k \mathcal{B}_{\boldsymbol{j}}]\mathbb{L}^{-e} = [E_{\boldsymbol{j}}^\circ](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{nk-s_{\boldsymbol{j}}-\langle \boldsymbol{\nu},\boldsymbol{j}\rangle}.$$

This shows $h_*\mathcal{B}_{\boldsymbol{j}}$ is also measurable, and $\mu_X(h_*\mathcal{B}_{\boldsymbol{j}}) = [E_{\boldsymbol{j}}^\circ](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{-n-s_{\boldsymbol{j}}-e}$. We set

$$\mathcal{B}_e(h) = \{\alpha \in \mathcal{L}(M) : \mathrm{ord}_\alpha \det(dh) = e\}, \text{ and } \mathcal{B}_e^{(m)}(h) = \{\alpha \in \mathcal{B}_e(h) : h \circ \alpha \in \mathcal{L}^{(m)}(X)\}.$$

Then

$$h_*\mathcal{B}_e^{(m)}(h) = \bigsqcup_{\boldsymbol{j}\in A^{(m)}:\langle \nu,\boldsymbol{j}\rangle = e} h_*\mathcal{B}_{\boldsymbol{j}}$$

[1] I feel stable is a proper word but this is already used for different meaning. Any other idea for naming? How aboiut pre-stable?

and $h_*\mathcal{B}_e^{(m)}(h)$ is measurable. In fact, we have

$$\mu_X(h_*\mathcal{B}_e^{(m)}(h)) = \mathbb{L}^{-n} \sum_{\boldsymbol{j}\in A^{(m)}:\langle \nu,\boldsymbol{j}\rangle = e} [E_{\boldsymbol{j}}^\circ](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{-s_{\boldsymbol{j}}-e}.$$

If $h : M \to X$ is a resolution of (X, Y), we have

$$\mu_X(\mathcal{L}^{(m)}(X,Y)) = \mathbb{L}^{-n} \sum_{\boldsymbol{j}\in A^{(m)}} [E_{\boldsymbol{j}}^\circ \cap h^{-1}(Y)](\mathbb{L}-1)^{|\boldsymbol{j}|}\mathbb{L}^{-s_{\boldsymbol{j}}-\langle \boldsymbol{\nu},\boldsymbol{j}\rangle}.$$

Tending m to ∞ in the right hand side, we obtain

$$\begin{aligned}
&\mathbb{L}^{-n}\sum_J [E_J^\circ \cap h^{-1}(Y)] \prod_{i\in J}(\mathbb{L}-1)\sum_{j_i=1}^{\infty}\mathbb{L}^{-j_i(\nu_i+1)}\\
=&\mathbb{L}^{-n}\sum_J [E_J^\circ \cap h^{-1}(Y)] \prod_{i\in J}(\mathbb{L}-1)\frac{\mathbb{L}^{-\nu_i-1}}{1-\mathbb{L}^{-\nu_i-1}}\\
=&\mathbb{L}^{-n}\sum_J [E_J^\circ \cap h^{-1}(Y)] \prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i+1}-1}.
\end{aligned}$$

This computation can be interpreted as a formula in $\widehat{\mathcal{M}}$ and we obtain

$$\mu_X(\mathcal{L}(X,Y)\setminus\mathcal{L}(X',Y')) = \mathbb{L}^{-n}\sum_J [E_J^\circ \cap h^{-1}(Y)] \prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i+1}-1}.$$

where $X' = X \setminus X^\circ$ and $Y' = X' \cap Y$. Here we denote by X° its regular locus of dimension n for $\mathcal{AS}$-set X of dimension n.

Remark 2.11. We can show the real version of Proposition 6.3.2 in [5]:

$$\mu_X(\mathcal{L}(X,Y)) = \mathbb{L}^{-n}\sum_J [E_J^\circ \cap h^{-1}(Y)] \prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i+1}-1}.$$

For a closed $\mathcal{AS}$-set X of dimension n, we consider the filtration:

$$X = X_0 \supset X_1 \supset \cdots \supset X_{n-1} \supset X_n, \quad \text{with } \dim X_i = n-i \text{ and } X_i^\circ = X_i \setminus X_{i+1}.$$

We apply the previous discussion for a resolution $h_i : M_i \to X_i$ of $(X_i, X_i \cap Y)$, and obtain that

$$[\pi_k(h_i)_*\mathcal{B}_e^{(m)}(h_i)]\mathbb{L}^{-(n-i)(k+1)}$$

do not depend on k for $k \geq \max\{2e, m\}$. This implies that $\mathcal{L}(X_i, X_i \cap Y) \setminus \mathcal{L}(X_{i+1}, X_{i+1} \cap Y)$ is measurable and its measure is zero in $\mathcal{L}(X, Y)$ if $i > 0$, since

$$[\pi_k h_*\mathcal{B}_e^{(m)}(h_i)]\mathbb{L}^{-n(k+1)} = \mathbb{L}^{-i(k+1)}[\pi_k(h_i)_*\mathcal{B}_e^{(m)}(h_i)]\mathbb{L}^{-(n-i)(k+1)} \to 0 \ (k \to \infty) \text{ in } \widehat{M}.$$

Remark 2.12. For a closed $\mathcal{AS}$-subset S in X such that $\dim S < \dim X$, the set $\mathcal{L}(S)$ is measurable of measure zero ([2, Proposition 6.22], [5, Appendix]). This is shown by a similar discussion to the previous remark.

2.3 Jacobian and behaviour of arcs

Theorem 2.13. *Let $f : (X, 0) \to (Y, 0)$ be a semi-algebraic homeomorphism germ between algebraic varieties X, Y with $\dim X = \dim Y$. Assume that $\mu_X(\mathcal{L}(X, 0)) = \mu_Y(\mathcal{L}(Y, 0))$. If f is generically arc-analytic and the jacobian $\mathcal{J}_f$ is bounded from below, then the invere map $f^{-1} : Y \to X$ is generically arc-analytic and the jacobian $\mathcal{J}_f$ is bounded from above.*

Under the assumption in the previous theorem, we have

Corollary 2.14. *If f is blow-Nash and the jacobian $\mathcal{J}_f$ is bounded from below, then the inverse f^{-1} is blow-Nash and the jacobian $\mathcal{J}_f$ is bounded from above.*

We remark that J.-B. Campesato ([4, Theorem 3.5]) gave this corollary when $X = Y$.

Proof. A consequence of Lemma 1.2. □

Proof of Theorem 2.13. Since f is generically arc-analytic, there exists $p : \Gamma \to X$ a composition of blow-ups so that the map $f \circ p : (\Gamma, 0) \to (Y, 0)$ is Nash by Lemma 1.2. Let $q = f \circ p : \Gamma \to Y$ denote the natural map. Let $\sigma : M \to \Gamma$ be a composition of blow ups with nonsingular algebraic centers such that

(a) M is nonsingular,
(b) $f \circ p \circ \sigma$ is analytic (thus Nash),
(c) $(\det d(p \circ \sigma))_0$, and $(\det d(q \circ \sigma))_0$ are normal crossing, and
(d) $(p \circ \sigma)^* \Omega^n_X$, $(q \circ \sigma)^* \Omega^n_Y$ are invertible, and $\mathcal{J}_{p \circ \sigma}$, $\mathcal{J}_{q \circ \sigma}$ are generated by monomials in some coordinates.

We have the following diagram:

$$\begin{tikzcd} & M \arrow[d, "\sigma"] & \\ & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Let S be an algebraic subset of X with $\dim S < \dim X$ so that $f \circ \gamma$ is analytic for any analytic map $\gamma : (-1, 1) \to X$ which is not entirely in S. We may assume that S contains the singular set of X and the critical point set of f, and $f(S)$ contains the singular set of Y. Set

$$\begin{aligned} \mathcal{L}^{\#}(X, 0) =& \{\gamma \in \mathcal{L}(X, 0) : \gamma \text{ is not entirely in } S\}, \\ \mathcal{L}^{\#}(Y, 0) =& \{\gamma \in \mathcal{L}(Y, 0) : \gamma \text{ is not entirely in } f(S)\}, \\ \mathcal{L}^{\#}(M, Z) =& \{\gamma \in \mathcal{L}(M, Z) : p \circ \sigma \circ \gamma \text{ is not entirely in } S\}, \end{aligned}$$

where $Z = (p \circ \sigma)^{-1}(0)$. By Remark 2.12, we have

$$\mu_X(\mathcal{L}^{\#}(X, 0)) = \mu_X(\mathcal{L}(X, 0)),\ \mu_Y(\mathcal{L}^{\#}(Y, 0)) = \mu_Y(\mathcal{L}(Y, 0)),\ \mu_M(\mathcal{L}^{\#}(M, Z)) = \mu_M(\mathcal{L}(M, Z)).$$

We remark that $\mathcal{L}^{\#}(X, 0) = (p \circ \sigma)_* \mathcal{L}^{\#}(M, Z)$, since f is generically arc-analytic. Moreover f^{-1} is generically arc-analytic provided $\mathcal{L}^{\#}(Y, 0) = (q \circ \sigma)_* \mathcal{L}^{\#}(M, Z)$.

We set

$$\mathcal{B}_{e,e'} = \mathcal{B}_e(p \circ \sigma) \cap \mathcal{B}_{e'}(q \circ \sigma) \cap \mathcal{L}(M, Z).$$

The jacobian $\mathcal{J}_f$ is bounded from below, if and only if $\mathcal{B}_{e,e'} = \emptyset$ $(e < e')$. We also remark that the jacobian $\mathcal{J}_f$ is bounded from above, if and only if $\mathcal{B}_{e,e'} = \emptyset$ $(e > e')$. It is thus enough to show the following implication:

$$\begin{aligned} &\mu_X(\mathcal{L}^\#(X,0) \smallsetminus (p \circ \sigma)_* \mathcal{L}^\#(M,Z)) = 0,\ \mathcal{B}_{e,e'} = \emptyset\ (e < e') \\ &\qquad \Longrightarrow\ \mu_Y(\mathcal{L}^\#(Y,0) \smallsetminus (q \circ \sigma)_* \mathcal{L}^\#(M,Z)) = 0,\ \mathcal{B}_{e,e'} = \emptyset\ (e > e'). \quad (*) \end{aligned}$$

Remark that $\mu_X(\mathcal{L}(X,0) \smallsetminus (p \circ \sigma)_* \mathcal{L}^\#(M,Z)) = 0$ if f is blow-Nash. We also see that $\mu_Y(\mathcal{L}^\#(Y,0) \smallsetminus (q \circ \sigma)_* \mathcal{L}^\#(M,Z)) = 0$ only if f^{-1} is blow-Nash, i.e., has generic arc-lifting property. To see the "only if" implication, we take a Nash rectilinearization $\psi : \widetilde{Y} \to Y$ of the product of all components g_i of f^{-1}, by [15, Theorem 2.7]. Then $g_i \circ \psi(\varepsilon_1 x_1^{r_1}, \ldots, \varepsilon_n x_n^{r_n})$ are normal crossing in suitable patches and $r_i \in \mathbb{Q}$, $r_i > 0$, $\varepsilon_i = \pm 1$. If the rectilinearizations of g_i are not analytic, then there is a set of arcs of strictly positive measure whose image by f^{-1} is not analytic. Then the image $\mathcal{A}$ of this set has a non-zero measure in $\mathcal{L}^\#(Y,0)$ and we conclude that there is an arc $\gamma \in ((q \circ \sigma)_* \mathcal{L}^\#(M,Z)) \cap \mathcal{A}$. By the definition of $\mathcal{A}$, γ does not have an analytic lift on X. But γ has an analytic lift on X, since it is in $(q \circ \sigma)_* \mathcal{L}^\#(M,Z)$, and a contradiction.

Remark that $\mathcal{B}_{e,e'} = \bigcup_{\boldsymbol{j}} \mathcal{B}_{\boldsymbol{j}} : \langle \boldsymbol{j}, \boldsymbol{\nu} \rangle = e, \langle \boldsymbol{j}, \boldsymbol{\nu}' \rangle = e'$. By Corollary 2.4 and Example 2.10, we have, for $k \geq 2\max\{e, e'\}$,

$$[\pi_k \mathcal{B}_{e,e'}] = [(p \circ \sigma)_{*.k} \pi_k \mathcal{B}_{e,e'}] \mathbb{L}^e, \qquad [\pi_k \mathcal{B}_{e,e'}] = [(q \circ \sigma)_{*,k} \pi_k \mathcal{B}_{e,e'}] \mathbb{L}^{e'},$$

and thus

$$\mu_M(\mathcal{B}_{e,e'}) = \mu_X((p \circ \sigma)_* \mathcal{B}_{e,e'}) \mathbb{L}^e, \qquad \mu_M(\mathcal{B}_{e,e'}) = \mu_Y((q \circ \sigma)_* \mathcal{B}_{e,e'}) \mathbb{L}^{e'}.$$

Therefore we have

$$\begin{aligned} \mu_X((p \circ \sigma)_* \mathcal{L}^\#(M,Z)) &= \sum_{e,e'} \mu_X((p \circ \sigma)_* \mathcal{B}_{e,e'}) = \sum_{e,e'} \mu_M(\mathcal{B}_{e,e'}) \mathbb{L}^{-e}, \\ \mu_Y((q \circ \sigma)_* \mathcal{L}^\#(M,Z)) &= \sum_{e,e'} \mu_Y((q \circ \sigma)_* \mathcal{B}_{e,e'}) = \sum_{e,e'} \mu_M(\mathcal{B}_{e,e'}) \mathbb{L}^{-e'}. \end{aligned}$$

In other words,

$$\mu_X(\mathcal{L}^\#(X,0)) = \mu_X(\mathcal{L}^\#(X,0) \smallsetminus (p \circ \sigma)_* \mathcal{L}^\#(M,Z)) + \sum_{e,e'} \mu_M(\mathcal{B}_{e,e'}) \mathbb{L}^{-e}, \tag{3}$$

$$\mu_Y(\mathcal{L}^\#(Y,0)) = \mu_Y(\mathcal{L}^\#(Y,0) \smallsetminus (q \circ \sigma)_* \mathcal{L}^\#(M,Z)) + \sum_{e,e'} \mu_M(\mathcal{B}_{e,e'}) \mathbb{L}^{-e'}. \tag{4}$$

Therefore, if $\mu(\mathcal{L}^\#(X,0)) = \mu(\mathcal{L}^\#(Y,0))$ and $\mathcal{B}_{e,e'} = \emptyset$ $(e < e')$, we have

$$\begin{aligned} &\mu_X(\mathcal{L}^\#(X,0) \smallsetminus (p \circ \sigma)_* \mathcal{L}^\#(M,Z)) \\ =& \mu_Y(\mathcal{L}^\#(Y,0) \smallsetminus (q \circ \sigma)_* \mathcal{L}^\#(M,Z)) + \sum_{e > e'} \mu_M(\mathcal{B}_{e,e'}) \mathbb{L}^{-e'}(1 - \mathbb{L}^{e'-e}). \end{aligned}$$

Now we replace the isomorphism classes $[X]$ by the virtual Poincaré polynomials $\beta(X)$ in the definition of measure μ. We remark that the coefficient of the top degree term of the left hand side is positive whenever $\mu_X(\mathcal{L}^*(X,0)\setminus(p\circ\sigma)_*\mathcal{L}^\#(M)) \neq 0$. Similarly the coefficients of the top degree term of the each term in the right hand side is positive whenever $\mu_Y(\mathcal{L}^\#(Y,0)\setminus(q\circ\sigma)_*\mathcal{L}^\#(M,Z)) \neq 0$ or $\mu_M(\mathcal{B}_{e,e'}) \neq 0$ for some e and e'. So we claim the implication $(*)$. □

We say $\mu_X(\mathcal{A}) \leq \mu_X(\mathcal{B})$ if the top degree term of $\mu_X(\mathcal{B}) - \mu_X(\mathcal{A})$ has a positive coefficient.

Theorem 2.15. *Let $f : (X,0) \to (Y,0)$ be an $\mathcal{AS}$-homeomorphism between $\mathcal{AS}$-closed sets X and Y with $\dim X = \dim Y$.*

- *If f is generically arc-analytic and $\mathcal{J}_f$ is bounded from below, then $\mu_X(\mathcal{L}(X,0)) \leq \mu_Y(\mathcal{L}(Y,0))$.*
- *If f^{-1} is generically arc-analytic and $\mathcal{J}_f$ is bounded (from above), then $\mu_X(\mathcal{L}(X,0)) \geq \mu_Y(\mathcal{L}(Y,0))$.*

Proof. We use the notation of the previous proof. First we assume that f is generically arc-analytic, and $\mathcal{J}_f$ is bounded from below. Then $\mathcal{L}^\#(X,0) = (p\circ\sigma)_*\mathcal{L}^\#(M,Z)$ and $\mathcal{B}_{e,e'} = \emptyset$ $(e < e')$. Subtracting (3) from (4), we thus have

$$\begin{aligned}\mu_Y(\mathcal{L}(Y,0)) - \mu_X(\mathcal{L}(X,0)) &= \mu_Y(\mathcal{L}^\#(Y,0)) - \mu_X(\mathcal{L}^\#(X,0)) \\ &= \mu_Y(\mathcal{L}^\#(Y,0)\setminus(q\circ\sigma)_*\mathcal{L}^\#(M,Z)) + \sum_{e>e'} \mu_M(\mathcal{B}_{e,e'})\mathbb{L}^{-e'}(1-\mathbb{L}^{e'-e})\end{aligned}$$

After replacing $[-]$ by $\beta(-)$, the right-hand side has a positive coefficient in the top degree term. So we have the first assertion.

Secondly we assume f^{-1} is generically arc-analytic, and $\mathcal{J}_f$ is bounded from (above). Then $\mathcal{L}^\#(Y,0) = (q\circ\sigma)_*\mathcal{L}^\#(M,Z)$ and $\mathcal{B}_{e,e'} = \emptyset$ $(e > e')$. Subtracting (4) from (3), we thus have

$$\begin{aligned}\mu_X(\mathcal{L}(X,0)) - \mu_Y(\mathcal{L}(Y,0)) =& \mu_X(\mathcal{L}^\#(X,0)) - \mu_Y(\mathcal{L}^\#(Y,0)) \\ =& \mu_X(\mathcal{L}^\#(X,0)\setminus(p\circ\sigma)_*\mathcal{L}^\#(M,Z)) + \sum_{e<e'} \mu_M(\mathcal{B}_{e,e'})(1-\mathbb{L}^{e-e'}).\end{aligned}$$

After replacing $[-]$ by $\beta(-)$, the right-hand side has a positive coefficient in the top degree term. So we have the second assertion. □

Corollary 2.16. *Assume that a semi-algebraic homeomorphism $f : (X,0) \to (Y,0)$ is generically arc-analytic and f^{-1} is also generically arc-analytic. If the jacobian $\mathcal{J}_f$ is bounded from above and below, then $\mu_X(\mathcal{L}(X,0)) = \mu_Y(\mathcal{L}(Y,0))$.*

3 Lipschitz property

3.1 Lipschitz property and differentials

Let X be a semi-algebraic connected subset of $\mathbb{R}^N$. Let d_X denote the inner distance of X. Thus by definition $d_X(p,q)$ is the infimum over the length of subanalytic curves of X joining p and q.

Let X and Y be locally closed semi-algebraic subsets of $\mathbb{R}^N$ and $\mathbb{R}^{N'}$, respectively. We assume that $0 \in X$ and $0 \in Y$. A semi-algebraic map germ $f : (X, 0) \to (Y, 0)$ is said to be **inner-Lipschitz** if there is a neighbourhood U of 0 in X and a positive constant L so that

$$d_Y(f(x), f(x')) \leq L d_X(x, x') \qquad \forall x, x' \in U.$$

We simply say that f is Lipschitz, when $X = \mathbb{R}^n$, $Y = \mathbb{R}^n$ and d_X and d_Y denote the Euclidean distance.

We denote by $\overline{X^\circ}$ the closure of X° with respect to the Euclidean topology of the ambient Euclidean soace .

Let $f : (X, 0) \to (Y, 0)$ be a semi-algebraic map between real algebraic sets X and Y. Set

$$\begin{aligned} Z =&(X \smallsetminus X^\circ) \cup f^{-1}(Y \smallsetminus Y^\circ) \cup B(f) \cup C(f) \text{ where} \\ B(f) =&\{x \in X^\circ : f \text{ is not analytic at } x\}, \text{ and} \\ C(f) =&\{x \in X^\circ \smallsetminus B(f) : df_x : T_x X \to T_{f(x)} Y \text{ is not an isomorphism}\}. \end{aligned}$$

For $x \in X \smallsetminus Z$ and a unit vector $v \in T_x X$, we can define the directional derivative $D_v f$ by

$$D_v f(x) = \lim_{s \to 0} \frac{f \circ \alpha(s) - f \circ \alpha(0)}{s}$$

where $\alpha(s)$ is an arc so that $x = \alpha(0)$ and $v = \alpha'(0)$.

Lemma 3.1. *The map $f|_{\overline{X^\circ}}$ is inner-Lipschitz if and only if the set*

$$\{|D_v f(x)| : x \in X \cap U \smallsetminus Z, \ v \in T_x X, |v| = 1\},$$

where U is a neighbourhood of 0 in X, is bounded.

Proof. Let $\alpha(s)$ denote a curve so that $x = \alpha(0)$ and $v = \lim_{s \to 0} \alpha'(s)$. Let s be the arc length parameter of α.

If $f|_{\overline{X^\circ}}$ is inner-Lipschitz and $f \circ \alpha(s)$ attains the length between $f \circ \alpha(s)$ and $f \circ \alpha(0)$, then

$$|f \circ \alpha(s) - f \circ \alpha(0))| \leq d_Y(f \circ \alpha(s), f \circ \alpha(0)) \leq L d_X(\alpha(s), \alpha(0)) \leq Ls.$$

Then we have

$$|D_v f(x)| = \lim_{s \to 0} \left| \frac{f \circ \alpha(s) - f \circ \alpha(0)}{s} \right| \leq L.$$

Assume that $|D_v f(x)| \leq L$ for $x \in X \smallsetminus Z$, $v \in T_x X$, $|v| = 1$. For any $\varepsilon > 0$ there is $\delta > 0$ such that

$$0 < s < \delta \Longrightarrow \frac{|f \circ \alpha(s) - f \circ \alpha(0)|}{s} \leq L + \varepsilon.$$

Since $|f \circ \alpha(s) - f \circ \alpha(0)| \leq t(s)$ where $t(s) = \int_0^s |(f \circ \alpha)'(s)| ds$, and $\lim_{s \to 0} \left| \frac{f \circ \alpha(s) - f \circ \alpha(0)}{t(s)} \right| = 1$, we can choose $\delta > 0$ so that

$$0 < s < \delta \Longrightarrow (1 - \varepsilon) t(s) < |f \circ \alpha(s) - f \circ \alpha(0)| \leq t(s).$$

Set $U_\delta = \{(x, x') \in \overline{X^\circ} \times \overline{X^\circ} : d_X(x, x') \leq \delta\}$. Assume that $(x, x') \in U_\delta$. If $\alpha(s)$ attains the length between $x' = \alpha(s_1) \in X^\circ$, for some $s_1 > 0$, and $x = \alpha(0) \in X^\circ$, then

$$
\begin{aligned}
d_Y(f \circ \alpha(s), f \circ \alpha(0)) &\leq \frac{1}{1-\varepsilon} |f \circ \alpha(s) - f \circ \alpha(0)| \\
&= \frac{1}{1-\varepsilon} \frac{|f \circ \alpha(s) - f \circ \alpha(0)|}{s} d_X(\alpha(s), \alpha(0)) \leq \frac{L+\varepsilon}{1-\varepsilon} d_X(\alpha(s), \alpha(0)).
\end{aligned}
$$

whenever $0 < s < \delta$. We may assume that $(x, x') \in U \times U$ where U is a neighbourhood of 0 in $\overline{X^\circ}$ so that $\overline{U}$ is compact. Then $d_Y(f(x), f(x'))/d_X(x, x')$ is bounded on $U \times U \setminus U_\delta$, and we are done. $\square$

3.2 Inner-Lipschitz maps and an arc-analytic inverse mapping

Theorem 3.2. *Let X and Y be algebraic sets with $\mu_X(\mathcal{L}(X, 0)) = \mu_Y(\mathcal{L}(Y, 0))$. If a semi-algebraic homeomorphism $f : (X, 0) \to (Y, 0)$ is generically arc-analytic and $f^{-1}|_{\overline{Y^\circ}}$ is inner-Lipschitz, then $f|_{\overline{X^\circ}}$ is inner-Lipschitz and f^{-1} is generically arc-analytic.*

Proof. We use the notation in the proof of Theorem 2.13. We assume that

$$
(\mathcal{J}_{p\circ\sigma})_0 = \sum_i \nu_i E_i, \quad (\mathcal{J}_{q\circ\sigma})_0 = \sum_i \nu_i' E_i.
$$

Since $f^{-1}|_{\overline{Y^\circ}}$ is inner-Lipschitz, we have

$$
\{|D_w f^{-1}| : w \in Y \setminus W,\ w \in T_y Y,\ |w| = 1\}
$$

is bounded where $W = f(X \setminus X^\circ) \cup (Y \setminus Y^\circ) \cup B(f^{-1}) \cup C(f^{-1})$. Let U denote a coordinate chart in M and $z : U \to \mathbb{R}^n$ denote the coordinate function. Then there exist local coordinate systems $x_I = (x_i)_{i\in I}$ for X and $y_J = (y_j)_{j\in J}$ for Y so that

$$
\left((p\circ\sigma)^* \det \frac{\partial x_I}{\partial z}\right)_0 = \sum_i \nu_i E_i, \quad \left((q\circ\sigma)^* \det \frac{\partial y_J}{\partial z}\right)_0 = \sum_i \nu_i' E_i \quad \text{on } U.
$$

Since the Jacobi matrix $\frac{\partial x_I}{\partial y_J}$ is bounded, we have $\nu_i' \leq \nu_i$. By Corollary 2.14, we have $\nu_i = \nu_i'$ and $\det \frac{\partial x_I}{\partial y_J}$ is a unit. Thus the Jacobi matrix

$$
\frac{\partial y_J}{\partial x_I} = \frac{1}{\det(\frac{\partial x_I}{\partial y_J})} \text{ cofactor matrix of } \frac{\partial x_I}{\partial y_J}
$$

is bounded and all directional derivative

$$
\{|D_v f| : x \in X \setminus Z,\ v \in T_x X,\ |v| = 1\}
$$

is bounded. Here Z is the set where f is not directional derivative. $\square$

Corollary 3.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semi-algebraic homeomorphism. Then the following conditions are equivalent.*

- *f is arc-analytic and f^{-1} is Lipschitz.*
- *f^{-1} is arc-analytic and f is Lipschitz.*

Remark 3.4. M. Kobayashi and T.-C. Kuo ([12]) constructed a map $h : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:

- h sends a cusp $\{x^2 = y^3\}$ to a line $\{x = 0\}$.
- h and h^{-1} are blow-analytic, and thus arc-analytic.

As pointed out in [6, Example 8.2], h and h^{-1} are not Lipschitz.

4 Complex case

There are natural complex analogues for the notion of Jacobian $\mathcal{J}_f$ and we have complex versions of Theorem 2.2 and Corollary 2.4 and thus Theorem 2.13 as consequences. We state them here, since we do not think they are trivial facts.

4.1 Hodge-Deligne polynomial

For a complex algebraic variety X, we consider the Hodge-Deligne polynomial

$$E(X; u, v) = \sum_{p,q} \sum_{i=0}^{2\dim X} (-1)^i h^{p,q}(H_c^i(X, \mathbb{C})) u^p v^q.$$

where $H_c^i(X, \mathbb{C})$ is the ith cohomology with compact support with coefficient $\mathbb{C}$. We may use the Hodge-Deligne polynomial instead of the virtial Poincaré polynomial it satisfies the following properties:

- When X is nonsingular and projective, we have
$$E(X; u, v) = \sum_{p,q} \sum_{i=0}^{2\dim X} (-1)^i h^{p,q}(X) u^p v^q.$$
- If $X = \sqcup_i X_i$ is a disjoint union of Zariski locally closed strata, then
$$E(X; u, v) = \sum_i E(X_i; u, v)$$
- The Hodge-Deligne polynomial is multiplicative, i.e., $E(X \times Y) = E(X)E(Y)$.
- If $f : Y \to X$ is a Zariski locally trivial fibration and F is the fibre over a closed point, then $E(Y) = E(X)E(F)$.

4.2 Weakly holomorphic map and arc lifting property

A map $f : X \to Y$ is said to be **weakly holomorphic** if it is bounded and holomorphic on the regular locus of X. If $h : \widetilde{X} \to X$ is a desingularization of X, then $f \circ h$ is a bounded holomorphic map except the exceptional locus which is of codimension at least 1 for any weakly holomorphic map f. So $f \circ h$ has a holomorphic extension by Riemann's extension theorem. If X is normal, then a weakly holomorphic map is holomorphic.

Theorem 4.1. *Let X and Y be complex algebraic varieties with $\mu_X(\mathcal{L}(X)) = \mu_Y(\mathcal{L}(Y))$. Let $f : X \to Y$ be a continuous map, which is weakly holomorphic. If the jacobian $\mathcal{J}_f$ is bounded from below, then $\mathcal{J}_f$ is bounded and f has generically arc lifting property.*

Theorem 4.2. *Let (X, x) and (Y, y) be germs of complex algebraic varieties with $\mu_X(\mathcal{L}(X, x)) = \mu_Y(\mathcal{L}(Y, y))$. Let $f : (X, x) \to (Y, y)$ be a continuous map, which is weakly holomorphic. If the jacobian $\mathcal{J}_f$ is bounded from below at x, then $\mathcal{J}_f$ is bounded and f has generically arc lifting property.*

It would be worth to add a complex version of Theorem 3.2 also.

Theorem 4.3. *Let X and Y be complex algebraic sets with $\mu_X(\mathcal{L}(X, 0)) = \mu_Y(\mathcal{L}(Y, 0))$. If a semi-algebraic homeomorphism $f : (X, 0) \to (Y, 0)$ is weakly holomorphic and f^{-1} is inner-Lipschitz, then f is inner-Lipschitz and f^{-1} is weakly holomorphic.*

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