Bifurcation Model for Nonlinear Equations

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Abstract

A bifurcation model for a nonlinear equation is introduced. Under the non-degeneracy condition (Definition 2.1), our bifurcation model describes the bifurcation of solutions to the nonlinear equation. We also show how these models work for Dirichlet problem on the square. We observe a perturbation of rectangles to a square creates new bifurcation, which is not a limit of the bifurcations on rectangles.

Introduction

In this paper, we investigate the bifurcation of trivial solution to the nonlinear elliptic equation

$$-\Delta u - \lambda u + h(\lambda, u) = 0, \quad u \in L^2(\Omega) \tag{0.1}$$

with Dirichlet boundary condition where Ω is a bounded domain with piecewise smooth boundary in $\mathbb{R}^n$, $L^2(\Omega)$ is the space of L^2-functions on Ω, Δ is Laplacian and $h(\lambda, u)$ is a smooth function in two variables λ and u with

$$h(\lambda, 0) = 0, \quad h_u(\lambda, 0) = 0. \tag{0.2}$$

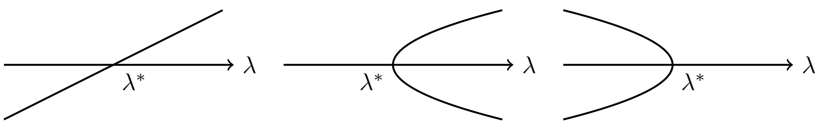
Let us write the equation (0.1) as $\Phi(\lambda, u) = 0$. We call $(\lambda^*, 0)$ **a bifurcation point**, if for any neighborhood U of $(\lambda^*, 0)$, there exists $(\lambda, u) \in U$ so that $\Phi(\lambda, u) = 0$, $u \neq 0$. It is well-known that if $(\lambda^*, 0)$ is a bifurcation point, then λ^* is an eigenvalue of $-\Delta$. Let m denote the multiplicity of the eigenvalue λ^*. If $m = 1$, and

$$h(\lambda, u) = a_k(\lambda)u^k/k! + o(u^k), \ u \to 0, \ a_k(\lambda^*) \neq 0, \tag{0.3}$$

where $a_k(\lambda)$ is a smooth function of λ, then the bifurcation of solutions is often described by

$$(\lambda^* - \lambda)x + ax^k = 0, \ a \neq 0. \tag{0.4}$$

In this case, the bifurcation of solutions is decided by k and a, as shown in the following figures.



Tanscrtical bifurcation (k' is even) Supercrtical bifurcation (k' is odd, $a > 0$) Subcrtical bifurcation (k' is odd, $a < 0$)

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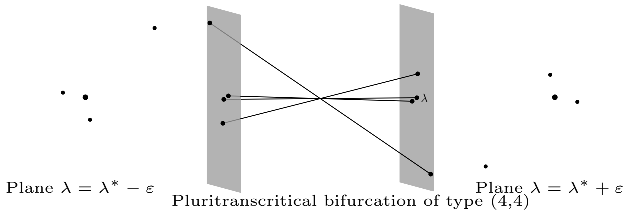
Such a figure is often called a bifurcation diagram, since it explains the bifurcation of solutions. A motivation of this paper is to generalize to draw such bifurcation diagrams to the case that multiplicity m of the eigenvalues λ^* is finite with $m > 1$. Ambrosetti ([2], [3, page 66]) showed the existence of bifurcation in suitable setup. Ambrosetti also showed several related results loc. cite., but they do not say how bifurcation diagram looks like, the number of branches, symmetry, etc., which is insufficient from the viewpoint of singularity theory. In the authors' knowledge, no other thing is known in the case $m > 1$. This motivates to generalize the equation (0.4) to a more wide setup, and this is the main subject of the paper.

We have already called the diagrams above bifurcation diagrams, but strictly speaking this is an abuse of the language. Precisely, a bifurcation diagram is the zero set of bifurcation equations, obtained by the Lyapunov-Schmidt reduction (see Definition 1.3). Since the Lyapunov-Schmidt reduction requires to be applied the implicit function theorem, the bifurcation equation contains implicit functions, and we should clarify their properties. This causes several difficulties to investigate the bifurcation in the case when $m > 1$. So the strategy is to reduce the bifurcation equation to certain normal forms. Assuming (0.3), we introduce a bifurcation model (Definition 2.2), which is often determined by the initial nonlinear term $a_k(\lambda)u^k/k!$. We show in Theorem 2.3 that our bifurcation model describes the bifurcations of solutions under certain non-degeneracy condition (which is introduced in Definition 2.1). Remark that our bifurcation model is defined by a weighted homogeneous system, whose weights are determined by the degree k' of certain homogeneous polynomial H, which is often equal to k, see Definitions 2.1, 2.2, and it is easier to investigate the bifurcation model than to investigate the bifurcation equation itself obtained by Lyapunov-Schmidt reduction. We also remark that H is often determined by the initial term of h. Our bifurcation model enables us to describe symmetry of bifurcation diagram caused by symmetry of the domain. Moreover, our method suggests that symmetry of domains creates new bifurcation, as we observe in section 3.2.

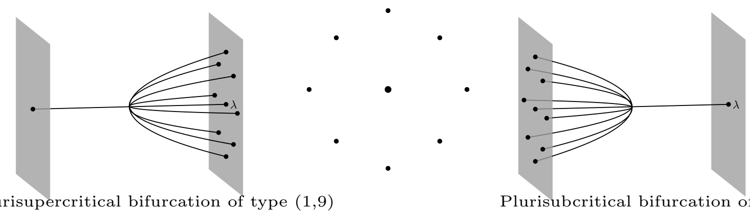
We will investigate the case $\Omega = [0, \pi]^2$ (a square) closely as a good example, assuming (0.3) and see how our bifurcation model works. In this case, an eigenvalues λ^* of $-\Delta$ is a sum of squares of finitely many positive integers, and the number of ways to express λ^* in such a sum is its multiplicity. We have the following consequences.

(i) When $(m, k) = (2, 2)$, and $\lambda^* = a^2 + b^2$ where a, b are positive integers with $ab \not\equiv 0 \bmod 2$, we have pluritranscritical bifurcation of type $(4, 4)$, that is, the bifurcation model in (λ, x, y) space looks like as follows: Here ε is a small positive number. The dot $\bullet$ represents the trivial solution and the dots $\cdot$, smaller than the previous dot, represent non-trivial solutions which bifurcate at $(\lambda^*, 0)$ from the trivial solution. This is a topological model which explains the bifurcation of the trivial solution in this case. These are stated in Theorem 3.2.
When $ab \equiv 0 \bmod 2$, the situation is more complicated. We show how our method works in the case. See Theorem 3.2 for details.



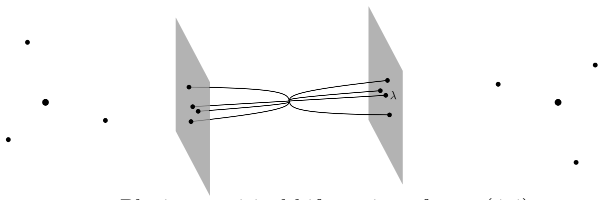
Pluritranscritical bifurcation of type (4,4)

(ii) When $(m,k) = (2,3)$, we have plurisupercritical (or plurisubcritical) bifurcation of type $(1,9)$ (or type $(9,1)$), which is shown as the following figures.

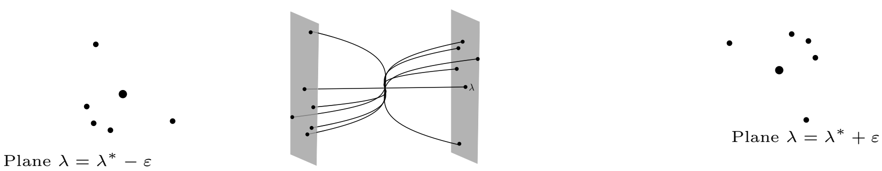

Plurisupercritical bifurcation of type (1,9) Plurisubcritical bifurcation of type (9,1)

The middle figure shows the configulation of solutions on the plane λ =constants which intersects the bifurcation model in (λ, x, y)-space with multiple points. This is also a topological model which explains the bifurcation of the trivial solution. We will show this fact in Theorem 3.1.

(iii) When $(m,k) = (2,4)$, and $\lambda^* = a^2 + b^2$ where a, b are positive integers with $ab \not\equiv 0 \bmod 2$, the bifurcation there is described by our bifurcation model. If $\lambda^* \leq 146$, then the bifurcation model is pluritranscritical bifurcation of type (4,4).


Pluritranscritical bifurcation of tyoe (4,4)

But when $\lambda^* = 178 = 3^2 + 13^2$, the situation becomes a bit different. In this case, the bifurcation is pluritranscritical of type (6,6) as shown by the figure below.

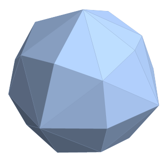


Pluritranscritical bifurcation of tyoe (6,6)

(iv) When $(m,k) = (2,5)$, we have plurisupercritical (or plurisubcritical) bifurcation of type (1,9) (or (9,1)) (Theorem 3.3).

(v) The first eigenvalue λ^* with $m = 3$ is 50 and the bifurcations there are transcrtical of type (8,8) if $k = 2, 4$; plurisupercrtical (or plurisubcritical) of type (1,27) (or (27,1)) if $k = 3, 5$. These are treated in subsection 3.3.

The figure shows the configuration of the solutions with $\lambda = \lambda^* + \varepsilon$ ($\varepsilon > 0$) for $k = 3$, $a_k(\lambda^*) > 0$, $\lambda^* = 50$. The center of the polyhedron corresponds to the trivial solution. The non-trivial solutions corresponds to the vertices of the polyhedron.



To show these results, we apply Lyapunov-Schmidt reduction to the differential equation (0.1), and reduce the problem to finite-dimensional set-up, and then apply singularity theory technique to conclude the results. Several people in singularity theory worked about bifurcation problem of the zero set of $\mathbb{R}^n \to \mathbb{R}^{n-1}$ ([5], [6], [7], etc.). A motivation of these investigations is originated in describing bifurcations of solutions to several differential equations.

The readers would observe several symmetry in configurations of solutions, and the authors believe it is a reflection of symmetry of the region we consider. Moreover, we observe in subsection 3.2 that solutions of the Dirichlet problem on the square are not, in general, limits of solutions for the Dirichlet problem on rectangles convergent to the square. This would be related with the fact that the Dirichlet problem on the square is D_4-invariant where D_4 stands the dihedral group of order 8, while the Dirichlet problem on the rectangle is invariant with respect to the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$. This would suggest that deforming the region to increase symmetry increases solution of the Dirichlet problem.

The paper is organized as follows. In section 1, we recall several basic materials (inverse function theorem and implicit function theorem) and Lyapunov-Schmidt reduction process. In section 2, we introduce a bifurcation model, which is often determined by the initial nonlinear term of nonlinear equations. Our main theorem is Theorem 2.3, which asserts that the bifurcation model describes the bifurcation of the solution to (0.1) under certain non-degeneracy condition (Definition 2.1). We also give a characterization of non-degeneracy condition in subsection 2.2. The proof of the main theorem is given in subsection 2.3. In section 3, we show how our method works for Dirichlet problem (0.1) on the square $\Omega = [0, \pi]^2$. We observe that there are new bifurcations on the square, which are not the limits of bifurcations on rectangles.

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1 Preliminary

We recall several basic theorems that we need in the paper later on.

1.1 Inverse function theorem and implicit function theorem

Let X, Y be Banach spaces. Let U, V be open subsets of X, Y, respectively, and $F : U \to V$ be a map. We denote their norms using the symbol $\|\cdot\|$. We say that F is (Fréchet) differentiable at $u \in U$ if there exists a linear continuous map $L_u : X \to Y$ such that

$$F(u+v) - F(u) = L_u[v] + o(\|v\|), \text{ as } \|v\| \to 0.$$

We say F is (Fréchet) differentiable if F is (Fréchet) differentiable at any $u \in U$. When F is Fréchet differentiable at $u \in X$, the map L_u is uniquely determined by F and u and is denoted by $dF(u)$, $d_uF(u)$, $F_u(u)$ or $F'(u)$. It is easy to see that if F is Fréchet differentiable, then it is also differentiable along any direction. Conversely, if F is differentiable along any directions, $L_u \in L(X, Y)$, where $L_u[v] = \lim_{t\to 0}(F(u+tv) - F(u))/t$, $L(X, Y)$

is the space of linear maps of X into Y, and the map $u \mapsto L_u$ is a continuous map from X to $L(X,Y)$, then F is Fréchet differentiable [3, page 5]. We say a function $F : X \to Y$ is a C^1-map, if it is Fréchet differentiable and its Fréchet derivative $X \to L(X,Y)$, $u \mapsto L_u$, is continuous.

Lemma 1.1 (Inverse function theorem (Theorem 3.1.1 in [3]))**.** *Let $P : U \to V$ be a C^1map between Banach spaces, U, V are open sets of X, Y, respectively. Suppose that for some $f_0 \in U$ the derivative $dP(f_0) : X \to Y$ is an invertible linear map. Then we can find neighborhoods $\tilde{U}$ of f_0 and $\tilde{V}$ of $g_0 = P(f_0)$ such that the map P gives a one-to-one map of $\tilde{U}$ onto $\tilde{V}$, and the inverse map $P^{-1} : \tilde{V} \subset Y \to \tilde{U} \subset X$ is C^1.*

Lemma 1.2 (Implicit function theorem (Theorem 3.2.1 in [3]))**.** *Let X, Y be Banach spaces and fix $(\lambda_0, u_0) \in \mathbb{R}^n \times X$. Assume that F is a C^1 map from a neighborhood of (λ_0, u_0) in $\mathbb{R}^n \times X$ into Y such that $F(\lambda_0, u_0) = 0$ and suppose that $d_u F(\lambda_0, u_0)$ is invertible. Then there exist a neighborhood Λ of λ_0 and a neighborhood U of u_0 such that the equation $F(\lambda, u) = 0$ has a unique solution $u = u(\lambda) \in U$ for all $\lambda \in \Lambda$. The function $u(\lambda)$ is of class C^1, and the following holds*

$$u'(\lambda_0) = -[d_u F(\lambda_0, u_0)]^{-1} d_\lambda F(\lambda_0, u_0).$$

1.2 Lyapunov-Schmidt reduction

Consider the map

$$\Phi : \mathbb{R} \times X \to X, \quad (\lambda, u) \mapsto -\Delta u - \lambda u + h(\lambda, u)$$

where X denote the space of L^2-functions on Ω with $u|_{\partial\Omega} = 0$, that is,

$$X = \{u \in L^2(\Omega) : u|_{\partial\Omega} = 0\}.$$

We assume that, for $k \geq 2$,

$$h(\lambda, u) = a_k(\lambda)\frac{u^k}{k!} + a_{k+1}(\lambda)\frac{u^{k+1}}{(k+1)!} + a_{k+2}(\lambda)\frac{u^{k+2}}{(k+2)!} + o(u^{k+2}),$$

with $a_k(\lambda^*) \neq 0$ as $u \to 0$. We assume that $a_k(\lambda)$, $a_{k+1}(\lambda)$, $a_{k+2}(\lambda)$ are smooth functions of λ. Set $V = \operatorname{Ker} \Phi_x(\lambda^*, 0)$ where Φ_x denote the differential of the map $X \to X$, $u \mapsto \Phi(\lambda, u)$, and W denotes the orthogonal complement of V. We assume that

- V is m-dimensional and $v_1, \ldots, v_m$ form an orthnormal basis of V,
- $w_1, w_2, \ldots$ form an orthnormal basis of W and $-\Delta w_j = \lambda_j w_j$, $j = 1, 2, \ldots$, and
- $\phi(u) = \int_\Omega u\, ds$ where ds denote the volume form induced from $\mathbb{R}^n$. (Remark that, since the volume of Ω is finite, say v, we have $\|u\|_{L^1} \leq v^{1/2}\|u\|_{L^2}$ by Hölder's inequality, and we obtain $L^2(\Omega) \subset L^1(\Omega)$.)

For any $u \in X$, u can be expressed as

$$u = P(u) + Q(u), \qquad P(u) = \sum_{i=1}^{m} x_i v_i, \quad Q(u) = \sum_{j=1}^{\infty} y_j w_j,$$

where $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, $y = (y_1, y_2, \dots) \in \ell^2$. Here ℓ^2 is the space of square-summable sequences. Then we can write

$$\Phi(\lambda, u) = \sum_{i=1}^{m}(\lambda^* - \lambda)x_i v_i + \sum_{j=1}^{\infty}(\lambda_j - \lambda)y_j w_j + h(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j).$$

We choose $v_i^* \in V^*$ and $w_j^* \in W^*$ such that $v_i^* v_s = \delta_{is}$, $w_j^* w_t = \delta_{jt}$, $v_i^* w_j = w_j^* v_i = 0$, $1 \leq j, s \leq m$, $1 \leq j, t < \infty$, where $\delta_{ij} = 1$ $(i = j)$; 0 $(i \neq j)$.

Let p_X denote the projection

$$p_X : X \longrightarrow \mathbb{R}^m \times \ell^2, \quad u \mapsto (v_i^* u, w_j^* u),$$

and ι_X denote the injection

$$\iota_X : p_X(X) \longrightarrow X, \quad (x_i, y_j) \mapsto \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} y_j w_j.$$

We have $p_X \circ \iota_X$ and $\iota_X \circ p_X$ are the identities. Then we define $\widetilde{\Phi}$ by $\widetilde{\Phi} = p_Y \circ \Phi \circ (\mathrm{id}_{\mathbb{R}} \times \iota_X)$, and have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R} \times X \arrow[r, "\Phi"] & X \arrow[d, "p_X"] \\
\mathbb{R} \times p_X(X) \arrow[u, "\mathrm{id}_{\mathbb{R}} \times \iota_X"] \arrow[r, "\widetilde{\Phi}"] & \mathbb{R}^m \times \ell^2
\end{tikzcd}$$

The function $\widetilde{\Phi}(\lambda, x, y)$ is written as

$$\widetilde{\Phi}(\lambda, x, y) = ((\lambda^* - \lambda)x_i + h_i, (\lambda_j - \lambda)y_j + h_j)_{i=1,\dots,m;, j=1,2,\dots} \tag{1.1}$$

near $(\lambda^*, 0)$, where $h_i = v_i^* h(\lambda, u)$, $i = 1, \dots, m$; $h_j = w_j^* h(\lambda, u)$, $j = 1, 2, \dots$.

By calculation, one can find the following derivatives directly,

$$\begin{aligned}
\widetilde{\Phi}_\lambda &= (x_p, y_q)_{p=1,\dots,m;\, q=1,2,\dots}, \\
\widetilde{\Phi}_{x_i} &= (\delta_{p,i}(\lambda^* - \lambda) + (h_p)_{x_i}, (h_q)_{x_i})_{p=1,\dots,m;\, q=1,2,\dots}, \\
\widetilde{\Phi}_{y_j} &= ((h_p)_{y_j}, \delta_{q,j}(\lambda_j - \lambda) + (h_q)_{y_j})_{p=1,\dots,m;\, q=1,2,\dots}.
\end{aligned}$$

Let

$$P : \mathbb{R}^m \times \ell^2 \longrightarrow \mathbb{R}^m, \quad Q : \mathbb{R}^m \times \ell^2 \longrightarrow \ell^2,$$

denote the natural projections. Since $\lambda_j \neq \lambda^*$, $j = 1, 2, \dots$, we can apply the implicit function theorem, and $F_{y_j}(\lambda^*, 0)$ is invertible. Thus there exists a unique map

$$\varphi = (\varphi_j)_{j=1,2,\dots} : \mathbb{R} \times \mathbb{R}^m \longrightarrow \ell^2,$$

such that $Q \circ \widetilde{\Phi}(\lambda, x, \varphi(\lambda, x)) = 0$.

Definition 1.3 (Bifurcation equation). *We call $P \circ \widetilde{\Phi}(\lambda, x, \varphi(\lambda, x)) = 0$ the* **bifurcation equation** *in the notation above.*

Remark 1.4. *There is no guarantee that the projection Q (or the natural projection $X \to W$) is bounded. If it is not bounded, we consider the composition of Q with*

$$\mathbb{R}^\infty \longrightarrow \ell^2, \qquad (y_j)_{j=1,2,\dots} \mapsto \left(\frac{1}{j^2(\lambda_j - \lambda^*)} y_j\right)_{j=1,2,\dots}$$

instead of Q, which is a bounded linear map. We can thus apply the implicit function theorem.

2 Bifurcation model

We continue to use the notation in the previous section. To define the key homogeneous polynomial H of our bifurcation model, we compute the first few terms of $P\circ\widetilde{\Phi}(\lambda, x, \varphi(\lambda, x))$.

If $Q\circ\widetilde{\Phi}(\lambda, x, y) = 0$, we have, for $j = 1, 2, \dots,$

$$
\begin{aligned}
0 =&(\lambda_j - \lambda)y_j + \phi(h(\lambda, P(u) + Q(u))w_j)\\
=&(\lambda_j - \lambda)y_j + \phi\big(\big(\tfrac{a_k(\lambda)}{k!}(P(u) + Q(u))^k + o(u^k)\big)w_j\big)\\
=&(\lambda_j - \lambda^*)y_j + \tfrac{a_k(\lambda^*)}{k!}\phi(P(u)^k w_j) + o(u^k, \lambda - \lambda^*),
\end{aligned}
$$

as $u \to 0$, $\lambda \to \lambda^*$. We thus obtain that, for $j = 1, 2 \dots,$

$$
\varphi_j(\lambda, x) = \frac{a_k(\lambda^*)}{k!(\lambda^* - \lambda_j)}\phi(P(u)^k w_j) + o(u^k, \lambda - \lambda^*) \quad \text{as } u \to 0,\ \lambda \to \lambda^*. \tag{2.1}
$$

Let $\widehat{F} : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a map defined by $\widehat{F}(\lambda, x) = P\circ\widetilde{\Phi}(\lambda, x, \varphi(\lambda, x))$. This is written as

$$
\widehat{F}(\lambda, x) = \Big((\lambda^* - \lambda)x_p + h_p\Big(\lambda, \sum_{i=1}^{m} x_i v_i + \sum_{j=1}^{\infty} \varphi_j(\lambda, x)w_j\Big)\Big)_{p=1,\dots,m}. \tag{2.2}
$$

These are written in the following form:

$$
(\lambda^* - \lambda)x_i + \phi\Big(h\Big(\lambda, \sum_{p=1}^{m} x_p v_p + \sum_{q=1}^{\infty} \varphi_q(\lambda, x)w_q\Big)v_i\Big) = 0, \quad i = 1, \dots, m. \tag{2.3}
$$

We then have, for $i = 1, \dots, m$,

$$
\begin{aligned}
0 =&\langle -\Delta u - \lambda u + h(\lambda, u), v_i\rangle\\
=&(\lambda^* - \lambda)x_i + \big\langle \tfrac{a_k(\lambda)}{k!}u^k + \tfrac{a_{k+1}(\lambda)}{(k+1)!}u^{k+1} + o(u^{k+1}, \lambda - \lambda^*), v_i\big\rangle\\
=&(\lambda^* - \lambda)x_i + \big\langle \tfrac{a_k(\lambda)}{k!}(P(u) + Q(u))^k + \tfrac{a_{k+1}(\lambda)}{(k+1)!}P(u)^{k+1} + o(u^{k+1}, \lambda - \lambda^*), v_i\big\rangle\\
=&(\lambda^* - \lambda)x_i + \tfrac{a_k(\lambda)}{k!}\langle P(u)^k, v_i\rangle + \tfrac{a_k(\lambda)}{(k-1)!}\langle P(u)^{k-1}Q(u), v_i\rangle\\
&\quad + \tfrac{a_{k+1}(\lambda)}{(k+1)!}\langle P(u)^{k+1}, v_i\rangle + o(u^{k+1}, \lambda - \lambda^*)\\
=&(\lambda^* - \lambda)x_i + \tfrac{a_k(\lambda^*)}{k!}\langle P(u)^k, v_i\rangle + \tfrac{a_k(\lambda^*)}{(k-1)!}\sum_{j=1}^{\infty} y_j\langle P(u)^{k-1}w_j, v_i\rangle\\
&\quad + \tfrac{a_{k+1}(\lambda^*)}{(k+1)!}\langle P(u)^{k+1}, v_i\rangle + o(u^{k+1}, \lambda - \lambda^*)\\
=&(\lambda^* - \lambda)x_i + \tfrac{a_k(\lambda^*)}{k!}\langle P(u)^k, v_i\rangle + \sum_{j=1}^{\infty} \tfrac{a_k(\lambda^*)^2\langle P(u)^k, w_j\rangle}{k!(\lambda_j - \lambda^*)}\tfrac{\langle P(u)^{k-1}w_j, v_i\rangle}{(k-1)!}\\
&\quad + \tfrac{a_{k+1}(\lambda^*)}{(k+1)!}\langle P(u)^{k+1}, v_i\rangle + o(u^{k+1}, \lambda - \lambda^*) \qquad \text{(by (2.1))}\\
=&\partial_{x_i}\Big[(\lambda^* - \lambda)\sum_{j=1}^{m}\tfrac{x_j^2}{2} + \tfrac{a_k(\lambda^*)}{(k+1)!}\phi(P(u)^{k+1}) + \tfrac{a_k(\lambda^*)^2}{2(k!)^2}\sum_{j=1}^{\infty}\tfrac{\langle P(u)^k, w_j\rangle^2}{\lambda_j - \lambda^*}\\
&\quad + \tfrac{a_{k+1}(\lambda^*)}{(k+2)!}\phi(P(u)^{k+2})\Big] + o(u^{k+1}, \lambda - \lambda^*),
\end{aligned}
$$

as $u \to 0$, $\lambda \to \lambda^*$.

2.1 Definition of bifurcation model

When $k = 2$, we set

$$H(x) = \begin{cases} \frac{a_2(\lambda^*)}{3!}\phi(P(u)^3) & (\text{if } a_2(\lambda^*)\phi(P(u)^3) \neq 0), \\ \frac{a_2(\lambda^*)^2}{8}\sum_{j=1}^{\infty}\frac{\phi(P(u)^2 w_j)^2}{\lambda_j - \lambda^*} + \frac{a_3(\lambda^*)}{24}\phi(P(u)^4) & (\text{if } a_2(\lambda)\phi(P(u)^3) = 0). \end{cases} \tag{2.4}$$

When $k \geq 3$, we set

$$H(x) = \begin{cases} \frac{a_k(\lambda^*)}{(k+1)!}\phi(P(u)^{k+1}) & (\text{if } a_k(\lambda^*)\phi(P(u)^{k+1}) \neq 0), \\ \frac{a_{k+1}(\lambda^*)}{(k+2)!}\phi(P(u)^{k+2}) & (\text{if } a_k(\lambda)\phi(P(u)^{k+1}) = 0). \end{cases}$$

Definition 2.1 (Non-degeneracy)**.** *Let H be a homogeneous polynomial of $x_1, \ldots, x_m$ of degree $k'+1$. We say that H is* **non-degenerate** *if the restriction of H to S is a Morse function, and 0 is not a critical value of the restriction of H to S. Here S is the sphere defined by $\sum_{i=1}^{m} x_i^2 = k'+1$.*

Definition 2.2 (Bifurcation model)**.** *We say the zero locus Z of*

$$F_i(\lambda, x) = (\lambda^* - \lambda)x_i + H_{x_i}, \quad i = 1, \ldots, m, \tag{2.5}$$

in $\mathbb{R} \times \mathbb{R}^m$ is the **bifurcation model** *when H is non-degenerate in the sense above.*

When $m = 1$ and k is finite, our bifurcation model is defined by (0.4) with $k' = k$.

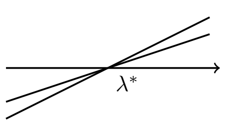
When our bifurcation model is defined, it has a singularity defined by a weighted homogeneous system with weight $(k'-1, 1, \ldots, 1; k', \ldots, k')$ defines an isolated singularity. There are $(k')^m$ complex branches of the bifurcation model, and the solution curves of the bifurcation model (Definition 2.2) are expressed in the following form:

$$t \mapsto (\lambda, x_1, x_2, \ldots, x_m) = (\lambda^* + a_0 t^{k'-1}, a_1 t, a_2 t, \ldots, a_m t).$$

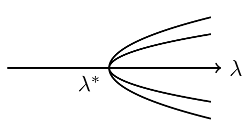
Assume that it represents a real branch, i.e., a_i are real for $i = 0, 1, \ldots, m$. We call the image of the interval $t \geq 0$ (or $t \leq 0$) a **real semi-branch** of the bifurcation model.

(i) If k' is even, then all real branches reach the region $\lambda > \lambda^*$ from the region $\lambda < \lambda^*$. Several transcritical bifurcations take place at the bifurcation point $(\lambda^*, 0)$. We call such a bifurcation **pluritranscritical bifurcation** (or multi-transcritical bifurcation). See the left figure below.

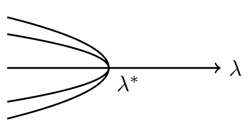
(ii) If k' is odd, then the real branches of each solution stay in the region $\lambda \leq \lambda^*$ or $\lambda \geq \lambda^*$. Then possible bifurcation scenarios are illustrated on the right three figures below. We call them **plurisubcritical bifurcation** (or multi-subcritical bifurcation), **plurisupercritical bifurcation** (or multi-supercritical bifurcation), **mixed critical bifurcation**, respectively.



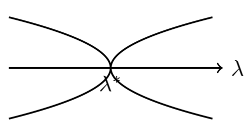
Pluritranscrtical bifurcation (k is even)



Plurisupercrtical bifurcation (k' is odd)



Plurisubcrtical bifurcation (k' is odd)



Mixed crtical bifurcation (k' is odd)

We also say such a bifurcation is of type (b_-, b_+) when b_- and b_+ are the number of local real semi-branches at $(\lambda^*, 0)$ in the region $\lambda < \lambda^*$ and $\lambda > \lambda^*$, respectively.

Let $\widehat{Z}$ denote the set defined by the bifurcation equation $\widehat{F} = 0$ in $\mathbb{R} \times \mathbb{R}^m$ (see (2.2)).

Theorem 2.3. *If H is non-degenerate, then the bifurcation equations $\widehat{F}_i = 0$ $(i = 1, \ldots, m)$ are equivalent to the bifurcation model $F_i = 0$ $(i = 1, \ldots, m)$, that is, there is a homeomorphism germ*

$$\Xi : (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^m, (\lambda^*, 0)),$$

preserving the hyperplane defined by $\lambda = \lambda^*$, with $\Xi(Z) = \widehat{Z}$.

In terms of singularity theory (cf. [8]), we can say that $F = (F_1, \ldots, F_m)$ is $\mathcal{K}$-equivalent to $\widehat{F}$ when the conclusion of the theorem holds.

The use of the function H has already appeared in [2, Theorem 1], [3, Page 66]. They showed $(\lambda^*, 0)$ is a branching point under non-degeneracy conditions. Since we use singularity theory, we are able to conclude the bifurcation model and its type, which gives more precise information for bifurcation.

Remark 2.4. We remark that our bifurcation model works under other suitable conditions for Neumann problem in a similar way to the case of Dirichlet problem.

Remark 2.5. *When $k = 3$, and $a_3(\lambda)\phi(P(u)^4)|_S$ is a constant, we can consider the bifurcation model using the following H:*

$$H(x) = \begin{cases} \frac{a_4(\lambda^*)}{5!}\phi(P(u)^5) & (\textit{if } a_4(\lambda^*)\phi(P(u)^5) \neq 0), \\ \frac{a_3(\lambda^*)^2}{72}\sum_{j=1}^{\infty} \frac{\phi(P(u)^3 w_j)^2}{\lambda_j - \lambda^*} + \frac{a_5(\lambda^*)}{720}\phi(P(u)^6) & (\textit{if } a_4(\lambda) = 0). \end{cases}$$

2.2 A characterization of non-degeneracy

The definition of non-degeneracy can be characterized by the following singularity conditions.

Lemma 2.6. *The homogeneous polynomial H of degree $k+1$ is non-degenerate in the sense of Definition 2.1 if and only if the following conditions* (i) *and* (ii) *hold.*

(i) *Any irreducible component of $F_i = 0$ $(i = 1, \ldots, n)$ is not in the hyperplane defined by $\lambda = \lambda^*$, that is, $\{\lambda = \lambda^*, H_{x_1} = \cdots = H_{x_m} = 0\} = \{(\lambda^*, 0)\}$.*

(ii) *$F_i = 0$ $(i = 1, \ldots, n)$ defines curves with an isolated singularity at $(\lambda^*, 0)$, that is,* $\operatorname{rank}(x_i, \delta_{ij}(\lambda^* - \lambda) + H_{x_i x_j}) = m$ *if $F_i = 0$ $(i = 1, \ldots, n)$ except $(\lambda^*, 0)$.*

Proof. First we remark that the conditions $F_i = 0$ $(i = 1, \ldots, m)$ is equivalent that $k(\lambda - \lambda^*)$ is an eigenvalue of $(H_{x_i x_j})_{i,j=1,\ldots,m}$ with an eigenvector x, since $H_{x_i} = \frac{1}{k}\sum_{j=1}^{m} x_j H_{x_i x_j}$. So, the condition (i) is equivalent to the condition that 0 is not an eigenvalue of the Hesse matrix $(H_{x_i x_j})$ with eigenvector x.

Next we observe that (ii) is equivalent to the following condition (ii′).

(ii′) $k(\lambda-\lambda^*)$ is an eigenvalue of $(H_{x_ix_j})$ with an eigenvector x, and $\lambda-\lambda^*$ is not an eigenvalue of $(H_{x_ix_j})$.

In fact, if the condition (ii) does not hold and $F_i=0$ $(i=1,\ldots,m)$, then $\lambda-\lambda^*$ is an eigenvalue of $(H_{x_ix_j})$. Conversely, if $\lambda-\lambda^*$ is a non-zero eigenvalue of $(H_{x_ix_j})$, then the corresponding eigenvector $y=(y_1,\ldots,y_m)$ is perpendicular to x, and

$$(y_1,\ldots,y_m)(x_i,\delta_{i,j}(\lambda^*-\lambda)+H_{x_ix_j})=0.$$

This implies that $\operatorname{rank}(x_i,\delta_{i,j}(\lambda^*-\lambda)+H_{x_ix_j})<m$ and the condition (i) does not hold.

Suppose that H is non-degenerate. The critical points set of the restriction of H to the sphere S defined by $\sum_{i=1}^m x_i^2=k+1$ is $Z\cap S$, and $\lambda-\lambda^*$ is the value of H there, since $(k+1)H=\sum_{i=1}^m x_iH_{x_i}=(\lambda-\lambda^*)\sum_{i=1}^m x_i^2$ on Z. We have

$$\begin{vmatrix} 0 & x_j \\ x_i & (\lambda^*-\lambda)\delta_{i,j}+H_{x_ix_j}\end{vmatrix}\neq 0 \text{ on } Z\cap S,$$

and the conditions (i) and (ii) hold.

Suppose that the conditions (i) and (ii) hold. If the restriction of H to S is not a Morse function, then $\operatorname{rank}(x_i,(\lambda^*-\lambda)\delta_{ij}+H_{x_ix_j})<m$. Thus the following equation

$$\begin{pmatrix} 0 & x_j \\ x_i & (\lambda^*-\lambda)\delta_{ij}+H_{x_ix_j}\end{pmatrix}\begin{pmatrix} y_0\\ y_1\\ \vdots\\ y_m\end{pmatrix}=0,$$

has a nonzero solution $(y_0,\ldots,y_m)$ and $x_1y_1+\cdots+x_my_m=0$. Let $v_1={}^t(x_1,\ldots,x_m)$, v_2, ..., v_m be the eigenvectors of $(H_{x_ix_j})$, which are perpendicular each other, and set $y={}^t(y_1,\ldots,y_m)=b_1v_1+\cdots+b_mv_m$. We have $b_1=0$, and

$$\begin{aligned}0&=y_0v_1+[(\lambda^*-\lambda)\delta_{ij}+H_{x_ix_j}]y\\ &=y_0v_1+[(\lambda^*-\lambda)\delta_{ij}+H_{x_ix_j}]\sum_{j=1}^m b_jv_j=y_0v_1+\sum_{j=1}^m b_j(\lambda^*-\lambda+\lambda_j)v_j.\end{aligned}$$

Thus $y_0=0$ and $b_j(\lambda^*-\lambda+\lambda_j)=0$, $j=2,\ldots,m$. Since y is not zero, there exists j such that $\lambda^*-\lambda+\lambda_j=0$, then $\lambda-\lambda^*$ is an eigenvalue of $(H_{x_ix_j})$, which contradict to (ii′). □

2.3 The proof of Theorem 2.3

Here we present the proof of Theorem 2.3 based on the usage of singularity theory.

Replacing $\lambda-\lambda^*$ by λ, it is enough to prove the theorem assuming $\lambda^*=0$. Set $\rho=(\lambda^2+x_1^{2(k'-1)}+\cdots+x_m^{2(k'-1)})^{\frac{1}{2(k'-1)}}$. Let M denote the minimum of

$$\rho^2\det((F_j)_{x_1},\ldots,(F_j)_{x_m})^2+\lambda^2\sum_{i=1}^m\det((F_j)_\lambda,(F_j)_{x_1},\ldots,\widehat{(F_j)_{x_i}},\ldots,(F_j)_{x_m})^2$$

on $\rho^{-1}(1)$. By the conditions (i) and (ii), we have $M>0$.

Let us consider a singular metric $\langle\ ,\ \rangle$ defined by

$$\langle \lambda\partial_\lambda, \lambda\partial_\lambda\rangle = 1,\ \langle \lambda\partial_\lambda, \rho\partial_{x_i}\rangle = 0,\ \langle \rho\partial_{x_i}, \rho\partial_{x_j}\rangle = \delta_{ij},\ i,j = 1,\dots,m. \tag{2.6}$$

We remark that the gradient of $f : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$, $(\lambda, x) \mapsto f(\lambda, x)$, with respect to this singular metric, is given by

$$\nabla f = \lambda^2 f_\lambda \partial_\lambda + \rho^2 \sum_{i=1}^m f_{x_i}\partial_{x_i}.$$

Then we have $\det(\langle \nabla F_i, \nabla F_j\rangle) + |F|^{2m} \geq M$ on $\rho^{-1}(1)$, since

$$\begin{aligned}\det(\langle \nabla F_i, \nabla F_j\rangle) =& \rho^{2m}\det((F_j)_{x_1},\dots,(F_j)_{x_m})^2 \\ &+ \lambda^2\rho^{2(m-1)}\sum_{i=1}^m \det((F_j)_\lambda, (F_j)_{x_1},\dots,\widehat{(F_j)_{x_i}},\dots,(F_j)_{x_m})^2.\end{aligned}$$

We thus have the following inequality on $\rho^{-1}(1)$ and therefore on $\mathbb{R}\times\mathbb{R}^m$,

$$\det(\langle \nabla F_i, \nabla F_j\rangle) + |F|^{2m} \geq M\rho^{2mk},$$

because of weighted homogeneity of both sides.

Define $K_i(\lambda, x)$ by $\widehat{F}_i = F_i + K_i$. There are positive constants C_i and δ so that

$$|K_i| \leq C_i \rho^{k'+\delta} \quad \text{near } 0. \tag{2.7}$$

Set $\tilde{F}_j(\lambda, x, t) = \lambda x_j + H_{x_j} + tK_j$ which are functions on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$. We set

$$\tilde{\nabla}\tilde{F}_j = \nabla\tilde{F}_j + (F_j)_t\partial_t, \quad \nabla\tilde{F}_j = \lambda^2(\tilde{F}_j)_\lambda\partial_\lambda + \rho^2\sum_{i=1}^m (\tilde{F}_j)_{x_i}\partial_{x_i}.$$

There is a positive constant C_i' so that

$$\|\nabla\tilde{F}_i\| \leq C_i'\rho^{k'} \quad \text{near } 0. \tag{2.8}$$

Set $A(\lambda, x, t) = \det(\langle \nabla\tilde{F}_i, \nabla\tilde{F}_j\rangle) + |\tilde{F}|^{2m}$ and $A_0(\lambda, x) = \det(\langle \nabla F_i, \nabla F_j\rangle) + |F|^{2m}$. Then there is a function $A_1(\lambda, x, t)$ with $A(\lambda, x, t) = A_0(\lambda, x) + tA_1(\lambda, x, t)$. By (2.7) and (2.8), $0 \leq |A_1(x,t)| \leq A_0(x)/2$ near $(\lambda, x) = (0, 0)$. Since

$$A_0(x) - tA_0(x)/2 \leq A_0(x) + tA_1(x,t) \text{ near } (\lambda, x) = (0,0) \text{ for } t \geq 0,$$

one has

$$\tfrac{1}{2}A_0(x) \leq (1 - \tfrac{t}{2})A_0(x) \leq A(x,t) \quad \text{near } (\lambda, x) = (0,0) \text{ for any } t \in [0,1].$$

Therefore we have

$$\det(\langle \nabla\tilde{F}_i, \nabla\tilde{F}_j\rangle) + |\tilde{F}|^{2m} \geq C_0\rho^{2k'm} \text{ near } 0. \tag{2.9}$$

Set

$$\xi = \frac{1}{\det(\langle \nabla\tilde{F}_i, \nabla\tilde{F}_j\rangle) + |\tilde{F}|^{2m}}\begin{vmatrix} \langle \tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \nabla\tilde{F}_i \\ \langle \partial_t, \tilde{\nabla}\tilde{F}_j\rangle & 0\end{vmatrix} + \partial_t.$$

We show that $\xi\tilde{F}_i = 0$ if $F_i(x) = 0$ except $(\lambda, x) = (0,0)$. To see this, we consider the orthogonal projection to the tangent space of $\tilde{F}_j = 0$, which is defined at its regular point, with respect to the singular metric induced by (2.6) and the Euclidean metric on t-axis. This is expressed by

$$v \mapsto \pi(v) = \frac{1}{\det\left(\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle\right)} \begin{vmatrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \tilde{\nabla}\tilde{F}_i \\ \langle v, \tilde{\nabla}\tilde{F}_j\rangle & v \end{vmatrix}.$$

Then we have

$$\begin{aligned}
\langle\pi(\partial_t), \partial_\lambda\rangle =& \frac{1}{\det(\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle)} \begin{vmatrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \langle\tilde{\nabla}\tilde{F}_i, \partial_\lambda\rangle \\ \langle\partial_t, \tilde{\nabla}\tilde{F}_j\rangle & 0 \end{vmatrix}, \\
\langle\pi(\partial_t), \partial_{x_i}\rangle =& \frac{1}{\det(\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle)} \begin{vmatrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \langle\tilde{\nabla}\tilde{F}_i, \partial_{x_i}\rangle \\ \langle\partial_t, \tilde{\nabla}\tilde{F}_j\rangle & 0 \end{vmatrix}, \\
\langle\pi(\partial_t), \partial_t\rangle =& \frac{1}{\det(\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle)} \begin{vmatrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \langle\tilde{\nabla}\tilde{F}_i, \partial_t\rangle \\ \langle\partial_t, \tilde{\nabla}\tilde{F}_j\rangle & \langle\partial_t, \partial_t\rangle \end{vmatrix} = \frac{\det(\langle\nabla F_i, \nabla F_j\rangle)}{\det(\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle)},
\end{aligned}$$

and conclude that $\xi = \frac{\det\langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle}{\det\langle\nabla F_i, \nabla F_j\rangle}\pi(\partial_t)$ if $\tilde{F}_i = 0$ $(i = 1, \dots, m)$. This shows $\xi\tilde{F}_i = 0$ whenever $\tilde{F}_i = 0$ and ξ is defined. Now we define $\tilde{\xi}$ by $\tilde{\xi} = \xi$ if $(\lambda, x) \neq (0,0)$; $\tilde{\xi} = \partial_t$ if $(\lambda, x) = (0,0)$. Let $\tilde{\xi} = \xi_0\partial_\lambda + \sum_{i=1}^{m} \xi_i\partial_{x_i} + \partial_t$. By (2.7), (2.8) and (2.9), there is a positive constant C so that

$$\begin{aligned}
|\xi_0| \leq& \frac{1}{\det(\langle\nabla\tilde{F}_i, \nabla\tilde{F}_j\rangle) + |\tilde{F}|^{2m}} \left\| \begin{matrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \tilde{F}_\lambda\lambda^2 \\ \langle\partial_t, \tilde{\nabla}\tilde{F}_j\rangle & 0 \end{matrix} \right\| \leq \frac{C\rho^{2k'm+\delta}|\lambda|}{\rho^{2k'm}} = C\rho^\delta|\lambda|, \\
|\xi_i| \leq& \frac{1}{\det(\langle\nabla\tilde{F}_i, \nabla\tilde{F}_j\rangle) + |\tilde{F}|^{2m}} \left\| \begin{matrix} \langle\tilde{\nabla}\tilde{F}_i, \tilde{\nabla}\tilde{F}_j\rangle & \tilde{F}_{x_i}\rho^2 \\ \langle\partial_t, \tilde{\nabla}\tilde{F}_j\rangle & 0 \end{matrix} \right\| \leq \frac{C\rho^{2k'm+\delta}\rho}{\rho^{2k'm}} = C\rho^{1+\delta},
\end{aligned}$$

near $(\lambda, x) = (0,0)$. These inequalities imply the uniqueness of the flow of $\tilde{\xi}$. (See [4, 2.2-4].) Thus the flows of $\tilde{\xi}$ yield a desired homeomorphism.

Remark 2.7. *By construction, Ξ is C^∞ diffeomorphism except $(\lambda^*, 0)$.*

2.4 Examples of H and the numbers of real semi-branches

Example 2.8. *When $H(x,y) = \alpha(x+y)^3 + \beta(x+y)xy$, we have $(m,k) = (2,2)$, and is non-degenerate in the sense of Definitiuon 2.1, whenever $\beta(2\alpha+\beta)(4\alpha+\beta)(12\alpha+5\beta) \neq 0$. Under this assumption, the locus Z in Definition 2.2 is described by the following equations:*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \frac{-\lambda}{3(4\alpha+\beta)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \frac{\lambda}{2\beta}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \frac{\sqrt{12\alpha+5\beta}}{\sqrt{\beta}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].$$

Example 2.9. *The homogeneous polynomial*

$$H(x,y) = \alpha(x^2+y^2)^2 + 2\beta x^2y^2 + 2\gamma xy(x^2+y^2)$$

is non-degenerate in the sense of Definitiuon 2.1, whenever $\beta \neq \pm\gamma$, $\alpha + \beta/2 \neq \pm\gamma$ *and* $2\alpha\beta \neq \gamma^2$. *The locus* Z *in Definition 2.2 is described by the following equations:*

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \frac{-\lambda}{4(2\alpha+\beta\pm 2\gamma)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \frac{\lambda}{4(\gamma^2-2\alpha\beta)}\left[\beta\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \varepsilon\sqrt{\beta^2-\gamma^2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

where ε *is the sign of* $\lambda(\gamma^2 - 2\alpha\beta)$.

3 Bifurcation models for Dirichlet problem on a square

As an application of the bifurcation model, we consider the bifurcation problem of the solutions of the Dirichlet problem (0.1) with $\Omega = [0,\pi]^2$. An eigenvalue λ^* of $-\Delta$ is represented by $\lambda^* = a^2 + b^2$, $(a,b) \in \mathbb{N}^2$. An orthnormal basis of the eigenspace is given by $(s,t) \mapsto \sin as \sin bt/(\pi/2)$, where $(a,b) \in \mathbb{N}^2$ with $a^2 + b^2 = \lambda^*$. So the eigenvalues for small a and b are given as follows (with multiplicities):

$$2, 5, 5, 8, 10, 10, 13, 13, 17, 17, 18, 20, 20, 25, 25, 26, 26, 29, 29, 32, 34, 34, 37, 37,$$
$$40, 40, 41, 41, 45, 45, 50, 50, 50, 52, 52, 53, 53, 58, 58, 61, 61, 65, 65, 65, 65, \ldots.$$

3.1 Bifurcation models for $(m,k) = (2,2), (2,3), (2,4), (2,5)$

We consider the bifurcation model (see Definition 2.2) and we denote by b_- (resp. b_+) the number of semi-branches of the solution curves to $H/a_k(\lambda^*) = 0$ with $\lambda < \lambda^*$ (resp. $\lambda > \lambda^*$), which coincides with the number of solutions to Dirichlet problem (0.1) in the region $0 > a_k(\lambda^*)(\lambda - \lambda^*) > -|a_k(\lambda^*)|\varepsilon$ (resp. $0 < a_k(\lambda^*)(\lambda - \lambda^*) < |a_k(\lambda^*)|\varepsilon$) where ε is a sufficiently small positive number.

Theorem 3.1. *Assume that* $k = 3$ *and* λ^* *is an eigenvalue of* $-\Delta$ *of multiplicity 2. If* $a_3(\lambda^*) \neq 0$, *then we have the bifurcation model with non-degenerate*

$$H = \frac{3\pi^2}{256} a_3(\lambda^*)(3x_1^4 + 8x_1^2x_2^2 + 3x_2^4).$$

If $a_3(\lambda^*) > 0$ *(resp.* $a_3(\lambda^*) < 0$*), then the bifurcation at the point* $(\lambda^*, 0)$ *is plurisupercritical (resp. plurisubcritical) bifurcation of type* $(b_-, b_+) = (9,1)$ *(resp.* $(b_-, b_+) = (1,9)$*).*

Before the proof, we introduce a notation: $I_{p,q}(a,b) = \int_0^\pi \sin^p at \sin^q bt\, dt$. Note that $I_{p,q}(a,b) = I_{q,p}(b,a)$.

Proof. For the eigenvalue $\lambda^* = a^2 + b^2$ of multiplicity 2 where a and b are positive integers with $a \neq b$, we have

$$\begin{aligned} &\int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^4 ds\, dt \\ =&(x^4 + y^4) I_{4,0}(a,b) I_{0,4}(a,b) + 4xy(x^2 + y^2) I_{3,1}(a,b) I_{1,3}(a,b) + 6x^2y^2 I_{2,2}(a,b)^2 \\ =&\frac{3\pi^2}{64}(3x^4 + 8x^2y^2 + 3y^4) = \frac{3\pi^2}{64}(3(x^2+y^2)^2 + 2x^2y^2) \end{aligned} \tag{3.1}$$

Here we use the following facts:

$$I_{4,0}(a,b) = \frac{3\pi}{8}, \qquad I_{3,1}(a,b) = \begin{cases} -\frac{\pi}{8} & (b=3a), \\ 0 & (\text{otherwise}), \end{cases} \qquad I_{2,2}(a,b) = \frac{\pi}{4}.$$

If $a_3(\lambda^*) > 0$ (resp. $a_3(\lambda^*) < 0$), then $(b_-, b_+) = (9,1)$ (resp. $(b_-, b_+) = (1,9)$). $\square$

Theorem 3.2. *Assume that $k=2$ and $\lambda^* = a^2+b^2$, $(a,b) \in \mathbb{N}^2$, is an eigenvalue of $-\Delta$ of multiplicity 2.*

(i) If ab is odd and $a_2(\lambda^*) \neq 0$, then the bifurcation model (Definition 2.2) is defined with

$$H = \frac{8}{\pi^3}\frac{16(x+y)}{9ab}\Big[x^2+y^2 - \frac{2(2a^2+b^2)(a^2+2b^2)}{(a^2-4b^2)(4a^2-b^2)}xy\Big].$$

This is non-degenerate and $(\lambda^*, 0)$ is a transcritical bifurcation point of type $(4,4)$.

(ii) If ab is even, then the bifurcation model (Definition 2.2) is defined with

$$H = \frac{8a_2(\lambda^*)}{3\pi^6}(16a^2b^2)^2 G + \frac{3a_3(\lambda^*)}{4\pi}(3(x^2+y^2)^2 + 2x^2y^2) \tag{3.2}$$

where

$$G = \sum_{p\equiv 1(2),\ q\equiv 1(2)} \frac{\left(\begin{array}{c}\frac{1}{pq}\Big(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\Big) \\ + \frac{2pqxy}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}\end{array}\right)^2}{p^2+q^2-a^2-b^2}$$

if $a+b$ is even; and

$$G = \sum_{p\equiv 1(2), q\equiv 1(2)} \frac{\Big(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\Big)^2}{(p^2+q^2-a^2-b^2)p^2q^2} + \sum_{p\equiv 0(2), q\equiv 0(2)} \frac{\Big(\frac{2pqxy}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}\Big)^2}{p^2+q^2-a^2-b^2},$$

if $a+b$ is odd.

We remark that H defined by (3.2) is non-degenerate for generic $a_2(\lambda^*)$, $a_3(\lambda^*)$. We compute the homogeneous polynomial H following the definition (2.4).

Proof. Remark that

$$\begin{aligned}&\int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^3 ds\, dt \\ =&\big[(x^3+y^3)I_{3,0}(a,b)I_{0,3}(a,b) + 3xy(x+y)I_{2,1}(a,b)I_{1,2}(a,b)\big].\end{aligned} \tag{3.3}$$

When ab is odd, this is

$$16\Big[\frac{x^3+y^3}{9ab} - \frac{3abxy(x+y)}{(4a^2-b^2)(a^2-4b^2)}\Big] = \frac{16(x+y)}{9ab}\Big[x^2+y^2 - \frac{2(2a^2+b^2)(a^2+2b^2)xy}{(a^2-4b^2)(4a^2-b^2)}\Big],$$

since $I_{3,0}(a,b) = 4/3a$ if a is odd; $I_{2,1}(a,b) = 4a^2/b(4a^2-b^2)$ if b is odd. By Example 2.8, we observe that they define transcritical bifurcation models of type $(4,4)$.

If ab is even, then (3.3) is zero, since $I_{3,0}(a,b) = 0$ (a is even) and $I_{2,1}(a,b) = 0$ (b is even). So we have the case that $\phi(P(u)^3) = 0$. We continue to compute H following (2.4). We have computed the term $\phi(P(u)^4)$ by (3.1). Setting $I(a,b,p) = \int_0^\pi \sin as \sin bs \sin ps\, ds$, we have

$$I(a,b,p) = \begin{cases} 0 & (a+b+p \text{ is even}), \\ \frac{4abp}{(a+b+p)(a+b-p)(a-b+p)(a-b-p)} & (a+b+p \text{ is odd}). \end{cases}$$

For (p,q) with $p^2+q^2 \neq \lambda^*$, we have

$$\begin{aligned}
&\int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^2 \sin ps \sin qt\, ds\, dt \\
=& x^2 I(a,a,p)I(b,b,q) + 2xy\, I(a,b,p)I(a,b,q) + y^2 I(b,b,p)I(a,a,q) \\
=& \frac{4(1-(-1)^p)(1-(-1)^q)a^2b^2}{pq}\Big(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\Big) \\
&+ \frac{8(1-(-1)^{a+b+p})(1-(-1)^{a+b+q})a^2b^2pq}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}xy
\end{aligned}$$

$$= \begin{cases} 0, & (a+b \text{ is even}, pq \text{ is even,}) \\ 16a^2b^2 \left(\begin{array}{l} \frac{1}{pq}\big(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\big) \\ + \frac{2a^2b^2pqxy}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)} \end{array} \right), & (a+b \text{ is even}, pq \text{ is odd,}) \\ \frac{32a^2b^2pq}{((a+b)^2-p^2)((a-b)^2-p^2)((a+b)^2-q^2)((a-b)^2-q^2)}xy, & (a+b \text{ is odd}, pq \text{ is even,}) \\ \frac{16a^2b^2}{pq}\big(\frac{x^2}{(4a^2-p^2)(4b^2-q^2)} + \frac{y^2}{(4a^2-q^2)(4b^2-p^2)}\big), & (a+b \text{ is odd}, pq \text{ is odd.}) \end{cases}$$

By (2.4), we obtain (3.2). Comparing with Example 2.9, we complete the proof. $\square$

Approximations of $(16a^2b^2)^2G$ are given by the following table:

λ^*	$(16a^2b^2)^2G$	(b_-, b_+)
$5 = 1^2+2^2$	$-0.437133(x^2+y^2)^2 + 1.08885x^2y^2$	$(1,9)$
$13 = 2^2+3^2$	$-0.296234(x^2+y^2)^2 - 0.00192049x^2y^2$	$(1,9)$
$17 = 1^2+4^2$	$-0.112539(x^2+y^2)^2 + 0.638932x^2y^2$	$(5,5)$
$20 = 2^2+4^2$	$-0.111457(x^2+y^2)^2 - 0.512649x^2y^2$ $-0.207558xy(x^2+y^2)$	$(1,9)$
$25 = 3^2+4^2$	$0.526489(x^2+y^2)^2 - 0.331983x^2y^2$	$(9,1)$
$29 = 2^2+5^2$	$-0.12589(x^2+y^2)^2 + 0.614732x^2y^2$	$(5,5)$
$37 = 1^2+6^2$	$-0.0548666(x^2+y^2)^2 + 0.215801x^2y^2$	$(1,9)$
$40 = 2^2+6^2$	$-0.0595494(x^2+y^2)^2 + 0.158775x^2y^2$ $+0.0276499xy(x^2+y^2)$	$(1,9)$
$41 = 4^2+5^2$	$0.0254434(x^2+y^2)^2 - 0.406293x^2y^2$	$(5,5)$
$45 = 3^2+6^2$	$-0.00459466(x^2+y^2)^2 - 0.126777x^2y^2$	$(1,9)$
$52 = 4^2+6^2$	$-0.22101(x^2+y^2)^2 + 0.106694x^2y^2$ $+0.185669xy(x^2+y^2)$	$(1,5)$

The polynomials in the table happen to be non-degenerate in the sense of Definition 2.1. The numbers of semi-branches of the bifurcation model with $H = (16a^2b^2)^2G$ are also given in the table. These data unable us to determine the bifurcation when $a_3(\lambda) = 0$.

Theorem 3.3. *For $k = 5$ and for an eigenvalue λ^* of multiplicity 2, the homogeneous polynomial H is non-degenerate. If $a_5(\lambda^*) > 0$ (resp. $a_5(\lambda^*) < 0$), then the bifurcation at the point $(\lambda^*, 0)$ is plurisupercritical (resp. plurisubcritical) bifurcation of type $(b_-, b_+) = (1, 9)$ (resp. $(b_-, b_+) = (9, 1)$).*

Proof. We first remark that $I_{6,0}(a, b) = \frac{5\pi}{16}$,

$$I_{5,1}(a,b) = \begin{cases} -\frac{5\pi}{32} \ (b = 3a), \\ \frac{\pi}{32} \quad (b = 5a), \\ 0 \ (\text{otherwise}), \end{cases} \quad I_{4,2}(a,b) = \begin{cases} \frac{5\pi}{32} \ (b = 2a), \\ \frac{3\pi}{16} \ (\text{otherwise}), \end{cases} \quad I_{3,3}(a,b) = \begin{cases} -\frac{3\pi}{32} \ (b = 3a), \\ 0 \ (\text{otherwise}). \end{cases}$$

These imply that

$$\begin{aligned} &\int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^6 ds\, dt \qquad (3.4) \\ =&(x^6 + y^6)I_{6,0}(a, b)I_{0,6}(a, b) + 6xy(x^4 + y^4)I_{5,1}(a, b)I_{1,5}(a, b) \\ &+ 15x^2y^2(x^2 + y^2)I_{4,2}(a, b)I_{2,4}(a, b) + 20x^3y^3I_{3,3}(a, b)^2 \\ =&\Big(\frac{5\pi}{16}\Big)^2 \begin{cases} x^6 + y^6 + \frac{9}{2}x^2y^2(x^2 + y^2) & (b = 2a \text{ or } a = 2b) \\ x^6 + y^6 + \frac{27}{5}x^2y^2(x^2 + y^2) + \frac{9}{5}x^3y^3 & (b = 3a \text{ or } a = 3b) \\ x^6 + y^6 + \frac{27}{5}x^2y^2(x^2 + y^2) & (\text{otherwise}) \end{cases} \\ =&\Big(\frac{5\pi}{16}\Big)^2 \begin{cases} (x^2 + y^2)[(x^2 + y^2)^2 + \frac{3}{2}x^2y^2] & (b = 2a \text{ or } a = 2b) \\ (x^2 + y^2)^3 + \frac{12}{5}x^2y^2(x^2 + y^2) + \frac{9}{5}x^3y^3 & (b = 3a \text{ or } a = 3b), \\ (x^2 + y^2)[(x^2 + y^2)^2 + \frac{12}{5}x^2y^2] & (\text{otherwise}). \end{cases} \end{aligned}$$

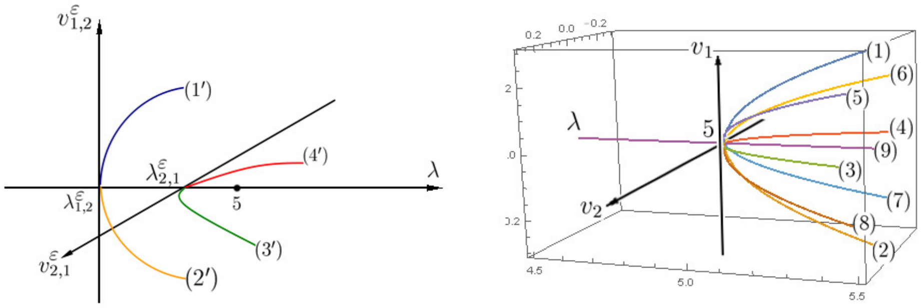
The remaining assertions are routine calculation. □

Remark 3.4. Assume that $k = 4$ and $\lambda^* = a^2 + b^2$, $(a, b) \in \mathbb{N}^2$, is an eigenvalue of $-\Delta$ with multiplicity 2. Remark that

$$\begin{aligned} &\int_0^\pi \int_0^\pi (x \sin as \sin bt + y \sin bs \sin at)^5 ds\, dt \qquad (3.5) \\ =&(x^5 + y^5)I_{5,0}(a, b)I_{0,5}(a, b) + 5xy(x^3 + y^3)I_{4,1}(a, b)I_{1,4}(a, b) + 20x^2y^2(x + y)I_{3,2}(a, b)I_{2,3}(a, b). \end{aligned}$$

When ab is odd, this is

$$\begin{aligned} &16 \left(\begin{array}{l} \frac{16}{15^2ab}(x^5 + y^5) + \frac{720a^2b^2xy(x^3+y^3)}{(4a^2-b^2)(16a^2-b^2)(a^2-4b^2)(a^2-16b^2)} \\ + \frac{160ab(5b^2-2a^2)(5a^2-2b^2)x^2y^2(x+y)}{9(4b^2-a^2)(9a^2-4b^2)(a+2b)(4a^2-b^2)(9b^2-4a^2)(2a+b)} \end{array} \right) \\ =&16 \left(\begin{array}{l} \frac{16}{15^2ab}(x + y)(x^2 + y^2)^2 + \frac{64(16a^8-325a^6b^2-1407a^4b^4-325a^2b^6+16b^8))}{15^2ab(a^2-4b^2)(4a^2-b^2)(16a^2-b^2)(a^2-16b^2)} xy(x + y)(x^2 + y^2) \\ + \frac{64(192a^{12}-1084a^{10}b^2-8642a^8b^4+35943a^6b^6-8642a^4b^8-1084a^2b^{10}+192b^{12})}{75ab(a^2-4b^2)(4a^2-b^2)(16a^2-b^2)(a^2-16b^2)(9a^2-4b^2)(4a^2-9b^2)}(x + y)x^2y^2 \end{array} \right), \end{aligned}$$



(a) Bifurcation model for the rectangle Ω_ε.

(b) Bifurcation model for the square Ω.

since $I_{5,0}(a,b) = 16/15a$ if a is odd; $I_{4,1}(a,b) = 3 \cdot 16a^4/b(4a^2-b^2)(16a^2-b^2)$ if b is even; $I_{3,2}(a,b) = 16b^2(5a^2-2b^2)/3a(4b^2-a^2)(9a^2-4b^2)(a+2b)$ if a is odd. Numerical experiences show that this homogeneous polynomial is non-degenerate and the bifurcation model is of type $(4,4)$ when $\lambda^* = 10, 26, 34, 58, 74, 82, 90, 106, 122, 146$. But this defines a bifurcation model of type $(6,6)$ when $\lambda^* = 178 = 3^2 + 13^2$.

If ab is even, then this integral (3.5) is 0, since $I_{5,0}(a,b) = 0$ (a even), $I_{4,1}(a,b) = 0$ (b even), $I_{3,2}(a,b) = 0$ (a even). In this case the integral (3.4) brings us a bifurcatuon model whenever $a_5(\lambda^*) \neq 0$.

3.2 Symmetry creates new bifurcation

In this section we are going to show that symmetry in the domain creates new bifurcations. To show it, let us consider the following Dirichlet problem

$$\Delta u = -\lambda u + a_3(\lambda)u^3 + o(u^3) \text{ on } \Omega_\varepsilon, \quad u|_{\partial\Omega_\varepsilon} = 0, \tag{3.6}$$

where $\Omega_\varepsilon = [0,\pi] \times [0,(1+\varepsilon)\pi]$. The rectangles Ω_ε converge to the square $\Omega = [0,\pi]^2$, and the eigenvalues $\lambda_1 = 1^2 + (\frac{2}{1+\varepsilon})^2$, $\lambda_2 = 2^2 + (\frac{1}{1+\varepsilon})^2$ on Ω_ε converge to the eigenvalue $\lambda^* = 5$ on Ω, as $\varepsilon \to 0$. For the bifurcation model, see the left figure below, where $v^\varepsilon_{1,2} = \sin(x)\sin(\frac{2}{1+\varepsilon}y)$ is orthogonal to the vector $v^\varepsilon_{2,1} = \sin(2x)\sin(\frac{1}{1+\varepsilon}y)$.

The eigenvalue 5 is of multiplicity 2 and our bifurcation model has the following solutions:

(1) $(\lambda, g_1(\lambda), 0)$,	(2) $(\lambda, -g_1(\lambda), 0)$,	(3) $(\lambda, 0, g_1(\lambda))$,	(4) $(\lambda, 0, -g_1(\lambda))$,
(5) $(\lambda, g_2(\lambda), g_2(\lambda))$,	(6) $(\lambda, -g_2(\lambda), g_2(\lambda))$,	(7) $(\lambda, -g_2(\lambda), -g_2(\lambda))$,	(8) $(\lambda, g_2(\lambda), -g_2(\lambda))$,

and the trivial solution (9) $(\lambda, 0, 0)$, where $g_1(\lambda) = \frac{4}{3\pi}\sqrt{\frac{\lambda-5}{a_3(\lambda^*)}}$, $g_2(\lambda) = \frac{4}{\pi}\sqrt{\frac{\lambda-5}{21a_3(\lambda^*)}}$. The following figures show the bifurcation of the solutions to (3.6) in Ω_ε and Ω, respectively.

Comparing the bifurcation model (a) with (b) as $\varepsilon \to 0$, there are 4 new semi-branches (5), (6), (7), (8) in (b), which do not come from the semi-branches in the model (a).

3.3 The first eigenvalue with $m = 3$

The first eigenvalue with multiplicity 3 is 50. Note that $50 = 1^2 + 7^2 = 2 \times 5^2$. When the order of the initial nonlinear term $a_k(\lambda^*)u^k$, $a_k(\lambda^*) \neq 0$, is $2, 3, 4, 5$, we show the data for the bifurcation models in the following table.

k	$\int_0^\pi \int_0^\pi (x \sin s \sin 7t + y \sin 7s \sin t + z \sin 5s \sin 5t)^{k+1} ds\, dt$	(b_-, b_+)
2	$\frac{16}{675} \begin{pmatrix} (x+y)(\frac{25(x^2+y^2)}{7} - \frac{374}{91}xy) + 7z(\frac{26250}{20449}xy - \frac{x^2+y^2}{19}) \\ + \frac{15625}{1309}(x+y)z^2 + z^3 \end{pmatrix}$	$(8,8)$
3	$\frac{3\pi^2}{256}(3(x^4+y^4+z^4) + 8(x^2y^2 + x^2z^2 + y^2z^2))$	$(1,27)$
4	$\frac{256}{441} \begin{pmatrix} 7(x+y)(\frac{x^4+y^4}{125} - \frac{99964xy(x^2+y^2)}{13996125} + \frac{421500766}{103977212625}x^2y^2) \\ +(\frac{16807(x^4+y^4)}{1081575} + \frac{790130684xy(x^2+y^2)}{1173261375} + \frac{4049858x^2y^2}{946785675})z \\ +2(x+y)(\frac{15625(x^2+y^2)}{63767} - \frac{58360350722xy}{135847622625})z^2 \\ +2(\frac{19531250xy}{97357689} - \frac{42189(x^2+y^2)}{3044275})z^3 + \frac{390625(x+y)}{1247103}z^4 + \frac{49z^5}{3125} \end{pmatrix}$	$(8,8)$
5	$\frac{5\pi^2}{1536} \begin{pmatrix} 5(x^6+y^6+z^6) + 27(x^4(y^2+z^2) + y^4(x^2+z^2) + z^4(x^2+y^2)) \\ +72x^2y^2z^2 - 9xy(x+y)z^3 \end{pmatrix}$	$(1,27)$

Here b_- (resp. b_+) is the number of semi-branches, with $\lambda < \lambda_*$ (resp. $\lambda > \lambda_*$), of the solution curves to zero of the bifurcation model corresponding to the homogeneous polynomials in the table. These data unable us to determine the number of solutions to Dirichelet problem (0.1) in the region $0 > a_k(\lambda)(\lambda - \lambda^*) > -|a_k(\lambda^*)|\varepsilon$ (resp. $0 < a_k(\lambda)(\lambda - \lambda^*) < |a_k(\lambda^*)|\varepsilon$) where ε is a sufficiently small positive number.

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