Curvature for curves in semi-Euclidean spaces

Toshizumi Fukui* and Donghe Pei

December 11, 2014

Abstract

We show a formula for curvatures of curves in a semi-Euclidean space (or pseudo-sphere) with respect to Frenet-Serre type frame in terms of volumes. We also investigate versality of height unfolding and distance squared unfolding for a curve.

Contents

Introducion

We consider curves in a semi-Euclidean space $\mathbb{R}^n_q$, i.e., vector spaces with pseudo inner product with index $(n-q,q)$. We construct Frenet-Serre type frame along the curve and define their curvatures as an analogy to Euclidean case. We present a formula for curvatures in terms of volumes (Theorem 2.2) and discuss limiting behaviour of curvatures for a 1-parameter family of curves (Remarks 2.5). We also consider a frame with respect to a pseudo-sphere

$$M(c)=\{\mathbf{x}\in\mathbb{R}^n_q : \langle\mathbf{x},\mathbf{x}\rangle=c\},$$

2010 *Mathematics subject classification.* 53A04,53A35,53B30

Key words and phrases. semi-Euclidean space, curvatures of curves, height unfolding, distance squared unfolding.

*Partially supported by Grant-in-Aid in Sciences 24540067.

and define curvatures using this frame. This notion is useful to investigate a curve in a pseudo-sphere. We present a formula for curvatures in terms of volumes (Theorem 2.6). When $c \neq 0$, it is possible to define curvatures $\hat{\kappa}_i$ $(i = 1, \dots, n-1)$ constructing a frame with respect to $M(c)$. When $c = 0$, the analogy is not possible. But we also show that we can define "higher order curvatures" $\hat{\kappa}_i$ $(i = 4, \dots, n-1)$ for a non-degenerate curve $\gamma : (-\varepsilon, \varepsilon) \to M(0)$, even though $\hat{\kappa}_1$ and $\hat{\kappa}_2$ are not defined (Remark 2.9).

In the last section, we investigate versality of height unfolding and distance squared unfolding for a curve in $\mathbb{R}^n_q$ or $M(c)$. We conclude Propositions 3.3, 3.6, 3.8, 3.10, 3.13, 3.14, 3.15, which assert that the height unfoldings, e.g., is versal for a generic curve in several contexts. They lead to criteria of singularity types of bifurcation and discriminant sets of these unfoldings.

1 Semi-Euclidean space

Let V denote a real n-dimensional vector space endowed with non-degenerate bilinear form $\langle\ ,\ \rangle$. That is, there is a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of V so that

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_n y_n$$

where $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$, $\mathbf{y} = y_1 \mathbf{e}_1 + \cdots + y_n \mathbf{e}_n$. Remark that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 & (i = j \leq p), \\ -1 & (i = j > p), \\ 0 & (i \neq j). \end{cases}$$

We often denote $(V, \langle\ ,\ \rangle)$ by $\mathbb{R}^n_q$ where $q = n - p$. We call $\mathbb{R}^n_q$ **semi-Euclidean space**. Consider the pseudo-sphere defined by

$$M_{\mathbf{p}}(c) = \{ \mathbf{x} \in V : \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle = c \}$$

and we call it by the pseudo-sphere centered at $\mathbf{p} \in V$ with radius $\sqrt{|c|}$.

We identify the tangent space $T_{\mathbf{p}}V$ of the vector space V at $\mathbf{p}$ with the vector space V and consider the pseudo-inner product of the tangent space induced by this identification.

We say that a non zero vector $\mathbf{v} \in V$ is $\begin{cases} \text{space-like if } \langle \mathbf{v}, \mathbf{v} \rangle > 0, \\ \text{light-like if } \langle \mathbf{v}, \mathbf{v} \rangle = 0, \\ \text{time-like if } \langle \mathbf{v}, \mathbf{v} \rangle < 0. \end{cases}$

1.1 Pseudo-volumes

We define k-dimensional pseudo-volume $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)$ of the parallelotope generated by $\mathbf{a}_1, \dots, \mathbf{a}_k$ by

$$\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 = \begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_k \rangle \end{vmatrix}.$$

We assume that

- $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)$ is a non-negative real number if $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 \geq 0$,
- $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)$ is a pure imaginary number with positive imaginary part if $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 < 0$.

Lemma 1.1. *Set $\mathbf{a}_j = a_{j,1}\mathbf{e}_1 + \cdots + a_{j,n}\mathbf{e}_n$, $j = 1, \dots, n$. Then*

$$\mathrm{Vol}_k(\mathbf{a}_1, \cdots, \mathbf{a}_k)^2 = \sum_{j_1 < \cdots < j_k} (-1)^{k-r} \begin{vmatrix} a_{1,j_1} & \cdots & a_{k,j_1} \\ \vdots & & \vdots \\ a_{1,j_k} & \cdots & a_{k,j_k} \end{vmatrix}^2 : r = \min\{i : j_i \leq p\}$$

Proof.

$$\begin{aligned}
&\mathrm{Vol}_k(\mathbf{a}_1, \cdots, \mathbf{a}_k)^2 \\
&= \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,p} & -a_{1,p+1} & \cdots & -a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,p} & -a_{k,p+1} & \cdots & -a_{k,n} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{k,1} \\ \vdots & & \vdots \\ a_{1,n} & \cdots & a_{k,n} \end{pmatrix} \\
&= \sum_{j_1 < \cdots < j_k} \det \begin{pmatrix} a_{1,j_1} & \cdots & a_{1,j_q} & -a_{1,j_{q+1}} & \cdots & -a_{1,j_k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k,j_1} & \cdots & a_{k,j_q} & -a_{k,j_{q+1}} & \cdots & -a_{k,j_k} \end{pmatrix} \begin{pmatrix} a_{1,j_1} & \cdots & a_{k,j_1} \\ \vdots & & \vdots \\ a_{1,j_k} & \cdots & a_{k,j_k} \end{pmatrix} \\
&= \sum_{j_1 < \cdots < j_k} (-1)^{k-r} \begin{vmatrix} a_{1,j_1} & \cdots & a_{k,j_1} \\ \vdots & & \vdots \\ a_{1,j_k} & \cdots & a_{k,j_k} \end{vmatrix}^2 : r = \min\{i : j_i \leq p\}
\end{aligned}$$

□

This shows that, if $\mathbf{a}_1, \dots, \mathbf{a}_k$ are linearly dependent, then the pseudo-volume $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)$ is zero.

1.2 Pseudo-orthonormal projections

Lemma 1.2. *Let $\mathbf{a}_1, \dots, \mathbf{a}_k$ be vectors of V with $\mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq 0$. Let W be the linear span of $\mathbf{a}_1, \dots, \mathbf{a}_k$ and $W^\perp$ denote its pseudo-orthogonal space. Define a linear map $\pi : V \to V$ by*

$$\pi(\mathbf{v}) = \frac{\begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_k \rangle & \mathbf{a}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_k \rangle & \mathbf{a}_k \\ \langle \mathbf{v}, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{v}, \mathbf{a}_k \rangle & \mathbf{v} \end{vmatrix}}{\begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_k \rangle \end{vmatrix}} \qquad \text{for } \mathbf{v} \in V.$$

Then π is the pseudo-orthogonal projection onto $W^\perp$.

Proof. If $\mathbf{v} = \mathbf{a}_j$, then each term in the numerator is zero and we obtain $\pi(\mathbf{v}) = 0$. This means $\pi|_W = 0$.

If $\mathbf{v} \in W^\perp$, then we obtain that

$$\pi(\mathbf{v}) = \frac{\begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_k \rangle & \mathbf{a}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_k \rangle & \mathbf{a}_k \\ 0 & \dots & 0 & \mathbf{v} \end{vmatrix}}{\begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_k \rangle \end{vmatrix}} = \mathbf{v}.$$

This means $\pi|_{W^\perp}$ is the identity, and we complete the proof. $\square$

We set $W_k = \langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle_{\mathbb{R}}$. Let $\pi_k : V \to W_k^\perp$ be the pseudo-orthogonal projection. Set

$$V_k = \mathrm{Vol}_k(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k) = \det(\langle \mathbf{a}_i, \mathbf{a}_j \rangle_{i,j=1,\dots,k})^{1/2}.$$

We set $V_0 = 1$, by convention.

Lemma 1.3. *Assume that $V_k \neq 0$ for $k = 1, \dots, n$. Let us put*

$$\mathbf{b}_k = \frac{1}{|V_k V_{k-1}|} \tilde{\mathbf{b}}_k \qquad \textit{where} \qquad \tilde{\mathbf{b}}_k = \begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_{k-1} \rangle & \mathbf{a}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{a}_{k-1}, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle & \mathbf{a}_{k-1} \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_{k-1} \rangle & \mathbf{a}_k \end{vmatrix}.$$

Then $\mathbf{b}_1, \dots, \mathbf{b}_n$ form a pseudo-orthonormal basis so that

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle_{\mathbb{R}} = \langle \mathbf{b}_1, \dots, \mathbf{b}_k \rangle_{\mathbb{R}}, \qquad k = 1, \dots, n.$$

Proof. Since

$$\langle \mathbf{a}_i, \tilde{\mathbf{b}}_k \rangle = \begin{cases} 0 & i = 1, 2, \dots, k-1, \\ {V_k}^2 & i = k, \end{cases}$$

we have

$$\langle \tilde{\mathbf{b}}_k, \tilde{\mathbf{b}}_k \rangle = \begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_{k-1} \rangle & \langle \mathbf{a}_1, \tilde{\mathbf{b}}_k \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{a}_{k-1}, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle & \langle \mathbf{a}_{k-1}, \tilde{\mathbf{b}}_k \rangle \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_{k-1} \rangle & \langle \mathbf{a}_k, \tilde{\mathbf{b}}_k \rangle \end{vmatrix} = \begin{vmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{a}_{k-1} \rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{a}_{k-1}, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle & 0 \\ \langle \mathbf{a}_k, \mathbf{a}_1 \rangle & \dots & \langle \mathbf{a}_k, \mathbf{a}_{k-1} \rangle & {V_k}^2 \end{vmatrix}$$
$$= {V_{k-1}}^2 {V_k}^2.$$

This completes the proof. $\square$

Let W be a subspace generated by linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$.

We say that the subspace W is $\begin{cases} \text{space-like if } \mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 > 0, \\ \text{light-like if } \mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 = 0, \\ \text{time-like if } \mathrm{Vol}_k(\mathbf{a}_1, \dots, \mathbf{a}_k)^2 < 0. \end{cases}$

This notion does not depend on the choice of basis $\mathbf{a}_1, \dots, \mathbf{a}_k$.

Lemma 1.4. *The following conditions are equivalent.*

- *The restriction of the non-degenerate bilinear form $\langle\ ,\ \rangle$ to W is degenerate.*
- *W is light-like.*

2 Frames along curves

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote a frame defined on a curve $\gamma(t)$. We define K^A by

$$\frac{d}{dt}A = K^A A \qquad \text{where } A = {}^t(\mathbf{a}_1\ \ldots\ \mathbf{a}_n).$$

If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ form a pseudo-orthonormal frame, then $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0$ if $i \neq j$ and $\langle \mathbf{a}_i, \mathbf{a}_i \rangle = \varepsilon_i$. So we have

$$0 = \frac{d}{dt}\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \frac{d}{dt}\mathbf{a}_i, \mathbf{a}_j \rangle + \langle \mathbf{a}_i, \frac{d}{dt}\mathbf{a}_j \rangle = \varepsilon_j {K^A}_{i,j} + \varepsilon_i {K^A}_{j,i}.$$

Lemma 2.1. *Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b_n}$ denote two frames defined along $\gamma(t)$. We assume that $A = PB$ where $A = {}^t(\mathbf{a}_1\ \ldots\ \mathbf{a}_n)$, $B = {}^t(\mathbf{b}_1\ \ldots\ \mathbf{b}_n)$ and P is a regular matrix. We define K^A, K^B by $\frac{d}{dt}A = K^A A$, $\frac{d}{dt}B = K^B B$. Then we obtain*

$$K^A = PK^B P^{-1} + \frac{dP}{dt}P^{-1}.$$

Proof. Since $K^A PB = K^A A = \frac{d}{dt}A = \frac{d}{dt}(PB) = \frac{dP}{dt}B + P\frac{dB}{dt} = \frac{dP}{dt}B + PK^B B$, we have

$$K^A P = \frac{dP}{dt} + PK^B.$$

Multiplying P^{-1} from the right, we obtain the result. $\square$

2.1 Frame in $\mathbb{R}^n_q$

We assume that $V_k = \mathrm{Vol}_k(\frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2}, \ldots, \frac{d^k\gamma}{dt^k}) \neq 0$ for $k = 1, 2, \ldots, n$. We consider pseudo-orthonormal frame $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ so that

$$\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle_{\mathbb{R}} = \left\langle \frac{d\gamma}{dt}, \ldots, \frac{d^k\gamma}{dt^k} \right\rangle_{\mathbb{R}}, \qquad k = 1, 2, \ldots, n-1.$$

and $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle_{\mathbb{R}} = \mathbb{R}^n_q$. We have

$$\mathbf{a}_k = \frac{\tilde{\mathbf{a}}_k}{|V_{k-1}V_k|}, \quad \tilde{\mathbf{a}}_k = \begin{vmatrix} \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \ldots & \langle \frac{d\gamma}{dt}, \frac{d^{k-1}\gamma}{dt^{k-1}} \rangle & \frac{d\gamma}{dt} \\ \vdots & \ddots & \vdots & \vdots \\ \langle \frac{d^{k-1}\gamma}{dt^{k-1}}, \frac{d\gamma}{dt} \rangle & \ldots & \langle \frac{d^{k-1}\gamma}{dt^{k-1}}, \frac{d^{k-1}\gamma}{dt^{k-1}} \rangle & \frac{d^{k-1}\gamma}{dt^{k-1}} \\ \langle \frac{d^k\gamma}{dt^k}, \frac{d\gamma}{dt} \rangle & \ldots & \langle \frac{d^k\gamma}{dt^k}, \frac{d^{k-1}\gamma}{dt^{k-1}} \rangle & \frac{d^k\gamma}{dt^k} \end{vmatrix} \quad (k = 1, \ldots, n-1)$$

and

$$\mathbf{a}_n = \frac{\tilde{\mathbf{a}}_n}{|\langle \tilde{\mathbf{a}}_n, \tilde{\mathbf{a}}_\mathbf{n} \rangle|^{1/2}}, \qquad \tilde{\mathbf{a}}_n = \begin{vmatrix} \langle \frac{d\gamma}{dt}, \mathbf{e}_1 \rangle & \ldots & \langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \mathbf{e}_1 \rangle & \mathbf{e}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \frac{d\gamma}{dt}, \mathbf{e}_n \rangle & \ldots & \langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \mathbf{e}_n \rangle & \mathbf{e}_n \end{vmatrix}.$$

We remark that $V_n{}^2 = (-1)^q|\frac{d\gamma}{dt}\ \frac{d^2\gamma}{dt^2}\ \dots\ \frac{d^n\gamma}{dt^n}|^2$ and $|\frac{d\gamma}{dt}\ \frac{d^2\gamma}{dt^2}\ \dots\ \frac{d^n\gamma}{dt^n}| = \sigma|V_n|$ where $\sigma = \operatorname{sign}|\frac{d\gamma}{dt}\ \frac{d^2\gamma}{dt^2}\ \dots\ \frac{d^n\gamma}{dt^n}|$.

Now we define the curvatures κ_k $(k = 1, \dots, n)$ by

$$\frac{d}{ds}\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2\kappa_1 & 0 & \dots & 0\\ -\varepsilon_1\kappa_1 & 0 & \varepsilon_3\kappa_2 & \ddots & \vdots\\ 0 & -\varepsilon_2\kappa_2 & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \varepsilon_n\kappa_{n-1}\\ 0 & \dots & 0 & -\varepsilon_{n-1}\kappa_{n-1} & 0 \end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_n\end{pmatrix}$$

where $\varepsilon_i = \langle \mathbf{a}_i, \mathbf{a}_i\rangle$.

Let k_i, $i = 1, \dots, q$, be positive integers with $1 \le k_1 < k_2 < \dots < k_q \le n$, We say that γ is a curve **of type** $(k_1, \dots, k_q)$, if

$$V_{k_j-1}{}^2 V_{k_j}{}^2 < 0\ (j = 1, \dots, q),\ \text{ or, equivalently }\ \varepsilon_i = \begin{cases} 1 & (i \notin \{k_1, \dots, k_q\}),\\ -1 & (i \in \{k_1, \dots, k_q\}),\end{cases}$$

since $\varepsilon_i = \operatorname{sign}\langle \tilde{\mathbf{a}}_i, \tilde{\mathbf{a}}_i\rangle = \operatorname{sign}(V_{i-1}{}^2 V_i{}^2)$.

Theorem 2.2. *Then we have*

$$\varepsilon_{i+1}\kappa_i = \varepsilon_i\frac{|V_{i-1}V_{i+1}|}{|V_0 V_i{}^2|} \quad (i = 1, \dots, n-2), \quad \varepsilon_n\kappa_{n-1} = \varepsilon_{n-1}\sigma\operatorname{sign}(V_n{}^2)\frac{|V_{n-2}V_n|}{|V_1 V_{n-1}{}^2|}.$$

Remark 2.3. It is also possible to show a similar formula for curves in Euclidean space. It was obtained in Gluck's paper [3]. The authors did not know Gluck's paper [3], when they first showed Theorem 2.2.

We first see the following

Lemma 2.4. $\langle \tilde{\mathbf{a}}_n, \tilde{\mathbf{a}}_n\rangle = V_{n-1}{}^2$ *and* $\langle \frac{d^n\gamma}{dt^n}, \tilde{\mathbf{a}}_n\rangle = (-1)^q\sigma|V_n|$.

Proof. Since

$$\langle \tilde{\mathbf{a}}_n, \mathbf{e}_i\rangle = (-1)^{n+i}\langle \mathbf{e}_i, \mathbf{e}_i\rangle M_i, \qquad M_i = \begin{vmatrix} \langle \frac{d\gamma}{dt}, \mathbf{e}_1\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_1\rangle\\ \vdots & & \vdots\\ \langle \frac{d\gamma}{dt}, \mathbf{e}_{i-1}\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_{i-1}\rangle\\ \langle \frac{d\gamma}{dt}, \mathbf{e}_{i+1}\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_{i+1}\rangle\\ \vdots & & \vdots\\ \langle \frac{d\gamma}{dt}, \mathbf{e}_n\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_n\rangle \end{vmatrix},$$

for $i = 1, \dots, n$, we have

$$\langle \tilde{\mathbf{a}}_n, \tilde{\mathbf{a}}_n\rangle = \begin{vmatrix} \langle \frac{d\gamma}{dt}, \mathbf{e}_1\rangle & \dots & \langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \mathbf{e}_1\rangle & \langle \tilde{\mathbf{a}}_n, \mathbf{e}_1\rangle\\ \vdots & \ddots & \vdots & \vdots\\ \langle \frac{d\gamma}{dt}, \mathbf{e}_n\rangle & \dots & \langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \mathbf{e}_n\rangle & \langle \tilde{\mathbf{a}}_n, \mathbf{e}_n\rangle \end{vmatrix} = \sum_{i=1}^n \langle \mathbf{e}_i, \mathbf{e}_i\rangle M_i{}^2 = V_{n-1}{}^2.$$

We also have

$$\left\langle \frac{d^n\gamma}{dt^n}, \tilde{\mathbf{a}}_n \right\rangle = \begin{vmatrix} \langle \frac{d\gamma}{dt}, \mathbf{e}_1\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \frac{d\gamma}{dt}, \mathbf{e}_n\rangle & \dots & \langle \frac{d^n\gamma}{dt^n}, \mathbf{e}_n\rangle \end{vmatrix} = (-1)^q \begin{vmatrix} \frac{d\gamma}{dt} & \cdots & \frac{d^n\gamma}{dt^n} \end{vmatrix}$$

and we completes the proof. □

Proof of Theorem 2.2. Since $\kappa_i = \langle \frac{d}{ds}\mathbf{a}_i, \mathbf{a}_{i+1}\rangle$ for $i = 1, \dots, n-2$, we obtain

$$\begin{aligned}
\kappa_i =& \left\langle \frac{d}{ds}\mathbf{a}_i, \mathbf{a}_{i+1} \right\rangle = \left\langle \frac{d}{ds}\frac{\tilde{\mathbf{a}}_i}{|V_{i-1}V_i|}, \ \frac{\tilde{\mathbf{a}}_{i+1}}{|V_iV_{i+1}|} \right\rangle \\
=& \left\langle \Big(\frac{d}{ds}\frac{1}{|V_{i-1}V_i|}\Big)\tilde{\mathbf{a}}_i, \ \frac{\tilde{\mathbf{a}}_{i+1}}{|V_iV_{i+1}|} \right\rangle + \left\langle \frac{1}{|V_{i-1}V_i|}\frac{d}{ds}\tilde{\mathbf{a}}_i, \ \frac{\tilde{\mathbf{a}}_{i+1}}{|V_iV_{i+1}|} \right\rangle \\
=& \frac{1}{|V_{i-1}V_i^2V_{i+1}|}\frac{dt}{ds}\left\langle \frac{d}{dt}\tilde{\mathbf{a}}_i, \ \tilde{\mathbf{a}}_{i+1} \right\rangle \quad (\text{since } \langle \tilde{\mathbf{a}}_i, \tilde{\mathbf{a}}_{i+1}\rangle = 0) \\
=& \frac{1}{|V_{i-1}V_i^2V_{i+1}|}\frac{1}{|\langle\gamma', \gamma'\rangle|^{1/2}} \begin{vmatrix} \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle & \dots & \langle \frac{d\gamma}{dt}, \frac{d^{i-1}\gamma}{dt^{i-1}}\rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \frac{d^{i-1}\gamma}{dt^{i-1}}, \frac{d\gamma}{dt}\rangle & \dots & \langle \frac{d^{i-1}\gamma}{d^{i-1}t}, \frac{d^{i-1}\gamma}{dt^{i-1}}\rangle & 0 \\ \frac{d}{dt}\langle \frac{d^{i}\gamma}{dt^{i}}, \frac{d\gamma}{dt}\rangle & \dots & \frac{d}{dt}\langle \frac{d^{i}\gamma}{dt^{i}}, \frac{d^{i-1}\gamma}{dt^{i-1}}\rangle & \langle \frac{d^{i+1}\gamma}{dt^{i+1}}, \tilde{\mathbf{a}}_{i+1}\rangle \end{vmatrix} \\
& \qquad (\text{since } \langle \frac{d^j\gamma}{dt^j}, \tilde{\mathbf{a}}_{i+1}\rangle = 0, \quad j = 1, 2, \dots, i-1) \\
=& \frac{1}{|V_1V_{i-1}V_i^2V_{i+1}|}{V_{i-1}}^2{V_{i+1}}^2 \\
=& \operatorname{sign}(({V_{i-1}}^2{V_i}^2)({V_i}^2{V_{i+1}}^2))\frac{|V_{i-1}V_{i+1}|}{|V_1{V_i}^2|} = \varepsilon_i\varepsilon_{i+1}\frac{|V_{i-1}V_{i+1}|}{|V_1{V_i}^2|}
\end{aligned}$$

We also have that

$$\begin{aligned}
\kappa_{n-1} =& \left\langle \frac{d}{ds}\mathbf{a}_{n-1}, \mathbf{a}_n \right\rangle = \left\langle \frac{d}{ds}\frac{\tilde{\mathbf{a}}_{n-1}}{|V_{n-2}V_{n-1}|}, \ \frac{\tilde{\mathbf{a}}_n}{|V_{n-1}|} \right\rangle = \left\langle \frac{1}{|V_{n-2}V_{n-1}|}\frac{d}{ds}\tilde{\mathbf{a}}_{n-1}, \ \frac{\tilde{\mathbf{a}}_n}{|V_{n-1}|} \right\rangle \\
=& \frac{1}{|V_{n-2}{V_{n-1}}^2|}\frac{1}{|\langle\gamma', \gamma'\rangle|^{1/2}} \begin{vmatrix} \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle & \dots & \langle \frac{d\gamma}{dt}, \frac{d^{n-2}\gamma}{dt^{n-2}}\rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle \frac{d^{n-2}\gamma}{dt^{n-2}}, \frac{d\gamma}{dt}\rangle & \dots & \langle \frac{d^{n-2}\gamma}{dt^{n-2}}, \frac{d^{n-2}\gamma}{dt^{n-2}}\rangle & 0 \\ \frac{d}{dt}\langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \frac{d\gamma}{dt}\rangle & \dots & \frac{d}{dt}\langle \frac{d^{n-1}\gamma}{dt^{n-1}}, \frac{d^{n-2}\gamma}{dt^{n-2}}\rangle & \langle \frac{d^{n}\gamma}{dt^{n}}, \tilde{\mathbf{a}}_n\rangle \end{vmatrix} \\
=& (-1)^q\frac{\operatorname{sign}({V_{n-2}}^2)|V_{n-2}|}{|V_1{V_{n-1}}^2|}\sigma|V_n| = (-1)^q\sigma\operatorname{sign}({V_n}^2)\frac{\varepsilon_{n-1}\varepsilon_n|V_{n-2}V_n|}{|V_1{V_{n-1}}^2|}
\end{aligned}$$

which completes the proof. □

As a consequence, we have, for $i = 1, 2, \dots, n-2$,

$$\varepsilon_{i+1}\kappa_1\kappa_2\cdots\kappa_i = \varepsilon_1\frac{|V_2|}{|V_1{V_1}^2|}\frac{|V_1V_3|}{|V_1{V_2}^2|}\frac{|V_2V_4|}{|V_1{V_3}^2|}\cdots\frac{|V_{i-1}V_{i+1}|}{|V_1{V_i}^2|} = \varepsilon_1\frac{|V_{i+1}|}{|{V_1}^{i+1}V_i|}$$

and $\varepsilon_n\kappa_1\kappa_2\cdots\kappa_{n-1} = \sigma\varepsilon_1\operatorname{sign}({V_n}^2)\frac{|V_n|}{|{V_1}^nV_{n-1}|}$.

Remark 2.5. *Assume that* $V_i = t^{e_i}(v_i + O(t))$, $v_i \neq 0$, *for* $i = 1, \dots, n$. *Then*

$$|\kappa_i| = |t|^{e_{i-1}+e_{i+1}-2e_i-e_1}\left(\frac{|v_{i-1}v_{i+1}|}{|v_1 {v_i}^2|} + O(t)\right).$$

We denote e_i *and* v_i *by* $e_i(\gamma)$ *and* $v_i(\gamma)$ *respectively when we want to mention the curve* $\gamma(t)$. *Let us consider a family of curves* $\gamma_a(t)$ *with parameter* $a \in (-\delta, \delta)$. *If* $e_i = e_i(\gamma_a)$ $(i = 1, \dots n-1)$ *do not depend on* a, *then we have*

$$\lim_{a\to 0}\lim_{t\to 0} t^{e_1+2e_i-e_{i-1}-e_{i+1}}\kappa_i(\gamma_a(t)) = \lim_{t\to 0}\lim_{a\to 0} t^{e_1+2e_i-e_{i-1}-e_{i+1}}\kappa_i(\gamma_a(t)).$$

Arc length parameter Let s be the arc length parameter, that is, the parameter s with $ds = |V_1|dt$. Then we have $s = \frac{|v_1|}{e_1+1}t^{e_1+1} + o(t)$. Next let us denote derivative by s by $'$. Then $|\langle\gamma', \gamma'\rangle| \equiv 1$, and consider the derivative of $\langle\gamma', \gamma'\rangle \equiv \pm 1$ by s. Then we have

$$\langle\gamma', \gamma''\rangle = 0, \quad \langle\gamma', \gamma'''\rangle + \langle\gamma'', \gamma''\rangle = 0, \quad \langle\gamma', \gamma^{(4)}\rangle + 2\langle\gamma'', \gamma'''\rangle + \langle\gamma'', \gamma'''\rangle = 0, \quad \dots$$

and obtain that ${V_1}^2 = 1$, ${V_2}^2 = \langle\gamma'', \gamma''\rangle$, ${V_3}^2 = \langle\gamma'', \gamma''\rangle\langle\gamma''', \gamma'''\rangle - \langle\gamma'', \gamma'''\rangle^2 - \langle\gamma'', \gamma''\rangle^3$,

$${V_4}^2 = \begin{vmatrix} 1 & 0 & -\langle\gamma'', \gamma''\rangle & -2\langle\gamma'', \gamma'''\rangle - \langle\gamma'', \gamma'''\rangle \\ 0 & \langle\gamma'', \gamma''\rangle & \langle\gamma'', \gamma'''\rangle & \langle\gamma'', \gamma^{(4)}\rangle \\ -\langle\gamma'', \gamma''\rangle & \langle\gamma''', \gamma''\rangle & \langle\gamma''', \gamma'''\rangle & \langle\gamma''', \gamma^{(4)}\rangle \\ -2\langle\gamma'', \gamma'''\rangle - \langle\gamma'', \gamma'''\rangle & \langle\gamma^{(4)}, \gamma''\rangle & \langle\gamma^{(4)}, \gamma'''\rangle & \langle\gamma^{(4)}, \gamma^{(4)}\rangle \end{vmatrix}.$$

Taylor expansion Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n_q$ be a curve whose Taylor expansion is

$$\sum_{i=0}^{\infty}\frac{c_i}{i!}t^i = c_0 + c_1 t + \frac{1}{2}c_2 t^2 + \frac{1}{3!}c_3 t^3 + \cdots, \qquad c_i \in \mathbb{R}^n_q.$$

Since $\frac{d^j\gamma}{dt^j} = \sum_{i=j}^{\infty}\frac{c_i}{(i-j)!}t^{i-j}$, we have

$$\left\langle\frac{d^{j_1}\gamma}{dt^{j_1}}, \frac{d^{j_2}\gamma}{dt^{j_2}}\right\rangle = \langle c_{j_1}, c_{j_2}\rangle + (\langle c_{j_1+1}, c_{j_2}\rangle + \langle c_{j_1}, c_{j_2+1}\rangle)t + \cdots \qquad j_1, j_2 = 1, 2, \dots$$

and obtain

$${V_k}^2 = |\langle c_{j_1}, c_{j_2}\rangle + (\langle c_{j_1+1}, c_{j_2}\rangle + \langle c_{j_1}, c_{j_2+1}\rangle)t + \cdots|_{j_1,j_2=1,2,\dots,k}.$$

In particular,

$$\begin{aligned} {V_1}^2 =& \langle\gamma, \gamma\rangle = \langle c_1, c_1\rangle + 2\langle c_1, c_2\rangle t + (2\langle c_1, c_3\rangle + \langle c_2, c_2\rangle)t^2 + \cdots \\ {V_2}^2 =& \begin{vmatrix} \langle\gamma', \gamma'\rangle & \langle\gamma', \gamma''\rangle \\ \langle\gamma'', \gamma'\rangle & \langle\gamma'', \gamma''\rangle \end{vmatrix} = \begin{vmatrix} \langle c_1, c_1\rangle & \langle c_1, c_2\rangle \\ \langle c_2, c_1\rangle & \langle c_2, c_2\rangle \end{vmatrix} \\ &+ \left(\begin{vmatrix} \langle c_1, c_2\rangle & \langle c_1, c_2\rangle \\ \langle c_1, c_3\rangle & \langle c_2, c_2\rangle \end{vmatrix} + \begin{vmatrix} \langle c_1, c_1\rangle & \langle c_1, c_3\rangle + \langle c_2, c_2\rangle \\ \langle c_2, c_1\rangle & 2\langle c_2, c_3\rangle \end{vmatrix}\right)t + \cdots \end{aligned}$$

2.2 Frame with respect to $M(c)$.

We assume that $\widehat{V}_k = \mathrm{Vol}_{k+1}(\gamma, \frac{d\gamma}{dt}, \dots, \frac{d^{k-1}\gamma}{dt^{k-1}}) \neq 0$ for $k = 1, \dots, n$. We consider pseudo-orthonormal frame $\mathbf{b}_1, \dots, \mathbf{b}_n$ so that

$$\langle \mathbf{b}_1, \dots, \mathbf{b}_k \rangle_{\mathbb{R}} = \left\langle \gamma, \frac{d\gamma}{dt}, \dots, \frac{d^{k-1}\gamma}{dt^{k-1}} \right\rangle_{\mathbb{R}}, \qquad k = 1, \dots, n-1,$$

and $\langle \mathbf{b}_1, \dots, \mathbf{b}_n \rangle_{\mathbb{R}} = \mathbb{R}^n_q$. We have

$$\mathbf{b}_k = \frac{\tilde{\mathbf{b}}_k}{|\widehat{V}_k \widehat{V}_{k-1}|}, \qquad \tilde{\mathbf{b}}_k = \begin{vmatrix} \langle \gamma, \gamma \rangle & \langle \gamma, \frac{d\gamma}{dt} \rangle & \dots & \langle \gamma, \frac{d^{k-2}\gamma}{dt^{k-2}} \rangle & \gamma \\ \langle \frac{d\gamma}{dt}, \gamma \rangle & \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \dots & \langle \frac{d\gamma}{dt}, \frac{d^{k-2}\gamma}{dt^{k-2}} \rangle & \frac{d\gamma}{dt} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle \frac{d^{k-1}\gamma}{dt^{k-1}}, \gamma \rangle & \langle \frac{d^{k-1}\gamma}{dt^{k-1}}, \frac{d\gamma}{dt} \rangle & \dots & \langle \frac{d^{k-1}\gamma}{dt^{k-1}}, \frac{d^{k-2}\gamma}{dt^{k-2}} \rangle & \frac{d^{k-1}\gamma}{dt^{k-1}} \end{vmatrix},$$

for $k = 1, \dots, n-1$, and

$$\mathbf{b}_n = \frac{\tilde{\mathbf{b}}_n}{|\tilde{\mathbf{b}}_n|}, \qquad \tilde{\mathbf{b}}_n = \begin{vmatrix} \langle \gamma, \mathbf{e}_1 \rangle & \langle \frac{d\gamma}{dt}, \mathbf{e}_1 \rangle & \dots & \langle \frac{d^{n-2}\gamma}{dt^{n-2}}, \mathbf{e}_1 \rangle & \mathbf{e}_1 \\ \vdots & \vdots & & \vdots & \vdots \\ \langle \gamma, \mathbf{e}_n \rangle & \langle \frac{d\gamma}{dt}, \mathbf{e}_n \rangle & \dots & \langle \frac{d^{n-2}\gamma}{dt^{n-2}}, \mathbf{e}_n \rangle & \mathbf{e}_n \end{vmatrix}.$$

We set $\widehat{V}_0 = 1$, by convention. We remark that $\widehat{V}_n^2 = (-1)^q |\gamma \ \frac{d\gamma}{dt} \ \frac{d^2\gamma}{dt^2} \ \dots \ \frac{d^{n-1}\gamma}{dt^{n-1}}|^2$ and $|\gamma \ \frac{d\gamma}{dt} \ \frac{d^2\gamma}{dt^2} \ \dots \ \frac{d^{n-1}\gamma}{dt^{n-1}}| = \hat{\sigma} |\widehat{V}_n|$ where $\hat{\sigma} = \mathrm{sign}\, |\gamma \ \frac{d\gamma}{dt} \ \frac{d^2\gamma}{dt^2} \ \dots \ \frac{d^{n-1}\gamma}{dt^{n-1}}|$.

Now we define the **curvatures** $\hat{\kappa}_k$ $(k = 1, \dots, n-1)$ by

$$\frac{d}{ds} \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 0 & \hat{\varepsilon}_2 \hat{\kappa}_1 & 0 & \dots & 0 \\ -\hat{\varepsilon}_1 \hat{\kappa}_1 & 0 & \hat{\varepsilon}_2 \hat{\kappa}_2 & \ddots & \vdots \\ 0 & -\hat{\varepsilon}_2 \hat{\kappa}_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \hat{\varepsilon}_n \hat{\kappa}_{n-1} \\ 0 & \dots & 0 & -\hat{\varepsilon}_{n-1} \hat{\kappa}_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$$

where $\hat{\varepsilon}_i = \mathrm{sign} \langle \mathbf{b}_i, \mathbf{b}_i \rangle$.

Let k_i, $i = 1, \dots, q$, be positive integers with $1 \leq k_1 < k_2 < \dots < k_q \leq n$. We say that γ is a curve **of type** $(k_1, \dots, k_q)$ with respect to $M(c)$ if

$$\widehat{V}_{i-1}^2 \widehat{V}_i^2 < 0 \ (i = k_1, \dots, k_q), \quad \text{or, equivalently} \quad \hat{\varepsilon}_i = \begin{cases} 1 & (i \notin \{k_1, \dots, k_q\}), \\ -1 & (i \in \{k_1, \dots, k_q\}), \end{cases}$$

since $\hat{\varepsilon}_i = \mathrm{sign} \langle \tilde{\mathbf{b}}_i, \tilde{\mathbf{b}}_i \rangle = \mathrm{sign}(\widehat{V}_{i-1}^2 \widehat{V}_i^2)$.

Theorem 2.6. *We have*

$$\hat{\varepsilon}_{i+1} \hat{\kappa}_i = \frac{\hat{\varepsilon}_i |\widehat{V}_{i-1} \widehat{V}_{i+1}|}{|\langle \gamma', \gamma' \rangle^{1/2} \widehat{V}_i^2|} \quad (i = 1, \dots, n-2), \quad \hat{\varepsilon}_n \hat{\kappa}_{n-1} = \frac{\hat{\varepsilon}_{n-1} \hat{\sigma} \, \mathrm{sign}(\widehat{V}_n^2) |\widehat{V}_{n-2} \widehat{V}_n|}{|\langle \gamma', \gamma' \rangle^{1/2} \widehat{V}_{n-1}^2|}.$$

The proof is similar to that of Theorem 2.2. A similar statement to Lemma 2.4 becomes as follows.

Lemma 2.7. *$\langle \tilde{\mathbf{b}}_n, \tilde{\mathbf{b}}_n\rangle = \widehat{V}_{n-1}^2$ and $\langle \frac{d^n\gamma}{dt^n}, \tilde{\mathbf{b}}_n\rangle = (-1)^q\hat{\sigma}|\widehat{V}_n|$.*

As a consequence, we have, for $i = 1, 2, \dots, n-2$,

$$\hat{\varepsilon}_i\hat{\kappa}_1\hat{\kappa}_2\cdots\hat{\kappa}_i = \frac{\hat{\varepsilon}_1}{|\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle|^{i/2}}\frac{|\widehat{V}_0\widehat{V}_2|}{|\widehat{V}_1^2|}\frac{|\widehat{V}_1\widehat{V}_3|}{|\widehat{V}_2^2|}\frac{|\widehat{V}_2\widehat{V}_4|}{|\widehat{V}_3^2|}\cdots\frac{|\widehat{V}_{i-1}\widehat{V}_{i+1}|}{|\widehat{V}_i^2|} = \frac{\hat{\varepsilon}_1}{|\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle|^{i/2}}\frac{|\widehat{V}_{i+1}|}{|\widehat{V}_1\widehat{V}_i|},$$

and $\hat{\varepsilon}_n\hat{\kappa}_1\hat{\kappa}_2\cdots\hat{\kappa}_{n-1} = \frac{\hat{\sigma}\hat{\varepsilon}_1\,\mathrm{sign}(\widehat{V}_n^2)}{|\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle|^{\frac{n-1}{2}}}\frac{|\widehat{V}_n|}{|\widehat{V}_1\widehat{V}_{n-1}|}$.

Remark 2.8. *Assume that $|\langle\gamma', \gamma'\rangle|^{1/2} = t^{\hat{e}_0}(\hat{v}_0 + O(t))$, $\hat{v}_0 \neq 0$, $\widehat{V}_i = t^{\hat{e}_i}(\hat{v}_i + O(t))$, $\hat{v}_i \neq 0$, for $i = 1, \dots, n$. Then*

$$|\hat{\kappa}_i| = |t|^{\hat{e}_{i-1}+\hat{e}_{i+1}-2\hat{e}_i-\hat{e}_0}\left(\frac{|\hat{v}_{i-1}\hat{v}_{i+1}|}{|\hat{v}_0\hat{v}_i^2|} + O(t)\right).$$

Let us consider a family of curves $\gamma_a(t)$ with parameter $a \in (-\delta, \delta)$. If $\hat{e}_i$ ($i = 0, 1, \dots n-1$) do not depend on a, then we have

$$\lim_{a\to 0}\lim_{t\to 0} t^{\hat{e}_0+2\hat{e}_i-\hat{e}_{i-1}-\hat{e}_{i+1}}\hat{\kappa}_i(\gamma_a(t)) = \lim_{t\to 0}\lim_{a\to 0} t^{\hat{e}_0+2\hat{e}_i-\hat{e}_{i-1}-\hat{e}_{i+1}}\hat{\kappa}_i(\gamma_a(t)).$$

Taylor expansion Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}_q^n$ be a curve whose Taylor expansion is

$$\sum_{i=0}^{\infty}\frac{c_i}{i!}t^i = c_0 + c_1 t + \frac{1}{2}c_2t^2 + \frac{1}{3!}c_3t^3 + \cdots, \qquad c_i \in \mathbb{R}_q^n.$$

Since $\frac{d^j\gamma}{dt^j} = \sum_{i=j}^{\infty}\frac{c_i}{(i-j)!}t^{i-j}$, we have

$$\left\langle\frac{d^{j_1}\gamma}{dt^{j_1}}, \frac{d^{j_2}\gamma}{dt^{j_2}}\right\rangle = \langle c_{j_1}, c_{j_2}\rangle + (\langle c_{j_1+1}, c_{j_2}\rangle + \langle c_{j_1}, c_{j_2+1}\rangle)t + \cdots \qquad j_1, j_2 = 0, 1, 2, \dots$$

and obtain

$$\widehat{V}_k^2 = |\langle c_{j_1}, c_{j_2}\rangle + (\langle c_{j_1+1}, c_{j_2}\rangle + \langle c_{j_1}, c_{j_2+1}\rangle)t + \cdots|_{j_1,j_2=0,1,\dots,k-1}.$$

2.3 Curves in $M(c)$.

In order to investigate a curve in $M(c)$, $c \neq 0$, it is natural to use the frame with respect to $M(c)$. If we consider a curve $\gamma : (-\varepsilon, \varepsilon) \to M(c)$, we have

$$\begin{aligned}
\langle\gamma, \gamma\rangle &= c\\
\left\langle\gamma, \frac{d\gamma}{dt}\right\rangle &= 0\\
\left\langle\gamma, \frac{d^2\gamma}{dt^2}\right\rangle + \left\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle &= 0\\
\left\langle\gamma, \frac{d^3\gamma}{dt^3}\right\rangle + 3\left\langle\frac{d\gamma}{dt}, \frac{d^2\gamma}{d^2t}\right\rangle &= 0\\
\left\langle\gamma, \frac{d^4\gamma}{dt^4}\right\rangle + 4\left\langle\frac{d\gamma}{dt}, \frac{d^3\gamma}{d^3t}\right\rangle + 3\left\langle\frac{d^2\gamma}{d^2t}, \frac{d^2\gamma}{d^2t}\right\rangle &= 0
\end{aligned}$$

$$\cdots$$

and

$$\widehat{V}_{k+1} = \begin{vmatrix} c & 0 & -\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & -3\langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle & \cdots & \langle \gamma, \frac{d^k\gamma}{dt^k} \rangle \\ 0 & \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3} \rangle & \cdots & \langle \frac{d\gamma}{dt}, \frac{d^k\gamma}{dt^k} \rangle \\ -\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d^3\gamma}{dt^3} \rangle & \cdots & \langle \frac{d^2\gamma}{dt^2}, \frac{d^k\gamma}{dt^k} \rangle \\ -3\langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d^3\gamma}{dt^3} \rangle & \cdots & \langle \frac{d^3\gamma}{dt^3}, \frac{d^k\gamma}{dt^k} \rangle \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \frac{d^k\gamma}{dt^k}, \gamma \rangle & \langle \frac{d^k\gamma}{dt^k}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^k\gamma}{dt^k}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^k\gamma}{dt^k}, \frac{d^3\gamma}{dt^3} \rangle & \cdots & \langle \frac{d^k\gamma}{dt^k}, \frac{d^k\gamma}{dt^k} \rangle \end{vmatrix}^{1/2}.$$

We remark that

$$\begin{aligned}
\widehat{V}_1^2 =& c, \\
\widehat{V}_2^{\,2} =& c\Big\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \Big\rangle, \\
\widehat{V}_3^{\,2} =& c\bigg(\Big\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \Big\rangle \Big\langle \frac{d^2\gamma}{dt^2}, \frac{d^2\gamma}{dt^2} \Big\rangle - \Big\langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \Big\rangle^2 \bigg) - \Big\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \Big\rangle^3, \\
\widehat{V}_4^{\,2} =& \begin{vmatrix} c & 0 & -\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & -3\langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle \\ 0 & \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3} \rangle \\ -\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^2\gamma}{dt^2}, \frac{d^3\gamma}{dt^3} \rangle \\ -3\langle \frac{d\gamma}{dt}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d\gamma}{dt} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d^2\gamma}{dt^2} \rangle & \langle \frac{d^3\gamma}{dt^3}, \frac{d^3\gamma}{dt^3} \rangle \end{vmatrix}.
\end{aligned}$$

We thus conclude that

$$\begin{aligned}
\hat{\kappa}_1 =& \operatorname{sign}(\widehat{V}_2^2)\frac{|\widehat{V}_2|}{|\langle\gamma',\gamma'\rangle|^{1/2}|\widehat{V}_1^2|} = \operatorname{sign}(c\langle\gamma',\gamma'\rangle)\frac{|c\langle\gamma',\gamma'\rangle|^{1/2}}{|\langle\gamma',\gamma'\rangle|^{1/2}|c|} = \operatorname{sign}(c\langle\gamma',\gamma'\rangle)\frac{1}{|c|^{1/2}} \\
\hat{\kappa}_2 =& \operatorname{sign}(\widehat{V}_1^2\widehat{V}_3^2)\frac{|\widehat{V}_1\widehat{V}_3|}{|\langle\gamma',\gamma'\rangle|^{1/2}|\widehat{V}_2^2|} \\
=& \operatorname{sign}(c\widehat{V}_3^2)\frac{|c|^{1/2}|c(\langle\gamma',\gamma'\rangle\langle\gamma'',\gamma''\rangle - \langle\gamma',\gamma''\rangle^2) - \langle\gamma',\gamma'\rangle^3|^{1/2}}{|\langle\gamma',\gamma'\rangle|^{3/2}|c|} \\
=& \operatorname{sign}(c\widehat{V}_3^2)\left(\frac{|c(\langle\gamma',\gamma'\rangle\langle\gamma'',\gamma''\rangle - \langle\gamma',\gamma''\rangle^2) - \langle\gamma',\gamma'\rangle^3|}{|\langle\gamma',\gamma'\rangle^3 c|}\right)^{1/2} \\
=& \operatorname{sign}(c\widehat{V}_3^2)\left|\frac{\langle\gamma'',\gamma''\rangle}{\langle\gamma',\gamma'\rangle^2} - \frac{\langle\gamma',\gamma''\rangle^2}{\langle\gamma',\gamma'\rangle^3} - \frac{1}{c}\right|^{1/2} \\
|\hat{\kappa}_3| =& \frac{|\widehat{V}_2\widehat{V}_4|}{|\langle\gamma',\gamma'\rangle|^{1/2}|\widehat{V}_3^2|} = \frac{|c|^{1/2}|\widehat{V}_4|}{|c(\langle\gamma',\gamma'\rangle\langle\gamma'',\gamma''\rangle - \langle\gamma',\gamma''\rangle^2) - \langle\gamma',\gamma'\rangle^3|} \\
|\hat{\kappa}_4| =& \frac{|\widehat{V}_3\widehat{V}_5|}{|\langle\gamma',\gamma'\rangle|^{1/2}|\widehat{V}_4^2|}
\end{aligned}$$

Remark 2.9. *Let $\gamma_c : (-\varepsilon, \varepsilon) \to M(c)$ be a family of curves for $c \in (-\delta, \delta)$ with $\gamma_0 = \gamma$.*

So we have $\widehat{V}_i(\gamma_c(t))^2 = O(c)$ when $c \to 0$ for $i = 1, 2$. This means that

$$\lim_{c\to 0} |\hat{\kappa}_i(\gamma_c(t))| = \begin{cases} \infty & (i = 1, 2) \\ 0 & (i = 3) \\ \frac{|\widehat{V}_{i-1}\widehat{V}_{i+1}|}{|\langle\gamma',\gamma'\rangle^{1/2}\widehat{V}_i^2|}\Big|_{c=0} & (4 \le i \le n-1) \end{cases}$$

assuming $\langle\gamma', \gamma'\rangle \neq 0$, $\widehat{V}_i \neq 0$. This implies even for a curve in the light cone $M(0)$, we can define the notion of curvature $\hat{\kappa}_i$ for $i \geq 4$, whenever $\langle\gamma', \gamma'\rangle \neq 0$ and $\widehat{V}_{i-1} \neq 0$.

3 Height functions, distance squared functions and unfoldings

For $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ we consider the inner product defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

We consider the semi-Euclidean space with this inner product and denote it by $(\mathbb{R}^n_q, \ \cdot\ )$ Set $S^{n-1} = \{\mathbf{y} \in \mathbb{R}^n_q : \mathbf{y} \cdot \mathbf{y} = 1\}$. We consider the map

$$g_c : M(c) \setminus \{0\} \to S^{n-1}, \quad \mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i \mapsto \frac{1}{\sqrt{\sum_{i=1}^n {x_i}^2}} \mathbf{x}.$$

We remark that

$$\operatorname{Im} g_c = \begin{cases} S_+ & (c > 0) \\ S_0 & (c = 0) \\ S_- & (c < 0) \end{cases} \quad \text{where} \quad \begin{aligned} S_+ &= \{\mathbf{y} \in S^{n-1} : \langle\mathbf{y}, \mathbf{y}\rangle > 0\} \\ S_0 &= \{\mathbf{y} \in S^{n-1} : \langle\mathbf{y}, \mathbf{y}\rangle = 0\} \\ S_- &= \{\mathbf{y} \in S^{n-1} : \langle\mathbf{y}, \mathbf{y}\rangle < 0\} \end{aligned}$$

When $c \neq 0$, the inverse is defined by

$$\operatorname{Im} g_c \to M(c) \setminus \{0\}, \quad \mathbf{y} \mapsto \frac{\sqrt{|c|}}{\sqrt{|\langle\mathbf{y}, \mathbf{y}\rangle|}} \mathbf{y}.$$

So $M(c)$, $c > 0$, is isomorphic to S_+, and $M(c)$, $c < 0$, is isomorphic to S_-.

Remark 3.1. *Consider Lorentz transformations*

$$\mathbb{R}^2_1 \to \mathbb{R}^2_1, \ \mathbf{x} \mapsto \mathbf{y} = P\mathbf{x}, \quad P = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}.$$

Remark that ${y_1}^2 + {y_2}^2 = ({x_1}^2 + {x_2}^2)\cosh 2\theta - 2x_1x_2 \sinh 2\theta$. So the definition of S^{n-1} (and thus $S_\pm$) does depend on the inner product $\mathbf{x} \cdot \mathbf{y}$.

Let $\gamma : (-\varepsilon, \varepsilon) \to X$ be a curve in $X = \mathbb{R}^n_q$ or $M(c)$. We define height function by $h_{\mathbf{y}}(t) = \langle\gamma(t), \mathbf{y}\rangle$ for $\mathbf{y} \in \mathbb{R}^n_q$. We consider the height unfoldings

$$H : (X \times P, (0, \mathbf{y}_0)) \to \mathbb{R}, \qquad (t, \mathbf{y}) \mapsto h_{\mathbf{y}}(t) - u_0,$$

$$\widehat{H} : (X \times P \times \mathbb{R}, (0, \mathbf{y}_0, u_0)) \to \mathbb{R}, \qquad (t, \mathbf{y}, u) \mapsto h_{\mathbf{y}}(t) - u,$$

where $P = \mathbb{R}^n_q$, $M(c)$, S^{n-1}, S_+, S_-, S_0, $T^c_{\mathbf{x}}M(c)$. Here

$$T^c_{\mathbf{x}}M(c) = \{\mathbf{y} \in \mathbb{R}^n_q : \langle \mathbf{y}, \mathbf{x}\rangle = 0,\ \langle \mathbf{y}, \mathbf{y}\rangle = c\}.$$

We remark that

$$\frac{d}{dt}h_{\mathbf{y}}(t) = \left\langle \mathbf{y}, \frac{d\gamma}{dt}\right\rangle = 0, \qquad \frac{d^2}{dt^2}h_{\mathbf{y}}(t) = \left\langle \mathbf{y}, \frac{d^2\gamma}{dt^2}\right\rangle = 0$$

defines a subspace in $\mathbb{R}^n_q$, which we call the binormal space. So the bifurcation set of H

$$B_H = \left\{\mathbf{y} \in P : \frac{d}{dt}h_{\mathbf{y}}(t) = \frac{d^2}{dt^2}h_{\mathbf{y}}(t) = 0\right\}$$

is the intersection of the union of binormal spaces with P. The discriminant of H

$$D_H = \left\{\mathbf{y} \in P : h_{\mathbf{y}}(t) - u_0 = \frac{d}{dt}h_{\mathbf{y}}(t) = 0\right\}$$

is the intersection of the union of normal spaces with P and $h_{\mathbf{y}}(t) = u_0$.

For a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n_q$, we also consider the distance squared function

$$\phi_{\mathbf{y}}(t) = -\frac{1}{2}\langle \mathbf{y} - \gamma(t), \mathbf{y} - \gamma(t)\rangle$$

for $\mathbf{y} \in \mathbb{R}^n_q$, and the distance squared unfoldings

$$\begin{aligned} &\Phi : (\mathbb{R} \times \mathbb{R}^n_q, (0, \mathbf{y}_0)) \to \mathbb{R}, && (t, \mathbf{y}) \mapsto \phi_{\mathbf{y}}(t) - u_0, \text{ and}\\ &\widehat{\Phi} : (\mathbb{R} \times \mathbb{R}^n_q \times \mathbb{R}, (0, \mathbf{y}_0, u_0)) \to \mathbb{R}, && (t, \mathbf{y}, u) \mapsto \phi_{\mathbf{y}}(t) - u. \end{aligned}$$

We remark that the bifurcation set of Φ

$$B_\Phi = \left\{\mathbf{y} \in \mathbb{R}^n_q : \frac{d}{dt}\phi_{\mathbf{y}}(t) = \frac{d^2}{dt^2}\phi_{\mathbf{y}}(t) = 0\right\}$$

is the focal set of γ. The discriminant of Φ

$$D_\Phi = \left\{\mathbf{y} \in \mathbb{R}^n_q : \phi_{\mathbf{y}}(t) - u_0 = \frac{d}{dt}\phi_{\mathbf{y}}(t) = 0\right\}$$

is the tube of γ.

Let T_P denote the tangent space of P at $\mathbf{y}$. We remark that at $\mathbf{y} = (y_1, \ldots, y_n)$ with $y_1 \neq 0$

$$T_P = \begin{cases} T_{\mathbf{y}}\mathbb{R}^n_q = \left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right\rangle_{\mathbb{R}} & P = \mathbb{R}^n_q \\ T_{\mathbf{y}}M(c) = \left\langle \langle \mathbf{e}_i, \mathbf{e}_i\rangle y_i \frac{\partial}{\partial y_1} - \langle \mathbf{e}_1, \mathbf{e}_1\rangle y_1 \frac{\partial}{\partial y_i} : i = 2, \ldots, n\right\rangle_{\mathbb{R}} & P = M(c) \\ T_{\mathbf{y}}S^{n-1} = \left\langle y_i \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_i} : i = 2, \ldots, n\right\rangle_{\mathbb{R}} & P = S^{n-1}, S_+, S_- \end{cases}$$

and, at a point $\mathbf{y} = (y_1, \ldots, y_n)$ with $y_1 \neq 0$, $y_{p+1} \neq 0$,

$$T_{\mathbf{y}}S_0 = \left\langle y_i \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_i},\ i = 2, \ldots, p;\ y_j \frac{\partial}{\partial y_{p+1}} - y_{p+1} \frac{\partial}{\partial y_j},\ j = p+2, \ldots, n\right\rangle_{\mathbb{R}}$$

and, at a point $\mathbf{y} = (y_1, \dots, y_n) \in T^c_{\mathbf{x}}M(c)$ with $x_1y_2 - x_2y_1 \neq 0$ the tangent space of $T^c_{\mathbf{x}}M(c)$ is spanned by

$$\mathbf{w}_i = \frac{1}{\langle \mathbf{e}_1, \mathbf{e}_1\rangle}\begin{vmatrix} x_2 & y_2 \\ x_i & y_i\end{vmatrix}\frac{\partial}{\partial y_1} - \frac{1}{\langle \mathbf{e}_2, \mathbf{e}_2\rangle}\begin{vmatrix} x_1 & y_1 \\ x_i & y_i\end{vmatrix}\frac{\partial}{\partial y_2} + \frac{1}{\langle \mathbf{e}_i, \mathbf{e}_i\rangle}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}\frac{\partial}{\partial y_i} \quad (i = 3, \dots, n).$$

Set $V_P = \langle \mathbf{v}h_{\mathbf{y}} : \mathbf{v} \in T_P\rangle_{\mathbb{R}}$. Since $h_{\mathbf{y}}(t) = \sum_{s=1}^n \langle \mathbf{e}_s, \mathbf{e}_s\rangle y_s\gamma_s(t)$, we have

$$V_P = \begin{cases} \langle \gamma_i\rangle_{\mathbb{R}} & P = \mathbb{R}^n_q \\ \big\langle \langle \mathbf{e}_1, \mathbf{e}_1\rangle y_i\gamma_1 - \langle \mathbf{e}_i, \mathbf{e}_i\rangle y_1\gamma_i\big\rangle_{\mathbb{R}} & P = S^{n-1}, S_+, S_- \\ \big\langle \langle \mathbf{e}_1, \mathbf{e}_1\rangle\langle \mathbf{e}_i, \mathbf{e}_i\rangle(y_i\gamma_1 - y_1\gamma_i)\big\rangle_{\mathbb{R}} = \langle y_i\gamma_1 - y_1\gamma_i\rangle_{\mathbb{R}} & P = M(c) \\ \big\langle y_i\gamma_1 - y_1\gamma_i,\ i = 2, \dots, p;\ y_j\gamma_{p+1} - y_{p+1}\gamma_j,\ j = p+2, \dots, n\big\rangle_{\mathbb{R}} & P = T_{\mathbf{y}}S_0 \\ \left\langle \begin{vmatrix} x_2 & y_2 \\ x_i & y_i\end{vmatrix}\gamma_1 - \begin{vmatrix} x_1 & y_1 \\ x_i & y_i\end{vmatrix}\gamma_2 + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}\gamma_i : i = 3, \dots, n\right\rangle & P = T^c_{\mathbf{x}}M(c) \end{cases}$$

Consider the linear map $\psi_P : \mathbb{R}^n_q \to \mathbb{R}^n_q$ defined by $\psi_P(\mathbf{z}) = Y_P\mathbf{z}$ where

$$Y_P = \begin{cases} \begin{pmatrix} \langle \mathbf{e}_1, \mathbf{e}_1\rangle y_2 & -\langle \mathbf{e}_2, \mathbf{e}_2\rangle y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{e}_1, \mathbf{e}_1\rangle y_n & 0 & \dots & -\langle \mathbf{e}_n, \mathbf{e}_n\rangle y_1 \end{pmatrix} & P = S^{n-1}, S_+, S_- \\ \begin{pmatrix} y_2 & -y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & -y_1 \end{pmatrix} & P = M(c) \\ \begin{pmatrix} y_2 & -y_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ y_p & 0 & \cdots & -y_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & y_{p+2} & -y_{p+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & y_n & 0 & \cdots & -y_{p+1} \end{pmatrix} & P = S_0 \\ \begin{pmatrix} \begin{vmatrix} x_2 & y_2 \\ x_i & y_i\end{vmatrix} & -\begin{vmatrix} x_1 & y_1 \\ x_i & y_i\end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} x_2 & y_2 \\ x_i & y_i\end{vmatrix} & -\begin{vmatrix} x_1 & y_1 \\ x_i & y_i\end{vmatrix} & 0 & \dots & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix} \end{pmatrix} & P = T^cM(c) \end{cases}$$

Then

$$\operatorname{Ker}\psi_P = \begin{cases} \langle \mathbf{y}^*\rangle_{\mathbb{R}} & P = S^{n-1}, S_+, S_- \\ \langle \mathbf{y}\rangle_{\mathbb{R}} & P = M(c) \\ \langle \mathbf{y}', \mathbf{y}''\rangle_{\mathbb{R}} & P = S_0 \\ \langle \mathbf{x}, \mathbf{y}\rangle_{\mathbb{R}} & P = T^c_{\mathbf{x}}M(c) \end{cases}$$

where $\mathbf{y}^* = (\langle \mathbf{e}_1, \mathbf{e}_1\rangle y_1, \dots, \langle \mathbf{e}_n, \mathbf{e}_n\rangle y_n)$, $\mathbf{y}' = (y_1, \dots, y_p, 0, \dots, 0)$, $\mathbf{y}'' = (0, \dots, 0, y_{p+1}, \dots, y_n)$.

3.1 Height function and unfoldings

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n_q$ be a curve whose Taylor expansion is

$$c_0 + c_1 t + \frac{1}{2} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \dots, \qquad c_i \in \mathbb{R}^n_q.$$

Assume that $X = \mathbb{R}^n_q$.

Proposition 3.2. *The following conditions are equivalent.*

(i) *There is* $\mathbf{y}$ *so that* $h_{\mathbf{y}}(t)$ *is* A_k *singularity at* $t = 0$.

(ii) $\operatorname{rank}(c_1, c_2, \dots, c_k) < \operatorname{rank}(c_1, c_2, \dots, c_k, c_{k+1})$.

Proof. Since

$$\frac{d^j}{dt^j} h_{\mathbf{y}} = \Big\langle \frac{d^j \gamma}{dt^j}, \mathbf{y} \Big\rangle, \qquad j = 1, 2, \dots,$$

we have that $h_{\mathbf{y}}(t)$ is A_k singularity at $t = 0$ if

$$\frac{dh_{\mathbf{y}}}{dt}(0) = \frac{d^2 h_{\mathbf{y}}}{dt^2}(0) = \dots = \frac{d^k h_{\mathbf{y}}}{dt^k}(0) = 0, \qquad \frac{d^{k+1} h_{\mathbf{y}}}{dt^{k+1}}(0) \neq 0.$$

This is equivalent that

$$\Big\langle \frac{d^j \gamma}{dt^j}(0), \mathbf{y} \Big\rangle = 0, \qquad j = 1, \dots, k, \qquad \Big\langle \frac{d^{k+1} \gamma}{dt^{k+1}}(0), \mathbf{y} \Big\rangle \neq 0.$$

Such $\mathbf{y}$ exists if and only if

$$\operatorname{rank}\Big(\frac{d\gamma}{dt}, \dots, \frac{d^k \gamma}{dt^k} \Big)(0) < \operatorname{rank}\Big(\frac{d\gamma}{dt}, \dots, \frac{d^k \gamma}{dt^k}, \frac{d^{k+1} \gamma}{dt^{k+1}} \Big)(0),$$

which completes the proof. $\square$

Proposition 3.3. *Assume that* $X = \mathbb{R}^n_q$, $P = \mathbb{R}^n_q, M(c)$ *or that* $X = (\mathbb{R}^n_q, \ \cdot \ )$, $P = S^{n-1}, S_+, S_-, S_0$. *If* $h_{\mathbf{y}_0}(t)$ *has* A_k *singularity at* $t = 0$, *then the following conditions are equivalent.*

(i) *The unfolding* H *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(ii) *The unfolding* $\widehat{H}$ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(iii) *The unfolding* $\widehat{H}$ *is* $\mathcal{R}$*-versal.*

(iv) *The unfolding* $\widehat{H}$ *is* $\mathcal{K}$*-versal.*

(v) $\operatorname{rank}(c_1, c_2, \dots, c_{k-1}) = k - 1$ *(when* $P = \mathbb{R}^n_q$*).*
$\operatorname{rank}(\mathbf{y}^*_0, c_1, c_2, \dots, c_{k-1}) = k$ *(when* $P = S^{n-1}, S_+, S_-$*).*
$\operatorname{rank}(\mathbf{y}_0, c_1, c_2, \dots, c_{k-1}) = k$ *(when* $P = M(c)$, $c \neq 0$*).*
$\operatorname{rank}(\mathbf{y}'_0, \mathbf{y}''_0, c_1, c_2, \dots, c_{k-1}) = k + 1$ *(when* $P = S_0$*).*

Remark 3.4. *The unfolding*

$$H : \mathbb{R} \times M(c) \to \mathbb{R}, \quad (t, \mathbf{y}) \mapsto \langle \gamma(t), \mathbf{y} \rangle$$

is $\mathcal{R}^{aug}$*-versal if and only if the unfolding*

$$\mathbb{R} \times S^{\pm} \to \mathbb{R}, \quad (t, \mathbf{y}) \mapsto \frac{\sqrt{|c|}}{\sqrt{|\langle \mathbf{y}, \mathbf{y} \rangle|}} \langle \gamma(t), \mathbf{y} \rangle$$

is $\mathcal{R}^{aug}$-versal, since $M(c)$, $c \neq 0$, is isomorphic to S_+ or S_-. This may not be equivalent to the $\mathcal{R}^{aug}$-versality of the unfolding

$$H : \mathbb{R} \times S^{\pm} \to \mathbb{R}, \quad (t, \mathbf{y}) \mapsto \langle \gamma(t), \mathbf{y} \rangle.$$

Proof. We may assume that $y_{0,1} \neq 0$ when $P = S^{n-1}$, S_+, S_-, $M(c)$; $y_{0,1} \neq 0$ and $y_{0,p+1} \neq 0$ when $P = S_0$. The unfolding H is $\mathcal{R}^{\text{aug}}$-versal, if and only if

$$\mathcal{E}_t = \left\langle \frac{dh_{\mathbf{y}}}{dt} \right\rangle_{\mathcal{E}_t} + V_P + \langle 1 \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

Since $\frac{\partial \widehat{H}}{\partial u} = -1$, this is equivalent that the unfolding $\widehat{H}$ is $\mathcal{R}^{\text{aug}}$-versal, and $\mathcal{R}$-versal. The unfolding $\widehat{H}$ is $\mathcal{K}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{dh_{\mathbf{y}}}{dt}, h_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} + V_P + \langle 1 \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

These two conditions are equivalent, since

$$\left\langle \frac{dh_{\mathbf{y}_0}}{dt}, h_{\mathbf{y}_0} \right\rangle_{\mathcal{E}_t} = \langle t^k, t^{k+1} \rangle_{\mathcal{E}_t} = \left\langle \frac{dh_{\mathbf{y}_0}}{dt} \right\rangle_{\mathcal{E}_t}.$$

Thus the condition is equivalent that

$$\mathcal{E}_t = V_P + \langle 1 \rangle_{\mathbb{R}} + \langle t^k \rangle_{\mathcal{E}_t}.$$

that is, $\operatorname{rank} A_P = k - 1$ where

$$A_P = \begin{cases} (c_1, \dots, c_{k-1}) & P = \mathbb{R}^n_q \\ Y_P(c_1, \dots, c_{k-1}) & P = M(c), S^{n-1}, S_+, S_-, S_0 \end{cases}$$

Let $W = \langle c_1, \dots, c_{k-1} \rangle_{\mathbb{R}}$. Since

$$\begin{aligned} \operatorname{rank} A_P &= \dim(\psi_P(W)) = \dim W - \dim(W \cap \operatorname{Ker} \psi_P) \\ &= \dim(W + \operatorname{Ker} \psi_P) - \dim \operatorname{Ker} \psi_P \end{aligned}$$

we have the result. $\square$

If this holds, then the bifurcation set B_H is locally diffeomorphic to the bifurcation set of a $\mathcal{R}^{\text{aug}}$-versal unfolding of A_k singularity with the same number of parameters.

Proposition 3.5. *The following conditions are equivalent.*
(i) *There is $\mathbf{y}$ so that $h_{\mathbf{y}}^{-1}(0)$ is A_k singularity at $t = 0$.*
(ii) $\operatorname{rank}(c_0, c_1, c_2, \dots, c_k) < \operatorname{rank}(c_0, c_1, c_2, \dots, c_k, c_{k+1})$.

Proof. We have that $h_{\mathbf{y}}^{-1}(0)$ is A_k singularity at $t = 0$ if

$$h_{\mathbf{y}}(0) = \frac{dh_{\mathbf{y}}}{dt}(0) = \frac{d^2 h_{\mathbf{y}}}{dt^2}(0) = \cdots = \frac{d^k h_{\mathbf{y}}}{dt^k}(0) = 0, \qquad \frac{d^{k+1} h_{\mathbf{y}}}{dt^{k+1}}(0) \neq 0.$$

This is equivalent that

$$\left\langle \frac{d^j\gamma}{dt^j}(0), \mathbf{y} \right\rangle = 0, \qquad j = 0, 1, \ldots, k, \qquad \left\langle \frac{d^{k+1}\gamma}{dt^{k+1}}(0), \mathbf{y} \right\rangle \neq 0.$$

Such $\mathbf{y}$ exists if and only if

$$\operatorname{rank}\Big(\gamma, \frac{d\gamma}{dt}, \ldots, \frac{d^k\gamma}{dt^k}\Big)(0) < \operatorname{rank}\Big(\gamma, \frac{d\gamma}{dt}, \ldots, \frac{d^k\gamma}{dt^k}, \frac{d^{k+1}\gamma}{dt^{k+1}}\Big)(0),$$

which completes the proof. $\square$

Proposition 3.6. *Assume that $X = \mathbb{R}^n_q$, $P = \mathbb{R}^n_q, M(c)$ or that $X = (\mathbb{R}^n_q, \ \cdot\ )$, $P = S^{n-1}, S_+, S_-, S_0$, and that $h_{\mathbf{y}_0}(t)$ has A_k singularity at $t = 0$. Then the following conditions are equivalent.*

(i) *The unfolding H is $\mathcal{R}$-versal.*
(ii) *The unfolding H is $\mathcal{K}$-versal.*
(iii) $\operatorname{rank}(c_0,\ c_1,\ \ldots,\ c_{k-1}) = k$ *(when $P = \mathbb{R}^n_q$)*
$\operatorname{rank}(\mathbf{y}^*_0, c_0,\ c_1,\ \ldots,\ c_{k-1}) = k + 1$ *(when $P = S^{n-1}, S_+, S_-$)*
$\operatorname{rank}(\mathbf{y}_0, c_0,\ c_1,\ \ldots,\ c_{k-1}) = k + 1$ *(when $P = M(c)$)*
$\operatorname{rank}(\mathbf{y}'_0, \mathbf{y}''_0, c_0,\ c_1,\ \ldots,\ c_{k-1}) = k + 2$ *(when $P = S_0$).*

Proof. We assume that $\mathbf{y}_{0,1} \neq 0$. We use the same notation as the proof of Proposition 3.3.

Then the unfolding H is $\mathcal{R}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{dh_{\mathbf{y}}}{dt} \right\rangle_{\mathcal{E}_t} + \langle \mathbf{v}h_{\mathbf{y}}|_{\mathbf{y}=\mathbf{y}_0} : \mathbf{v} \in T_P \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

The unfolding H is $\mathcal{K}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{dh_{\mathbf{y}}}{dt}, h_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} + \langle \mathbf{v}h_{\mathbf{y}}|_{\mathbf{y}=\mathbf{y}_0} : \mathbf{v} \in T_P \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

These two conditions are equivalent, since

$$\left\langle \frac{dh_{\mathbf{y}}}{dt}, h_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} = \langle t^k, t^{k+1} \rangle_{\mathcal{E}_t} = \left\langle \frac{dh_{\mathbf{y}}}{dt} \right\rangle_{\mathcal{E}_t}.$$

Thus this versality is equivalent to the condition:

$$\mathcal{E}_t = \langle \mathbf{v}h_{\mathbf{y}}(t)|_{\mathbf{y}=\mathbf{y}_0} : \mathbf{v} \in T_P \rangle_{\mathbb{R}} + \langle t^k \rangle_{\mathcal{E}_t}.$$

The remaining proof is similar to that of Proposition 3.3. $\square$

If H (resp. $\widehat{H}$) is $\mathcal{K}$-versal, then the discriminant sets D_H and $D_{\widehat{H}}$ is locally diffeomorphic to the discriminant set of a $\mathcal{K}$-versal unfolding of A_k singularity with the same number of parameters.

3.2 Distance squared function and unfoldings

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n_q$ be a curve whose Taylor expansion is

$$c_0 + c_1 t + \frac{1}{2} c_2 t^2 + \frac{1}{3!} c_3 t^3 + \ldots, \qquad c_i \in \mathbb{R}^n_q.$$

We first remark that

$$\frac{d^j}{dt^j} \phi_{\mathbf{y}} = \left\langle \frac{d^j \gamma}{dt^j}, \mathbf{y} - \gamma(t) \right\rangle - \varphi_j(t), \qquad j = 1, 2, \ldots$$

where $\varphi_1(t) = 0$ and $\varphi_{j+1}(t) = \frac{d}{dt}\varphi_j(t) + \langle \frac{d^j\gamma}{dt^j}, \frac{d\gamma}{dt} \rangle$ $(j = 1, 2, \ldots)$. We set $\varphi_0(t) = 0$, by convention. We remark that

$$\begin{aligned}
\varphi_0(0) =& \varphi_1(0) = 0 \\
\varphi_2(0) =& \langle c_1, c_1 \rangle \\
\varphi_3(0) =& 3\langle c_1, c_2 \rangle \\
\varphi_4(0) =& 3\langle c_2, c_2 \rangle + 4\langle c_1, c_3 \rangle \\
\varphi_5(0) =& 10\langle c_2, c_3 \rangle + 5\langle c_1, c_4 \rangle \\
\varphi_6(0) =& 10\langle c_3, c_3 \rangle + 15\langle c_2, c_4 \rangle + 6\langle c_1, c_5 \rangle \\
\varphi_7(0) =& 35\langle c_3, c_4 \rangle + 21\langle c_2, c_5 \rangle + 7\langle c_1, c_6 \rangle \\
& \cdots
\end{aligned}$$

We set $\widetilde{\gamma}^{(j)}(t) = (\frac{d^j\gamma}{dt^j}(t), \varphi_j(t))$.

Proposition 3.7. *The following conditions are equivalent.*
(i) *There is* $\mathbf{y}$ *so that* $\phi_{\mathbf{y}}$ *is* A_k *singularity at* $t = 0$.
(ii) $\operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k)})(0) < \operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k+1)})(0)$, *and* $\operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k)})(0) = \operatorname{rank}(c_1, c_2, \ldots, c_k)$.

In particular, if $\operatorname{rank}(c_1, c_2, \ldots, c_{k+1}) = k + 1$, *there is* $\mathbf{y}$ *so that* $\phi_{\mathbf{y}}(t)$ *is* A_k *singularity at* $t = 0$.

Proof. Now we have that $\phi_{\mathbf{y}}$ is A_k singularity at $t = 0$ if

$$\frac{d^j}{dt^j}\phi_{\mathbf{y}}(0) = 0, \quad (j = 1, 2, \ldots, k), \qquad \frac{d^{k+1}}{dt^{k+1}}\phi_{\mathbf{y}}(0) \neq 0.$$

Such $\mathbf{y}$ exists if and only if

$$\begin{aligned}
\operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k)})(0) <& \operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k+1)})(0) \\
\operatorname{rank}(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \ldots, \widetilde{\gamma}^{(k)})(0) =& \operatorname{rank}(c_1, c_2, \ldots, c_k)
\end{aligned}$$

and we complete the proof. □

Proposition 3.8. *Assume that* $\phi_{\mathbf{y}_0}$ *has* A_k *singularity at* $t = 0$. *Then the following conditions are equivalent.*
(i) *The unfolding* Φ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*
(ii) *The unfolding* $\widehat{\Phi}$ *is* $\mathcal{R}$*-versal.*
(iii) *The unfolding* $\widehat{\Phi}$ *is* $\mathcal{K}$*-versal.*
(iv) $\operatorname{rank}(c_1\ c_2\ \ldots\ c_{k-1}) = k - 1$.

Proof. The unfolding Φ is $\mathcal{R}_{\rm aug}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{d\phi_{\mathbf{y}}}{dt} \right\rangle_{\mathcal{E}_t} + \left\langle \frac{\partial \phi_{\mathbf{y}}}{\partial y_1}\Big|_{\mathbf{y}=\mathbf{y}_0}, \dots, \frac{\partial \phi_{\mathbf{y}}}{\partial y_n}\Big|_{\mathbf{y}=\mathbf{y}_0} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}$$

Since $\frac{\partial \widehat{\Phi}}{\partial u} = -1$, this is equivalent that the unfolding $\widehat{\Phi}$ is $\mathcal{R}$-versal. This also is equivalent that the unfolding $\widehat{\Phi}$ is $\mathcal{K}$-versal, since

$$\left\langle \frac{d}{dt}\phi_{\mathbf{y}}, \phi_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} = \langle t^k, t^{k+1} \rangle_{\mathcal{E}_t} = \left\langle \frac{d}{dt}\phi_{\mathbf{y}} \right\rangle_{\mathcal{E}_t}.$$

This condition is equivalent to the condition:

$$\mathcal{E}_t = \langle \gamma_1(t) - y_1, \dots, \gamma_n(t) - y_n \rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \langle t^k \rangle_{\mathcal{E}_t} = \langle \gamma_1(t), \dots, \gamma_n(t) \rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \langle t^k \rangle_{\mathcal{E}_t}.$$

This means that any polynomial in t of degree $k-1$ without constant term can be expressed as a linear combination of $\gamma_1(t) - \gamma_1(0)$, $\dots$, $\gamma_n(t) - \gamma_n(0)$ modulo t^k. □

If this holds, then the bifurcation set B_Φ is locally diffeomorphic to the bifurcation set of a $\mathcal{R}^{\rm aug}$-versal unfolding of A_k singularity with the same number of parameters.

Proposition 3.9. *The following conditions are equivalent.*
(i) *There is* $\mathbf{y}$ *so that* $\phi_{\mathbf{y}}^{-1}(0)$ *is* A_k *singularity at* $t = 0$.
(ii) $\operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \dots, \widetilde{\gamma}^{(k)})(0) < \operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \dots, \widetilde{\gamma}^{(k+1)})(0)$, *and* $\operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \dots, \widetilde{\gamma}^{(k)})(0) = \operatorname{rank}(c_0, c_1, \dots, c_k)$.
In particular, if $\operatorname{rank}(c_0, c_1, \dots, c_{k+1}) = k + 2$, *there is* $\mathbf{y}$ *so that* $\phi_{\mathbf{y}}^{-1}(0)$ *is* A_k *singularity at* $t = 0$.

Proof. The function $\phi_{\mathbf{y}}$ is A_k singularity at $t = 0$ if and only if

$$\phi_{\mathbf{y}}(0) = \frac{d\phi_{\mathbf{y}}}{dt}(0) = \frac{d^2\phi_{\mathbf{y}}}{dt^2}(0) = \dots = \frac{d^k\phi_{\mathbf{y}}}{dt^k}(0) = 0, \qquad \frac{d^{k+1}\phi_{\mathbf{y}}}{dt^{k+1}}(0) \neq 0.$$

This is equivalent that

$$\left\langle \frac{d^j\gamma}{dt^j}(0), \mathbf{y} - \gamma(t) \right\rangle = 0, \qquad j = 0, 1, \dots, k, \qquad \left\langle \frac{d^{k+1}\gamma}{dt^{k+1}}(0), \mathbf{y} - \gamma(t) \right\rangle \neq 0.$$

$$\begin{aligned} \operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \dots, \widetilde{\gamma}^{(k)})(0) &< \operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \dots, \widetilde{\gamma}^{(k+1)})(0) \\ \operatorname{rank}(\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(2)}, \dots, \widetilde{\gamma}^{(k)})(0) &= \operatorname{rank}(c_0, c_1, c_2, \dots, c_k) \end{aligned}$$

and we complete the proof. □

Proposition 3.10. *Assume that* $\phi_{\mathbf{y}_0}^{-1}(0)$ *has* A_k *singularity at* $t = 0$. *Then the following conditions are equivalent.*

(i) *The unfolding* Φ *is* $\mathcal{R}$*-versal.*
(ii) *The unfolding* Φ *is* $\mathcal{K}$*-versal.*
(iii) $\operatorname{rank}(c_0\ c_1\ \dots\ c_{k-1}) = k$.

Proof. The unfolding Φ is $\mathcal{R}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{d\phi_{\mathbf{y}}}{dt} \right\rangle_{\mathcal{E}_t} + \left\langle \frac{\partial \phi_{\mathbf{y}}}{\partial y_1}\Big|_{\mathbf{y}=\mathbf{y}_0}, \ldots, \frac{\partial \phi_{\mathbf{y}}}{\partial y_n}\Big|_{\mathbf{y}=\mathbf{y}_0} \right\rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

The unfolding Φ is $\mathcal{K}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{d}{dt}\phi_{\mathbf{y}}, \phi_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} + \left\langle \frac{\partial \phi_{\mathbf{y}}}{\partial y_1}\Big|_{\mathbf{y}=\mathbf{y}_0}, \ldots, \frac{\partial \phi_{\mathbf{y}}}{\partial y_n}\Big|_{\mathbf{y}=\mathbf{y}_0} \right\rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

These two conditions are equivalent, since

$$\left\langle \frac{d}{dt}\phi_{\mathbf{y}}, \phi_{\mathbf{y}} \right\rangle_{\mathcal{E}_t} = \langle t^k, t^{k+1} \rangle_{\mathcal{E}_t} = \left\langle \frac{d}{dt}\phi_{\mathbf{y}} \right\rangle_{\mathcal{E}_t}.$$

Thus this versality is equivalent to the condition:

$$\mathcal{E}_t = \langle \gamma_1(t), \ldots, \gamma_n(t) \rangle_{\mathbb{R}} + \langle t^k \rangle_{\mathcal{E}_t}.$$

This means that any polynomial in t of degree $k-1$ can be expressed as a linear combination of $\gamma_1(t)$,..., $\gamma_n(t)$ modulo t^k. □

If Φ (resp. $\widehat{\Phi}$) is $\mathcal{K}$-versal, then the discriminant set D_Φ (resp. $D_{\widehat{\Phi}}$) is locally diffeomorphic to the discriminant set of a $\mathcal{K}$-versal unfolding of A_k singularity with the same number of parameters.

3.3 Height unfolding for a curve in $M(c)$

Let $\gamma : (-\varepsilon, \varepsilon) \to X = M(c)$ be a curve whose Taylor expansion is

$$c_0 + c_1 + \frac{1}{2}c_2 t^2 + \frac{1}{3!}t^3 + \cdots, \qquad c_i \in \mathbb{R}^n_q.$$

If $\gamma(t) \in M(c)$ and $\mathbf{y} \in M(c')$, then

$$\phi_{\mathbf{y}}(t) = -\frac{1}{2}\langle \mathbf{y} - \gamma(t), \mathbf{y} - \gamma(t) \rangle = \langle \mathbf{y}, \gamma(t) \rangle - \frac{c+c'}{2} = h_{\mathbf{y}}(t) - \frac{c+c'}{2}$$

where $h_{\mathbf{y}}(t) = \langle \gamma(t), \mathbf{y} \rangle$, and the height function with $\mathbf{y} \in M(c')$ is exactly the distance squared function.

We consider the height unfolding

$$\begin{aligned} \Phi &: (\mathbb{R} \times T^c_{\mathbf{x}} M(c), (0, \mathbf{y}_0)) \to \mathbb{R}, & (t, \mathbf{y}) &\mapsto h_{\mathbf{y}}(t) - u_0, \\ \widehat{\Phi} &: (\mathbb{R} \times T^c_{\mathbf{x}} M(c) \times \mathbb{R}, (0, \mathbf{y}_0, u_0)) \to \mathbb{R}, & (t, \mathbf{y}, u) &\mapsto h_{\mathbf{y}}(t) - u. \end{aligned}$$

We also consider the distance-squared unfolding

$$\begin{aligned} \Psi &: (\mathbb{R} \times M(c'), (0, \mathbf{y}_0)) \to \mathbb{R}, & (t, \mathbf{y}) &\mapsto \phi_{\mathbf{y}}(t) - u_0, \\ \widehat{\Psi} &: (\mathbb{R} \times M(c') \times \mathbb{R}, (0, \mathbf{y}_0, u_0)) \to \mathbb{R}, & (t, \mathbf{y}, u) &\mapsto \phi_{\mathbf{y}}(t) - u. \end{aligned}$$

By Propositions 3.2, 3.5, we have the followings

Proposition 3.11. *The following conditions are equivalent.*

(i) *There is* $\mathbf{y} \neq 0$ *so that* $\phi_{\mathbf{y}}(t)$ *is* A_k *singularity at* $t = 0$

(ii) $\operatorname{rank}(c_1,\ c_2,\ \ldots,\ c_k) < \operatorname{rank}(c_1,\ c_2,\ \ldots,\ c_k,\ c_{k+1})$.

Proposition 3.12. *The following conditions are equivalent.*

(i) *There is* $\mathbf{y} \neq 0$ *so that* $\phi_{\mathbf{y}}^{-1}(0)$ *is* A_k *singularity at* $t = 0$

(ii) $\operatorname{rank}(c_0, c_1, c_2, \ldots, c_k) < \operatorname{rank}(c_0, c_1, c_2,\ \ldots, c_k, c_{k+1})$.

Proposition 3.13. *Assume that* $c \neq 0$*. Assume that* $\phi_{\mathbf{y}}(t)$ $(\mathbf{y} \neq 0)$ *has* A_k *singularity at* $t = 0$*. Then the following conditions are equivalent.*

(i) *The unfolding* Φ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(ii) *The unfolding* $\widehat{\Phi}$ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(iii) *The unfolding* $\widehat{\Phi}$ *is* $\mathcal{R}$*-versal.*

(iv) *The unfolding* $\widehat{\Phi}$ *is* $\mathcal{K}$*-versal.*

(v) $\operatorname{rank}(\mathbf{x}, \mathbf{y}_0, c_1, c_2, \ldots, c_{k-1}) = k + 1$.

Proof. We show that (i) $\iff$ (v). The equivalence to the other conditions is shown in a similar way and we omit the details. Set $P = T^c_{\mathbf{x}}M(c)$ and assume $x_1y_2 - x_2y_1 \neq 0$ Remark that Φ is $\mathcal{R}^{\mathrm{aug}}$-versal, if and only if

$$\mathcal{E}_t = \left\langle \frac{d\phi_{\mathbf{y}}}{dt} \right\rangle + V_P + \langle t^{k+2} \rangle_{\mathcal{E}_t}$$

Since $\left\langle \frac{\partial \phi_{\mathbf{y}}}{\partial t} \right\rangle = \langle t^k \rangle$, this is equivalent that the matrix

$$A = \left(\begin{vmatrix} x_1 & y_1 & c_{j,1} \\ x_2 & y_2 & c_{j,2} \\ x_i & y_i & c_{j,i} \end{vmatrix} \right)_{i=3,\ldots,n;\ j=1,\ldots,k-1}$$

is of rank $k-1$. Remark that $A = Y_P(c_1, \ldots, c_{k-1})$. Set $W = \langle c_1, \ldots, c_{k-1} \rangle_{\mathbb{R}}$. Since

$$\begin{aligned} \operatorname{rank} A &= \dim(\psi_P(W))) - \dim W - \dim \operatorname{Ker} \psi \\ &= \dim(W + \operatorname{Ker} \psi_P) - \dim \operatorname{Ker} \psi_P = \operatorname{rank}(\mathbf{x}, \mathbf{y}, c_1, \ldots, c_{k-1}) - 2 \end{aligned}$$

we have the result. $\square$

The case $c = 0$ is similar when we assume that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

Proposition 3.14. *Assume that* $\phi_{\mathbf{y}}(t)$ $(\mathbf{y} \neq 0)$ *has* A_k *singularity at* $t = 0$*. Then the following conditions are equivalent.*

(i) *The unfolding* Ψ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(ii) *The unfolding* $\widehat{\Psi}$ *is* $\mathcal{R}^{\mathrm{aug}}$*-versal.*

(iii) *The unfolding* $\widehat{\Psi}$ *is* $\mathcal{R}$*-versal.*

(iv) *The unfolding* $\widehat{\Psi}$ *is* $\mathcal{K}$*-versal.*

(v) $\operatorname{rank}(c_1, c_2, \ldots, c_{k-1}) = k - 1$.

Proof. We only show (i) $\iff$ (v), since the other part is similar to the proof of Lemma 3.3. By Lemma 3.2, $\phi_{\mathbf{y}}(t)$ has A_k singularity at $t = 0$, if and only if

$$\mathbf{y}(c_1\ c_2\ \ldots\ c_k\ c_{k+1}) = (0\ \ldots\ 0\ l), \qquad l \neq 0$$

Then Ψ is $\mathcal{R}^{\text{aug}}$-versal if and only if

$$\mathcal{E}_t = \left\langle \frac{d\phi_{\mathbf{y}}}{dt} \right\rangle + V_P + \langle 1 \rangle_{\mathbb{R}} + \langle t^{k+2} \rangle_{\mathcal{E}_t}.$$

We assume that $y_1 \neq 0$. Since $\langle \frac{d\phi_{\mathbf{y}}}{dt} \rangle = \langle t^k \rangle$, this condition is equivalent that the matrix A_P is of rank $k-1$. Because $\langle \mathbf{y}, c_i \rangle = 0$ for $i = 1, \dots, k$, we have

$$Y_1(c_1 \ c_2 \ \dots \ c_{k-1}) = \begin{pmatrix} 0 \\ A \end{pmatrix}, \quad \text{where} \quad Y_1 = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_2 & -y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & -y_1 \end{pmatrix},$$

and, by $\det Y_1 = {y_1}^{n-2} \sum_{i=1}^n {y_i}^2 \neq 0$, we thus obtain $\operatorname{rank}(c_1 \ \dots \ c_{k-1}) = \operatorname{rank} A$. $\square$

If Ψ is $\mathcal{R}^{\text{aug}}$-versal, then the bifurcation set B_Ψ is locally diffeomorphic to the bifurcation set of a $\mathcal{R}^{\text{aug}}$-versal unfolding of A_k singularity with the same number of parameters. This leads to criteria of singularity types of focal set of curves.

Proposition 3.15. *Assume that ${\phi_{\mathbf{y}_0}}^{-1}(0)$ $(\mathbf{y}_0 \neq 0)$ has A_k singularity at $t = 0$. Then the following conditions are equivalent.*
(i) *The unfolding Φ is $\mathcal{R}$-versal.*
(ii) *The unfolding Φ is $\mathcal{K}$-versal.*
(iii) $\operatorname{rank}(c_0, c_1, c_2, \dots, c_{k-1}) = k$.

Proof. Similar to that of Lemma 3.6. $\square$

If Φ (resp. $\widehat{\Phi}$) is $\mathcal{K}$-versal, then the discriminant set D_Φ is locally diffeomorphic to the discriminant set of a $\mathcal{K}$-versal unfolding of A_k singularity with the same number of parameters. This leads to criteria of singularity types of tubes of curves.

References

[1] B. O'Neil, Semi-Riemannien geometry, with application to relativity, Academic Press, 1983.

[2] K. L. Duggal, A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and its Applications, 364, Kluwer Academic Publishers Group, Dordrecht, 1996.

[3] H. Gluck, Higher curvatures of curves in euclidean space. Amer. Math. Monthly 73 (1966) 699 – 704.

[4] C. T. C. Wall, Finite-determinacy of smooth map-germs, Bull. London Math. Soc.. 13 (1981), 481 – 539.