Properness of polynomial maps with Newton polyhedrons

(joint work with Takeki Tsuchiya)

Toshi Fukui (Saitama University)

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Definition

Let $f = (f^1, \ldots, f^n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map.

Definition

$y_0 \in \mathbb{C}^n$ is a proper point of f, if
for any arc $x(t) : \mathbb{C}^* \to \mathbb{C}^n$,

$$\lim_{t\to 0} f(x(t)) = y_0 \implies \lim_{t\to 0} x(t) \text{ exists in } \mathbb{C}^n.$$

We consider the non-properness set of f:

$$\begin{aligned} S_f =& \{y_0 \in \mathbb{C}^n : y_0 \text{ is not a proper point of } f\} \\ =& \left\{ y_0 \in \mathbb{C}^n : \begin{array}{c} \exists x(t) : \mathbb{C}^* \to \mathbb{C}^n \text{ such that} \\ \lim_{t\to 0} f(x(t)) = y_0, \ \lim_{t\to 0} x(t) = \infty \end{array} \right\} \end{aligned}$$

A preliminary consideration

Compactify $f : \mathbb{C}^n \to \mathbb{C}^n$ as

$$\bar{f} : X \longrightarrow Y$$

where X and Y are suitable projective manifolds.
Set $X_\infty = X \setminus \mathbb{C}^n$, $Y_\infty = Y \setminus \mathbb{C}^n$.
Assume that X_∞ and Y_∞ are simple normal crossing divisors. We then have

$$S_f = \bar{f}(X_\infty) \cap \mathbb{C}^n$$

Since $\bar{f}$ is proper, the set $\bar{f}(X_\infty)$ is closed in Y and we obtain that S_f is closed.

Jelonek's results

Theorem [Jelonek 1993]

If a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is dominant, then S_f is a $\mathbb{C}$-uniruled hypersurface or empty.

A variety is a $\mathbb{C}$-uniruled if it is the image of $X \times \mathbb{C}$, for some X, by a birational map.

Jelonek showed several degree estimates of S_f.

Z. Jelonek, The set of points at which a polynomial map is not proper, Ann. Polon. Math. 58 (1993), 259–266.
Z. Jelonek, Testing sets for properness of polynomial mappings, Math. Ann., 315(1), 1–35, 1999.

Z. Jelonek and M. Lasoń, Quantitative properties of the non-properness set of a polynomial map, manuscripta math. 156, 383–397 (2018)

Newton polyhedron

Set $f^j = \sum_\nu c^j_\nu x^\nu$, $j = 1, \dots, n$.
Newton polyhedron of f^j is defined by

$$\Delta(f^j) = \text{convex hull of } \{\boldsymbol{\nu} \in \mathbb{R}^n : c^j_\nu \neq 0\}.$$

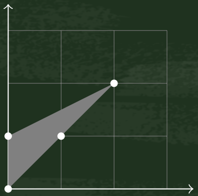
For $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$

$$d_j(\boldsymbol{p}) = -\min\{\langle \boldsymbol{p}, \boldsymbol{\nu} \rangle : \boldsymbol{\nu} \in \Delta(f^j)\}$$
$$\gamma_j(\boldsymbol{p}) = \{\boldsymbol{\nu} \in \Delta(f^j) : \langle \boldsymbol{p}, \boldsymbol{\nu} \rangle = -d_j(\boldsymbol{p})\}$$

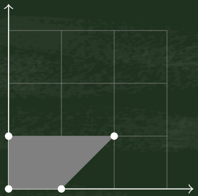
where $\langle \boldsymbol{p}, \boldsymbol{\nu} \rangle = p_1\nu_1 + \cdots + p_n\nu_n$.
We say $\gamma_j(\boldsymbol{p})$ the face of $\Delta(f^j)$ supported by $\boldsymbol{p}$.

Example

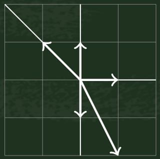
$$(f^1(x,y), f^2(x,y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$$



$\Delta(f^1)$



$\Delta(f^2)$



Vectors supporting faces

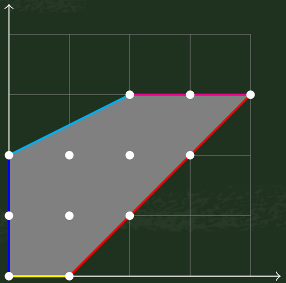
$d_1(0,-1) = 2$ $\qquad$ $d_2(0,-1) = 1$

$d_1(-1,1) = 0$ $\qquad$ $d_2(-1,1) = 1$

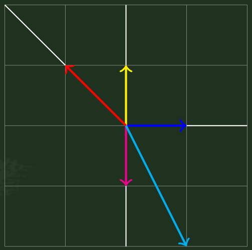
Minkowski sum

- $\Delta(\boldsymbol{f}) = \Delta(f^1) + \cdots + \Delta(f^n)$
- Δ^* is the dual fan of $\Delta(\boldsymbol{f})$
- $\boldsymbol{f}_{\boldsymbol{\gamma}} = (f^1_{\gamma_1}, \ldots, f^n_{\gamma_n})$ where $f^j_{\gamma_j} = \sum_{\boldsymbol{\nu}\in\gamma_j} c^j_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$

$$(f^1(x,y), f^2(x,y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$$



$\Delta(\boldsymbol{f})$



generators of 1-cones of Δ^*

Assume that f^j $(j = 1, \dots, n)$ are non-constant polynomials with constant terms.

- For $p \in \mathbb{Z}^n$, $\gamma(p) = (\gamma_1(p), \dots, \gamma_n(p))$.
- For $p \in \mathbb{Z}^n$, $J_{\gamma(p)} = \{j \in \{1, \dots, n\} : 0 \notin \gamma_j\}$.
- $\Delta_{\mathrm{nc}}(f) = \{\gamma(p) : p \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq 0})^n\}$
  the set of non-coordinate faces
- For $J \subset \{1, \dots, n\}$, we denote

$$\mathbb{C}^n = \mathbb{C}^J \times \mathbb{C}^{J^c}, \quad J^c = \{1, \dots, n\} \setminus J$$

- For $J \subset \{1, \dots, n\}$, we denote $f^J = (f^j)_{j \in J}$

$$f^J : \mathbb{C}^n \to \mathbb{C}^J$$

- f is dominant if $\det \mathrm{Jac}(f)$ is not identically zero.

Main Theorem

Set

$$Z(f_{\gamma}^{J}) = \{x \in (\mathbb{C}^*)^n : f_{\gamma}^{J}(x) = 0\},$$
$$\Sigma(f_{\gamma}^{J}) = \{x \in (\mathbb{C}^*)^n : \mathrm{rank}\,\mathrm{Jac}(f_{\gamma}^{J}) < \#J\}.$$

We say that Z has a dense nonsingular locus if $Z \setminus \Sigma$ is dense in Z. .

Theorem 1

If f is dominant and $Z(f_{\gamma}^{J}) \setminus \Sigma(f_{\gamma}^{J})$ is dense in $Z(f_{\gamma}^{J})$ $\forall \gamma \in \Delta_{\mathrm{nc}}(f)$, $J = J_{\gamma}$, then

$$S_f = \bigcup_{\gamma \in \Delta_{\mathrm{nc}}(f)} S_{\gamma}$$

where $S_{\gamma} = f^{J^c}(Z(f_{\gamma}^{J})) \times \mathbb{C}^{J}$.

Non-degeneracy condition

Y. Chen, L.R.G. Dias, K. Takeuchi and M. Tibar, Invertible polynomial mappings via Newton non-degeneracy, Ann. Inst. Fourier, Grenoble 64, 5 (2014), 1807–1822.
define non-degeneracy condtion as

$$Z(f_\gamma^J) \cap \Sigma(f_\gamma^J) = \emptyset, \; J = J_\gamma \quad \text{for any } \gamma.$$

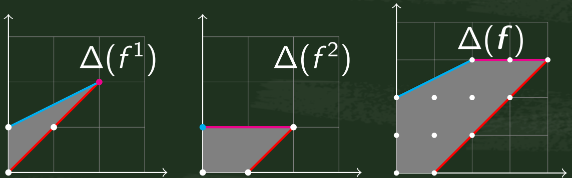
Under this condition, they show that

$$S_f \subset \bigcup_{\gamma \in \Delta_{\mathrm{nc}}(f)} S_\gamma.$$

They actually treated the bifurcation set of a polynomial map $f : \mathbb{K}^m \to \mathbb{K}^n$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $m \geq n$.

Example (continued)

$$(f^1(x,y), f^2(x,y)) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$$



$$f_{\gamma(-1,1)} = (x^2y^2 + xy + 1, x(xy+1)), \quad Z(f^{\{2\}}_{\gamma(1,-1)}) = \{xy + 1 = 0\}$$

$$f_{\gamma(0,-1)} = (x^2y^2, (x^2+1)y)), \quad Z(f^{\{1,2\}}_{\gamma(0,-1)}) = \emptyset$$

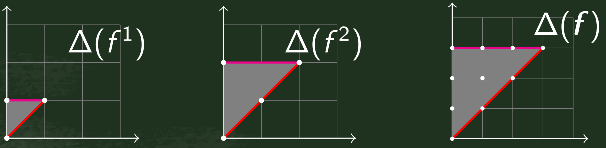
$$f_{\gamma(1,-2)} = (x^2y^2 + y, y), \quad Z(f^{\{1,2\}}_{\gamma(1,-2)}) = \emptyset$$

Therefore

$$\begin{aligned} S_f &= \{x^2y^2 + xy + 1 : xy + 1 = 0\} \times \mathbb{C} \\ &= \{t^2 + t + 1 : t + 1 = 0\} \times \mathbb{C} = \{1\} \times \mathbb{C} \end{aligned}$$

Another example

$$(f^1(x,y), f^2(x,y)) = (xy + y + 1, x^2y^2 + y^2 + xy + 1)$$



$$f_{\gamma(-1,1)} = (xy + 1, x^2y^2 + xy + 1), \quad Z(f^{\emptyset}_{\gamma(1,-1)}) = (\mathbb{C}^*)^2$$

$$f_{\gamma(0,-1)} = ((x+1)y, (x^2+1)y^2), \quad Z(f^{\{1,2\}}_{\gamma(0,-1)}) = \emptyset$$

Therefore

$$\begin{aligned} S_f &= \{(X,Y) : \exists (x,y) \text{ s.t. } (X,Y) = (xy+1, x^2y^2+xy+1)\} \\ &= \{(X,Y) : \exists t \text{ s.t. } (X,Y) = (t+1, t^2+t+1)\} \\ &= \{Y - X^2 + X - 1 = 0\} \end{aligned}$$

Proof of Main Theorem

It is enough to prove the following

Theorem 2.

If f is dominant, then

$$\bigcup_{\boldsymbol{\gamma}\in\boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S'_{\boldsymbol{\gamma}} \subset S_{\boldsymbol{f}} \subset \bigcup_{\boldsymbol{\gamma}\in\boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S_{\boldsymbol{\gamma}} \tag{1}$$

where

$$S'_{\boldsymbol{\gamma}} = \boldsymbol{f}^{J^c}\left(\overline{Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \setminus \Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})}\right) \times \mathbb{C}^J,\ J = J_{\boldsymbol{\gamma}}.$$

Here $\overline{Z}$ denotes the closure of Z.

Proof of Main Theorem

For $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, consider an arc

$$\begin{aligned} \boldsymbol{x}(t) =& (t^{p_1} v^1(t), \ldots, t^{p_n} v^n(t)), \text{ where} \\ \boldsymbol{v}(t) =& (v^1(t), \ldots, v^n(t)) = \sum_{i=0}^{\infty} \boldsymbol{v}_i t^i, \\ \boldsymbol{v}_i =& (v_i^1, \ldots, v_i^n), \ \ \boldsymbol{v}_0 \in (\mathbb{K}^*)^n. \end{aligned}$$

We denote $\mathcal{A}(\boldsymbol{p})$ the set of such arcs.

$$\mathcal{A}_{\boldsymbol{f}}(\boldsymbol{p}) = \{\boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}) : \exists \boldsymbol{y} \in S_{\boldsymbol{f}} \ \ \boldsymbol{y} = \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(t))\}$$

Proof of $S_{\boldsymbol{f}} \subset \bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S_{\boldsymbol{\gamma}}$

Express $f^j(\boldsymbol{x}(t))$ as

$$f^j(\boldsymbol{x}(t)) = t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \cdots + \hat{f}_{d_j-1}^j t^{d_j-1} + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j}))$$

where $d_j = d_j(\boldsymbol{p})$. If $\boldsymbol{y} \in S_{\boldsymbol{f}}$, then $\exists \boldsymbol{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$ and

$$\exists \boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}) \text{ such that } \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{y},$$

$$\hat{f}_0^j = \hat{f}_1^j = \cdots = \hat{f}_{d_j-1}^j = 0 \quad (j \in J = J_{\boldsymbol{\gamma}}).$$

In particular, $\hat{f}_0^j = f_{\gamma_j(\boldsymbol{p})}^j(\boldsymbol{v}_0) = 0$ ($j \in J$).
As $t \to 0$, $f^j(\boldsymbol{x}(t)) \to \hat{f}_{d_j}^j = \hat{f}_0^j = f_{\gamma_j(\boldsymbol{p})}^j(\boldsymbol{v}_0)$ ($j \notin J$). $\square$

Proof of $\bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})} S'_{\boldsymbol{\gamma}} \subset S_{\boldsymbol{f}}$

We have

$$\hat{f}^j_i = (df^j_{\gamma_j})_{\boldsymbol{v}_0}(\boldsymbol{v}_i) + r^j_i(\boldsymbol{v}_0, \dots, \boldsymbol{v}_{i-1}) \quad (i = 1, 2, \dots)$$

where $r^j_i(\boldsymbol{v}_0, \dots, \boldsymbol{v}_{i-1})$ is a suitable polynomial of $\boldsymbol{v}_0, \dots, \boldsymbol{v}_{i-1}$. Let $(a^j_k)_{j\in J; k\geq 1}$ be any sequence. Assume that we already take $\boldsymbol{v}_0$, $\boldsymbol{v}_1, \dots,$ $\boldsymbol{v}_{i-1}$ so that

$$\hat{f}^j_k = a^j_k \quad (1 \leq k < i, j \in J).$$

There exists $\boldsymbol{v}_i$ so that $\hat{f}^j_i = a^j_i$ for $j \in J$, whenever $(df^j_{\gamma_j})_{j\in J}$ is of full rank at $\boldsymbol{v}_0$. Thus

$$\begin{aligned} S'_{\boldsymbol{\gamma}} &= \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}\left(\overline{Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \setminus \Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}})}\right) \times \mathbb{C}^J \\ &\subset \overline{\boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}}(Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}) \setminus \Sigma(\boldsymbol{f}^J_{\boldsymbol{\gamma}}))} \times \mathbb{C}^J \subset \overline{S_{\boldsymbol{f}}} = S_{\boldsymbol{f}}. \qquad \square \end{aligned}$$

Generalizations

- to a polynomial map $f : \mathbb{C}^m \to \mathbb{C}^n$ (our discussion works when f is generically proper, that is, there is a nowhere dense subset S of its image so that $f^{-1}(y)$ is compact if $y \notin S$.) But, when $m > n$, $S_f = \overline{\mathrm{Im}\, f}$, automatically.
- to relative version $f = (f', f'') : \mathbb{C}^m \to \mathbb{C}^{n-k} \times \mathbb{C}^k$

$$f : X \to \mathbb{C}^{n-k} \times \{y_0''\} \subset \mathbb{C}^n$$

  where $X = (f'')^{-1}(y_0'')$.
- to degenerate case:

$$Z(f_\gamma^J) \setminus \overline{Z(f_\gamma^J) \setminus \Sigma(f_\gamma^J)} \neq \emptyset \text{ for some } \gamma \in \Delta_{\mathrm{nc}(f)}$$

- to real case

Relative version

Set $f' = (f^1, \dots, f^{n-k})$ and $f'' = (f^{n-k+1}, \dots, f^n)$.

Consider $f'|_X : X \to \mathbb{C}^{n-k}$ where $X = (f'')^{-1}(y_0'')$.

Assume that the regular locus of X is dense in X. Identify $f'|_X$ with $f|_X$. For a face $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ of $\boldsymbol{\Delta}(f)$, define $\boldsymbol{\gamma}' = (\gamma_1, \dots, \gamma_{n-k})$, $\boldsymbol{\gamma}'' = (\gamma_{n-k+1}, \dots, \gamma_n)$. Set $J = J_{\boldsymbol{\gamma}'} = \{j \in \{1, \dots, n-k\} : 0 \in \gamma_j\}$, and

$$S_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} = f^{J^c}_{\boldsymbol{\gamma}'}(Z(f^J_{\boldsymbol{\gamma}}, f''_{\boldsymbol{\gamma}''} - y_0'')) \times \mathbb{C}^J,$$

$$S'_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} = f^{J^c}_{\boldsymbol{\gamma}'}(\overline{Z(f^J_{\boldsymbol{\gamma}}, f''_{\boldsymbol{\gamma}''} - y_0'') \setminus \Sigma(f^J_{\boldsymbol{\gamma}})}) \times \mathbb{C}^J.$$

Relative version (Theorem)

Theorem 3

If X does not have a component in $\{x_1 \cdots x_n = 0\}$, then

$$\bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})} S'_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''} \subset S_{\boldsymbol{f}|X} \subset \bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})} S_{\boldsymbol{\gamma}';\boldsymbol{\gamma}''}.$$

If $Z(\boldsymbol{f}^J_{\boldsymbol{\gamma}}, \boldsymbol{f}''_{\boldsymbol{\gamma}''} - \boldsymbol{y}''_0)$ has a dense nonsingular locus for all $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\rm nc}(\boldsymbol{f})$, we have equalities.
If X has a component X_1 in $\{x_1 \cdots x_n = 0\}$, then we could proceed a similar computation for $\boldsymbol{f}|_{X_1}$ which is a polynomial map with less number of variables and obtain that $S_{\boldsymbol{f}|_{X_1}} \subset S_{\boldsymbol{f}|X}$.

A trick for degenerate case

Let $\boldsymbol{h}: \mathbb{K}^{m+k} \to \mathbb{K}^{n+k}$ be a polynomial map with

$$h^{n+i}(x) = \varphi_1(x_1, \ldots, x_m) - x_{m+i} \text{ for } i = 1, \ldots, k.$$

Let X be a subset defined by

$$x_{m+i} = \varphi_i(x_1, \ldots, x_m), \quad i = 1, \ldots, k.$$

The set X is isomotrphic to $\mathbb{K}^n$ by $\boldsymbol{x} \mapsto (\boldsymbol{x}, \varphi_i(\boldsymbol{x}))$ where $\boldsymbol{x} = (x_1, \ldots, x_m)$. If $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}, \varphi_1(\boldsymbol{x}), \ldots, \varphi_k(\boldsymbol{x}))$, then we have

$$S_{\boldsymbol{f}} = S_{\boldsymbol{h}|X}$$

via the identification of $\mathbb{K}^n$ with $\mathbb{K}^n \times \{0\}$.

Degenerate example

Let $h : \mathbb{K}^3 \to \mathbb{K}^3$ be the map defined by

$$h(x, y, z) = (1 + x + z^2, 1 + x^2 + z^3, y^2 - x^3 - z).$$

Let X be the subset of $\mathbb{K}^3$ defined by $z = y^2 - x^3$. Then the map $h|_X$ is isomorphic to the map $f : \mathbb{K}^2 \to \mathbb{K}^2$ defined by

$$f(x, y) = (1 + x + (y^2 - x^3)^2, 1 + x^2 + (y^2 - x^3)^3).$$

Applying Theorem 3, we conclude that h is proper, and thus so is f.

Another idea for deg case

Assume that, for some $\boldsymbol{\gamma} \in \boldsymbol{\Delta}_{\mathrm{nc}}(\boldsymbol{f})$,

$$V_{\boldsymbol{\gamma}} := Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^{J}) \setminus \overline{Z(\boldsymbol{f}_{\boldsymbol{\gamma}}^{J}) \setminus \Sigma(\boldsymbol{f}_{\boldsymbol{\gamma}}^{J})} \neq \emptyset, \quad J = J_{\boldsymbol{\gamma}}.$$

Remark that

$$S_{\boldsymbol{f}} = \bigcup_{\boldsymbol{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n} S_{\boldsymbol{f}}(\boldsymbol{p}), \quad S_{\boldsymbol{f}}(\boldsymbol{p}) = \left\{ \boldsymbol{y} \in \mathbb{C}^n : \begin{array}{c} \exists \boldsymbol{x}(t) \in \mathcal{A}(\boldsymbol{p}) \\ \lim_{t \to 0} \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t})) = \boldsymbol{y} \end{array} \right\}.$$

Set

$$f^j(\boldsymbol{x}) = f_0^j(\boldsymbol{x}) + f_1^j(\boldsymbol{x}) + \cdots, \quad f_m^j(t^{p_1} x_1, \ldots, t^{p_n} x_n) = t^{-d_j + m} f_m^j(\boldsymbol{x}).$$

Remark that $f_0^j = f_{\gamma_j(\boldsymbol{p})}^j$ and thus $\boldsymbol{f}_0 = \boldsymbol{f}_{\boldsymbol{\gamma}(\boldsymbol{p})}$.

Another idea for deg case (continued)

$$W(\boldsymbol{p}) = \left\{ \boldsymbol{y} \in \mathbb{C}^n : \begin{array}{c} \exists \boldsymbol{x} \in V_{\boldsymbol{\gamma}(\boldsymbol{p})}, \quad \boldsymbol{y}^{J^c} = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{x}) \\ \mathrm{rank} \begin{bmatrix} \partial_{x_i} \boldsymbol{f}_0^{J'}(\boldsymbol{x}) \\ \partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x}) \end{bmatrix} = \mathrm{rank} \begin{bmatrix} \partial_{x_i} \boldsymbol{f}_0^{J'}(\boldsymbol{x}) & \boldsymbol{f}_1^{J'}(\boldsymbol{x}) - y^{J'} \\ \partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x}) & \boldsymbol{f}_1^{J''}(\boldsymbol{x}) \end{bmatrix} \end{array} \right\},$$

$$W'(\boldsymbol{p}) = \left\{ \boldsymbol{y} \in W(\boldsymbol{p}) : \begin{array}{c} \exists \boldsymbol{x} \in V_{\boldsymbol{\gamma}(\boldsymbol{p})}, \; \boldsymbol{y}^{J^c} = \boldsymbol{f}^{J^c}_{\boldsymbol{\gamma}(\boldsymbol{p})}(\boldsymbol{x}) \\ \mathrm{rank}\left(\partial_{x_i} \boldsymbol{f}_0^{J''}(\boldsymbol{x})\right)_{i=1,\dots,n} = \# J'' \end{array} \right\}.$$

where $J = J_{\boldsymbol{\gamma}(\boldsymbol{p})} = \{j; d_j(\boldsymbol{p}) > 0\}$,

$J' = \{j \in J : d_j(\boldsymbol{p}) = 1\}, \qquad J'' = \{j \in J : d_j(\boldsymbol{p}) \geq 2\}.$

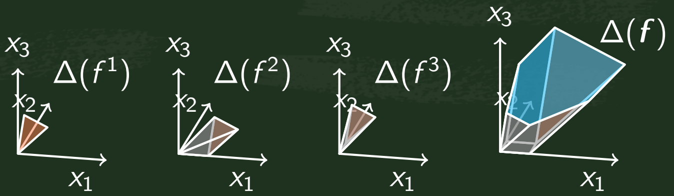
Theorem 4

$$V_{\boldsymbol{\gamma}(\boldsymbol{p})} \neq \emptyset \implies W'(\boldsymbol{p}) \subset S_{\boldsymbol{f}}(\boldsymbol{p}) \subset W(\boldsymbol{p})$$

A degenerate example

Consider the map $f : \mathbb{K}^3 \to \mathbb{K}^3$ defined by

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 + x_1x_3 + x_2x_3 \\ 1 + x_1(1 - x_1x_3 + x_2x_3) \\ 1 + x_2(1 - x_1x_3 + x_2x_3) \end{pmatrix}.$$



We easily see that

$$S_{\gamma(1,1,-1)}(f) = \{y_1 = y_3 = 1\},\ S_{\gamma(1,1,-2)}(f) = \{y_2 + y_3 = 2\},$$
$$S_{\gamma(-1,0,1)}(f) = \{y_1 = 2, y_3 = 1\},\ S_{\gamma(0,-1,1)}(f) = \{y_1 = 0, y_2 = 1\}.$$

Example (continued)

$f_{\gamma(-1,-1,1)}$ is degenerate. $W(-1,-1,1)$ is given by

$$\left\{ \begin{array}{c} \exists \mathbf{x}\ 1 - x_1x_3 + x_2x_3 = 0, y_1 = 1 + x_1x_3 + x_2x_3, \\ \text{rank} \begin{pmatrix} 1 - 2x_1x_3 + x_2x_3 & x_1x_3 & x_1(x_2 - x_1) & 1 - y_2 \\ x_2x_3 & 1 - x_1x_3 + 2x_2x_3 & x_2(x_2 - x_1) & 1 - y_3 \end{pmatrix} = 1 \end{array} \right\}$$

$$= \{y_1(y_2 - y_3) - 2y_2 + 2 = 0\}.$$

$$\Phi : \mathbb{K}^3 \times \mathbb{K}^3 \to \mathbb{K}^3, \ (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{z} = \left(y_1, y_2, \frac{x_2}{x_1} y_2 + y_3 \right).$$

is well-defined along arc in $\mathcal{A}(-1,-1,1)$ when $t \to 0$. $g(x) = \Phi(x, f(x))$ is non-degenerate and we manage to show that

$$W(-1,-1,1) = S_{\gamma(-1,-1,1)}.$$

We conclude that

$$S_f = S_{\gamma(1,1,-2)} \cup S_{\gamma(-1,-1,1)}.$$

Real version

Change $\mathbb{C}$ by $\mathbb{R}$.
Everything works!

Causion: S_f is semi-algebraic, in general.
For $f : \mathbb{R}^m \to \mathbb{R}^n$, $m > n$, S_f may not $\mathbb{R}^n$.

Thank you very much for your attention!

Dzię kuję bardzo za uwagę!