Bifurcation of Euler buckling problem, revisited

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Abstract

We present a precisation of Golubitsky and Schaefer's treatment on bifurcation of Euler buckling problem. We discuss smoothness of the problem and derive the equations describing bifurcation set B and hysteresis set H up to order 3, which unable us to draw their figures approximately under suitable set-up.

Bifurcation of solutions of partial differential equations or variational problems are one attractive field for application of singularity theory. M. Golubitsky and D. Schaefer ([2], [3]) showed how singularity theory works to investigate them.

We consider buckling of rod which is subjected to compressed force λ. In 1757, L. Euler found the critical load of this system, and it is often called Euler buckling problem. This is actually a famous example of pitchfolk bifurcation.

One mathematical formulation of this problem is the problem minimizing the energy

$$E = S + \lambda T, \quad S = \frac{1}{2}\int_0^l \left[\frac{u''}{(1-u'^2)^{1/2}}\right]^2 ds, \quad T = \int_0^l \sqrt{1-(u')^2}ds$$

on $U = \{u \in X : \|u'\|_\infty < 1\}$ where X is Sobolev space

$$X = \{u \in H^2[0,l] : u(0) = u(l) = 0\}.$$

Here S is the strain energy given by the integral of the square of curvature (remark that the curvature of the curve $s \mapsto (x(s), u(s))$ is $\frac{u''}{\sqrt{1-u'^2}}$), and T is the potential energy (the distance between two ends of the rod). This describes buckling of the rods with pinned ends. As the knowledge of the authors, this formulation was first appeared in [5, pages 27-29] without using Sobolev space, and the formulation using Sobolev space was appeared in [2, page 76].

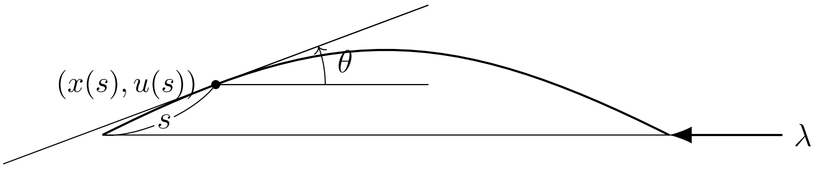


Figure 1: Buckling with pinned ends

We are interested in the bifurcation of the zero set of the directional derivatives:

$$(D_\phi E)_u = \lim_{t\to 0} \frac{E|_{u+t\phi} - E|_u}{t} = \int_0^l \Big[\frac{u''\phi''}{1-(u')^2} + \Big(\frac{u'(u'')^2}{(1-(u')^2)^2} - \frac{\lambda u'}{(1-(u')^2)^{1/2}}\Big)\phi'\Big] ds, \quad (0.1)$$

of directions $\phi \in X$ which may attain extreme of the total energy E. Clearly the function, which is identically zero, is a solution, and we often refer it as trivial solution, We are going to discuss the bifurcation from the trivial solution. By differentiating (0.1) by the direction v and evaluating at $u = 0$, we obtain

$$\int_0^l v''\phi'' ds - \lambda \int_0^l v'\phi' ds = \int_0^l (v'' + \lambda v)\phi'' ds.$$

The last equality is obtained by integration by parts. So the inverse function theorem implies there are no other solution near by the trivial solution whenever $v'' + \lambda v \neq 0$, that is, $\lambda \neq \pi^2 n^2/l^2$, $n \in \mathbb{Z}$. If $\lambda = \pi^2 n^2/l^2$, we apply Lyapunov-Schmidt reduction, and reduce the equation to finite dimensional set up.

M. Golubitsky and D. Schaeffer have also considered a modified version of this problem in [2, (6.1)], namely, the problem minimizing the modified energy

$$\frac{1}{2}\int_0^l \left[\frac{u''}{(1-u'^2)^{1/2}} - \alpha_1\right]^2 ds + \lambda \int_0^l \sqrt{1-(u')^2} ds + \alpha_2 u(\tfrac{l}{2}) \quad (0.2)$$

on U where the first term is a modified strain energy with minimum when curvature is constant α_1, i.e., the rod is a circular arc, and the third term represents a central load of size α_2. They showed that this modified problem represents a versal unfolding of the bifurcation equation of the original problem. To apply their criterion of versality ([2, Lemma 4.3]), we need to ensure the equation describing the problem is smooth (C^∞). Since we are in the context of a variational problem, it is not a priori clear, and proof was not discussed in loc. cite..

We actually consider a bit general problem, the variational problem minimizing the energy (2.1), obtained by replacing α_1 by $\alpha_1\kappa$ in (0.2) where κ is defined in (2.2), since we do not have any reason to assume that a circular arc minimizes the strain energy. This problem also has a term for modified strain energy, which has minimum at a curve with given curvature $\alpha_1\kappa$. After stating the first variational formula (Lemma 2.1), the problem becomes to describe the change of bifurcation of the zero of the function

$$\Phi : U \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow X^*, \ \Phi(u, \lambda, \alpha_1, \alpha_2) = [\phi \mapsto (\Psi - \lambda\Lambda - \alpha_1 K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2})].$$

See Lemma 2.1 for the definitions of Ψ, Λ, K.

We first show that

Theorem 0.1. *The function Φ is smooth.*

This theorem allows us to apply Lyapunov-Schmidt reduction to reduce the bifurcation problem to that of finite dimensional set-up. This theorem enables us to discuss the values of higher order differentials of Φ and we are going to apply the criterion of bifurcation type. In the paper, we compute Taylor coefficients of F, which is defined in (5.2), and which describes the bifurcation of $\Phi = 0$. This is an unfolding of pitchfolk bifurcation near $(x, \lambda, \alpha) = (0, \lambda^*, 0)$, $\lambda^* = \pi^2 n^2 / l^2$, and we show

Theorem 0.2. *If n is odd, then F is p-$\mathcal{K}$-versal.*

Roughly speaking, this implies all nearby bifurcation of a pitchfolk bifurcation can be realized by Φ near $(0, \lambda^*, 0)$. See Definition 7.1 for the precise definition on p-$\mathcal{K}$-versality. Remark that M. Golubitsky and D. Schaefer showed this theorem when $n = 1$

To describe how the pitchfolk bifurcation changes nearby the origin, we recall the bifurcation set B and hysteresis set H, which are defined by

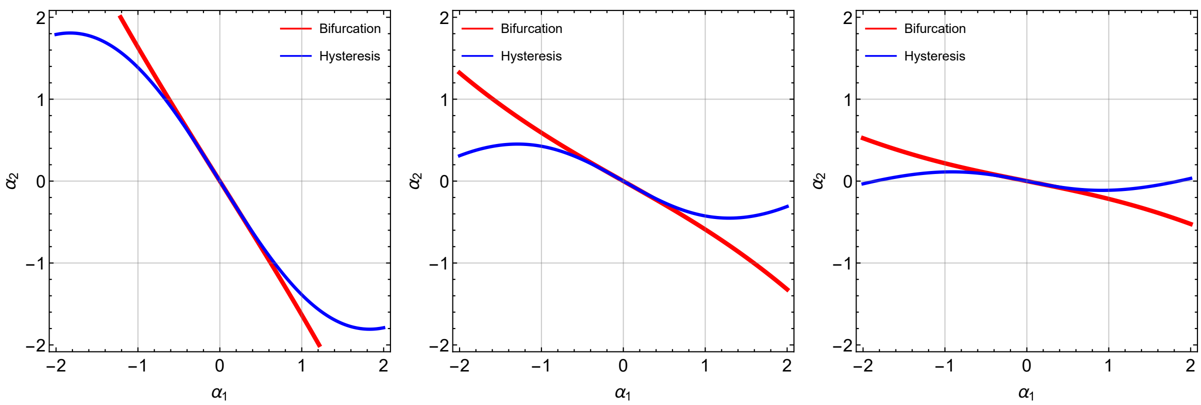
$$B = \{\alpha : \exists(x, \lambda)\ F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = F_\lambda(x, \lambda, \alpha) = 0\}, \tag{0.3}$$

$$H = \{\alpha : \exists(x, \lambda)\ F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = F_{xx}(x, \lambda, \alpha) = 0\} \tag{0.4}$$

in our situation. If n is odd, these sets are zeros of certain functions with the following 1-jet:

$$\left(\frac{4\pi n^2}{l^2} \sum_{i=0}^{\infty} \frac{n a_i}{n^2 - 4i^2}\right)\alpha_1 + \left((-1)^{\frac{n-1}{2}} \sqrt{\tfrac{2}{l}}\right)\alpha_2 \ (= \bar{F}_1 \alpha_1 + \bar{F}_2 \alpha_2, \quad \text{in later notation (§6)}).$$

In Proposition 8.1, we describe their 3-jets as (8.1) and (8.2), respectively, which enables us to draw B and H approximately near the origin. For example, $\kappa = 1/\sqrt{\pi/2}$, $n = 1$, the zeros of these 3-jets look like:



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 2: Approximations of B and H ($a_0 = 1, a_{i \geq 1} = 0$, $b_i = 0$)

The paper is organized as follows. In §1, we recall some basics on differentials on functions on Sobolev space etc. In §2, we derive Φ as Lemma 2.1, and show continuity of Φ. Theorems 0.1 is proved in §3. We compute the differential coefficients of Φ up to order 3 in §4. In §5, we apply Lyapunov-Schmidt reduction in our set-up. Theorem 0.2 is shown in §7. In §8, we show the 3-jet of certain functions which describe B and H. We present several numerical results, which are useful to describe the figures of B and H, in later half of §8. Lastly, we show several figures of zero of 3-jet of the above-mentioned functions which describe B and H near 0.

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1 Preliminaries

Let X, Y, Z, ... be Banach spaces and let X^*, Y^*, Z^*, ... be the dual spaces of X, Y, Z, ..., respectively. We denote by X', Y', ... the topological dual spaces of X, Y, ..., respectively. A multi-linear map

$$\psi : X \times \cdots \times X \longrightarrow Y^*, \quad (x_1, \dots, x_k) \mapsto (y \mapsto \psi[x_1, \dots, x_k] \cdot y)$$

is continuous and the image is in Y', if there is a positive constant C such that

$$|\psi[x_1, \dots, x_k] \cdot y| \leq C\|x_1\|_X \cdots \|x_k\|_X \|y\|_Y \quad \text{for any } x_1, \dots, x_k \in X, \text{ and } y \in Y.$$

Let $L(X \times \cdots \times X, Y')$ denote the set of such linear maps. A map $Z \to L(X \times \cdots \times X, Y')$, $z \mapsto \psi_z$, is continuous if there is a positive constant C such that

$$|(\psi_{z_1}[x_1, \dots, x_k] - \psi_{z_2}[x_1, \dots, x_k]) \cdot y| \leq C\|z_1 - z_2\|_Z \|x_1\|_X \cdots \|x_k\|_X \|y\|_Y$$

for any $x_1, \dots, x_k \in X$, $y \in Y$, and $z_1, z_2 \in Z$.

Let $\mathcal{F}[0, l]$ denote the set of function $[0, l] \to \mathbb{R}$ modulo the equivalence relation $\underset{\text{a.e.}}{\sim}$. Here $f \underset{\text{a.e.}}{\sim} g$ means f and g coincide except measure zero set for a function $f, g : [0, l] \to \mathbb{R}$. We consider Sobolev space $W^{k,p}[0, l] = \{u \in \mathcal{F}[0, l] : \|u\|_{k,p} < \infty\}$ equipped with Sobolev norm

$$\|u\|_{k,p} = \Big(\sum_{i=0}^{k} \binom{k}{i} \|D^i u\|_p^2\Big)^{\frac{1}{2}}, \quad \|u\|_p = \begin{cases} \big(\int_0^l |u|^p ds\big)^{1/p}, & 1 \leq p < \infty, \\ \max\{|u(s)| : s \in [0, l]\}, & p = \infty, \end{cases}$$

where $D^i u$ denote the ith order distributional derivatives of u. We denote by $L^p[0, l]$ for the set $\{u : [0, l] \to \mathbb{R} : \|u\|_p < \infty\}$. We denote by $H^k[0, l]$ for $W^{k,2}[0, l]$, which is a Hilbert space with the inner product

$$\langle u_i, u_j \rangle_k = \begin{cases} (1 + \frac{\pi^2 i^2}{l^2})^k, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Lemma 1.1. (i) If $1 \leq p \leq q < \infty$, then $\|u\|_p \leq l^{\frac{1}{p} - \frac{1}{q}} \|u\|_q$ for $u \in L^q[0, l]$.

(ii) If $p \geq 1$, then $\|u\|_\infty \leq C_p \|u\|_{1,p}$ for $u \in L^p[0, l]$ with $\|u\|_\infty < \infty$ where $C_p = \big(\frac{l}{2}\big)^{-\frac{1}{2}} \big(\sum_{m=1}^{\infty} (1 + \pi^2 m^2 / l^2)^{-p}\big)^{\frac{1}{2}}$.

Proof. (i): Since $q/p \geq 1$, $1 - p/q = (q-p)/q$, we obtain that, by Hölder's inequality,

$$\int_0^l |u^p|\, ds \leq \Big(\int_0^l |u^p|^{\frac{q}{p}}\, ds\Big)^{\frac{p}{q}} \Big(\int_0^l 1 ds\Big)^{1-\frac{p}{q}}.$$

Taking p-th roots of both sides, we obtain the result.
(ii): For $u = \sum_{m=1}^{\infty} y_m u_m$, we have

$$\begin{aligned}\Big|\sum_{m=1}^{\infty} y_m u_m\Big| &\leq \big(\tfrac{l}{2}\big)^{-\frac{1}{2}} \sum_{m=1}^{\infty} |y_m| = \big(\tfrac{l}{2}\big)^{-\frac{1}{2}} \sum_{m=1}^{\infty} (1+\pi^2 m^2/l^2)^{-\frac{p}{2}} (1+\pi^2 m^2/l^2)^{\frac{p}{2}} |y_m| \\ &\leq \big(\tfrac{l}{2}\big)^{-\frac{1}{2}} \Big(\sum_{m=1}^{\infty} (1+\pi^2 m^2/l^2)^{-p}\Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\infty} (1+\pi^2 m^2/l^2)^{p} |y_m|^2\Big)^{\frac{1}{2}} \leq C_p \|u\|_{1,p}. \qquad \square\end{aligned}$$

Let $C^k[0,l]$ be the set of C^k-functions defined on $[0,l]$.

Remark 1.2. (i) By Sobolev embedding theorem, we have $H^k[0,l] \subset C^{k-1}[0,l]$. In particular, we can choose C^{k-1}-represetative to express an element of $H^k[0,l]$.

(ii) We have a natural embedding $C^{k+1}[0,l] \subset H^k[0,l]$.

(iii) If $u = \sum_{m=0}^{\infty} y_m u_m \in C^k[0,l]$ where $u_m = \frac{1}{\sqrt{l/2}} \sin \frac{m\pi s}{l}$, then $|y_m| \leq M_k/m^k$ where $M_k = \sup\{|u^{(k)}(s)| : s \in [0,l]\}$.

(iv) If u is of C^2-class, then $\|u\|_\infty = \sup\{|u(s)|\} \leq \sum_{m=1}^{\infty} \frac{|y_m|}{\sqrt{l/2}} \leq \sum_{m=1}^{\infty} \frac{M_2}{m^2\sqrt{l/2}} < \infty$.

2 Calculation of variation

We consider the problem minimizing the functional

$$E : U \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad u \mapsto E(u,\lambda,\alpha), \quad \alpha = (\alpha_1, \alpha_2)$$

where $U = \{u \in X : \|u'\|_\infty < 1\}$, defined by

$$E(u,\lambda,\alpha) = \frac{1}{2}\int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1 \kappa\Big)^2 ds + \lambda \int_0^l \sqrt{1-(u')^2} ds + \alpha_2 u(l/2). \tag{2.1}$$

Here κ is a function defined by

$$\kappa = \frac{1}{\sqrt{l/2}}\Big[a_0 + \sum_{i=1}^{\infty}\Big(a_i \cos\frac{2i\pi s}{l} + b_i \sin\frac{2i\pi s}{l}\Big)\Big] \tag{2.2}$$

with $\|\kappa\|_\infty < \infty$. Since $u \in H^2[0,l]$, we can choose that u is C^1, and there is a constant ε_1, $0 < \varepsilon_1 < 1$, so that $|u'(s)| < 1 - \varepsilon_1$.

We consider the directional derivatives of E at u defined by

$$(dE)_{(u,\lambda,\alpha)} \cdot \phi = \lim_{t \to 0} \tfrac{1}{t}(E(u+t\phi,\lambda,\alpha) - E(u,\lambda,\alpha)).$$

Lemma 2.1. *We obtain that*

$$(dE)_{(u,\lambda,\alpha)} \cdot \phi = ((\Psi)_u - \lambda(\Lambda)_u) \cdot \phi - \alpha_1 (K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2})$$

where

$$\begin{aligned}
(\Psi)_u \cdot \phi &= \int_0^l \Big(\frac{u''\phi''}{(1-(u')^2)} + \frac{u'(u'')^2\phi'}{(1-(u')^2)^2} \Big) ds, \\
(\Lambda)_u \cdot \phi &= \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds, \\
(K)_u \cdot \phi &= \int_0^l \kappa \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds.
\end{aligned}$$

Proof. $(dE)_{(u,\lambda,\alpha)} \cdot \phi$

$$\begin{aligned}
=& \int_0^l \bigg(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1 \kappa \bigg) \Big(\frac{\phi'}{(1-(u')^2)^{1/2}} \Big)' ds - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds + \alpha_2 \phi(\frac{l}{2}) \\
=& \int_0^l \bigg(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1 \kappa \bigg) \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds \\
& - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds + \alpha_2 \phi(\frac{l}{2}) \\
=& \int_0^l \Big(\frac{u''\phi''}{1-(u')^2} + \frac{u'(u'')^2\phi'}{(1-(u')^2)^2} \Big) ds - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds \\
& - \alpha_1 \int_0^l \kappa \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds + \alpha_2 \phi(\tfrac{l}{2}) \\
=& ((\Psi)_u - \lambda(\Lambda)_u) \cdot \phi - \alpha_1 (K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2}).
\end{aligned}$$

□

Consider the map $\Phi = dE : X \times \mathbb{R} \times \mathbb{R}^2 \to X^*$ defined by

$$(u, \lambda, \alpha) \mapsto [\phi \mapsto (dE)_{(u,\lambda,\alpha)} \cdot \phi].$$

We first show

Lemma 2.2. *The image of Φ is in X'.*

Proof. Since

$$\begin{aligned}
|(\Psi)_u \cdot \phi| \leq& \int_0^l \Big(\big| \tfrac{u''\phi''}{1-(u')^2} \big| + \big| \tfrac{u'(u'')^2\phi'}{(1-(u')^2)^2} \big| \Big) ds \\
\leq& \big\| \tfrac{1}{1-(u')^2} \big\|_\infty \|u''\phi''\|_1 + \big\| \tfrac{u'\phi'}{(1-(u')^2)^2} \big\|_\infty \|(u'')^2\|_1 \\
\leq& \big\| \tfrac{1}{1-(u')^2} \big\|_\infty \|u''\|_2 \|\phi''\|_2 + \big\| \tfrac{u'}{(1-(u')^2)^2} \big\|_\infty \|u''\|_2^2 \|\phi'\|_\infty, \qquad (2.3) \\
|(\Lambda)_u \cdot \phi| \leq& \int_0^l \big| \tfrac{u'\phi'}{(1-(u')^2)^{1/2}} \big| ds \leq \big\| \tfrac{u'}{(1-(u')^2)^{1/2}} \big\|_2 \|\phi'\|_2, \qquad (2.4)
\end{aligned}$$

$$
\begin{aligned}
|(K)_u\cdot\phi| \le & \int_0^l \Big(\big|\tfrac{\kappa\phi''}{(1-(u')^2)^{1/2}}\big| + \big|\tfrac{\kappa u'u''\phi'}{(1-(u')^2)^{3/2}}\big|\Big)ds \\
\le & \big\|\tfrac{\kappa}{(1-(u')^2)^{1/2}}\big\|_\infty \|\phi''\|_1 + \big\|\tfrac{\kappa u'\phi'}{(1-(u')^2)^{3/2}}\big\|_2 \|u''\|_2, \\
\le & l^{1/2}\big\|\tfrac{\kappa}{(1-(u')^2)^{1/2}}\big\|_\infty \|\phi''\|_2 + \big\|\tfrac{\kappa u'}{(1-(u')^2)^{3/2}}\big\|_\infty \|\phi'\|_2\|u''\|_2, \qquad (2.5)\\
|\phi(l/2)| \le & \|\phi\|_\infty \le C_2\|\phi\|_{1,2} \quad \text{(since } \phi \text{ can be choosen continuously)}, \qquad (2.6)
\end{aligned}
$$

there is a positive constant C (may depend on u) such that

$$
|\Phi_{(u,\lambda,\alpha)}\cdot\phi| \le C\|\phi\|_{2,2}. \qquad \square
$$

Remark that a function is locally Lipschitz continuous if it is C^1. So the result in the following section implies Φ is locally Lipschitz continuous.

3 Smoothness of Φ

In this section, we show the following

Theorem 3.1. Φ *is* C^∞.

We first discuss several estimate before the proof of this theorem. First set

$$
A(x) = (1-x^2)^{-\frac{d}{2}} \sum_{i=0}^n a_i x^i, \qquad \text{where } a_i \in \mathbb{R}. \tag{3.1}
$$

We also set $|A|(x) = (1-x^2)^{-\frac{d}{2}}\sum_{i=0}^n |a_i|x^i$.

Lemma 3.2. *If* $\|u'\|_\infty \le \varepsilon < 1$, *then* $\|A(u')\|_\infty \le |A|(\varepsilon)$.

Proof. The estimate is obtained by

$$
\|A(u')\|_\infty \le \Big\|\frac{1}{1-(u')^2}\Big\|_\infty^{\frac{d}{2}} \sum_{i=1}^n |a_i|\|u'\|_\infty^i \le \frac{\sum_{i=0}^n |a_i|\varepsilon^i}{(1-\varepsilon^2)^{\frac{d}{2}}} = |A|(\varepsilon). \qquad \square
$$

Lemma 3.3. *If* $\|u_i'\|_\infty \le \varepsilon < 1$ *for* $i=1,2$, *then* $\|A(u_1') - A(u_2')\|_\infty \le C(A,\varepsilon)\|u_1'-u_2'\|_\infty$ *where*

$$
C(A,\varepsilon) = \frac{|A|(\varepsilon)}{(1-\varepsilon^2)^{\frac{d}{2}}}\Bigg[\sum_{s:2s<\max\{i,k\}} \Big|\binom{\frac{d}{2}}{s}\Big||2s-i|\varepsilon^{2s-1} + \frac{2}{(1-\varepsilon^2)^2}\Bigg].
$$

Proof. Since $\|A(u_1') - A(u_2')\|_\infty$

$$
\begin{aligned}
&\le \sum_{i=1}^n |a_i|\|(1-(u_2')^2)^{\frac{d}{2}}(u_1')^i - (1-(u_1')^2)^{\frac{d}{2}}(u_2')^i\|_\infty \Big\|\frac{1}{1-(u_1')^2}\Big\|_\infty^{\frac{d}{2}} \Big\|\frac{1}{1-(u_2')^2}\Big\|_\infty^{\frac{d}{2}} \\
&\le \sum_{i=1}^n |a_i| \Big\|\sum_{s=0}^\infty \binom{\frac{d}{2}}{s}\big[(u_1')^i(u_2')^{2s} - (u_1')^{2s}(u_2')^i\big]\Big\|_\infty \frac{1}{(1-\varepsilon^2)^d},
\end{aligned}
$$

the following estimate gives the result.

$$
\begin{aligned}
&\Big\|\sum_{s=0}^{\infty}\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)[(u_1')^i(u_2')^{2s}-(u_1')^{2s}(u_2')^i]\Big\|_\infty \le \sum_{s=0}^{\infty}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|(u_1')^i(u_2')^{2s}-(u_1')^{2s}(u_2')^i\|_\infty\\
=&\sum_{2s<i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|(u_1')^i(u_2')^{2s}-(u_1')^{2s}(u_2')^i\|_\infty+\sum_{2s>i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|(u_1')^i(u_2')^{2s}-(u_1')^{2s}(u_2')^i\|_\infty\\
=&\sum_{2s<i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|u_1'u_2'\|_\infty^{2s}\|(u_1')^{i-2s}-(u_2')^{i-2s}\|_\infty+\|u_1'u_2'\|_\infty^i\sum_{2s>i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|(u_1')^{2s-i}-(u_2')^{2s-i}\|_\infty\\
=&\|u_1'-u_2'\|_\infty\Big[\sum_{2s<i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\|u_1'\|_\infty^{2s}\|u_2'\|_\infty^{2s}\sum_{p+q=i-2s-1}\|u_1'\|_\infty^p\|u_2'\|_\infty^q\\
&\qquad\qquad+\|u_1'\|_\infty^i\|u_2'\|_\infty^i\sum_{2s>i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\sum_{p+q=2s-i-1}\|u_1'\|_\infty^p\|u_2'\|_\infty^q\Big]\\
=&\|u_1'-u_2'\|_\infty\Big[\sum_{2s<i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|\varepsilon^{4s}(i-2s)\varepsilon^{i-2s-1}+\varepsilon^{2i}\sum_{2s>i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|(2s-i)\varepsilon^{2s-i-1}\Big]\\
\le&\|u_1'-u_2'\|_\infty\varepsilon^i\Big[\sum_{2s<i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|(i-2s)\varepsilon^{2s-1}+\sum_{2s>i}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big|(2s-i)\varepsilon^{2s-1}\Big]\\
\le&\|u_1'-u_2'\|_\infty\varepsilon^i\Big[\sum_{2s<i}\Big|\binom{\frac{d}{2}}{s}\Big|(i-2s)\varepsilon^{2s-1}+\sum_{s:i<2s<d}\Big|\binom{\frac{d}{2}}{s}\Big|(2s-i)\varepsilon^{2s-1}+\sum_{2s>i,2s\ge d}(2s-i)\varepsilon^{2s-1}\Big]\\
\le&\|u_1'-u_2'\|_\infty\varepsilon^i\Big[\sum_{s:2s<\max\{i,d\}}\Big|\left(\begin{smallmatrix}\frac{d}{2}\\ s\end{smallmatrix}\right)\Big||2s-i|\varepsilon^{2s-1}+\frac{2}{(1-\varepsilon^2)^2}\Big].
\end{aligned}
$$

For the last inequality, we use the following inequality:

$$
\sum_{2s>i,2s\ge d}(2s-i)\varepsilon^{2s-1}=\left\{\begin{matrix}\frac{\varepsilon^i(1+\varepsilon^2)}{(1-\varepsilon^2)^2} & (i:\text{odd})\\ \frac{2\varepsilon^{i+1}}{(1-\varepsilon^2)^2} & (i:\text{even})\end{matrix}\right\}\le\frac{2\varepsilon^i}{(1-\varepsilon^2)^2}. \qquad \square
$$

We condsider a k-linear form $X\times\cdots\times X\to\mathbb{R}$, $(v_1,\dots,v_k)\mapsto I(u)[v_1,\dots,v_k]$, defined by

$$
I(u)[v_1,\dots,v_k]=\int_0^l A(u')\,(u'')^j v_1^{(i_1)}\cdots v_k^{(i_k)}ds
$$

where $A(x)$ is given by (3.1), and $i_1,\dots,i_k=1,2$.

Lemma 3.4. *If $j+i_1+\cdots+i_k\le k+2$, then there is a positive constant C such that*

$$
|I(u)[v_1,\dots,v_k]|\le C\|A(u')\|_\infty\|u\|_{2,2}^j\|v_1\|_{2,2}\cdots\|v_k\|_{2,2}. \tag{3.2}
$$

Proof. If $i_1,i_2=1,2$, then

$$
\Big|\int_0^l A(u')f_1^{(i_1)}f_2^{(i_2)}f_3'\cdots f_k'\,ds\Big|\le\|f_1^{(i_1)}\|_2\|A(u')f_2^{(i_2)}f_3'\cdots f_k'\|_2
$$

$$
\begin{aligned}
&\le \|f_1^{(i_1)}\|_2\|f_2^{(i_2)}\|_2\|A(u')f_3'\cdots f_k'\|_\infty\\
&\le \|f_1^{(i_1)}\|_2\|f_2^{(i_2)}\|_2\|A(u')\|_\infty\|f_3'\|_\infty\cdots\|f_k'\|_\infty\\
&\le C_2^{k-2}\|f_1^{(i_1)}\|_2\|f_2^{(i_2)}\|_2\|A(u')\|_\infty\|f_3'\|_{1,2}\cdots\|f_k'\|_{1,2}\\
&\le C_2^{k-2}\|A(u')\|_\infty\|f_1\|_{2,2}\cdots\|f_k\|_{2,2}.
\end{aligned}
$$

If $j+i_1+\cdots+i_k\le k+2$, then the number $\#\{a : i_a = 2\}$ is at most $2-j$, and we complete the proof by the estimate above. □

Remark 3.5. Set $B(x)=x^j$. Since $D_v(A(u')B(u''))=A_u'[v']B(u'')+A(u')B_u'[v'']$, we obtain that

$$I_1(u)[v_1,\dots,v_k,v]=\lim_{t\to 0}\frac{1}{t}(I(u+tv)[v_1,\dots,v_k]-I(u)[v_1,\dots,v_k])$$

is a $(k+1)$-linear form, which is a linear combination of integrals of the type

$$\int_0^l A(u')(u'')^{j'}v_1^{(i_1')}\cdots v_k^{(i_k')}v^{(i_{k+1}')}ds \quad (j',i_1',\dots,i_{k+1}'=1,2)$$

with $j'+i_1'+\cdots+i_k'+i_{k+1}'\le k+3$, whenever $j+i_1+\cdots+i_k\le k+2$. We thus obtain that

$$|I_1(u)[v_1,\dots,v_k,v]|\le C\|v_1\|_{2,2}\cdots\|v_k\|_{2,2}\|v\|_{2,2}$$

where C is a constant (depending on only u), by Lemma 3.4.

Lemma 3.6. *If $j+i_1+\cdots+i_k\le k+2$, then*

$$|I(u_1)[v_1,\dots,v_k]-I(u_2)[v_1,\dots,v_k]|\le C\|u_1-u_2\|_{2,2}\|v_1\|_{2,2}\cdots\|v_k\|_{2,2}$$

for some constant C which may depend on u_1, u_2 only.

Proof. If $j=2$, then we can assume that $i_1=\cdots=i_k=1$, and we obtain

$$
\begin{aligned}
\text{LHS}=&\Big|\int_0^l \big([A(u_1')-A(u_2')](u_2'')^2+A(u_1')[(u_1'')^2-(u_2'')^2]\big)v_1'\cdots v_k'ds\Big|\\
\le&\int_0^l\Big|\big([A(u_1')-A(u_2')](u_2'')^2+A(u_1')(u_1''-u_2'')(u_1''+u_2'')\big)v_1'\cdots v_k'\Big|ds\\
\le&C_2^k(\|A(u_1')-A(u_2')\|_\infty\|u_2\|_{2,2}^2+\|A(u_1')\|_\infty\|u_1-u_2\|_{2,2}\|u_1+u_2\|_{2,2})\|v_1\|_{2,2}\cdots\|v_k\|_{2,2},
\end{aligned}
$$

by Lemma 3.4. So the result follows by Lemmas 3.2 and 3.3.

If $j=1$, then we can assume that $i_2=\cdots=i_k=1$, and we obtain

$$
\begin{aligned}
\text{LHS}=&\Big|\int_0^l\big([A(u_1')-A(u_2')]u_2''+A(u_1')[u_1''-u_2'']\big)v_1^{(i_1)}v_2'\cdots v_k'ds\Big|\\
\le&C_2^{k-1}(\|A(u_1')-A(u_2')\|_\infty\|u_2\|_{2,2}+\|A(u_1')\|_\infty\|u_1-u_2\|_{2,2})\|v_1\|_{2,2}\cdots\|v_k\|_{2,2},
\end{aligned}
$$

by Lemma 3.4. So the result follows by Lemmas 3.2 and 3.3.

If $j=0$, then we can assume that $i_3=\cdots=i_k=1$, and we obtain

$$\text{LHS}=\Big|\int_0^l\big(A(u_1')-A(u_2')\big)v_1^{(i_1)}\cdots v_k^{(i_k)}ds\Big|\le C_2^{k-2}\|A(u_1')-A(u_2')\|_\infty\|v_1\|_{2,2}\cdots\|v_k\|_{2,2},$$

by Lemma 3.4. So the result follows by Lemma 3.3. □

Proof of Theorem 3.1. As in Remark 3.5, we see that the k-th order differential of Φ by u is of the form

$$\phi \mapsto (\Psi_k)_u[v_1, \dots, v_k] \cdot \phi = \int_0^l \sum_{j=0}^{2} \sum_{i_1,\dots,i_{k+1}=1}^{2} A^{\Psi}_{j,i_1,\dots,i_{k+1}}(u')(u'')^j v_1^{(i_1)} \cdots v_k^{(i_k)} \phi^{(i_{k+1})} ds$$

where $A^{\Psi}_{j,i_1,\dots,i_k,i_{k+1}}(u)$ are of the form (3.1) so that $j + i_1 + \cdots + i_k + i_{k+1} \leq k+3$. As Lemma 3.6, this is continuous.

The continuity of the higher order differentials of Φ containing differentiations by one of λ, α_1, α_2 can be shown similarly and we omit the details. □

Remark 3.7. In Lemma 3.4, it is important to assume that $j + i_1 + \cdots + i_k \leq k+2$. We do not know that (3.2) holds true when $j + i_1 + \cdots + i_k > k+2$, despite of the fact that the inequality changing $\| \cdot \|_{2,2}$ by $\| \cdot \|_{3,2}$ holds true.

4 Taylor coefficients of Φ

From now on, we denote by $(\Psi_k)_u$ the k-th order differential coefficient of $(\Psi)_u$ at u, and by Ψ_k the k-th order differential of Φ at $u = 0$. We denote by $(L_k)_u$ the k-th order differential coefficient of $(L)_u$ at u, and by L_k the k-th order differential of L at $u = 0$, and so on.

Lemma 4.1. *Set $(L)_u = (\Psi)_u - \lambda(\Lambda)_u$. The first derivative of $(L)_u$ at $u = 0$ is given by*

$$L_1[v] \cdot \phi = \int_0^l (v'' + \lambda v)\phi'' ds.$$

Proof.

$$\begin{aligned} L_1[v] \cdot \phi =&(\Psi_1[v] - \lambda\Lambda_1[v]) \cdot \phi = \int_0^l v''\phi'' ds - \lambda \int_0^l v'\phi' ds \\ =& \int_0^l v''\phi'' ds - \Big[v\phi'\Big]_0^l + \lambda \int_0^l v\phi'' ds = \int_0^l (v'' + \lambda v)\phi'' ds. \end{aligned}$$ □

Set $u_m = \sqrt{2/l}\sin(m\pi s/l)$ and u_m^* is an element in X^* defined by $u_m^*(u_j) = \delta_{i,j}$. If $\lambda^* = n^2\pi^2/l^2$, then

$$\begin{aligned} L_1[u_m] &= \frac{\pi^4}{l^4} m^2(m^2 - n^2)u_m^*, \\ L_1^{-1}[u_m^*], &= \frac{l^4}{\pi^4} \frac{1}{m^2(m^2 - n^2)} u_m^* \quad (m \neq n). \end{aligned}$$

Lemma 4.2. *When $\kappa = \frac{1}{\sqrt{l/2}}[a_0 + \sum_{i=1}^{\infty}\big(a_i \cos(2i\pi s/l) + b_i \sin(2i\pi s/l)\big)]$, we have*

$$K_0 = -\frac{\pi^2}{l^2}\Big[\sum_{m:\ odd} \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{m^3 a_i}{m^2 - 4i^2} u_m^* + \sum_{m:\ even} m^2 b_{m/2} u_m^*\Big], \quad K_1[u_a] = 0 \quad \textit{and}$$

$$K_2[u_a,u_b] = -\frac{ab\pi^3}{l^5}\sum_{i=0}^{\infty}\Big(a_i\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2=\pm1}\frac{(\varepsilon_1 a+\varepsilon_2 b+m)^2}{(\varepsilon_1 a+\varepsilon_2 b+m)^2-4i^2}\Big)u_m^*$$
$$+ b_i\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1}\varepsilon_3 i(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)u^*_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i}\Big).$$

Proof. For K_0, we have

$$K_0 = \sum_{m=1}^{\infty}\Big(\int_0^l \kappa u_m'' ds\Big)u_m^* = \sum_m\Big(\int_0^l \frac{a_0+\sum_{i=1}^{\infty}[a_i\cos(2i\pi s/l)+b_i\sin(2i\pi s/l)]}{\sqrt{l/2}}u_m'' ds\Big)u_m^*$$
$$= -\frac{\pi^2}{l^2}\Big[\sum_{m\equiv 1(2)}\frac{4}{\pi}\sum_{i=0}^{\infty}\frac{m^3 a_i}{m^2-4i^2}u_m^* + \sum_{m\colon \text{even}} m^2 b_{m/2}u_m^*\Big].$$

We remark that the second order differential of K at $u=0$ is given by $K_2[v_1,v_2]$: $\phi\mapsto\int_0^l \kappa(v_1'v_2'\phi''+(v_1'v_2''+v_1''v_2')\phi')ds$, that is,

$$K_2[v_1,v_2] = \sum_m\left(\int_0^l \kappa(v_1'v_2'u_m''+(v_1'v_2''+v_1''v_2')u_m')ds\right)u_m^*.$$

This implies that, if $\kappa=\frac{1}{\sqrt{l/2}}$, then $K_2[u_a,u_b] = -\frac{8ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} m u_m^*$;
if $\kappa=\frac{\cos\frac{2is\pi}{l}}{\sqrt{l/2}}$, then

$$K_2[u_a,u_b] = \sum_m\Big(\int_0^l\frac{\cos\frac{2is\pi}{l}}{\sqrt{l/2}}(u_a'u_b'u_m''+(u_a'u_b''+u_a''u_b')u_m')ds\Big)u_m^*$$
$$=\frac{ab\pi^3}{2l^5}\Big(\sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m\neq 0}((-1)^{a+b+m}-1)m\frac{\varepsilon_1 a+\varepsilon_2 b+m}{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m}\Big)u_m^*$$
$$=-\frac{ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1}\frac{\varepsilon_1 a+\varepsilon_2 b+m}{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m}\Big)u_m^*$$
$$=-\frac{ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2=\pm1}\frac{(\varepsilon_1 a+\varepsilon_2 b+m)^2}{(\varepsilon_1 a+\varepsilon_2 b+m)^2-4i^2}\Big)u_m^*;$$

and if $\kappa=\frac{\sin\frac{2is\pi}{l}}{\sqrt{l/2}}$, then

$$K_2[u_a,u_b] = \sum_m\Big(\int_0^l\frac{\sin\frac{2is\pi}{l}}{\sqrt{l/2}}(u_a'u_b'u_m''+(u_a'u_b''+u_a''u_b')u_m')ds\Big)u_m^*$$
$$=-\frac{ab\pi^3}{2l^5}\sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m=0}\varepsilon_3 m(\varepsilon_1 a+\varepsilon_2 b+m)u_m^*$$
$$=-\frac{ab\pi^3}{2l^5}\sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m=0}\varepsilon_3(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)2iu^*_{-\varepsilon_1 a-\varepsilon_2 b-2\varepsilon_3 i}$$
$$=-\frac{ab\pi^3}{l^5}\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1}\varepsilon_3 i(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)u^*_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i}.$$ □

Since $K_2[u_n, u_n] = \sum_m u_m^* \int_0^l \kappa \big[u_n' u_n' u_m'' + (u_n' u_n'' + u_n'' u_n') u_m'\big] ds$, we have

$$\begin{aligned}
K_2[u_n, u_n] \cdot u_n &= \int_0^l \kappa \Big[u_n' u_n' u_n'' + (u_n' u_n'' + u_n'' u_n') u_n'\Big] ds = 3 \int_0^l \kappa (u_n')^2 u_n'' ds \\
&= \begin{cases} -\frac{3\pi^3}{l^5} \sum_{i=0}^{\infty} \frac{8n^5(3n^2-4i^2)}{(n^2-4i^2)(9n^2-4i^2)} a_i, & n : \text{odd}, \\ -\frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2}), & n : \text{even}. \end{cases}
\end{aligned}$$

Lemma 4.3. *The second derivative of $(L)_u$ is*

$$(L_2)_u[v_1, v_2] \cdot \phi = \int_0^l [2u' v_2' v_1'' \phi'' + 2u'' \phi'' v_2' v_1' + 2u' v_1' v_2'' \phi'' + 2u'' v_1'' v_2' \phi' + 2v_2'' v_1'' \phi' u' + 2u'' v_2'' v_1' \phi'] ds.$$

So, setting $u = 0$, we obtain $L_2[v_1, v_2] \cdot \phi = 0$.

For the third order differential coefficient L_3 of L at $u = 0$, we have

$$L_3[v_1, v_2, v_3] = \sum_m \left[\int_0^l \begin{pmatrix} 2(v_1'' v_2' v_3' + v_1' v_2'' v_3' + v_1' v_2' v_3'') u_m'' \\ +(2(v_1'' v_2'' v_3' + v_1'' v_2' v_3'' + v_1' v_2'' v_3'') - 3\lambda v_1' v_2' v_3') u_m' \end{pmatrix} ds \right] u_m^*.$$

Lemma 4.4. *We have*

$$L_3[u_a, u_b, u_c] \cdot u_n = \frac{abcn^2\pi^5}{l^7} \sum_{\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \left[1 - \frac{3n}{2} + abcn\varepsilon_1\varepsilon_2\varepsilon_3 \Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c} \Big) \right]$$

where $\varepsilon_i = \pm 1$, $i = 1, 2, 3$.

Proof. Computing several integrals, we have

$$\begin{aligned}
L_3[u_a, u_b, u_c] \cdot u_n =& 2\frac{abcn^2\pi^5}{2l^7} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{-1, 1\}^3 : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
&+ 2\frac{a^2b^2c^2n\pi^5}{2l^7} \sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \varepsilon_1\varepsilon_2\varepsilon_3 \Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c} \Big) \\
&- 3\lambda^* \frac{abcn\pi^3}{2l^5} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
=& \frac{abcn^2\pi^5(2 - 3n)}{2l^7} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
&+ \frac{2a^2b^2c^2n^3\pi^5}{2l^7} \sum_{\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \varepsilon_1\varepsilon_2\varepsilon_3 \Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c} \Big) \\
=& \frac{abcn^2\pi^5}{l^7} \sum_{\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \left[1 - \frac{3n}{2} + abcn\varepsilon_1\varepsilon_2\varepsilon_3 \Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c} \Big) \right]. \qquad \square
\end{aligned}$$

Lemma 4.5. *We have $PL_3[u_n, u_n, v] = \frac{3n^6\pi^6}{2l^7}(u_n + 9u_{3n}) \cdot v$.*

Proof. $PL_3[u_n, u_n, v]$

$$
\begin{aligned}
&=\int_0^l [2(u_n'u_n'v''+2u_n'u_n''v')u_n''+(2(u_n''u_n''v'+2u_n'u_n''v'')-3\lambda^*u_n'u_n'v')u_n']ds\\
&=3\int_0^l [2(u_n')^2u_n''v''+(2u_n'(u_n'')^2-\lambda^*(u_n')^3)v']ds\\
&=\frac{6n^6\pi^6}{2l^7}(u_n+9u_{3n})\cdot v+\frac{6n^6\pi^6}{2l^7}(u_n-3u_{3n})\cdot v-3\lambda^*\frac{3n^4\pi^4}{2l^5}(u_n+u_{3n})\cdot v\\
&=\frac{3n^6\pi^6}{2l^7}(u_n+9u_{3n})\cdot v.
\end{aligned}
$$

□

5 Lyapunov-Schmidt reduction

When $\lambda^* = (n\pi/l)^2$, $u_n = \sqrt{2/l}\sin(n\pi s/l)$ is a non-zero function which generates the kernel of $L_1 = \Psi_1 - \lambda^*\Lambda_1$. Thus the orthogonal projection of X' to $\operatorname{Ker} L_1$ is

$$
P : X \longrightarrow X, \quad u \mapsto \frac{\langle u, u_n\rangle_2}{\langle u_n, u_n\rangle_2}u_n, \quad \text{and} \quad Q : X \longrightarrow X, \quad u \mapsto u - P(u),
$$

is the orthogonal projection of X' to $(\operatorname{Ker} L_1)^\perp$, the orthogonal complement to $\operatorname{Ker} L_1$.

The equation $\Phi(u, \lambda, \alpha) = 0$ is equivalent that

$$
P\Phi(u, \lambda, \alpha) = 0, \text{ and } Q\Phi(u, \lambda, \alpha) = 0.
$$

Observe that the differential map $(\operatorname{Ker} L_1)^\perp \to (\operatorname{Ker} L_1)^\perp$, $v \mapsto D_vQ\Phi$, at $(0, \lambda^*, 0)$ is given by

$$
v \mapsto \Big[\phi \mapsto \int_0^l (v'' + \lambda^* v)\phi'' ds\Big],
$$

which is an isomorphism. By implicit function theorem [1, 2.5.7], the later equation defines a function

$$
W : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to (\operatorname{Ker} L_1)^\perp \subset X, \qquad (x, \lambda, \alpha) \mapsto W(x, \lambda, \alpha)
$$

by

$$
Q\Phi(xu_n + W(x, \lambda, \alpha), \lambda, \alpha) = 0 \quad \text{near } (0, \lambda^*, 0). \tag{5.1}
$$

Lyapunov-Schmidt reduction says that the bifurcation of zero of $\Phi(u, \lambda, \alpha)$ is described by the zero of $F(x, \lambda, \alpha)$ where

$$
F(x, \lambda, \alpha) = P\Phi(xu_n + W(x, \lambda, \alpha), \lambda, \alpha). \tag{5.2}
$$

5.1 The first order derivatives of W

Lemma 5.1. *The differential coefficients of W at $(0, \lambda^*, 0)$ are given as follows: $\bar{W}_x = 0$, $\bar{W}_\lambda = 0$,*

$$
\bar{W}_1 = -\frac{l^2}{\pi^2}\Big[\frac{4}{\pi}\sum_{\substack{m:\,odd\\ m\neq n}}\frac{m}{m^2-n^2}\sum_{i=0}^{\infty}\frac{a_i}{m^2-4i^2}u_m^* + \sum_{\substack{m:\,even\\ m\neq n}}\frac{b_{m/2}}{m^2-n^2}u_m^*\Big], \quad \textit{and}
$$

$$\bar{W}_2 = -\frac{l^4}{\pi^4}\frac{1}{\sqrt{l/2}}\sum_{\substack{m:\,odd\\ m\neq n}}\frac{1}{m^2(m^2-n^2)}u_m^*.$$

Here we put a bar above a function to indicate evaluation at $(0,\lambda^*,0)$. *We also have that*

$$\begin{aligned}\bar{W}_1\alpha_1+\bar{W}_2\alpha_2 = &-\frac{l^2}{\pi^2}\sum_{\substack{m:\,odd\\ m\neq n}}\frac{1}{m^2-n^2}\Big(\frac{4\alpha_1}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2}+\frac{l^2}{\pi^2}\frac{\alpha_2}{m^2\sqrt{l/2}}\Big)u_m^*\\ &-\frac{l^2}{\pi^2}\sum_{\substack{m:\,even\\ m\neq n}}\frac{b_{m/2}}{m^2-n^2}u_m^*.\end{aligned}$$

Proof. We remark that

$$Q\Phi(u,\lambda,\alpha) = Q(L)_u - \alpha_1 Q(K)_u + \alpha_2 Q\delta, \quad Q(L)_u = Q(\Psi)_u - \lambda Q(\Lambda)_u \tag{5.3}$$

where $u = xu_n + W(x,\lambda,\alpha)$. Differentiating (5.1) by x, λ, α_1, α_2, we obtain that

$$Q(L_1)_u[u_n+W_x] - \alpha_1 Q(K_1)_u[u_n+W_x] = 0 \tag{5.4}$$

$$Q(L_1)_u[W_\lambda] - \alpha_1 Q(K_1)_u[W_\lambda] - Q\Lambda(u) = 0 \tag{5.5}$$

$$Q(L_1)_u[W_1] - \alpha_1 Q(K_1)_u[W_1] - Q(K)_u = 0 \tag{5.6}$$

$$Q(L_1)_u[W_2] - \alpha_1 Q(K_1)_u[W_2] + Q\delta = 0 \tag{5.7}$$

where $u = xu_n + W(x,\lambda,\alpha)$. We denote W_i for W_{α_i}, for shortness. We evaluate them at $(0,\lambda^*,0)$ and obtain

$$L_1[\bar{W}_x] = 0, \quad L_1[\bar{W}_\lambda] = 0, \quad L_1[\bar{W}_1] = QK_0, \quad L_1[\bar{W}_2] = -Q\delta.$$

Since $L_1 u_n = 0$. Thus we obtain that $\bar{W}_x = 0$, $\bar{W}_\lambda = 0$, $\bar{W}_1 = L_1^{-1}QK_0$, and $\bar{W}_2 = -L_1^{-1}Q\delta$, which conclude the results. □

5.2 The second order derivatives of W

By differentiating (5.4) by x, λ, α_1, α_2, we obtain

$$\begin{aligned}&Q(L_2)_u[u_n+W_x,u_n+W_x] + Q(L_1)_u[W_{xx}] - Q(K_2)_u[u_n+W_x,u_n+W_x]\\ &\qquad -Q(K_1)_u[W_{xx}] = 0,\\ &Q(L_2)_u[u_n+W_x,W_\lambda] + Q(L_1)_u[W_{x\lambda}] - Q(K_2)_u[u_n+W_x,W_\lambda] - Q(K_1)_u[W_{x\lambda}]\\ &\qquad -Q(\Lambda_1)_u[u_n+W_x] = 0,\\ &Q(L_2)_u[u_n+W_x,W_1] + Q(L_1)_u[W_{x1}] - Q(K_2)_u[u_n+W_x,W_1] - Q(K_1)_u[W_{x1}]\\ &\qquad -Q(K_1)_u[u_n+W_x] = 0,\\ &Q(L_2)_u[u_n+W_x,W_2] + Q(L_1)_u[W_{x2}] - Q(K_2)_u[u_n+W_x,W_2] - Q(K_1)_u[W_{x2}] = 0,\end{aligned}$$

and, by evaluating them at $(0,\lambda^*,0)$, we conclude

$$QL_1[\bar{W}_{xx}] = 0, \qquad QL_1[\bar{W}_{x\lambda}] = Q\Lambda_1[u_n] = 0, \qquad QL_1[\bar{W}_{x1}] = 0, \qquad QL_1[\bar{W}_{x2}] = 0.$$

By differentiating (5.5) by λ, α_1, α_2, we obatin

$$
\begin{aligned}
&Q(L_2)_u[W_\lambda, W_\lambda] + Q(L_1)_u[W_{\lambda\lambda}] - Q(K_2)_u[W_\lambda, W_\lambda] - Q(K_1)_u[W_{\lambda\lambda}] = Q(\Lambda_1)_u[W_\lambda],\\
&Q(L_2)_u[W_\lambda, W_1] + Q(L_1)_u[W_{\lambda 1}] - Q(K_1)_u[W_\lambda] - Q(K_2)_u[W_\lambda, W_{\lambda 1}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -Q(K_1)_u[W_1] = Q(\Lambda_1)_u[W_1],\\
&Q(L_2)_u[W_\lambda, W_2] + Q(L_1)_u[W_{\lambda 2}] - Q(K_2)_u[W_\lambda, W_2] - Q(K_1)_u[W_{\lambda 2}] = Q(\Lambda_1)_u[W_2],
\end{aligned}
$$

and, by evaluating them at $(0, \lambda^*, 0)$, we conclude

$$
QL_1[\bar{W}_{\lambda\lambda}] = 0, \quad QL_1[\bar{W}_{\lambda 1}] = Q\Lambda_1[\bar{W}_1] = Q\Lambda_1[\bar{W}_1], \quad QL_1[\bar{W}_{\lambda 2}] = Q\Lambda_1[\bar{W}_2] = Q\Lambda_1[\bar{W}_2].
$$

By differentiating (5.6) and (5.7) by α_1, α_2, we obtain

$$
\begin{aligned}
&Q(L_2)_u[W_1, W_1] + Q(L_1)_u[W_{11}] - Q(K_2)_u[W_1, W_1] - Q(K_1)_u[W_1] - Q(K_1)_u[W_{11}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -Q(K_1)_u[W_1] = 0,\\
&Q(L_2)_u[W_1, W_2] + Q(L_1)_u[W_{12}] - Q(K_2)_u[W_1, W_2] - Q(K_1)_u[W_{12}] - Q(K_1)_u[W_2] = 0,\\
&\qquad\quad Q(L_2)_u[W_2, W_2] + Q(L_1)_u[W_{22}] - Q(K_2)_u[W_2, W_2] - Q(K_1)_u[W_{22}] = 0,
\end{aligned}
$$

and, by evaluating them at $(0, \lambda^*, 0)$, we conclude

$$
QL_1[\bar{W}_{11}] - Q(K_1)[\bar{W}_1] - Q(K_1)[\bar{W}_1] = 0, \qquad QL_1[\bar{W}_{12}] = 0, \qquad QL_1[\bar{W}_{22}] = 0.
$$

We thus conclude that

$$
\begin{array}{lll}
\bar{W}_{xx} = 0, \qquad \bar{W}_{x\lambda} = 0, & \bar{W}_{x1} = 0, \qquad \bar{W}_{x2} = 0, & \bar{W}_{\lambda\lambda} = 0,\\
\bar{W}_{\lambda 1} = L_1^{-1} Q\Lambda_1[\bar{W}_1], & \bar{W}_{\lambda 2} = L_1^{-1} Q\Lambda_1[\bar{W}_2], & \\
\bar{W}_{11} = 0 \text{ (since } K_1 = 0), & \bar{W}_{12} = 0, & \bar{W}_{22} = 0.
\end{array}
$$

Set $k_m = \frac{4}{\pi}\sum_{i=0}^{\infty} \frac{m a_i}{m^2 - 4i^2}$, m odd; $b_{m/2}$, if m even. We look $\bar{W}_{\lambda 1}$ and $\bar{W}_{\lambda 2}$ closely and obtain

$$
\begin{aligned}
\bar{W}_{\lambda 1} =& L_1^{-1} Q\Lambda_1[\bar{W}_1] = L_1^{-1} Q\Lambda_1[\bar{W}_1] = \frac{\sqrt{2l}}{\pi} \sum_{m \neq n} \frac{k_m}{m^2 - n^2} L_1^{-1} \Lambda_1[u_m^*]\\
=& \frac{\sqrt{2l}}{\pi} \sum_{m \neq n} \frac{k_m}{m^2 - n^2} L_1^{-1} (m\pi/l)^2 u_m^* = \frac{\sqrt{2l}}{\pi} \sum_{m \neq n} \frac{k_m}{m^2 - n^2} \frac{(l/\pi)^2}{n^2 - m^2} (m\pi/l)^2 u_m^*\\
=& -\frac{\sqrt{2l}}{\pi} \sum_{m \neq n} \frac{m^2 k_m}{(m^2 - n^2)^2} u_m^*,\\
\bar{W}_{\lambda 2} =& L_1^{-1} Q\Lambda_1[\bar{W}_2] = L_1^{-1} Q\Lambda_1[\bar{W}_2] = (l/\pi)^2 \sqrt{2/l} \sum_{m: m \neq n, m:\ \text{odd}} L_1^{-1} Q\Lambda_1[u_m^*]\\
=& \sqrt{2/l} \sum_{m: m \neq n, m:\ \text{odd}} L_1^{-1} Q m^2 u_m^* = \sqrt{2/l} \sum_{m: m \neq n, m:\ \text{odd}} \frac{m^2}{n^2 - m^2} u_m^*.
\end{aligned}
$$

6 Bifurcation equation $F = 0$ and its Taylor coefficients

Now we consider the function

$$F(x,\lambda,\alpha) = P\Phi(xu_n + W(x,\lambda,\alpha),\lambda,\alpha) = P(L)_u - \alpha_1 P(K)_u + \alpha_2 P\delta \tag{6.1}$$

where $(L)_u = (\Psi)_u - \lambda(\Lambda)_u$, $u = xu_n + W(x,\lambda,\alpha)$. We denote F_i for F_{α_i}, F_{xi} for $F_{x\alpha_i}$, and so on.

6.1 The first order derivatives of F

Differentiating (6.1) by x, λ, α_1, α_2, we obtain that

$$F_x = P(L_1)_u[u_n + W_x] - \alpha_1 P(K_1)_u[u_n + W_x], \tag{6.2}$$
$$F_\lambda = P(L_1)_u[W_\lambda] - \alpha_1 P(K_1)_u[W_\lambda] - P(\Lambda)_u, \tag{6.3}$$
$$F_1 = P(L_1)_u[W_1] - \alpha_1 P(K_1)_u[W_1] - P(K)_u, \tag{6.4}$$
$$F_2 = P(L_1)_u[W_2] - \alpha_1 P(K_1)_u[W_2] + P\delta. \tag{6.5}$$

Evaluating them at $(0,\lambda^*,0)$, we have $\bar{F}_x = PL_1[u_n] = 0$, $\bar{F}_\lambda = 0$,

$$\bar{F}_1 = -PK_0 = \frac{\pi^2}{l^2}\Big[\sum_{m:\,\mathrm{odd}} \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{m^3 a_i}{m^2-4i^2}u_m^* + \sum_{m:\,\mathrm{even}} m^2 b_{m/2}u_m^*\Big]\cdot u_n \quad \text{(by Lemma 4.2)}$$
$$\bar{F}_2 = P\delta = u_n(l/2) = \begin{cases} (-1)^{\frac{n-1}{2}}\sqrt{2/l}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Here put a bar above a function to indicate evaluation at $(0,\lambda^*,0)$.

6.2 The second order derivatives of F

Differentiating (6.2) by x, λ, α_1, α_2, we obtain that

$$\begin{aligned} F_{xx} =& P(L_2)_u[u_n + W_x, u_n + W_x] + P(L_1)_u[W_{xx}] \\ &- \alpha_1 P(K_2)_u[u_n + W_x, u_n + W_x] - \alpha_1 P(K_1)_u[W_{xx}], \end{aligned} \tag{6.6}$$
$$\begin{aligned} F_{x\lambda} =& P(L_2)_u[u_n + W_x, W_\lambda] + P(L_1)_u[W_{x\lambda}] - P(\Lambda_1)_u[u_n + W_x] \\ &- \alpha_1 P(K_2)_u[u_n + W_x, W_\lambda] - \alpha_1 P(K_1)_u[W_{x\lambda}], \end{aligned} \tag{6.7}$$
$$\begin{aligned} F_{x1} =& P(L_2)_u[u_n + W_x, W_1] + P(L_1)_u[W_{x1}] \\ &- P(K_1)_u[u_n + W_x] - \alpha_1 P(K_2)_u[u_n + W_x, W_1] - \alpha_1 P(K_1)_u[W_{x1}], \end{aligned} \tag{6.8}$$
$$F_{x2} = P(L_2)_u[u_n + W_x, W_2] + P(L_1)_u[W_{x2}] - \alpha_1 P(K_2)_u[u_n + W_x, W_2] - \alpha_1 P(K_1)_u[W_{x2}]. \tag{6.9}$$

Evaluating them at $(0,\lambda^*,0)$, we have

$$\bar{F}_{xx} = 0, \qquad \bar{F}_{x\lambda} = -P\Lambda_1[u_n] = -\frac{n^2\pi^2}{l^2}, \qquad \bar{F}_{x1} = 0, \qquad \bar{F}_{x2} = 0.$$

Differentiating (6.3) by λ, κ, α_1, α_2, we obtain that

$$
\begin{aligned}
F_{\lambda\lambda} =& P(L_2)_u[W_\lambda, W_\lambda] + P(L_1)_u[W_{\lambda\lambda}] - \alpha_1 P(K_1)_u[W_{\lambda\lambda}] - P(\Lambda_1)_u[W_\lambda], && (6.10)\\
F_{\lambda 1} =& P(L_2)_u[W_\lambda, W_\lambda] + P(L_1)_u[W_{\lambda\lambda}] - P(\Lambda_1)_u[W_1]\\
&- P(K_1)_u[W_\lambda] - \alpha_1 P(K_2)_u[W_\lambda, W_1] - \alpha_1 P(K_1)_u[W_{\lambda\alpha_1}], && (6.11)\\
F_{\lambda 2} =& P(L_2)_u[W_\lambda, W_2] + P(L_1)_u[W_{\lambda 2}] - \alpha_1 P(K_2)_u[W_\lambda, W_2] - \alpha_1 P(K_1)_u[W_{\lambda 2}] - P(\Lambda_1)_u[W_2]. && (6.12)
\end{aligned}
$$

By evaluate them at $(0, \lambda^*, 0)$, we have

$$
\bar{F}_{\lambda\lambda} = 0, \qquad \bar{F}_{\lambda 1} = - P\Lambda_1[\bar{W}_1] = 0, \qquad \bar{F}_{\lambda 2} = P\Lambda_1[\bar{W}_2] = 0.
$$

Differentiating (6.3) and (6.4) by α_1, α_2, we obtain that

$$
\begin{aligned}
F_{11} =& P(L_2)_u[W_1, W_1] + P(L_1)_u[W_{11}] - \alpha_1 P(K_2)_u[W_1, W_1] - \alpha_1 P(K_1)_u[W_{11}] - 2P(K_1)_u[W_1], && (6.13)\\
F_{12} =& P(L_2)_u[W_1, W_2] + P(L_1)_u[W_{12}] - \alpha_1 P(K_2)_u[W_1, W_2] - \alpha_1 P(K_1)_u[W_{12}] - P(K_1)_u[W_2], && (6.14)\\
F_{22} =& P(L_2)_u[W_2, W_2] + P(L_1)_u[W_{22}] - \alpha_1 P(K_2)_u[W_2, W_2] - \alpha_1 P(K_1)_u[W_{22}]. && (6.15)
\end{aligned}
$$

Evaluating them at $(0, \lambda^*, 0)$, we have $\bar{F}_{11} = 0$, $\bar{F}_{12} = 0$, and $\bar{F}_{22} = 0$.

6.3 The third order derivatives of F

Lemma 6.1. $\bar{F}_{xxx} = \frac{3n^6\pi^6}{2l^7}$.

Proof. Differentiating (6.6) by x and evaluating them at $(0, \lambda^*, 0)$, we obtain that

$$
\begin{aligned}
\bar{F}_{xxx} =& PL_3[u_n, u_n, u_n] = \int_0^l 6((u_n')^2 u_n'' u_n'' + (u_n'(u_n'')^2)u_n')ds - 3\lambda^* \int_0^l (u_n')^3 u_n' ds\\
=& 12\int_0^l (u_n' u_n'')^2 ds - 3\lambda^* \int_0^l (u_n')^4 ds = 12\frac{n^6\pi^6}{2l^7} - 3\lambda^* \frac{3n^4\pi^4}{2l^5} = \frac{3n^6\pi^6}{2l^7}. \qquad \square
\end{aligned}
$$

Similarly, we obtain that

$$
\bar{F}_{xx\lambda} = 0,\ \bar{F}_{x\lambda 1} = 0,\ \bar{F}_{x\lambda 2} = 0,\ \bar{F}_{\lambda\lambda\lambda} = 0,\ \bar{F}_{\lambda 11} = 0,\ \bar{F}_{\lambda 12} = 0,\ \bar{F}_{\lambda 22} = 0,
$$

$$
\bar{F}_{x\lambda\lambda} = P\Lambda_1[L_1^{-1}Q\Lambda_1[u_n]] = 0,\ \bar{F}_{\lambda\lambda 1} = -P\Lambda_1[L_1^{-1}Q\Lambda_1[QK_0]] = 0,\ \bar{F}_{\lambda\lambda 2} = P\Lambda_1[L_1^{-1}Q\Lambda_1[Q\delta]] = 0.
$$

Lemma 6.2. *When n is odd,* $\bar{F}_{xx1} = \frac{3n^5\pi^3}{4l^5}\sum_{i=0}^{\infty}\frac{69n^2-20i^2}{(9n^2-4i^2)(n^2-4i^2)}a_i$*, and* $\bar{F}_{xx2} = -\frac{3n^2\pi^2}{16l^3}\sqrt{\frac{2}{l}}$*. When n is even,* $\bar{F}_{xx1} = \frac{3n^4\pi^4}{2l^5}b_{n/2} - \frac{3n^4\pi^4}{16l^5}b_{3n/2}$*, and* $\bar{F}_{xx2} = 0$.

Proof. Differentiating (6.6) by α_1 and evaluating them at $(0, \lambda^*, 0)$, we obtain that

$$
\bar{F}_{xx1} = PL_3[u_n, u_n, \bar{W}_1] - PK_2[u_n, u_n].
$$

When n is odd,

$$
\bar{F}_{xx1} = PL_3[u_n, u_n, \bar{W}_1] - PK_2[u_n, u_n] \quad \text{(by Lemma 4.5)}
$$

$$
\begin{aligned}
=&\frac{3n^6\pi^6}{2l^7}(\langle u_n, \bar{W}_1\rangle + 9\langle u_{3n}, \bar{W}_1\rangle) + \frac{3\pi^3}{l^5}\sum_{i=0}^{\infty}\frac{8n^5(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\
=&-\frac{27n^6\pi^6}{2l^7}\frac{l^2}{\pi^2}\sum_{m=3n}\frac{k_m}{m^2-n^2} + \frac{3\pi^3}{l^5}\sum_{i=0}^{\infty}\frac{8n^5(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\
=&-\frac{27n^4\pi^4}{2l^5}\frac{k_{3n}}{8} + \frac{3n^4\pi^3}{l^5}\sum_{i=0}^{\infty}\frac{8n(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\
=&\frac{3n^5\pi^3}{l^5}\left[-\frac{9}{4}\sum_{i=0}^{\infty}\frac{3a_i}{9n^2-4i^2} + 8\sum_{i=0}^{\infty}\frac{(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\right]\\
=&\frac{3n^5\pi^3}{4l^5}\sum_{i=0}^{\infty}\frac{69n^2-20i^2}{(9n^2-4i^2)(n^2-4i^2)}a_i.
\end{aligned}
$$

When n is even,

$$
\begin{aligned}
\bar{F}_{xx1} =&PL_3[u_n, u_n, \bar{W}_1] - PK_2[u_n, u_n] \quad \text{(by Lemma 4.5)}\\
=&\frac{3n^6\pi^6}{2l^7}(\langle u_n, \bar{W}_1\rangle + 9\langle u_{3n}, \bar{W}_1\rangle) + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2})\\
=&\frac{3n^6\pi^6}{2l^7}(-\frac{9l^2}{\pi^2}\frac{b_{3n/2}}{9n^2-n^2}) + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2})\\
=&-\frac{27n^4\pi^4}{16l^5}b_{3n/2} + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2}) = \frac{3n^4\pi^4}{2l^5}b_{n/2} - \frac{3n^4\pi^4}{16l^5}b_{3n/2}.
\end{aligned}
$$

Since $\bar{W}_2 = -\frac{l^4}{\pi^4}\sum_{\substack{m:\text{odd}\\ m\neq n}}\frac{\sqrt{2/l}}{m^2(m^2-n^2)}u_m^*$,

$$
\begin{aligned}
\bar{F}_{xx2} =&PL_3[u_n, u_n, \bar{W}_2] = \frac{3n^6\pi^6}{2l^7}(\langle u_n, \bar{W}_2\rangle + 9\langle u_{3n}, \bar{W}_2\rangle) \quad \text{(by Lemma 4.5)}\\
=&-\frac{3n^6\pi^6}{2l^7}\frac{l^4}{\pi^4}\sum_{m=3n,\, m:\text{odd}}\frac{\sqrt{2/l}}{m^2(m^2-n^2)} = \begin{cases} -\frac{3n^2\pi^2}{16l^3}\sqrt{\frac{2}{l}}, & n:\text{odd};\\ 0, & n:\text{even}.\end{cases}
\end{aligned}
$$

□

Lemma 6.3. *If we set* $C(\alpha) = \frac{1}{6}(\bar{F}_{111}\alpha_1^3 + 3\bar{F}_{112}\alpha_1^2\alpha_2 + 3\bar{F}_{122}\alpha_1\alpha_2^2 + \bar{F}_{222}\alpha_2^3)$*, then*

$$
C(\alpha) = \Big(\frac{1}{6}PL_3[u,u,u] - \frac{1}{2}\alpha_1 PK_2[u,u]\Big)\Big|_{u=\alpha_1\bar{W}_1+\alpha_2\bar{W}_2}.
$$

Proof. Differentiating (6.13), (6.14), (6.12) by α_1 and α_2 and evaluating them at $(0,\lambda^*,0)$, we obtain that

$$
\begin{aligned}
\bar{F}_{111} =&PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_1] - 3PK_2[\bar{W}_1,\bar{W}_1], & \bar{F}_{112} =&PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_2] - 2PK_2[\bar{W}_1,\bar{W}_2],\\
\bar{F}_{122} =&PL_3[\bar{W}_1,\bar{W}_2,\bar{W}_2] - PK_2[\bar{W}_2,\bar{W}_2], & \bar{F}_{222} =&PL_3[\bar{W}_2,\bar{W}_2,\bar{W}_2].
\end{aligned}
$$

□

Remark 6.4. As we will see in §8 the differential coefficients $\bar{F}_{x11}$, $\bar{F}_{x12}$, $\bar{F}_{x22}$ are not important to describe the equation of bifurcation set and hysteresis set up to order 3, and we will not investigate their exact values.

7 Versality

As shown in [2, (6.8)] (or lately in [4, 1.5 Theorem]), we have

$$\bar{F} = \bar{F}_x = \bar{F}_{xx} = \bar{F}_\lambda = 0, \quad \bar{F}_{xxx} \neq 0, \qquad \bar{F}_{x\lambda} \neq 0,$$

the bifurcation of $f(x, \lambda) = 0$ at $(0, \lambda^*)$, where $f(x, \lambda) = F(x, \lambda, 0)$, is a pitchfolk.

Definition 7.1. We say that an unfolding $F : (\mathbb{R}\times\mathbb{R}\times\mathbb{R}^k, (0, \lambda^*, 0)) \to (\mathbb{R}, 0)$, $(x, \lambda, \alpha) \mapsto F(x, \lambda, \alpha)$, of $f : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, $(x, \lambda) \mapsto f(x, \lambda)$, is **p-$\mathcal{K}$-versal**, if

$$\mathcal{E}_{x,\lambda}F + \mathcal{E}_{x,\lambda}F_x + \mathcal{E}_\lambda F_\lambda + \langle F_i|_{(x,\lambda,\alpha)=(0,\lambda^*,0)} : i = 1, \ldots, k\rangle_{\mathbb{R}} = \mathcal{E}_{x,\lambda}.$$

Here $\mathcal{E}_{x,\lambda}$, $\mathcal{E}_\lambda$ denote the ring of C^∞-function germs on $(\mathbb{R}^2, (0, \lambda^*))$, $(\mathbb{R}, \lambda^*)$ with variables (x, λ), and variable λ, respectively.

M. Golubitsky and D. Schaefer used the term "a universal unfolding" for this definition. We prefer to use the word "p-$\mathcal{K}$-versal", because it fits recent usage of terminologies in singularity theory.

Example 7.2. When $F(x, \lambda, \alpha_1, \alpha_2) = x^3 - \lambda x + \alpha_1 x^2 + \alpha_2$, we have $B = \{\alpha_2 = 0\}$ and $H = \{\alpha_1^3 = 27\alpha_2\}$. The bifurcation diagrams of the zeros of $f_\alpha(x, \lambda) = F(x, \lambda, \alpha)$ are shown as follows:

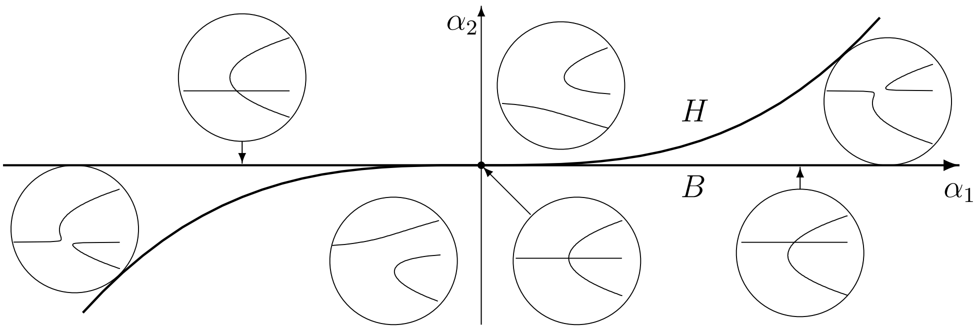


Figure 3: Bifurcation set B and hysteresis set H for Example 7.2

Lemma 7.3. *If n is odd, F is p-$\mathcal{K}$-versal.*

Proof. Since

$$\begin{vmatrix} \bar{F}_x & \bar{F}_{xx} & \bar{F}_{xxx} & \bar{F}_{x\lambda} \\ \bar{F}_\lambda & \bar{F}_{x\lambda} & \bar{F}_{xx\lambda} & \bar{F}_{\lambda\lambda} \\ \bar{F}_1 & \bar{F}_{x1} & \bar{F}_{xx1} & \bar{F}_{\lambda 1} \\ \bar{F}_2 & \bar{F}_{x2} & \bar{F}_{xx2} & \bar{F}_{\lambda 2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \bar{F}_{xxx} & \bar{F}_{x\lambda} \\ 0 & \bar{F}_{x\lambda} & \bar{F}_{xx\lambda} & \bar{F}_{\lambda\lambda} \\ \bar{F}_1 & 0 & \bar{F}_{xx1} & 0 \\ \bar{F}_2 & 0 & \bar{F}_{xx2} & 0 \end{vmatrix} = (\bar{F}_{x\lambda})^2 \begin{vmatrix} \bar{F}_1 & \bar{F}_{xx1} \\ \bar{F}_2 & \bar{F}_{xx2} \end{vmatrix} \neq 0,$$

F is p-$\mathcal{K}$-versal, by [2, Lemma 4.3] (see also (6.9) loc. cite.). $\square$

8 Bifurcation set and hysteresis set

Now we consider the bifurcation set of the zero of

$$F = \tfrac{x^3}{6}\bar{F}_{xxx} + \bar{F}_{x\lambda}\lambda x + \bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + \tfrac{x^2}{2}\ell(\alpha) + xQ(\alpha) + C(\alpha) + O(4),$$

where $\ell(\alpha) = \bar{F}_{xx1}\alpha_1 + \bar{F}_{xx2}\alpha_2$, $Q(\alpha) = \frac{1}{2}(\bar{F}_{x11}\alpha_1^2 + 2\bar{F}_{x12}\alpha_1\alpha_2 + \bar{F}_{x22}\alpha_2^2)$, and $C(\alpha)$ is defined in Lemma 6.3.

Proposition 8.1. *The bifurcation set B ((0.3)) and the hysteresis set H ((0.4)) are zeros of smooth functions with the following 3-jets*

$$\bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + C(\alpha), \text{ and} \tag{8.1}$$
$$\bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + C(\alpha) - \tfrac{2l^{14}}{27n^{12}\pi^{12}}\ell(\alpha)^3, \tag{8.2}$$

respectively.

Proof. Since $\bar{F}_{xxx} = \frac{3n^6\pi^6}{2l^7}$, we have

$$F_x = \tfrac{3n^6\pi^6}{4l^7}x^2 - \tfrac{n^2\pi^2}{l^2}\lambda + x\ell(\alpha) + Q(\alpha) + O(3),$$
$$F_\lambda = -\tfrac{n^2\pi^2}{l^2}x + O(3), \quad F_{xx} = \tfrac{3n^6\pi^6}{2l^7}x + \ell(\alpha) + O(3).$$

$F_x = F_\lambda = 0$ defines (x, λ) as a function of α and we obtain that

$$x = O(3), \qquad \lambda = \tfrac{l^2}{n^2\pi^2}Q(\alpha) + O(3).$$

Since $F - xF_x = -\frac{n^6\pi^6}{2l^5}x^3 + \bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 - \frac{x^2}{2}\ell(\alpha) + C(\alpha) + O(4)$, we obtain that the 3-jet of the equation for bifurcation set is (8.1).

Similarly $F_x = F_{xx} = 0$ defines (x, λ) as a function of α and we obtain that

$$x = -\tfrac{2l^7}{3n^6\pi^6}\ell(\alpha) + O(3), \qquad \lambda = O(2),$$

and thus the 3-jet of the equation for hysteresis points is (8.2). □

We present the data for $C(\alpha)$ (see Lemma 6.3) as follows: Set $k_m = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2}$, m odd; $b_{m/2}$, if m even.

$$PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_1] = -\frac{l^6}{\pi^6}\sum_{a,b,c\neq n}\frac{k_ak_bk_c}{(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c],$$

$$PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_2] = -\frac{l^6}{\pi^6}\sum_{c:\mathrm{odd},a,b,c\neq n}\frac{k_ak_b(l^2/\pi^2)\sqrt{2/l}}{c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c],$$

$$PL_3[\bar{W}_1, \bar{W}_2, \bar{W}_2] = -\frac{l^6}{\pi^6}\sum_{b,c:\mathrm{odd},a,b,c\neq n}\frac{k_a(l^4/\pi^4)2/l}{b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c],$$

$$PL_3[\bar{W}_2, \bar{W}_2, \bar{W}_2] = -\frac{l^6}{\pi^6}\sum_{a,b,c:\mathrm{odd},a,b,c\neq n}\frac{(l^6/\pi^6)(2/l)^{3/2}}{a^2b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c],$$

$$
\begin{aligned}
PK_2[\bar{W}_1, \bar{W}_1] =& \frac{l^4}{\pi^4} \sum_{a,b \neq n} \frac{k_a k_b}{(a^2-n^2)(b^2-n^2)} PK_2[u_a, u_b], \\
PK_2[\bar{W}_1, \bar{W}_2] =& \frac{l^6}{\pi^6} \sqrt{\frac{2}{l}} \sum_{a,b \neq n, b:\text{odd}} \frac{k_a}{b^2(a^2-n^2)(b^2-n^2)} PK_2[u_a, u_b], \\
PK_2[\bar{W}_2, \bar{W}_2] =& \frac{l^8}{\pi^8} \frac{2}{l} \sum_{a,b \neq n, \text{odd}} \frac{1}{a^2 b^2 (a^2-n^2)(b^2-n^2)} PK_2[u_a, u_b].
\end{aligned}
$$

Remaining part of this section, we describe numerical result on the data above to describe $C(\alpha)$ assuming $b_i = 0$ ($i \geq 1$) and $n = 1$. Remark that $k_m = \frac{4m}{\pi} \sum_{i=0}^{\infty} a_i/(m^2 - 4i^2)$, if m is odd; 0, if m is even, and we have

$$
\bar{W}_1 = -\frac{4l^2}{\pi^3} \sum_{m:\text{odd}, \neq n} \frac{m}{m^2-n^2} \sum_{i=0}^{\infty} \frac{a_i}{m^2-4i^2} u_m^*, \qquad \bar{W}_2 = -\frac{l^4}{\pi^4} \sum_{m:\text{odd}, \neq n} \frac{\sqrt{2/l}}{m^2(m^2-n^2)} u_m^*.
$$

We have

$$
\begin{aligned}
PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_1] =& \frac{c_0}{l\pi} \left(\frac{4}{\pi}\right)^3, & PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_2] =& \frac{lc_1}{\pi^3} \left(\frac{4}{\pi}\right)^2 \left(\frac{2}{l}\right)^{1/2}, \\
PL_3[\bar{W}_1, \bar{W}_2, \bar{W}_2] =& \frac{l^3 c_2}{\pi^5} \frac{4}{\pi} \frac{2}{l}, & PL_3[\bar{W}_2, \bar{W}_2, \bar{W}_2] =& \frac{l^5 c_3}{\pi^7} \left(\frac{2}{l}\right)^{3/2},
\end{aligned}
$$

where c_0, c_1, c_2, c_3 are constants. The approximate values of c_i are given by

$$
\begin{aligned}
c_0 \simeq& 0.305307 a_0^3 + 1.20457 a_0^2 a_1 + 0.556055 a_0^2 a_2 + 0.449847 a_0^2 a_3 + \cdots \\
&+ 1.5754 a_0 a_1^2 + 1.60049 a_0 a_1 a_2 + 1.23451 a_0 a_1 a_3 + \cdots \\
&+ 0.0536143 a_0 a_2^2 + 0.410507 a_0 a_2 a_3 - 0.0983358 a_0 a_3^2 + \cdots \\
&+ 0.683785 a_1^3 + 1.15217 a_1^2 a_2 + 0.821541 a_1^2 a_3 + \cdots \\
&- 0.121613 a_1 a_2^2 + 0.763853 a_1 a_2 a_3 - 0.154765 a_1 a_3^2 + \cdots \\
&+ 0.0918374 a_2^3 - 0.322925 a_2^2 a_3 + 0.0171554 a_2 a_3^2 + 0.0409826 a_3^2 + \cdots \\
c_1 \simeq& (0.0560462 a_0 + 0.147036 a_1 + 0.078606 a_2 + 0.0592183 a_3 + \cdots) a_0 \\
&+ (0.0965134 a_1 + 0.112754 a_2 + 0.0758876 a_3 + \cdots) a_1 \\
&+ (0.00853948 a_2 + 0.0472655 a_3 + \cdots) a_2 - 0.00887054 a_3^2 + \cdots \\
c_2 \simeq& 0.0105423 a_0 + 0.0141242 a_1 + 0.00815088 a_2 + 0.00496213 a_3 + \cdots, \\
c_3 \simeq& 0.00218564
\end{aligned}
$$

This numerical result follows computing the summations above with $m \leq 500$. We remark that the convergence of c_0 is very slow, and we are not sure how many digits are correct for this approximate value.

Since

$$
PK_2[u_a, u_b] = -\frac{abn\pi^3}{l^5} \sum_{i=0}^{\infty} a_i \sum_{a+b \not\equiv n(2)} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \frac{(\varepsilon_1 a + \varepsilon_2 b + n)^2}{(\varepsilon_1 a + \varepsilon_2 b + n)^2 - 4i^2}
$$

$$= -\frac{4abn\pi^3}{l^5}\sum_{i=0}^{\infty} a_i \sum_{a+b\not\equiv n(2)} \Big(1 + \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big),$$

we have, if n is odd,

$$\begin{aligned}
PK_2[\bar{W}_1,\bar{W}_1] =& \frac{l^4}{\pi^4}\sum_{a,b:\mathrm{odd},\neq n} \frac{(4/\pi)^2 ab}{(a^2-n^2)(b^2-n^2)} \sum_{i_1,i_2=0}^{\infty} \frac{a_{i_1}a_{i_2}}{(a^2-4i_1^2)(b^2-4{i_2}^2)} PK_2[u_a,u_b], \\
=& -\frac{64n}{\pi^3 l}\sum_{a,b:\mathrm{odd},\neq n} \frac{a^2b^2}{(a^2-n^2)(b^2-n^2)} \sum_{i,i_1,i_2=0}^{\infty} \frac{a_i a_{i_1}a_{i_2}}{(a^2-4i_1^2)(b^2-4{i_2}^2)} \\
& \times \Big(1 + \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big), \\
PK_2[\bar{W}_1,\bar{W}_2] =& \frac{l^6}{\pi^6}\sqrt{\frac{2}{l}}\sum_{a,b:\mathrm{odd},\neq n} \frac{(4/\pi)a}{b^2(a^2-n^2)(b^2-n^2)} \sum_{i_1=0}^{\infty} \frac{a_{i_1}}{a^2-4i_1^2} PK_2[u_a,u_b] \\
=& -\frac{16nl}{\pi^4}\sqrt{\tfrac{2}{l}}\sum_{a,b:\mathrm{odd},\neq n} \frac{a^2}{b(a^2-n^2)(b^2-n^2)} \sum_{i,i_1=0}^{\infty} \frac{a_{i_1}a_i}{a^2-4i_1^2}\Big(1 + \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \tfrac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big), \\
PK_2[\bar{W}_2,\bar{W}_2] =& \frac{l^8}{\pi^8}\frac{2}{l}\sum_{a,b\neq 1,\mathrm{odd}} \frac{1}{a^2b^2(a^2-n^2)(b^2-n^2)} PK_2[u_a,u_b] \\
=& -\frac{8nl^2}{\pi^5}\sum_{i=0}^{\infty} a_i \sum_{a,b\neq 1,\mathrm{odd}} \frac{1}{ab(a^2-n^2)(b^2-n^2)}\Big(1 + \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big).
\end{aligned}$$

Assume that $n=1$. Since

$$\sum_{\substack{a:\,\mathrm{odd}\\ a\neq 1}} \frac{a^2}{(a^2-1)(a^2-4i^2)} = \frac{12i^2+1}{4(4i^2-1)^2}, \qquad \sum_{a:\,\mathrm{odd},\,>1} \frac{1}{a(a^2-1)} = \frac{3}{4} - \log 2,$$

we obtain

$$\begin{aligned}
PK_2[\bar{W}_1,\bar{W}_1] =& -\frac{4}{\pi^3 l}\sum_{i,i_1,i_2=0}^{\infty} a_i a_{i_1} a_{i_2}\Big[\frac{(12i_1^2+1)(12i_2^2+1)}{(4i_1^2-1)^2(4i_2^2-1)^2} \\
& + \sum_{\substack{a,b:\mathrm{odd}\\ a,b\neq 1}} \frac{4^2 i}{(a^2-4i_1^2)(b^2-4{i_2}^2)}\frac{a^2b^2}{(a^2-1)(b^2-1)}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+1)^2-4i^2}\Big], \\
=& -\frac{4}{\pi^3 l}\sum_{i,i_1,i_2=0}^{\infty}\Big[a_i a_{i_1} a_{i_2}\frac{(12i_1^2+1)(12i_2^2+1)}{(4i_1^2-1)^2(4i_2^2-1)^2}
\end{aligned}$$

$$+a_1(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.0098018 & -0.0171753 & 0.00800604 & 0.00824388 & 0.00228896 & 0.00100968 & \cdots \\
-0.0171753 & -0.0306015 & 0.0169267 & 0.0144463 & 0.00298161 & 0.00129348 & \cdots \\
0.00800617 & 0.0169269 & -0.0241598 & -0.00438159 & 0.00500669 & 0.00110166 & \cdots \\
0.00824403 & 0.0144466 & -0.00438157 & -0.013504 & -0.00250348 & 0.00220992 & \cdots \\
0.00228912 & 0.00298184 & 0.00500672 & -0.00250347 & -0.00698926 & -0.00127408 & \cdots \\
0.00100984 & 0.00129372 & 0.00110169 & 0.00220993 & -0.00127408 & -0.00433055 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

$$
\begin{aligned}
&+a_2(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.00585423 & -0.00798357 & -0.0107953 & 0.00882541 & 0.0107354 & 0.00248558 & \cdots \\
-0.00798387 & -0.0107902 & -0.0175634 & 0.0166762 & 0.0177164 & 0.00217028 & \cdots \\
-0.0107948 & -0.0175626 & 0.00990841 & -0.01239 & -0.000800288 & 0.010955 & \cdots \\
0.00882601 & 0.0166771 & -0.0123899 & -0.00630149 & -0.011251 & -0.00120814 & \cdots \\
0.010736 & 0.0177173 & -0.000800176 & -0.011251 & -0.00633698 & -0.00721143 & \cdots \\
0.00248622 & 0.00217122 & 0.0109551 & -0.00120811 & -0.00721141 & -0.00368129 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} \\
&+a_3(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.010169 & -0.0173908 & 0.0098944 & -0.00878926 & 0.00988501 & 0.0125713 & \cdots \\
-0.0173914 & -0.0308041 & 0.0218814 & -0.0139982 & 0.0187884 & 0.0208632 & \cdots \\
0.00989557 & 0.0218833 & -0.0334693 & 0.00254736 & -0.0174167 & -0.00322272 & \cdots \\
-0.0087879 & -0.0139962 & 0.00254755 & 0.00573298 & -0.00157069 & -0.0105505 & \cdots \\
0.00988643 & 0.0187905 & -0.0174165 & -0.00157063 & -0.00275234 & -0.00445332 & \cdots \\
0.0125728 & 0.0208654 & -0.00322244 & -0.0105504 & -0.0044533 & -0.00391895 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} \\
&\quad + \cdots ,
\end{aligned}
$$

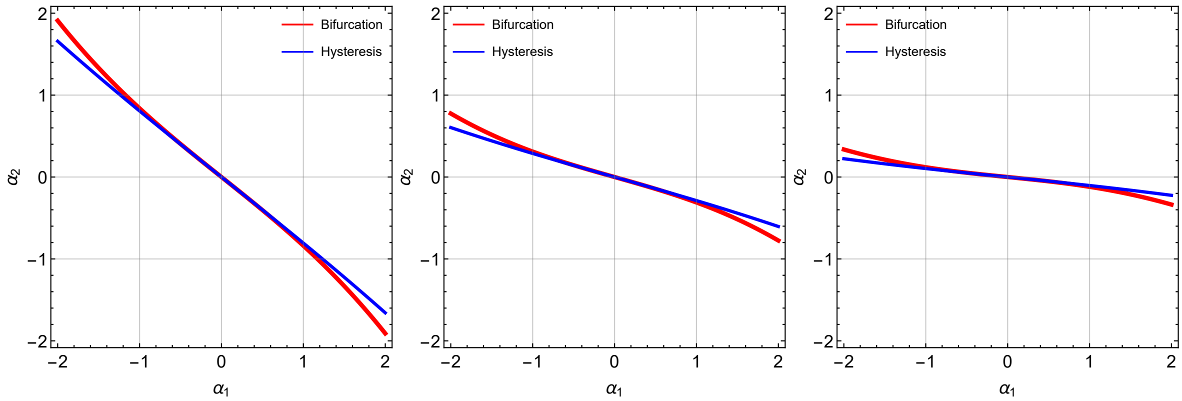
$$
\begin{aligned}
PK_2[\bar{W}_1, \bar{W}_2] = &- \frac{16l}{\pi^4}\sqrt{\tfrac{2}{l}} \sum_{i,i_1=0}^{\infty} a_i a_{i_1} \Big[\frac{12 i_1^2 + 1}{4(4i_1^2-1)^2}\Big(\frac{3}{4} - \log 2\Big) \\
&+ \sum_{\substack{a,b:\text{odd} \\ a,b\neq 1}} \frac{1}{a^2 - 4i_1^2} \frac{a^2}{b(a^2-1)(b^2-1)} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a + \varepsilon_2 b + 1)^2 - 4i^2}\Big], \\
\simeq &- \frac{16l}{\pi^4}\sqrt{\tfrac{2}{l}} \Big[\sum_{i,i_1=0}^{\infty} a_i a_{i_1} \frac{12 i_1^2 + 1}{4(4i_1^2-1)^2}\Big(\frac{3}{4} - \log 2\Big) \\
&- a_1 \begin{pmatrix} 0.00319975 a_0 + 0.00572494 a_1 - 0.00326835 a_2 \\ -0.00273496 a_3 - 0.000536221 a_4 - 0.00021051 a_5 + \cdots \end{pmatrix} \\
&- a_2 \begin{pmatrix} 0.00145234 a_0 + 0.00201291 a_1 + 0.00301828 a_2 \\ -0.00322987 a_3 - 0.00330068 a_4 - 0.000328207 a_5 + \cdots \end{pmatrix} \\
&- a_3 \begin{pmatrix} 0.00319406 a_0 + 0.00574215 a_1 - 0.00437078 a_2 \\ +0.00241329 a_3 + 0.00358896 a_4 - 0.00385563 a_5 + \cdots \end{pmatrix} + \cdots \Big],
\end{aligned}
$$

$$
\begin{aligned}
PK_2[\bar{W}_2, \bar{W}_2] = &- \frac{8l^2}{\pi^5} \sum_{i=0}^{\infty} a_i \Big[\Big(\frac{3}{4} - \log 2\Big)^2 + \sum_{\substack{a,b:\text{odd} \\ a,b\neq 1}} \frac{1}{ab(a^2-1)(b^2-1)} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a + \varepsilon_2 b + 1)^2 - 4i^2} \Big] \\
\simeq &- \frac{8l^2}{\pi^5} \Big[\sum_{i=0}^{\infty} a_i \Big(\frac{3}{4} - \log 2\Big)^2 - \begin{pmatrix} 0.00107265 a_1 + 0.000380642 a_2 + 0.00107825 a_3 \\ +0.00380099 a_4 + 0.00403667 a_5 + \cdots \end{pmatrix} \Big].
\end{aligned}
$$

Remark 8.2. If $a_0 = 1$, $a_i = 0$ $(i \geq 1)$, then

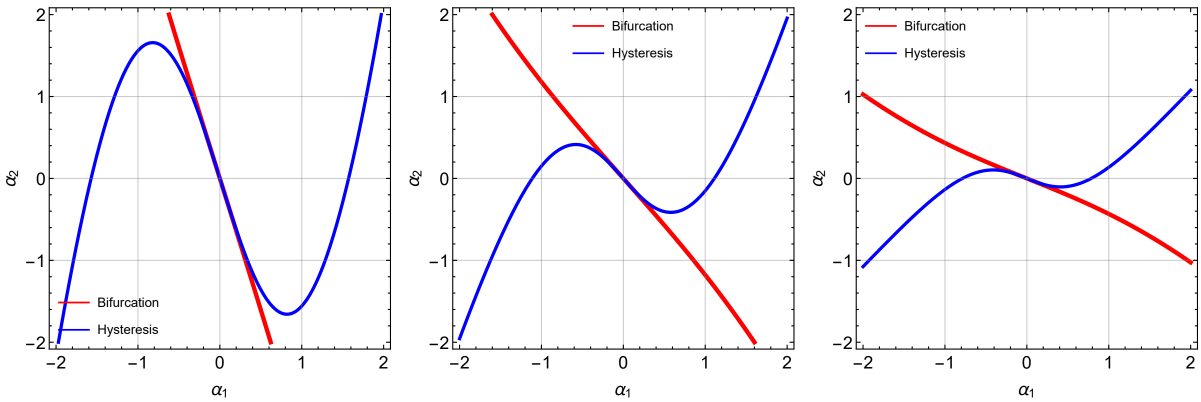
$$
PK_2[\alpha_1 \bar{W}_1 + \alpha_2 \bar{W}_2, \alpha_1 \bar{W}_1 + \alpha_2 \bar{W}_2] = -\frac{8}{l\pi}\Big(\frac{1}{\pi}\alpha_1 + \frac{3 - 4\log 2}{4} \frac{l^2}{\pi^2}\sqrt{\frac{2}{l}}\alpha_2 \Big)^2.
$$

We show below the figures of the zeros of (8.1) and (8.2) in several cases,



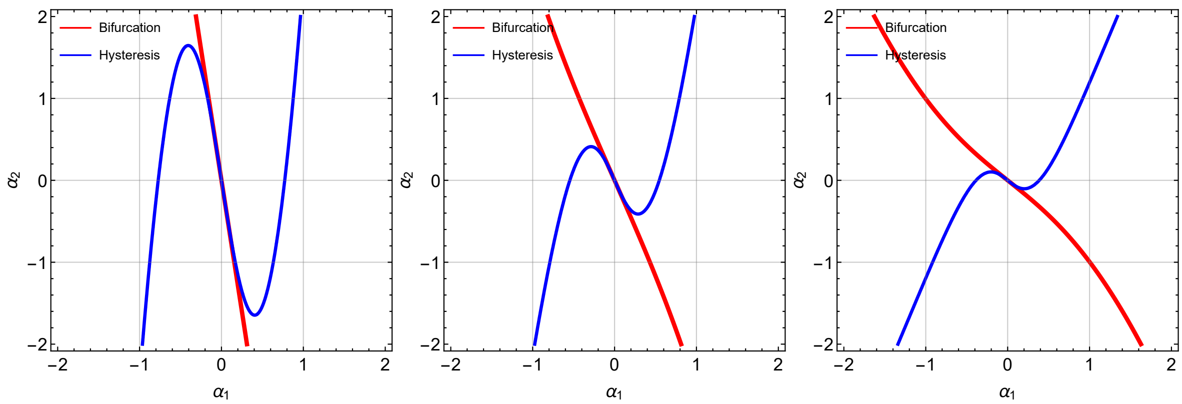
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 4: Approximations of B and H $(a_0 = 0.5, a_{i\geq 1} = 0,\ b_i = 0)$



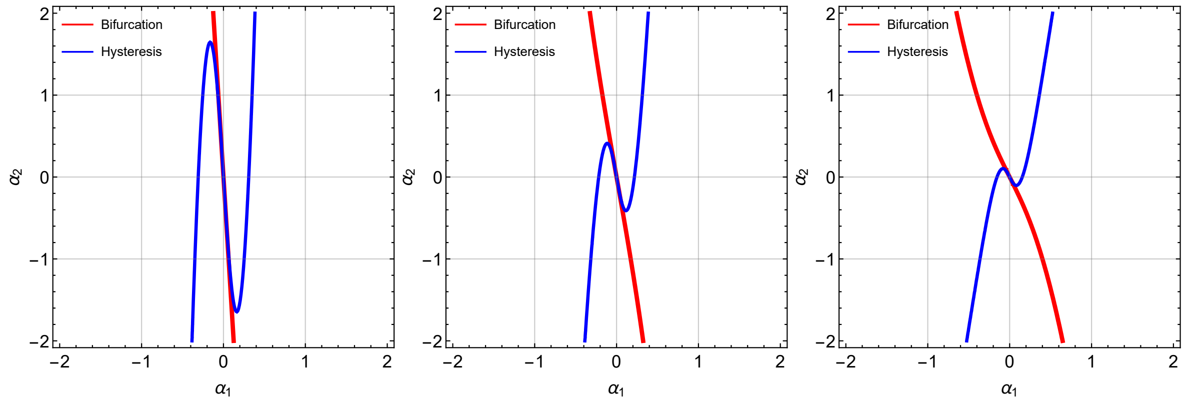
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 5: Approximations of B and H $(a_0 = 2, a_{i\geq 1} = 0)$



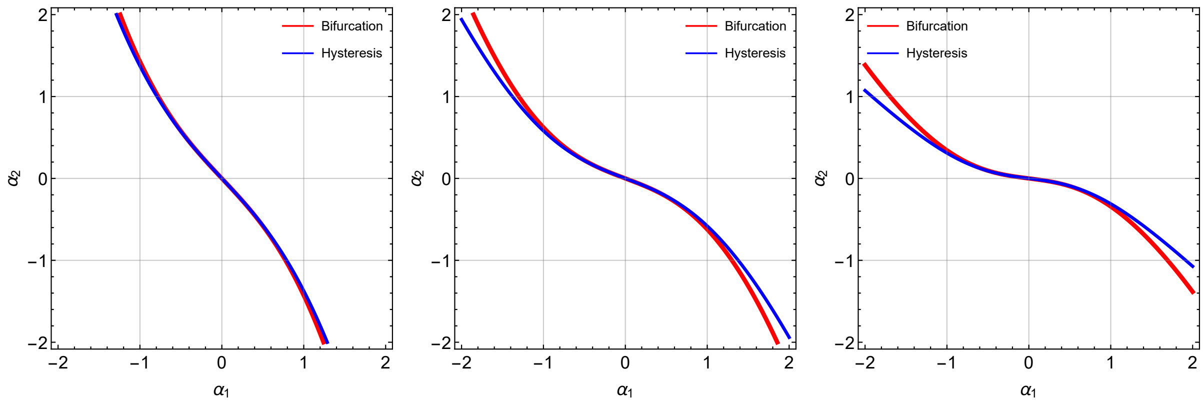
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 6: Approximations of B and H $(a_0 = 4, a_{i\geq 1} = 0)$



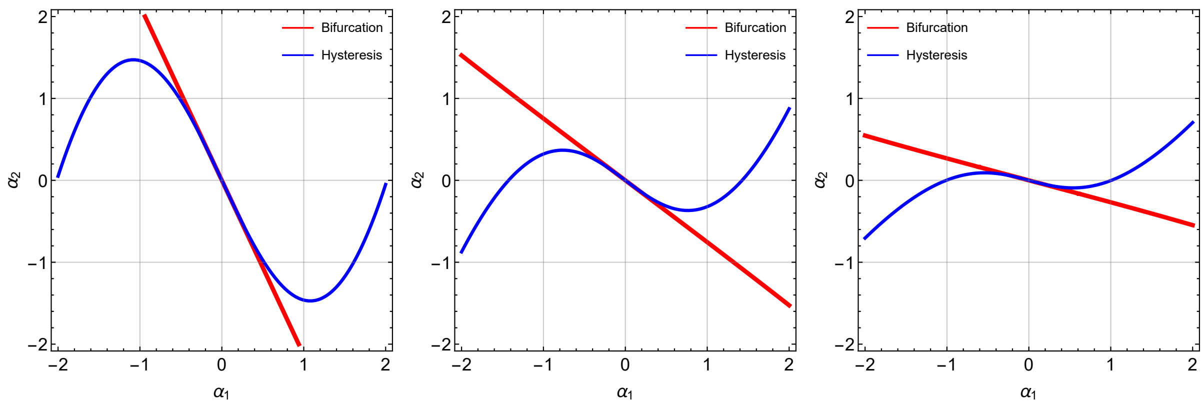
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 7: Approximations of B and H $(a_0 = 10, a_{i\geq 1} = 0)$



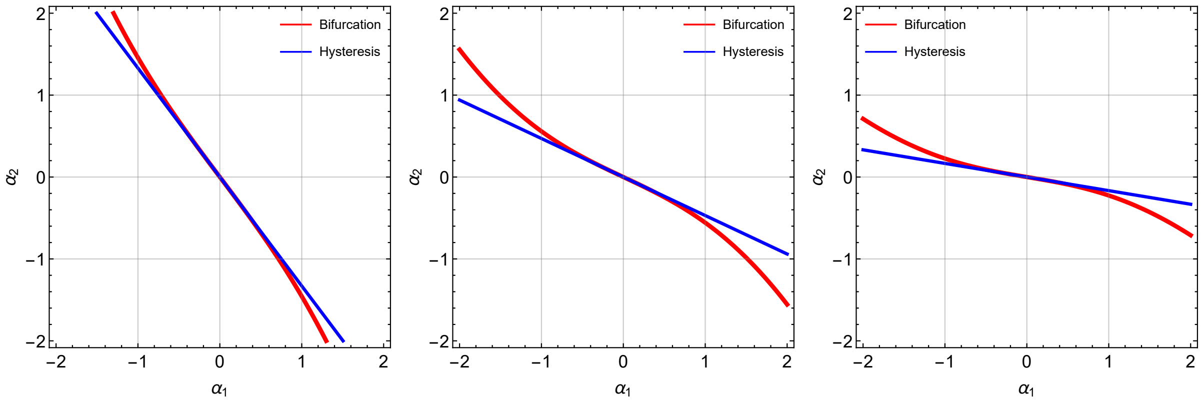
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 8: Approximations of B and H $(a_0 = 1, a_1 = 1, a_{i\geq 2} = 0)$



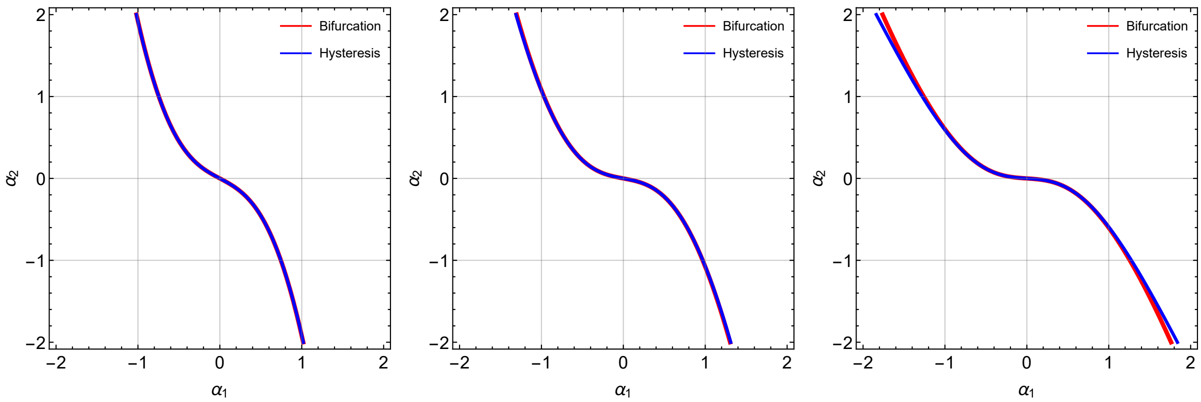
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 9: Approximations of B and H $(a_0 = 1, a_1 = -1, a_{i\geq 2} = 0)$



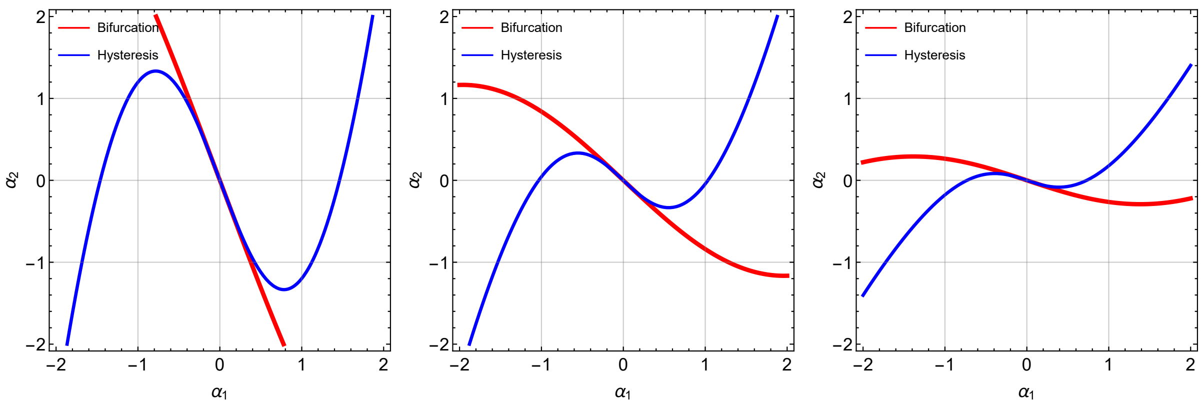
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 10: Approximations of B and H $(a_0 = 1, a_1 = 0.5, a_{i\geq 2} = 0)$



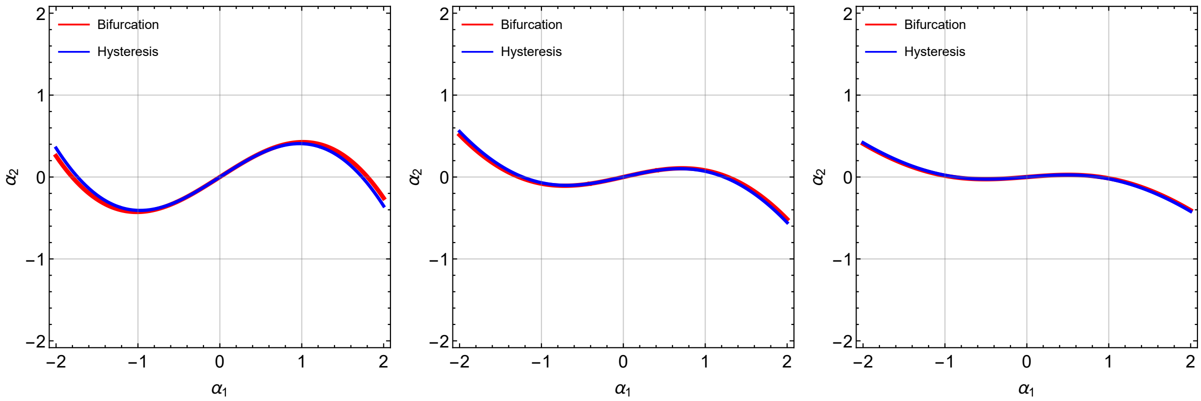
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 11: Approximations of B and H $(a_0 = 1, a_1 = 2, a_{i\geq 2} = 0)$



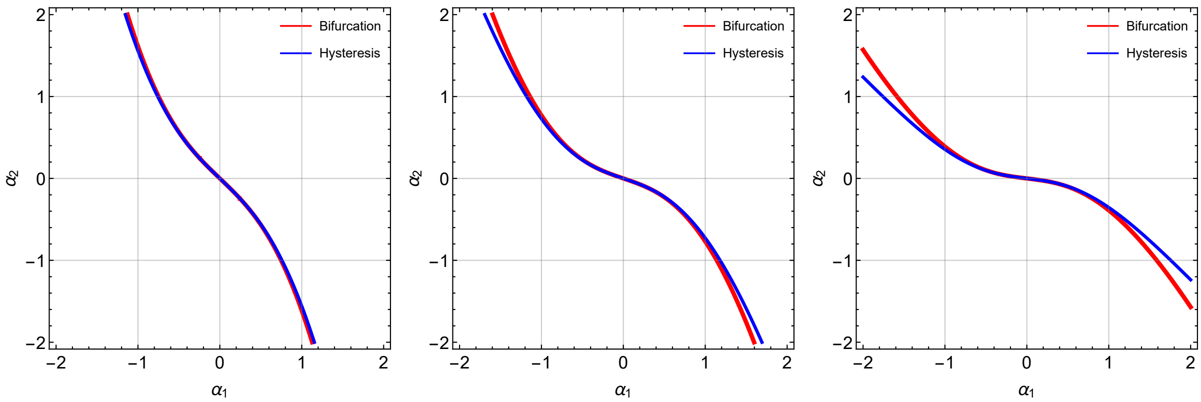
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 12: Approximations of B and H $(a_0 = 1, a_1 = -2, a_{i\geq 2} = 0)$



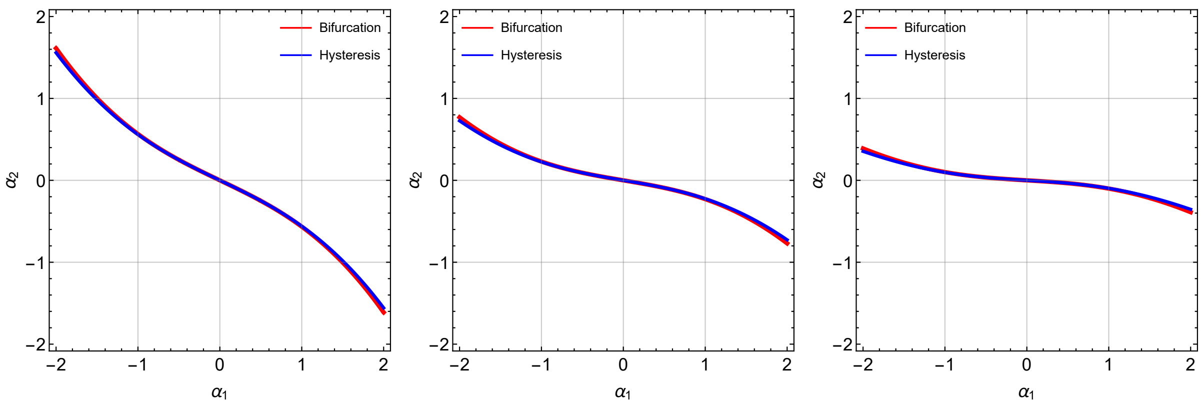
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 13: Approximations of B and H $(a_0 = 0, a_1 = 1, a_2 = 1, a_{i\geq 3} = 0)$



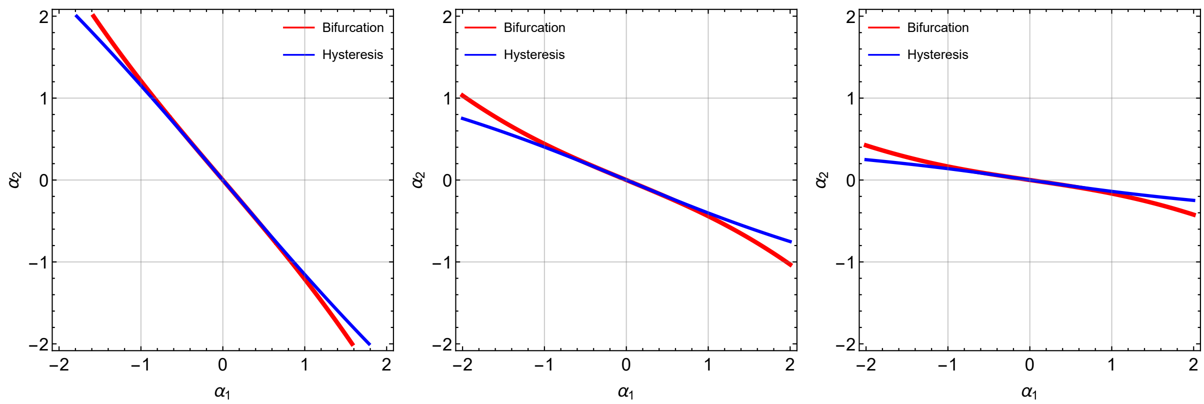
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 14: Approximations of B and H $(a_0 = 1, a_1 = 1, a_2 = 1, a_{i\geq 3} = 0)$



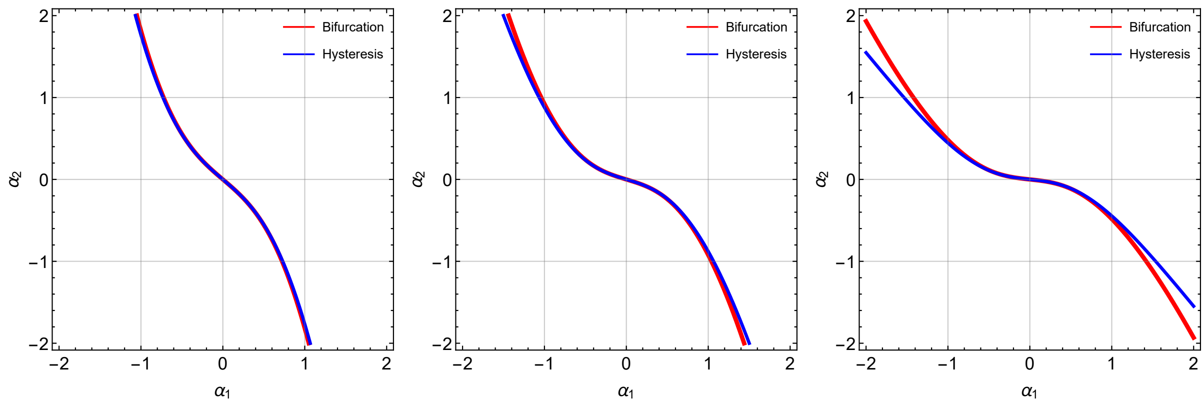
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 15: Approximations of B and H $(a_0 = 0.5, a_1 = 0.5, a_2 = 0.5, a_{i\geq 3} = 0)$



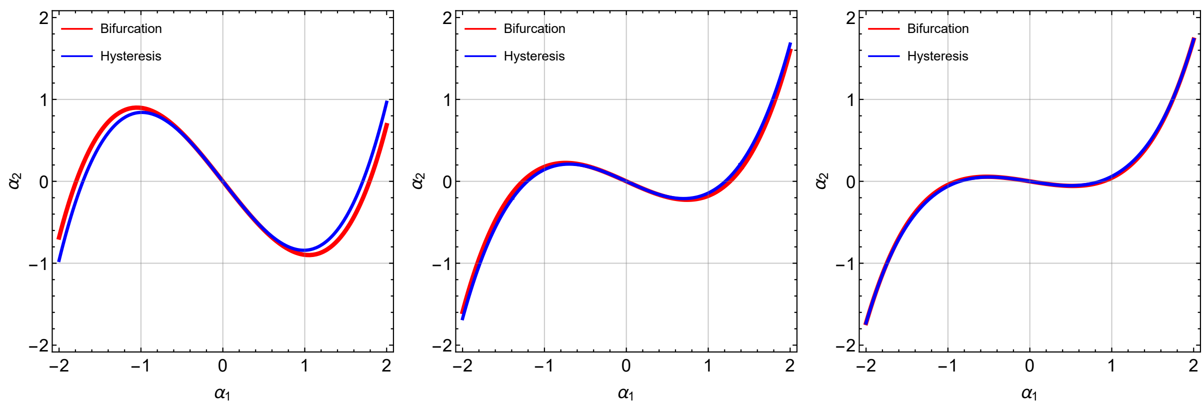
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 16: Approximations of B and H $(a_0 = 1, a_1 = 1, a_2 = -1, a_{i\geq 3} = 0)$



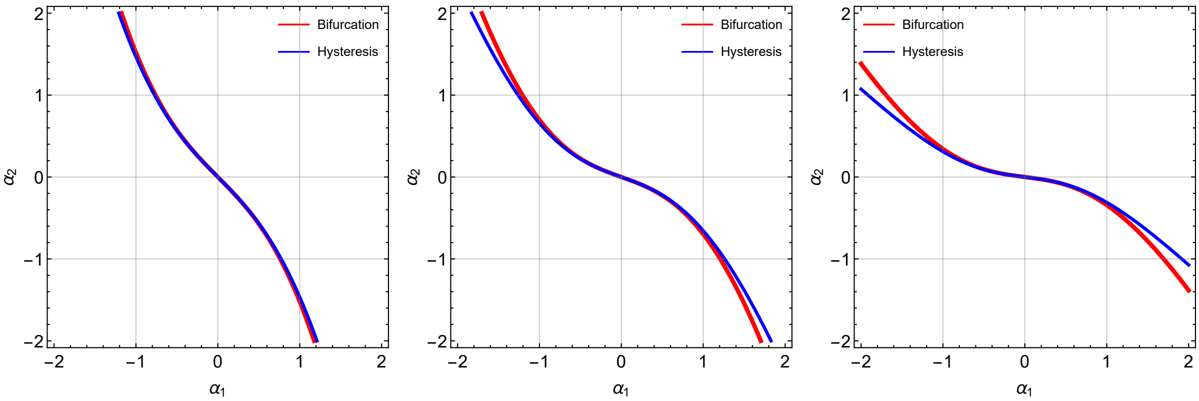
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 17: Approximations of B and H $(a_0 = 1, a_1 = 1, a_2 = 2, a_{i\geq 3} = 0)$



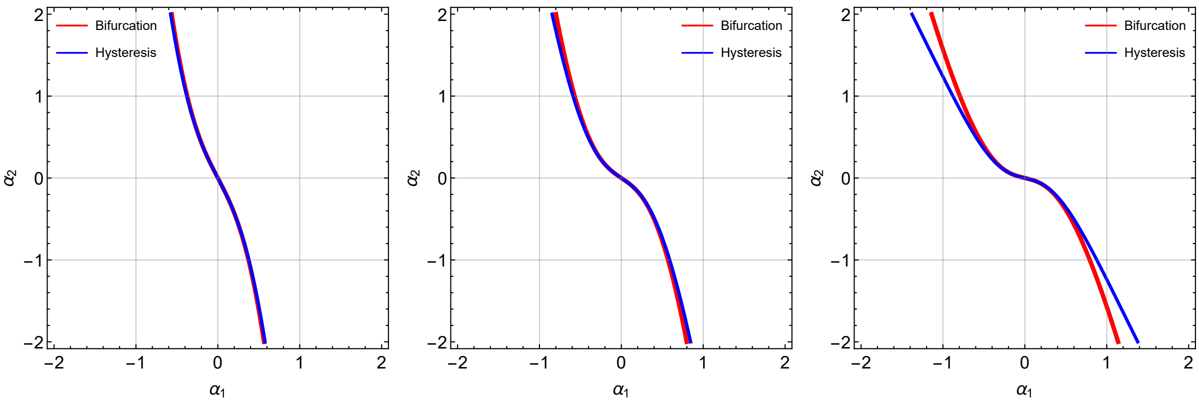
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 18: Approximations of B and H $(a_0 = 1, a_1 = 1, a_2 = -2, a_{i\geq 3} = 0)$



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 19: Approximations of B and H $(a_0 = 1, a_1 = 1, a_2 = 0.5, a_{i\geq 3} = 0)$



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

Figure 20: Approximations of B and H $(a_0 = 2, a_1 = 2, a_2 = 2, a_{i\geq 3} = 0)$

We observe from the figures above that bifurcation and hysteresis sets change as we change κ. When $a_i = 0$ for $i \geq 1$, the aspect of bifurcation sets is changing slightly, and the aspect of hysteresis sets is changing considerably according to the change of a_0. We often observe that the bifurcation set and the hysteresis set are close near the origin when the coefficients a_0, a_1 and a_2 are big.

References

[1] Ralph Abraham, Jerrold E. Marsden and Tudor S. Raţiu, Manifolds, tensor analysis, and applications, second edition, Applied Mathematical Sciences 75, Springer 1988.

[2] M. Golubitsky and D. Schaeffer, A theory for imperfect bifurcation via singularity theory, Comm. Pure Appl. Math. 32 (1979), 21–98.

[3] M. Golubitsky and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory, I, II, Springer-Verlag, 1985, 1988.

[4] Jerrold E. Marsden and Thomas J.R. Hughes, Mathematical Foundations of Elasticity, Courier Corporation, 1994.

[5] J. M. T. Thompson and G. W. Hunt, A General Theory of Elastic Stability, Wiley, London, 1973.