Bifurcation Model from Initial Nonlinear Term of Nonlinear Equations

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Abstract

The bifurcation model from the initial nonlinear term of nonlinear equation is introduced. Under the non-degeneracy condition (Definition 3.2), our bifurcation model describes the bifurcation of solutions to the nonlinear equation. We also show how these models work for Dirichlet problem on the square. We observe a perturbation of rectangles to a square creates new bifurcation, which is not a limit of the bifurcations on rectangles.

1 Introduction

Let $L : X \to X$ be a linear continuous operator of a Banach space $X$. In this paper, we investigate the bifurcation of solutions of the nonlinear equation

$$\Phi(\lambda, u) = Lu - \lambda u + H(\lambda, u) = 0, \ u \in X,$$

where $h(\lambda, u) \in C^\infty(\mathbb{R} \times X, X)$, $h(\lambda, 0) = 0$, $h_u(\lambda, 0) = 0$. We call $(\lambda^*, 0)$ a bifurcation point, if for any neighborhood $U$ of $(\lambda^*, 0)$, there exists $(\lambda, u) \in U$ so that $\Phi(\lambda, u) = 0$, $u \neq 0$. It is well-known that if $(\lambda^*, 0)$ is a bifurcation point, then $\lambda^*$ is an eigenvalue of $L$, i.e., $V_{\lambda^*} = \text{Ker}(L - \lambda^* I) \neq 0$. Set $m = \text{dim}_{\mathbb{R}} V_{\lambda^*}$.

If $m = 1$, and $H(\lambda, u) = a_k(\lambda)u^k + o(u^k)$, $a(\lambda^*) \neq 0$, then the bifurcation is described by

$$(\lambda^* - \lambda)u + au^k = 0, \ a = a(\lambda^*),$$

and the bifurcation of solutions is decided by $k$ and $a$, as shown in the following figures.

- Transcritical bifurcation ($k$ is even).
- Subcritical bifurcation ($k$ is odd, $a < 0$).
- Supercritical bifurcation ($k$ is odd, $a > 0$).

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A motivation of this paper is to generalize this phenomenon to the case that $m$ is finite.

To conclude the model (1.2), one uses Lyapunov-Schmidt reduction to reduce the problem to the bifurcation equation in a finite dimension vector space, which works well whenever $m = \dim_{\mathbb{R}} V_{\lambda^*} < \infty$. Since this process requires the implicit function theorem, the bifurcation equation contains implicit functions, which we do not know their properties. This causes several difficulty to investigate the bifurcation in the case that $m > 1$. So the strategy is to reduce the bifurcation equation to certain normal forms. Assuming that $H(\lambda, u) = a_k(\lambda)u^k + o(u^k)$, $a_k(\lambda^*) \neq 0$, we introduce the bifurcation model (Definition 3.1), which is determined by the initial nonlinear term $a_k(\lambda)u^k$, and we show in Theorem 3.3 that it is equivalent to the bifurcation equations of Lyapunov-Schmidt reduction whenever the equation (1.1) is non-degenerate (Definition 3.2). Our bifurcation model is a weighted homogeneous system and the weights are determined by $m$ and $k$.

As an application, we investigate the bifurcation from the trivial solution of Dirichlet problem:

$$\Delta u = -\mu u + h(\lambda, u) \text{ on } \Omega, \quad u|_{\partial\Omega} = 0,$$

(1.3)

where $\Delta$ is Laplacian and $\Omega = [0, \pi]^2$. Setting $X = \{u \in H_0^2(\Omega) : u|_{\partial\Omega} = 0\}$ and $L$ the inverse of $-\Delta$, $\lambda = \mu^{-1}$, $H(\lambda, u) = \lambda K(\lambda^{-1}, u)$, we have

$$Lu - \lambda u + H(\lambda, u) = 0,$$

and

$$\dim_{\mathbb{R}} V_{\lambda^*} = \#\{(a, b) \in \mathbb{N}^2 : 1/(a^2 + b^2) = \lambda^*\},$$

(1.4)

where $\mathbb{N}$ denotes the set of positive integers.

$$H(\lambda^*, 0) = \lambda K(h((\lambda^*)^{-1}, 0)) = 0$$

$$H_u(\lambda^*, 0) = \lambda K(h_u((\lambda^*)^{-1}, 0)) = 0$$

$$H_{uu}(\lambda^*, 0) = \lambda K(h_{uu}((\lambda^*)^{-1}, 0)) = 0 \quad (i = 1, \ldots, k - 1)$$

$$H_{uu^k}(\lambda^*, 0) = \lambda K(h_{uu^k}((\lambda^*)^{-1}, 0)) = \lambda k! K(a_k^2((\lambda^*)^{-1}))$$

$$-\Delta(u^3) = -(au^3)_{xx} - (au^3)_{yy} = -(a_xu^3 + 3au^2u_x)x - (a_yu^3 + 3au^2u_y)y$$

$$= -(a_{xx} + b_{yy})u^3 - 6u^2(a_xu_x + a_yu_y) - 3au^2(u_{xx} + u_{yy}) - 6u((u_x)^2 + (u_y)^2)$$

Applying $L$

$$au^3 = -L[(a_{xx} + b_{yy})u^3] - 6L[u^2(a_xu_x + a_yu_y)] - 3L[au^2(u_{xx} + u_{yy})] - 6L[u((u_x)^2 + (u_y)^2)]$$

Assuming that $h(\lambda, u) = a_k(\lambda)u^k + o(u^k)$, $a_k(\lambda^*) \neq 0$, we see that

$$H(\lambda, \mu) = \lambda K(a_k(\lambda)u^k) + K(o(u^k)),$$

We will see that our bifurcation models work well in many cases. For example,
When \((m, k) = (2, 2)\), and \(ab \not\equiv 0 \mod 2\), we have pluritranscritical bifurcation of type \((4, 4)\). When \(ab \equiv 0 \mod 2\), our bifurcation model does not work, since it is degenerate. These are stated in Theorem 5.1. 

When \((m, k) = (2, 3)\), we have plurisupercritical (or plurisubcritical) bifurcation (Theorem 5.2). 

When \((m, k) = (2, 4)\), and \(ab \not\equiv 0 \mod 2\), the bifurcation there are stated in Remark 5.3 when \(\lambda^*\) is small. When \(ab \equiv 0 \mod 2\), our bifurcation model does not work, since it is degenerate. 

When \((m, k) = (2, 5)\), we have plurisupercritical (or plurisubcritical) bifurcation (Theorem 5.4). 

The first eigenvalue \(\lambda^*\) with \(m = 3\) is 50 and the bifurcation there are stated in Remark 5.5.

The paper is organized as follows. In section 2, we recall several basic materials (inverse function theorem, implicit function theorem, and Schauder bases) and Lyapunov-Schmidt reduction in terms of Schauder basis. In section 3, the bifurcation model from the initial nonlinear term of nonlinear equations is defined. In section 4, an equivalent condition of non-degeneracy is stated and the main theorem is proved. In section 5, we show how our method works on the square. In section 6, when the rectangles degenerate to square, we observe that there are new bifurcations on the square, which are not the limits of bifurcations on rectangles.

2 Preliminary

We recall several basic theorems that we need in the paper later on.

2.1 Inverse function theorem and implicit function theorem

Let \(F\) be a map \(X \rightarrow Y\) between Banach spaces \(X, Y\). We say that \(F\) is (Fréchet) differentiable at \(u \in X\) if there exists a linear continuous map \(L_u : X \rightarrow Y\) such that

\[ F(u + v) - F(u) = L_u[v] + o(||v||), \text{ as } ||v|| \rightarrow 0. \]

When \(F\) is Fréchet differentiable at \(u \in X\), the map \(L_u\) is uniquely determined by \(F\) and \(u\) and is denoted by \(dF(u)\), \(d_uF(u)\), \(F_u(u)\) or \(F'(u)\). It is easy to see that if \(F\) is Fréchet differentiable, then it is also differentiable along any direction. Conversely, if \(F\) is differentiable along any directions, \(L_u \in L(X, Y)\) and the map \(u \mapsto L_u\) is a continuous map from \(X\) to \(L(X, Y)\), then \(F\) is Fréchet differentiable [2, 3].

**Lemma 2.1** (Inverse function theorem (Theorem 3.1.1 in [3], [5])). Let \(P : U \rightarrow V\) be a smooth map between Banach spaces, \(U, V\) are open sets of \(X, Y\), respectively. Suppose that for some \(f_0 \in U\) the derivative \(dP(f_0) : X \rightarrow Y\) is an invertible linear map. Then we can find neighborhoods \(\hat{U}\) of \(f_0\) and \(\hat{V}\) of \(g_0 = P(f_0)\) such that the map \(P\) gives a one-to-one map of \(\hat{U}\) onto \(\hat{V}\), and the inverse map \(P^{-1} : \hat{V} \subseteq Y \rightarrow \hat{U} \subseteq X\) is smooth.

\[ F(u + v) - F(u) = L_u[v] + o(||v||), \text{ as } ||v|| \rightarrow 0. \]
Lemma 2.2 (Implicit function theorem (Theorem 3.2.1 in [3], [2, 5])). Let $X, Y$ be Banach spaces and fix $(\lambda_0, u_0) \in \mathbb{R}^n \times X$. Assume that $F$ is a $C^1$ map from a neighborhood of $(\lambda_0, u_0)$ in $\mathbb{R}^n \times X$ into $Y$ such that $F(\lambda_0, u_0) = 0$ and suppose that $d_u F(\lambda_0, u_0)$ is invertible. Then there exist a neighborhood $\Lambda$ of $\lambda_0$ and a neighborhood $U$ of $u_0$ such that the equation $F(\lambda, u) = 0$ has a unique solution $u = u(\lambda) \in U$ for all $\lambda \in \Lambda$. The function $u(\lambda)$ is of class $C^1$, and the following holds

$$ u'(\lambda_0) = -[d_u F(\lambda_0, u_0)]^{-1}d_\lambda F(\lambda_0, u_0). $$

2.2 Schauder basis

A sequence $\{x_n\}$ of elements of a Banach space $X$ is said to be a Schauder basis for $X$ if for every $x$ of $X$ there is a unique sequence of numbers $\{a_n\}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$ in the sense that $\lim_{n \to \infty} \|x - \sum_{i=1}^{n} a_i x_i\| = 0$ (see [7, 9]).

- Every orthonormal basis in a separable Hilbert spaces is a Schauder basis (see [8, Example on the Page 134], [6, Theorem 1]).
- Let $E, F$ be two Banach spaces with Schauder bases $\{x_n\}, \{y_n\}$, respectively. Then the system of all products $x_i \otimes y_j$ is a Schauder basis of $E \otimes F$ (see [9, Theorem 18.1]).

2.3 Lyapunov-Schmidt reduction

Suppose that $X = V \oplus W$, where $V = \ker(L - \lambda^* I) = \operatorname{span}\{v_1, v_2, \ldots, v_m\}$, $W$ is the closure of $\operatorname{span}\{w_1, w_2, \ldots\}$. We assume that the sequence $\{v_1, \ldots, v_m, w_1, w_2, \ldots\}$ is a Schauder basis of $X$. For any $u \in X$, $u$ can be expressed as

$$ u = \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j, $$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $y = (y_1, y_2, \ldots) \in U \subset \mathbb{R}^\infty$, $U$ is an open neighborhood of 0. Then we have that

$$ \Phi(\lambda, u) = L\left(\sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j\right) - \lambda \left(\sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j\right) + h\left(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j\right) $$
$$ = \sum_{i=1}^{m} (\lambda^* - \lambda) x_i v_i + \sum_{j=1}^{\infty} (\lambda_j - \lambda) y_j w_j + h\left(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j\right). $$

We choose $v_i^* \in V^*$ and $w_j^* \in W^*$ such that $v_i^* v_s = \delta_{is}$, $w_t^* w_t = \delta_{jt}$, $v_i^* w_j = w_j^* v_i = 0$ where

$$ \delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}, \quad 1 \leq j, s \leq m, \quad 1 \leq j, t \leq \infty. $$
Let $p_X$ denote the projection
\[ p_X : X \to \mathbb{R}^m \times U, \quad u \mapsto (u^*u, w_j u)_{i=1,\ldots,m; j=1,2,\ldots}, \]
and $\iota_X$ denote the injection
\[ \iota_X : p_X(X) \to X, \quad (x, y) \mapsto \sum_{i=1}^m x_i v_i + \sum_{j=1}^\infty y_j w_j. \]
We remark that $p_X \circ \iota_X$ and $\iota_X \circ p_X$ are the identities. Then we define $\hat{\Phi}$ by $\hat{\Phi} = p_Y \circ \Phi \circ \iota_X$, and have the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{R} \times X & \xrightarrow{\Phi} & X \\
\downarrow p_X & & \downarrow p_Y \\
\mathbb{R} \times \mathbb{R}^m \times U & \xrightarrow{\hat{\Phi}} & \mathbb{R}^m \times \mathbb{R}^\infty
\end{array}
\]
The zero set of the function
\[ \hat{\Phi}(\lambda, x, y) = ((\lambda^* - \lambda)x_p + h_p, (\lambda_j - \lambda)y_j + h_q)_{p=1,\ldots,m; q=1,2,\ldots} \tag{2.1} \]
has the same bifurcations at $(\lambda^*, 0)$ as those of $\Phi(\lambda, u)=0$, where
\[
\begin{align*}
\hat{\Phi}_\lambda &= -(x_p, y_q)_{p=1,\ldots,m; q=1,2,\ldots}, \\
\hat{\Phi}_{x_i} &= (\delta_{p,i}(\lambda^* - \lambda) + (h_p)_{x_i}, (h_q)_{x_i})_{p=1,\ldots,m; q=1,2,\ldots}, \\
\hat{\Phi}_{y_j} &= ((h_p)y_j, \delta_{q,j}(\lambda_j - \lambda) + (h_q)y_j)_{p=1,\ldots,m; q=1,2,\ldots},
\end{align*}
\]
Since $\lambda^*$ is an eigenvalue of $L$, $\lambda_j \neq \lambda^*$, the component $\lambda_j - \lambda$ of (2.1) is non-zero at $(\lambda^*, 0)$, and $\hat{\Phi}_y(\lambda^*, 0)$ is invertible. By implicit function theorem, there exists a unique map
\[ \varphi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^\infty, \quad (\lambda, x) \mapsto (\varphi_j(\lambda, x))_{j=1,2,\ldots} \]
such that $\hat{\Phi}(\lambda, x, \varphi(\lambda, x)) = 0$. Since $(\varphi_j)_{\lambda}(\lambda^*, 0) = 0$, $(\varphi_j)_{x_i}(\lambda^*, 0) = 0$, we have
\[ \varphi_j(\lambda, x) = o(\lambda - \lambda^*, x). \]
Let $\hat{F} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a map defined by
\[
\hat{F}(\lambda, x) = \left((\lambda^* - \lambda)x_p + v^*_p h(\lambda, \sum_{i=1}^m x_i v_i + \sum_{j=1}^\infty \varphi_j(\lambda, x) w_j)\right)_{p=1,\ldots,m}.
\]
By Lyapunov-Schmidt reduction, $\hat{F}(\lambda, x) = 0$ is the bifurcation equation of $\hat{\Phi}(\lambda, x, y) = 0$. 

5
3 Bifurcation model from the initial nonlinear term

In this section, we are going to establish a bifurcation model for the nonlinear equation (1.1). We assume the following conditions:

(A1) \( \dim \mathbb{R}^V = m \).
(A2) \( h(\lambda, u) = a_k(\lambda)u^k + o(u^k), \ a_k(\lambda^*) \neq 0 \).
(A3) Let \( v_i \) be the basis of \( V \), and \( w_j \) be the eigenfunctions of \( L \) with eigenvalues \( \lambda_j \neq \lambda^* \), then \( \{v_i, w_j\} \) form a Schauder basis of \( X \), where \( 1 \leq i \leq m, 1 \leq j < \infty \).
(A4) There exists a linear function \( \phi : X \to \mathbb{R} \), such that \( v_x = \phi(vx), \ v \in V^*, \ x \in X \).

If \( X \) is a subspace of \( L^2 \)-space \( L^2(\Omega) \) where \( \Omega \) is a domain, then condition (A4) is satisfied for \( \phi(u) = \int_\Omega u \).

Definition 3.1 (Bifurcation model). Set \( F_i = (\lambda^* - \lambda)x_i + H_{x_i} \ (i = 1, \ldots, m) \) where
\[
H = \frac{a_k(\lambda^*)}{k+1} \phi((x_1v_1 + \cdots + x_mv_m)^{k+1})
\]
We say the set \( Z \) defined by \( F_i = 0 \ (i = 1, \ldots, m) \) in \( \mathbb{R} \times \mathbb{R}^m \) is bifurcation model from the initial nonlinear term.

Definition 3.2 (Non-degeneracy). We say that the equation (1.1) is non-degenerate if the restriction of \( H \) to \( S \) is a Morse function, and 0 is not a critical value of the restriction of \( H \) to \( S \). Here \( S \) is the sphere defined by \( \sum_{i=1}^m x_i^2 = k + 1 \).

When \( m = 1 \) and \( k \) is finite, our bifurcation model is defined by (1.2).

When our bifurcation model is non-degenerate, it has a singularity defined by a weighted homogeneous system with weight \( (k-1, 1, \ldots, 1; k, \ldots, k) \) which defines an isolated singularity. There are \( k^m \) complex branches of the bifurcation model, and the solution curves of the bifurcation model (Definition 3.1) are expressed in the following form:

\[
t \mapsto (\lambda, x_1, x_2, \ldots, x_m) = (\lambda^* + a_0 t^{k-1}, a_1 t, a_2 t, \ldots, a_m t).
\]

Assume that it represents a real branch, i.e., \( a_i \) are real for \( i = 0, 1, \ldots, m \). We call the image of the interval \( t \geq 0 \) (or \( t \leq 0 \)) a real semi-branch of the bifurcation model.

(i) If \( k \) is even, then all real branches go through from the region \( \lambda < \lambda^* \) to the region \( \lambda > \lambda^* \). Several transcritical bifurcations take place at the bifurcation point \( (\lambda^*, 0) \). We say such a bifurcation pluritranscritical bifurcation (or multi-transcritical bifurcation). See the left figure below.

(ii) If \( k \) is odd, then the real branches of each solution stay in the region \( \lambda \leq \lambda^* \) or \( \lambda \geq \lambda^* \). Then one of the following types is possible as the bifurcation at \( (\lambda^*, 0) \). See the right three figures below. We call them plurisubcritical bifurcation (or multi-subcritical bifurcation), plurisupercritical bifurcation (or multi-supercritical bifurcation), mixed critical bifurcation, respectively.
We also say such a bifurcation is of type \((b_-, b_+)\) when \(b_-\) and \(b_+\) are the number of local real semi-branches at \((\lambda^*, 0)\) in the region \(\lambda < \lambda^*\) and \(\lambda > \lambda^*\), respectively.

Let \(\widehat{Z}\) denote the set defined by the bifurcation equation \(\widehat{F} = 0\) in \(\mathbb{R} \times \mathbb{R}^m\) (see subsection 2.3).

**Theorem 3.3.** If the equation (1.1) is non-degenerate, then the bifurcation equations \(\widehat{F}_i = 0\) \((i = 1, \ldots, m)\) are equivalent to the bifurcation model \(F_i = 0\) \((i = 1, \ldots, m)\), that is, there is a homeomorphism germ

\[
\Xi : (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)) \to (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)),
\]

preserving the hyperplane defined by \(\lambda = \lambda^*\), with \(\Xi(Z) = \widehat{Z}\).

In terms of singularity theory, we can say that \(F = (F_1, \ldots, F_m)\) is \(\mathcal{K}\)-equivalent to \(\widehat{F}\) when the conclusion of the theorem holds.

The use of the function \(H\) has already appeared in [1, Theorem 1], [3, Page 66]. They showed \((\lambda^*, 0)\) is a branching point under non-degeneracy conditions. Since we use singularity theory, we are able to conclude the bifurcation model and its type, which gives more precise information for bifurcation.

### 4 The proof of the main theorem

In this section we are going to present the characterization of non-degeneracy and the proof of Theorem 3.3.

#### 4.1 A characterization of non-degeneracy

The definition of non-degeneracy can be characterized by the following singularity conditions.

**Lemma 4.1.** The system (1.1) is non-degenerate if and only if the following conditions (i) and (ii) hold.

(i) Any irreducible component of \(F_i = 0\) \((i = 1, \ldots, n)\) is not in the hyperplane defined by \(\lambda = \lambda^*\), that is, \(\{\lambda = \lambda^*, H_{x_1} = \cdots = H_{x_m} = 0\} = \{0\}\).
(ii) \( F_i = 0 \ (i = 1, \ldots, n) \) defines curves with an isolated singularity at \((\lambda^*, 0)\), that is, \(\text{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x,x_j}) = m\) if \(F_i = 0 \ (i = 1, \ldots, n) \) except \((\lambda^*, 0)\).

**Proof.** First we remark that the conditions \(F_i = 0 \ (i = 1, \ldots, m)\) is equivalent that \(k(\lambda - \lambda^*)\) is an eigenvalue of \((H_{x,x_j})_{i,j=1,...,m}\) with an eigenvector \(x\), since \(H_{x_i} = \frac{1}{k} \sum_{j=1}^{m} x_j H_{x,x_j}\).

So, the condition (i) is equivalent that 0 is not an eigenvalue of \((H_{x,x_j})\) with eigenvector \(x\).

Next we observe that (ii) is equivalent to the following condition (ii').

(iii') \(k(\lambda - \lambda^*)\) is an eigenvalue of \((H_{x,x_j})\) with an eigenvector \(x\), and \(\lambda - \lambda^*\) is not an eigenvalue of \((H_{x,x_j})\).

In fact, if the condition (ii) does not hold and \(F_i = 0 \ (i = 1, \ldots, m)\), then \(\lambda - \lambda^*\) is an eigenvalue of \((H_{x,x_j})\). Conversely, if \(\lambda - \lambda^*\) is a non-zero eigenvalue of \((H_{x,x_j})\), then the corresponding eigenvector \(y = (y_1, \ldots, y_m)\) is perpendicular to \(x\), and

\[
(y_1, \ldots, y_m)(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x,x_j}) = 0.
\]

This implies that \(\text{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x,x_j}) < m\) and the condition (i) does not hold.

Suppose that the equation (1.1) is non-degenerate. The critical points set of the restriction of \(H\) to the sphere \(S\) defined by \(\sum_{i=1}^{m} x_i^2 = k + 1\) is \(Z \cap S\), and \(\lambda - \lambda^*\) is the value of \(H\) there, since \((k + 1)H = \sum_{i=1}^{m} x_i H_{x_i} = (\lambda - \lambda^*) \sum_{i=1}^{m} x_i^2\) on \(Z\). We have

\[
\begin{vmatrix}
0 & x_j \\
x_i & (\lambda^* - \lambda)\delta_{ij} + H_{x,x_j}
\end{vmatrix} \neq 0 \text{ on } Z \cap S,
\]

and the conditions (i) and (ii) hold.

Suppose that the conditions (i) and (ii) hold. If the restriction of \(H\) to \(S\) is not a Morse function, then \(\text{rank}(x_i, (\lambda^* - \lambda)\delta_{ij} + H_{x,x_j}) < m\). Thus the following equation

\[
\begin{pmatrix}
0 \\
x_i \\
(\lambda^* - \lambda)\delta_{ij} + H_{x,x_j}
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
\vdots \\
y_m
\end{pmatrix} = 0,
\]

has a nonzero solution \((y_0, \ldots, y_m)\) and \(x_1 y_1 + \cdots + x_m y_m = 0\). Let \(v_1 = t(x_1, \ldots, x_m), v_2, \ldots, v_m\) be the eigenvectors of \((H_{x,x_j})\), which are perpendicular each other, and set \(y = t(y_1, \ldots, y_m) = b_1 v_1 + \cdots + b_m v_m\). We have \(b_1 = 0\), and

\[
0 = y_0 v_1 + [(\lambda^* - \lambda)\delta_{ij} + H_{x,x_j}]y = y_0 v_1 + \sum_{j=1}^{m} b_j (\lambda^* - \lambda + \lambda_j) v_j.
\]

Thus \(y_0 = 0\) and \(b_j (\lambda^* - \lambda + \lambda_j) = 0, j = 2, \ldots, m\). Since \(y\) is not zero, there exists \(j\) such that \(\lambda^* - \lambda + \lambda_j = 0\), then \(\lambda - \lambda^*\) is an eigenvalue of \((H_{x,x_j})\), which contradict to (iii'). \(\square\)
4.2 The proof of Theorem 3.3

Here we present the proof of Theorem 3.3 by singularity theory.

Replacing \( \lambda - \lambda^* \) by \( \lambda \), it is enough to show the theorem assuming \( \lambda^* = 0 \). Set \( \rho = (\lambda^2 + x_1^{2(k-1)} + \cdots + x_m^{2(k-1)})^{1/(k-1)} \). Let \( M \) denote the minimum of

\[
\rho^2 \det((F_j)_{x_1}, \ldots, (F_j)_{x_m})^2 + \lambda^2 \sum_{i=1}^m \det((F_j)_{x_1}, \ldots, (F_j)_{x_i}, \ldots, (F_j)_{x_m})^2
\]
on \( \rho^{-1}(1) \). By the conditions (i) and (ii), we have \( M > 0 \).

Let us consider a singular metric \( \langle \ , \ \rangle \) defined by

\[
\langle \lambda \partial_{\lambda}, \lambda \partial_{\lambda} \rangle = 1, \ \langle \lambda \partial_{\lambda}, \rho \partial_{x_i} \rangle = 0, \ \langle \rho \partial_{x_i}, \rho \partial_{x_j} \rangle = \delta_{ij}, \ i, j = 1, \ldots, m. \tag{4.1}
\]

We remark that the gradient of \( f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}, (\lambda, x) \mapsto f(\lambda, x) \), is given by

\[
\nabla f = \lambda^2 f_{\lambda} \partial_{\lambda} + \rho^2 \sum_{i=1}^m f_x \partial_{x_i}.
\]

Then we have

\[
det(\langle \nabla F_i, \nabla F_j \rangle) + |F|^{2m} \geq M \text{ on } \rho^{-1}(1), \text{ since}
\]

\[
det(\langle \nabla F_i, \nabla F_j \rangle) = \rho^{2m} \det((F_j)_{x_1}, \ldots, (F_j)_{x_m})^2 + \lambda^2 \rho^{2(m-1)} \sum_{i=1}^m \det((F_j)_{x_1}, \ldots, (F_j)_{x_i}, \ldots, (F_j)_{x_m})^2.
\]

We thus have the following inequality on \( \rho^{-1}(1) \) and therefore on \( \mathbb{R} \times \mathbb{R}^m \),

\[
det(\langle \nabla F_i, \nabla F_j \rangle) + |F|^{2m} \geq M \rho^{2mk},
\]
because of weighted homogeneity of both sides.

Define \( K_i(\lambda, x) \) by \( \tilde{F}_i = F_i + K_i \). There is a positive constant \( C_i \) and \( \delta \) so that

\[
|K_i| \leq C_i \rho^{k+\delta} \text{ near } 0. \tag{4.2}
\]

Set \( \tilde{F}_j(\lambda, x, t) = \lambda x_j + H_{x_j} + tK_j \) which are functions on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \). We set \( \tilde{\nabla} \tilde{F}_j = \nabla \tilde{F}_j + (F_j)_{x} \partial_t, \nabla \tilde{F}_j = \lambda^2 (\tilde{F}_j)_{\lambda} \partial_{\lambda} + \rho^2 \sum_{i=1}^m (F_i)_{x_i} \partial_{x_i} \). There is a positive constant \( C'_i \) so that

\[
|\nabla \tilde{F}_i| \leq C'_i \rho^k \text{ near } 0. \tag{4.3}
\]

Set \( A(\lambda, x, t) = \det(\nabla \tilde{F}_i, \nabla \tilde{F}_j) + |\tilde{F}|^{2m} \) and \( A_0(\lambda, x) = \det(\nabla F_i, \nabla F_j) + |F|^{2m} \). Then there is a function \( A_1(\lambda, x, t) \) with \( A(\lambda, x, t) = A_0(\lambda, x) + tA_1(\lambda, x, t) \). By (4.2) and (4.3),

\[
0 \leq |A_1(x, t)| \leq A_0(x)/2 \text{ near } (\lambda, x) = (0, 0).
\]

\[
A_0(x) - tA_0(x)/2 \leq A_0(x) + tA_1(x, t) \text{ near } (\lambda, x) = (0, 0) \text{ for } t \geq 0,
\]

and thus

\[
\frac{1}{2} A_0(x) \leq (1 - \frac{1}{2}) A_0(x) \leq A(x, t) \text{ near } (\lambda, x) = (0, 0) \text{ for any } t \in [0, 1].
\]
Therefore we have
\[ \det(\nabla \tilde{F}_i, \nabla \tilde{F}_j) + |\tilde{F}|^{2m} \geq C_0 \rho^{2km} \text{ near } 0. \] (4.4)

Set
\[ \xi = \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j) + |\tilde{F}|^{2m}} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \nabla \tilde{F}_i \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right| + \partial_t. \]

We show that \( \xi \tilde{F}_i = 0 \) if \( F_i(x) = 0 \) except \( (\lambda, x) = (0, 0) \). To see this, we consider the orthogonal projection to the tangent space of \( \tilde{F}_j = 0 \), which is defined at its regular point, with respect to the singular metric induced by (4.1) and the Euclidean metric of \( t \)-axis. This is expressed by
\[ v \mapsto \pi(v) = \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \nabla \tilde{F}_i \\ \langle v, \nabla \tilde{F}_j \rangle & v \end{array} \right|. \]

Then we have
\[ \langle \pi(\partial_i), \partial_\lambda \rangle = \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \langle \nabla \tilde{F}_i, \partial_\lambda \rangle \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right|, \]
\[ \langle \pi(\partial_i), \partial_{x_i} \rangle = \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \langle \nabla \tilde{F}_i, \partial_{x_i} \rangle \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right|, \]
\[ \langle \pi(\partial_i), \partial_t \rangle = \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \langle \nabla \tilde{F}_i, \partial_t \rangle \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right| = \det(\nabla F_i, \nabla F_j), \]

and conclude that \( \xi = \frac{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j)} \pi(\partial_i) \) if \( \tilde{F}_i = 0 \) \( (i = 1, \ldots, m) \). This shows \( \xi \tilde{F}_i = 0 \) whenever \( \tilde{F}_i = 0 \) and \( \xi \) is defined. Now we define \( \tilde{\xi} \) by \( \tilde{\xi} = \xi \) if \( (\lambda, x) \neq (0, 0) \); \( \tilde{\xi} = \partial_t \) if \( (\lambda, x) = (0, 0) \). Let \( \tilde{\xi} = \xi_0 \partial_\lambda + \sum_{i=1}^m \xi_i \partial_{x_i} + \partial_t \). By (4.2), (4.3) and (4.4), there is a positive constant \( C \) so that
\[ |\xi_0| \leq \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j) + |\tilde{F}|^{2m}} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \tilde{F}_i \lambda^2 \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right| \leq \frac{C \rho^{2km+\delta} |\lambda|}{\rho^{2km}} = C \rho^\delta |\lambda|, \]
\[ |\xi_i| \leq \frac{1}{\det(\nabla \tilde{F}_i, \nabla \tilde{F}_j) + |\tilde{F}|^{2m}} \left| \begin{array}{cc} \langle \nabla \tilde{F}_i, \nabla \tilde{F}_j \rangle & \tilde{F}_{x_i} \rho^2 \\ \langle \partial_i, \nabla \tilde{F}_j \rangle & 0 \end{array} \right| \leq \frac{C \rho^{2km+\delta} \rho}{\rho^{2km}} = C \rho^{1+\delta}, \]

near \( (\lambda, x) = (0, 0) \). These inequalities imply the uniqueness of the flow of \( \tilde{\xi} \). (See [4, §2.2-4]) Thus the flow of \( \tilde{\xi} \) yield a desired homeomorphism.

### 4.3 Examples of \( H \) and the numbers of real semi-branches

Let \( b_- \) (resp. \( b_+ \)) denote the number of real semi-branches of \( Z \) (in Definition 3.1) in the region \( \lambda^* - \varepsilon < \lambda \leq \lambda^* \) (resp. \( \lambda^* \leq \lambda < \lambda^* + \varepsilon \)) where \( \varepsilon \) is a small positive number.

**Example 4.2.** When \( H(x, y) = y((y - px)^2 + qx^2) \), we have \( (m, k) = (2, 2) \), and the bifurcation model is defined by
\[ \begin{cases} (\lambda^* - \lambda)x + y(2qx - 2p(-px + y)) = 0 \\ (\lambda^* - \lambda)y + qx^2 + 2y(-px + y) + (-px + y)^2 = 0 \end{cases}. \] (4.5)
The bifurcation of solutions are decided by \( p \) and \( q \). See the left figure below. The boundary of inner part is defined by \( f(p, q) = 0 \), where

\[
f(p, q) = 9p^2 + 26p^4 + 25p^6 + 8p^8 - 27q - 48p^2q + 14p^4q + 32p^6q + 54q^2 - 47p^2q^2 + 48p^4q^2 - 36q^3 + 32p^2q^3 + 8q^4.
\]

Example 4.2, \((b_-, b_+\)) in \((p, q)\)-plane.

Example 4.3. \((b_-, b_+)\) in \((p, q)\)-plane.

**Example 4.3.** When \( H(x, y) = p(x + y)^3 + q(x + y)xy \), we have \((m, k) = (2, 2)\), and the number of real semi-branches of bifurcation model is decided by \( p \) and \( q \). See the right figure above. The lines in the figure are \( q = -4p \), and \( q = -\frac{12}{5}p \).

**Example 4.4.** When \( H(x, y) = (px^2 + y^2)(qx^2 + y^2) \), we have \((m, k) = (2, 3)\) and the number of real semi-branches of bifurcation model is decided by \( p \) and \( q \). See the figure below. The curves in the figure are \( q = 2 - p \) and \( q = \frac{p}{2p - 1} \).

**Example 4.5.** When \( H_1(x, y) = p'(x + y)^4 + q'(x + y)^2xy + r'x^2y^2 \), we have \((m, k) = (2, 3)\). If \( r' = 0 \), then the function \( H \) is degenerate. Assume that \( r' \neq 0 \), let \( H(x, y) = p(x + y)^4 + q(x + y)^2xy + x^2y^2 \), where \( p = \frac{q}{r}, q = \frac{q}{r} \). The number of real semi-branches of bifurcation model is decided by \( p \) and \( q \). See the figure below. The curves in the figure are \( q^2 = 4p, 3q + 8p + 1 = 0, q = -4p, q = -1, \) and \( 4q + 16p + 1 = 0 \).
5 Bifurcation models for Dirichlet problem on square

As an application of the bifurcation model, we consider the bifurcation problem of the solutions of the Dirichlet problem (1.3) with $\Omega = [0, \pi]^2$. An eigenvalue $\lambda^*$ of $-\Delta$ is represented by $\lambda^* = a^2 + b^2$, $(a, b) \in \mathbb{N}^2$. The basis of the eigenspace is given by $(s, t) \mapsto \sin as \sin bt$ where $(a, b) \in \mathbb{N}^2$ with $a^2 + b^2 = \lambda^*$. So the first several eigenvalues are given as follows (with multiplicities):

$$2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 49, 50, 52, 53, 58, 61, 65, 66, 65, 66, \ldots$$

Let $v_i$ ($i = 1, \ldots, m$) be the basis above of the eigenspace $V$ for the eigenvalue $\lambda^*$, and let $w_j$ ($j = 1, 2, \ldots$) be the eigenfunctions of eigenvalue $\lambda_j$ ($\neq \lambda^*$) so that $v_i, w_j$ form a Schauder basis of $X = \{u \in L^2(\Omega): u|_{\partial \Omega} = 0\}$. Setting $W = \text{span}\{w_j\}$, we obtain that $X = V \oplus W$.

So we can consider the bifurcation model (Definition 3.1) and we denote by $b_-$ (resp. $b_+$) the number of half branches of the solution curves to $H/a_k(\lambda^*) = 0$ with $\lambda < \lambda^*$ (resp. $\lambda > \lambda^*$), which coincides with the number of solutions to Dirichlet problem (1.3) in the region $0 > a_k(\lambda^*)(\lambda - \lambda^*) > -|a_k(\lambda^*)|\varepsilon$ (resp. $0 < a_k(\lambda^*)(\lambda - \lambda^*) < |a_k(\lambda^*)|\varepsilon$) where $\varepsilon$ is a sufficiently small positive number.

**Theorem 5.1.** Assume that $k = 2$ and $\lambda^* = a^2 + b^2$, $(a, b) \in \mathbb{N}^2$, is an eigenvalue of $-\Delta$ with multiplicity 2.

(i) If $ab$ is odd, then our bifurcation model is non-degenerate and $(\lambda^*, 0)$ is a transcritical bifurcation point of type $(4, 4)$.  

Example 4.5, $(b_-, b_+)$ in $(p, q)$-plane.
Check the bifurcation model is non-degenerate with Theorem 5.2.

Proof. Remark that

$$H = \frac{1}{3} \int_0^\pi \int_0^\pi (x \sin a \sin b t + y \sin b \sin a) \, ds \, dt$$

$$= \frac{x^3 + y^3}{3} I_{3,0}(a, b) + x y(x + y) I_{2,1}(a, b) I_{1,2}(a, b).$$

If $ab$ is even, then $H = 0$, since $I_{3,0}(a, b) = 0$ (a is even) and $I_{2,1}(a, b) = 0$ (b is even). When $ab$ is odd, we have

$$H = \frac{1}{27ab} (x^3 + y^3) - \frac{ab}{4a^2 - 17a^2b^2 + 4b^4} xy(x + y)$$

$$= \frac{1}{27ab} (x + y)^3 - \frac{4(a - b)^2(a + b)^2}{9(4a^4 - 17a^2b^2 + 4b^4)} xy(x + y),$$

since $I_{3,0}(a, b) = 4/3a$ if $a$ is odd; $I_{2,1}(a, b) = 4a^2/b(4a^2 - b^2)$ if $b$ is odd. By Example 4.3, checking $q(12p + 5q) > 0$ and $4p + q \neq 0$ with $(p, q) = (1/27ab, 4(a - b)^2(a + b)^2/9(4a^4 - 17a^2b^2 + 4b^4)$, we observe that they define transcritical bifurcation models of type (4, 4).

**Theorem 5.2.** Assume that $k = 3$ and $\lambda^*$ is an eigenvalue of $-\Delta$ with multiplicity 2. Then the bifurcation model is non-degenerate with

$$H = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^3x_2^2 + 3x_2^4).$$

If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then the bifurcation at the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation of type $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$).

Proof. For the eigenvalue $\lambda^* = a^2 + b^2$ with multiplicity 2 where $a$ and $b$ are positive integers with $a \neq b$, we see

$$H = \frac{1}{4} \int_0^\pi \int_0^\pi (x \sin a \sin b t + y \sin b \sin a) \, ds \, dt$$

$$= \frac{x^4 + y^4}{4} I_{4,0}(a, b) + xy(x^2 + y^2) I_{3,1}(a, b) I_{1,2}(a, b) + \frac{3x^2y^2}{2} I_{2,2}(a, b)^2$$

$$= \frac{9\pi^2}{64} x^4 + \frac{\pi^2}{16} y^4 \frac{256}{2} 3\pi^2 (3x^4 + 8x^2y^2 + 3y^4).$$

Here we use the following facts:

$$I_{4,0}(a, b) = \frac{3\pi}{8} \quad I_{3,1}(a, b) = \begin{cases} -\frac{\pi}{8} & (b = 3a), \\ 0 & (\text{otherwise}), \end{cases} \quad I_{2,2}(a, b) = \frac{\pi}{4}.$$
Since
\[ H(x, y) = \frac{9\pi^2}{256}a_3(\lambda^*)(4+\sqrt[3]{2}x^2 + y^2)(4-\sqrt[3]{2}x^2 + y^2), \]
this is the case that \((p, q) = \left(\frac{4+\sqrt[3]{2}}{3}, \frac{4-\sqrt[3]{2}}{3}\right)\) in Example 4.4. If \(a_3(\lambda^*) > 0\) (resp. \(a_3(\lambda^*) < 0\)), then \((b_-, b_+) = (1, 9)\) (resp. \((b_-, b_+) = (9, 1)\)).

**Remark 5.3.** Assume that \(k = 4\) and \(\lambda^* = a^2 + b^2\), \((a, b) \in \mathbb{N}^2\), is an eigenvalue of \(-\Delta\) with multiplicity 2. Remark that

\[
\frac{H}{a_4(\lambda^*)} = \frac{1}{15} \int_0^\pi \int_0^\pi \left( x \sin as \sin bt + y \sin bs \sin at \right)^6 ds \, dt
\]

\[
= \frac{x^5 + y^5}{5} I_{5,0}(a, b) I_{0,5}(a, b) + xy(x^3 + y^3) I_{4,1}(a, b) I_{1,4}(a, b) + 4x^2y^2(x + y) I_{3,2}(a, b) I_{2,3}(a, b).
\]

If \(ab\) is even, then \(H = 0\), since \(I_{5,0}(a, b) = 0\) \((a\) even), \(I_{4,1}(a, b) = 0\) \((b\) even), \(I_{3,2}(a, b) = 0\) \((a\) even). When \(ab\) is odd, we have

\[
\frac{H}{16^2 a_4(\lambda^*)} = \frac{1}{15^2ab} \frac{x^5 + y^5}{5} + \frac{3^2 a^2 b^2 xy(x^3 + y^3)}{(4a^2 - b^2)(16a^2 - b^2)(a^2 - 4b^2)(a^2 - 16b^2)}
\]

\[
+ \frac{4ab(5b^2 - 2a^2)(5a^2 - 2b^2)x^2 y^2(x + y)}{(9(4b^2 - a^2) - 4a^2)(9a^2 - 4b^2)(4a^2 - b^2)(9b^2 - 4a^2)(2a + b)},
\]

since \(I_{5,0}(a, b) = 16/15a\) if \(a\) is odd; \(I_{4,1}(a, b) = 3 \cdot 16a^4/b(4a^2 - b^2)(16a^2 - b^2)\) if \(b\) is even; \(I_{3,2}(a, b) = 16b^2(5a^2 - 2b^2)/3a(4b^2 - a^2)(9a^2 - 4b^2)(a + 2b)\) if \(a\) is odd. Numerical experiences show that this is a non-degenerate bifurcation model of type \((4, 4)\) when \(\lambda^* = 10, 26, 34, 58, 74, 82, 90, 106, 122, 146\). But this is a non-degenerate bifurcation model of type \((6, 6\) when \(\lambda^* = 178\).

**Theorem 5.4.** For \(k = 5\) and for an eigenvalue \(\lambda^*\) with multiplicity 2, the bifurcation model is non-degenerate. If \(a_5(\lambda^*) > 0\) (resp. \(a_5(\lambda^*) < 0\)), then the bifurcation at the point \((\lambda^*, 0)\) is plurisupercritical (resp. plurisubcritical) bifurcation of type \((b_-, b_+) = (1, 9)\) (resp. \((b_-, b_+) = (9, 1)\)).

**Proof.** We first remark that \(I_{6,0}(a, b) = \frac{5\pi}{16}\),

\[
I_{5,1}(a, b) = \begin{cases} 
-\frac{5\pi}{32} & (b = 3a), \\
\frac{5\pi}{32} & (b = 5a), \\
0 & \text{(otherwise)},
\end{cases}
I_{4,2}(a, b) = \begin{cases} 
\frac{5\pi}{32} & (b = 2a), \\
\frac{3\pi}{16} & \text{(otherwise)},
\end{cases}
I_{3,3}(a, b) = \begin{cases} 
-\frac{3\pi}{32} & (b = 3a), \\
0 & \text{(otherwise)}.
\end{cases}
\]

These imply that

\[
\frac{H}{a_5(\lambda^*)} = \frac{1}{6} \int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^6 ds \, dt
\]

\[
= \left( \frac{x^5 + y^5}{6} I_{6,0}(a, b) I_{0,6}(a, b) + xy(x^4 + y^4) I_{5,1}(a, b) I_{1,5}(a, b) + 5x^2y^2(x^2 + y^2) I_{4,2}(a, b) I_{2,4}(a, b) + 20x^3y^3 I_{3,3}(a, b) \right)^2.
\]
show the data for the bifurcation models in the following table.

To show it, let us consider the following Dirichlet problem

\[ \begin{align*}
(1) \quad (\frac{\pi}{a})^2 x^4 + y^4 &= 0 \quad (b = 2a \text{ or } a = 2b) \\
(2) \quad (\frac{\pi}{a})^2 x^6 + (\frac{\pi}{a})^2 x^4 y^2 &= 0 \quad (b = 3a \text{ or } a = 3b) \\
(3) \quad (\frac{\pi}{a})^2 x^8 + x^6 y^2 &= 0 \\
(4) \quad (\frac{\pi}{a})^2 x^8 + x^6 y^2 &= 0 \\
\end{align*} \]

The remaining assertions are routine calculation. \(\square\)

**Remark 5.5.** The first eigenvalue with multiplicity 3 is 50. Note that 50 = 1^2 + 7^2 = 2 \times 5^2. When the order of the initial nonlinear term \(a_k(\lambda^*)u^k\), \(a_k(\lambda^*) \neq 0\), is 2, 3, 4, 5, we show the data for the bifurcation models in the following table.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(H/a_k(\lambda^*))</th>
<th>((b_-, b_+))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{1}{3} \left( \frac{16}{63} (x_1^4 + x_2^4) - \frac{142}{2025} x_1 x_2 (x_1 + x_2) - \frac{360}{12125} x_1^2 x_2^2 \right) )</td>
<td>(8, 8)</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{3 \pi^2}{256} \left[ 3 (x_1^4 + x_2^4) + 8 (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) \right] )</td>
<td>(1, 27)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{2 \pi^2}{8708} (x_1^4 + x_2^4) - \frac{655121}{2025825} x_1^2 x_2^3 (x_1 + x_2) + \frac{40956}{1000000} x_1 x_2 x_3 (x_1^2 + x_2^2) )</td>
<td>(8, 8)</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{5 \pi^2}{1536} \left( 5 (x_1^4 + x_2^4 + x_3^4) + 72 x_1 x_2 x_3 (x_1 + x_2) + 27 \right) )</td>
<td>(1, 27)</td>
</tr>
</tbody>
</table>

Here \(b_-\) (resp. \(b_+\)) is the number of half branches of the solution curves to \(H/a_k(\lambda^*) = 0\) with \(\lambda < \lambda_0\) (resp. \(\lambda > \lambda_0\)), which coincides with the number of solutions to Dirichlet problem (1.3) in the region \(0 > a_k(\lambda)(\lambda - \lambda^*) > 0\). \(\epsilon\) (resp. \(0 < a_k(\lambda)(\lambda - \lambda^*) < \epsilon\)) where \(\epsilon\) is a sufficiently small positive number.

### 6 Symmetry creates new bifurcation

In this section we are going to show that symmetry in the domain creates new bifurcations. To show it, let us consider the following Dirichlet problem

\[ \Delta u = -\lambda u + a_3(\lambda)u^3 + o(u^3) \text{ on } \Omega_\epsilon, \quad u|_{\partial \Omega_\epsilon} = 0, \quad (6.1) \]

where \(\Omega_\epsilon = [0, \pi] \times [0, (1 + \epsilon)\pi]\). The rectangles \(\Omega_\epsilon\) converge to the square \(\Omega = [0, \pi]^2\), and the eigenvalues \(\lambda_1 = 1^2 + (\frac{2}{1+\epsilon})^2\), \(\lambda_2 = 2^2 + (\frac{1}{1+\epsilon})^2\) on \(\Omega_\epsilon\) converge to the eigenvalue \(\lambda^* = 5\) on \(\Omega\), as \(\epsilon \to 0\). For the bifurcation model, see the left figure below, where \(v_{1,2}^\epsilon = \sin(x) \sin(\frac{2}{1+\epsilon}y)\) is orthogonal to the vector \(v_{2,1}^\epsilon = \sin(2\epsilon) \sin(\frac{1}{1+\epsilon}y)\).

The eigenvalue 5 is of multiplicity 2 and our bifurcation model has the following solutions:

1. \((\lambda, g_1(\lambda), 0)\),  
2. \((\lambda, -g_1(\lambda), 0)\),  
3. \((\lambda, 0, g_1(\lambda))\),  
4. \((\lambda, 0, -g_1(\lambda))\),  

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and the trivial solution (9) \((\lambda, 0, 0)\), where \(g_1(\lambda) = \frac{1}{2} \sqrt{\frac{\lambda - 5}{a_3(\lambda^*)}}\), \(g_2(\lambda) = \frac{1}{2} \sqrt{\frac{\lambda - 5}{21a_3(\lambda^*)}}\). The following figures show the bifurcation of the solutions to (6.1) in \(\Omega_\varepsilon\) and \(\Omega\), respectively.

Comparing the bifurcation model (a) with (b) as \(\varepsilon \to 0\), there are 4 new semi-branches (5), (6), (7), (8) in (b), which do not come from the semi-branches in the model (a).

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