Singularities of mixed polynomials with Newton polyhedrons

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August 25, 2023

Abstract

Oka introduced the concept of mixed polynomials and started to investigate how a study for singularities of mixed polynomials similar to the study of singularities of polynomials is possible. We introduce a mixed toric modification as a mixed analogy of toric modifications and discuss when this provides an analogy of resolutions of singularities defined by mixed polynomials. A mixed toric modification is associated with a mixed fan, which is a notion we introduce in the paper. They provide several combinatorial data for singularities of mixed polynomials. We define the notion of mixed Newton non-degeneracy for mixed polynomials and show that a mixed toric modification provides a semi-algebraic or real algebraic analogue of resolutions of singularities under mixed Newton non-degeneracy condition. Our approach allows us a combinatorial description of the topology of singularities of mixed polynomials, which are mixed Newton non-degenerate, and we show a formulas for the Euler characteristics and the monodromy zeta function of nearby fibers. We also show how the dual graphs of analogy of resolution of singularities of such mixed polynomial are obtained in low dimensions.

J. Milnor ([11]) started to investigate topology of complex polynomials and he shows there cone structure and fibration structure. These have been provided a wealth of examples in differential topology and basic language for describing how topological type changes under deformations. In [10], Khovanskii introduced a powerful method to construct a resolution of singularities. He introduced a suitable non-degeneracy condition with respect to Newton polyhedrons of holomorphic functions, and showed that a toric modification provides a resolution of singularities of a holomorphic function under this non-degenerate condition. This allows us several combinatorial descriptions of topological objects of the holomorphic functions like:

2020 Mathematics subject classification. 57R45, 14S70, 14M25

Key words and phrases. singularities of mixed polynomial, Newton polyhedrons, mixed toric modification, mixed Newton non-degeneracy.

*The author is supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (C) Grant Number 23K03106.

• topology of nearby fibers, in particular the Euler characteristics of the nearby fiber, and zeta function of monodromy ([17]),

• topology of links of singularity (for example, see [2] for surface singularities).

Such a description of phenomenon should be understood as a broder class of phenomenon that includees singularities of polynomials. For example, Pichon and Seade [15] showed a kind of Milnor fibration for functions of the form $f\bar{g}$. Oka ([12]) has launched to investigate the topology of the maps defined by a mixed polynomial introducing the notion of Newton polyhedron for a mixed polynomial (0.1). Oka introduced the notion of non-degeneracy for mixed polynomial ([12, 2.3]) and discussed how topology of mixed polynomials are described in terms of combinatorics of Newton polyhedrons. He showed fibration theorems [12, Theorems 29, 33] and describe zeta functions of monodromies [12, Theorem 60], etc. for singularities defined by several mixed polynomials. He also investigate when a toric modification provides a resolution of singularities for mixed functions ([13, Theorem 11]). Inaba, Kawashima and Oka also have investigated ([8]) the topology of links for special mixed polynomials. We also remark that Chen, Dias, Takeuchi and Tibăr ([3], [4]) discussed asymptotically critical values or bifurcation values for mixed polynomials using Newton polyhedrons.

In this paper, we introduce the notion of mixed Newton non-degeneracy (Definition 0.7) for mixed polynomials, and define the notion of mixed toric modifications (Definition 1.22) associated with mixed fans (Σ, β) (Definition 1.2) constructed from it, and show that they provide a semi-algebraic analogue of resolution of the singularities defined by the mixed polynomials. Our approach allows us to analyze considerably wide cases of mixed polynomials compared to Oka's results.

Let us state several definitions related to mixed polynomials to fix the terminology. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be the coordinate system of $\mathbb{C}^n$ and let $\bar{\boldsymbol{x}} = (\overline{x_1}, \ldots, \overline{x_n})$ where $\overline{x_i}$ denotes the complex conjugate of x_i. Setting $\boldsymbol{x}^{\boldsymbol{\nu}} = x_1^{\nu_1} \cdots x_n^{\nu_n}$, $\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}} = \overline{x_1}^{\bar{\nu}_1} \cdots \overline{x_n}^{\bar{\nu}_n}$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$, $\bar{\boldsymbol{\nu}} = (\bar{\nu}_1, \ldots, \bar{\nu}_n) \in \mathbb{Z}^n$, we consider a $\mathbb{C}$-linear combination of $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$:

$$\sum_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \boldsymbol{x}^{\boldsymbol{\nu}} \bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}, \quad c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \in \mathbb{C}. \tag{0.1}$$

• We call this (0.1) by a **mixed polynomial** if all the exponents $\boldsymbol{\nu}$ and $\bar{\boldsymbol{\nu}}$ are non-negative.

• In general, we say (0.1) is a **mixed Laurent polynomial**, since we allow negative exponents in the expression (0.1).

• We call $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$ a **mixed monomial** if the exponents $\boldsymbol{\nu}$ and $\bar{\boldsymbol{\nu}}$ are non-negative integers.

• We call $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$ a **mixed semi-monomial** if the exponents are non-negative half integers.

• We also call $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$ a **mixed Laurent monomial** if we allow negative integer exponents.

• We call $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$ a **mixed Laurent semi-monomial** if the exponents are half integers.

By comparison, we sometimes call polynomials in $\boldsymbol{x}$ are called **pure polynomials**, and monomials in $\boldsymbol{x}$ are called **pure monomials**.

For a mixed polynomial (0.1), we consider the map

$$f : \mathbb{C}^n \longrightarrow \mathbb{C}, \quad \text{defined by } \boldsymbol{x} \longmapsto f(\boldsymbol{x}) = \sum_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \boldsymbol{x}^{\boldsymbol{\nu}} \bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}.$$

For the abuse of language, we often call the map f a mixed polynomial.

Definition 0.2. We say that a mixed polynomial (0.1) is a **mixed weighted homogeneous polynomial** of type $(\boldsymbol{a},\boldsymbol{b};\ell,m)$, $\boldsymbol{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n_{\geq}$, $\boldsymbol{b}=(b_1,\dots,b_n)\in\mathbb{Z}^n$, if f is a $\mathbb{C}$-linear combination of mixed monomials $\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$ with

$$\langle \boldsymbol{\nu}+\bar{\boldsymbol{\nu}},\boldsymbol{a}\rangle=\ell,\ \langle \boldsymbol{\nu}-\bar{\boldsymbol{\nu}},\boldsymbol{b}\rangle=m. \tag{0.3}$$

We easily see that for such a polynomial we have

$$f(r^{a_1}e^{b_1\theta\mathfrak{i}}x_1,\dots,r^{a_n}e^{b_n\theta\mathfrak{i}}x_n)=r^{\ell}e^{m\theta\mathfrak{i}}f(\boldsymbol{x}) \tag{0.4}$$

for $r\geq 0$, and $\theta\in\mathbb{R}$.

Definition 0.5. We define **(absolute) Newton polyhedron** (or **radial Newton polyhedron** as in [12]) of the mixed polynomial (0.1) (as power series) by

$$\Gamma_+(f)=\text{convex hull of }\{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}+\mathbb{R}^n_{\geq} : c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}}\neq 0\}. \tag{0.6}$$

For a subset γ of $\mathbb{R}^n$, we define f_γ by $f_\gamma=\sum_{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\in\gamma}c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}}\boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$.

Definition 0.7. We say that a mixed polynomial f is **mixed Newton non-degenerate** if the following conditions hold.
(a) For each compact face γ of $\Gamma_+(f)$, we have

$$\Sigma(f_\gamma)\cap f_\gamma^{-1}(0)\subset\{x_1\cdots x_n=0\}. \tag{0.8}$$

Here $\Sigma(f_\gamma)$ denotes the singular set of the map $f_\gamma:\mathbb{C}^n\longrightarrow\mathbb{C}$ defined by $\boldsymbol{x}\longmapsto f_\gamma(\boldsymbol{x})$.
(b) The polynomial f_γ is mixed weighted homogeneous for each compact face γ of $\Gamma_+(f)$.

As in [12, 2.2], we say that a mixed polynomial f is **non-degenerate** if the condition (a) above holds.

Now we can state our main theorem.

Theorem 0.9. *If a mixed polynomial f is mixed Newton non-degenerate in the sense of Definition 0.7, then there exists a mixed toric modification $\pi: M\longrightarrow\mathbb{C}^n$ (see Definition 1.22) which provides a semi-algebraic analogy of "resolution of singularities" of $f^{-1}(0)$ near 0, that is, $(f\circ\pi)^{-1}(0)$ is a semi-algebraically normal crossing variety (see Definition 3.1) near $\pi^{-1}(0)$.*

As a by-product of this construction, we are able to discuss an analogy of A'Campo's formula for zeta functions, Euler characteristics of nearby fibers, and resolution graphs of singularities of $f^{-1}(0)$, as holomorphic function case. In particular, we have a formula (Theorem 3.19) for the monodromy zeta function for mixed polynomials.

The paper is organized as follows: In §1, we introduce the notion of mixed toric modification and describe their fundamental properties. The key idea is to construct mixed toric manifolds using the notion of mixed fan. Using the notion of Newton polyhedron in §2.1, we introduce mixed version of normal crossing properties, which is a semi-algebraic analogue of normal crossing properties (§3.1), and discuss when a mixed toric modification provides a semi-algebraic analogue of resolution of singularities for a mixed polynomial f. We also discuss a formula for monodromy zeta function of singularities in §3.2 and a way to compute the intersection numbers associated with resolution of singularities in

§3.3. In §4.1, we show a version of topologically triviality theorem, which comes from real-analytic isomorphism of the resolution spaces.

It is also important to look at phenomena specific to mixed polynomials, despite of the fact that it is important to pursue analogies of the study of polynomial in the study of topologies of mixed polynomials. We thus present an attempt to analyze a mixed polynomial, which is not mixed Newton non-degenerate, but which is non-degenerate, in §4.2.

Throughout the paper, we denote by $\mathbb{Z}$ (resp. $\mathbb{R}$, $\mathbb{C}$) the set of integers (resp. real numbers, complex numbers). We denote by $\mathbb{Z}_{\geq}$ (resp. $\mathbb{R}_{\geq}$) the set of non-negative integers (resp. real numbers). We set $\mathbb{C}^* = \mathbb{C} \setminus \{\mathbf{0}\}$ and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

The author is grateful to Masaharu Ishikawa for helpful comments on this topic. The author also thanks to H. Shimizu, since the paper is motivated by seminar works with him.

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1 Mixed toric manifold

1.1 Fan and mixed fan

We say a subset σ of $\mathbb{R}^n$ is rational polyhedral cone if there are integer vectors $\boldsymbol{a}^1, \ldots, \boldsymbol{a}^k$ so that

$$\sigma = \{c_1\boldsymbol{a}^1 + \cdots + c_k\boldsymbol{a}^k : c_i \geq 0,\ i = 1, \ldots, k\}.$$

Let Σ denote a **fan**, that is, a finite collection of rational polyhedral cones in $\mathbb{R}^n$ with the following properties.

- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.

- If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of σ and also is a face of σ'.

Let $\Sigma(k)$ denote the set of k-dimensional cones (k-cones, for short) in Σ. Let $\sigma(k)$ denote the set of k-dimensional faces of σ.

We say a fan Σ is **simplicial** if each cone in Σ is simplicial cones, that is, $\sigma \in \Sigma(k)$ is generated by linearly independent k vectors.

We say a fan Σ is **nonsingular** if each cone in Σ is generated by a part of $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Remark 1.1. A toric variety is constructed from a fan. This toric variety is nonsingular if the fan is nonsingular. Refer the survey article [5] for a fundamental facts for toric varieties. We remark that a nonsingular fan is called a regular fan in [5].

We say a vector $\boldsymbol{a} \in \mathbb{Z}^n$ is **primitive** if the greatest common divisor of non zero components of $\boldsymbol{a}$ is 1.

Definition 1.2 (**Mixed fan**)**.** Let Σ be a simplicial fan, and let β be a map

$$\Sigma(1) \longrightarrow \mathbb{Z}^n \times \mathbb{Z}^n, \quad \tau \longmapsto (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau).$$

We say that (Σ, β) is a **mixed fan**, if the following conditions hold.

(i) $\{\boldsymbol{a}^\tau : \tau \in \Sigma(1)\}$ is the set of the primitive generators of $\tau \in \Sigma(1)$. Thus, for all $\sigma \in \Sigma$,

$$\sigma = \{ \sum_{\tau \in \sigma(1)} c_\tau \boldsymbol{a}^\tau : c_\tau \geq 0\}.$$

(ii) $\{\boldsymbol{b}^\tau : \tau \in \sigma(1)\}$ forms a part of $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for all $\sigma \in \Sigma$.

If $\boldsymbol{a}^\tau = \boldsymbol{b}^\tau$ for $\tau \in \Sigma(1)$, the mixed fan (Σ, β) is a nonsingular fan with mentioning primitive generators $\boldsymbol{a}^\tau$ for $\tau \in \Sigma(1)$.

Mixed fans form a subclass of the set of topological fans introduced in [9]. We are going to define mixed toric manifold associated to a mixed fan, which also form a subclass of topological toric manifolds introduced in loc.cite..

The following lemma ensures the existence of nonsingular mixed fan for several cases.

Lemma 1.3. *Let Σ_0 denote a simplicial fan, which is a subdivision of the positive orthant $\mathbb{R}^n_{\geq}$. For $\tau \in \Sigma_0(1)$, set $\beta_0(\tau) = (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$ where $\boldsymbol{a}^\tau$ is the primitive vector generating τ. We assume that*

- *For $\sigma \in \Sigma_0(n)$, $\boldsymbol{b}^\tau$, $\tau \in \sigma(1)$, are linearly independent.*

Then there exists a mixed fan (Σ, β) with the following properties.

- *Σ is a subdivision of Σ_0.*
- *$\beta(\tau) = \beta_0(\tau)$ for $\tau \in \Sigma_0(1)$.*

Proof. We construct a simplicial subdivision Σ of Σ_0 and

$$\beta : \Sigma(1) \longrightarrow \mathbb{Z}^n \times \mathbb{Z}^n, \quad \tau \longmapsto \beta(\tau) = (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$$

with the following properties:

- For $\tau \in \Sigma_0(1)$, $\beta(\tau) = \beta_0(\tau)$.
- For $\tau \in \Sigma(1)$, $\boldsymbol{a}^\tau$ generates τ.
- For $\sigma \in \Sigma(n)$, $\boldsymbol{b}^\tau$, $\tau \in \sigma(1)$, generate a nonsingular n-cone.
- For $\sigma \in \Sigma(n)$ and $\tau \in \Sigma_1(1) \setminus \Sigma_0(1)$, $\boldsymbol{a}^\tau$ is a primitive generator of the 1-cone generated by $\sum_{\tau' \in \sigma(1)} c_{\tau'} \boldsymbol{a}^{\tau'}$ if $\boldsymbol{b}^\tau = \sum_{\tau' \in \sigma(1)} c_{\tau'} \boldsymbol{b}^{\tau'}$.

This Σ is obtained constructing a nonsingular subdivision of $\langle \boldsymbol{b}^\tau : \tau \in \sigma(1) \rangle_{\mathbb{R}_\geq}$ for $\sigma \in \Sigma_0(n)$ (see [5, 8.1]). $\square$

1.2 Mixed toric manifold

Definition 1.4 (**Group** G)**.** Let us define the group G as the kernel of the group morphism:

$$\lambda_{\Sigma,\beta} : (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow (\mathbb{C}^*)^n,\ (z_\tau)_{\tau\in\Sigma(1)} \longmapsto \left(\prod_{\tau\in\Sigma(1)} |z_\tau|^{a_i^\tau}\left(\frac{z_\tau}{|z_\tau|}\right)^{b_i^\tau}\right)_{i=1,\dots,n}. \tag{1.5}$$

We clearly have

$$(z_\tau)_{\tau\in\Sigma(1)} \in G \iff \begin{cases} \sum_{\tau\in\Sigma(1)} a_i^\tau \log|z_\tau| = 0, \\ \sum_{\tau\in\Sigma(1)} b_i^\tau \arg z_\tau \equiv 0 \pmod{2\pi}. \end{cases}$$

For $\sigma \in \Sigma(n)$, these equations can be written as

$$\begin{aligned} \sum_{\tau\in\sigma(1)} a_i^\tau \log|z_\tau| &= -\sum_{\tau'\notin\sigma(1)} a_i^{\tau'} \log|z_{\tau'}|, \\ \sum_{\tau\in\sigma(1)} b_i^\tau \arg z_\tau &\equiv -\sum_{\tau'\notin\sigma(1)} b_i^{\tau'} \arg z_{\tau'} \pmod{2\pi}, \end{aligned}$$

and we conclude that an element $(z_\tau)_{\tau\in\sigma(1)} \in G$ is determined by $(z_{\tau'})_{\tau'\notin\sigma(1)}$.

Notation 1.6. Let $\mathbb{i}$ denote the imaginary unit. We do not use i for the imaginary unit, since we may use i as an index of vectors, etc.. Then a complex number z can be written as in the following polar form

$$z = re^{\theta\mathbb{i}} \quad \text{where} \quad r = |z|,\ \theta = \arg z.$$

Since $\bar{z} = re^{-\theta\mathbb{i}}$, we have

$$r\partial_r = z\partial_z + \bar{z}\partial_{\bar{z}}, \quad \partial_\theta = \mathbb{i}(z\partial_z - \bar{z}\partial_{\bar{z}}). \tag{1.7}$$

by chain rule. To save symbols, we often denote them as

$$|z|\partial_{|z|} = z\partial_z + \bar{z}\partial_{\bar{z}}, \quad \partial_{\arg z} = \mathbb{i}(z\partial_z - \bar{z}\partial_{\bar{z}}). \tag{1.8}$$

Since the polar coordinate (r,θ) is a coordinate system of $\widehat{\mathbb{C}} = \mathbb{R}_{\geq} \times S^1$, we often regard the formula (1.7), or equivalently, (1.8), as a formula on $\widehat{\mathbb{C}}$ via the polar blow-up:

$$\widehat{\mathbb{C}} \longrightarrow \mathbb{C}, \quad \hat{z} = (r,\theta) \longmapsto z = re^{\theta\mathbb{i}}.$$

In [7, page 222], this map is called a simple oriented blowing-up.

Setting $x = \operatorname{Re} z = \frac{1}{2}(z+\bar{z})$, $y = \operatorname{Im} z = \frac{1}{2\mathbb{i}}(z-\bar{z})$, we have

$$J = \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix}\begin{pmatrix}\operatorname{Re} f & \operatorname{Im} f\end{pmatrix} = \begin{pmatrix}\partial_{\operatorname{Re} z} \\ \partial_{\operatorname{Im} z}\end{pmatrix}\begin{pmatrix}\operatorname{Re} f & \operatorname{Im} f\end{pmatrix} = \begin{pmatrix}1 & 1 \\ \mathbb{i} & -\mathbb{i}\end{pmatrix}\begin{pmatrix}\partial_z \\ \partial_{\bar{z}}\end{pmatrix}\begin{pmatrix}f & \bar{f}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -\mathbb{i} \\ 1 & \mathbb{i}\end{pmatrix}.$$

By (1.8), we have

$$\begin{pmatrix}\partial_z \\ \partial_{\bar{z}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1/z & -\mathbb{i}/z \\ 1/\bar{z} & \mathbb{i}/\bar{z}\end{pmatrix}\begin{pmatrix}|z|\partial_{|z|} \\ \partial_{\arg z}\end{pmatrix}$$

and

$$\det J = \begin{vmatrix} f_z & f_{\bar{z}} \\ \bar{f}_z & \bar{f}_{\bar{z}} \end{vmatrix} = \frac{\mathbb{i}}{2|z|}\begin{vmatrix} \partial_{|z|} f & \partial_{\arg z} f \\ \partial_{|z|} \bar{f} & \partial_{\arg z} \bar{f} \end{vmatrix}. \tag{1.9}$$

Definition 1.10 (Mixed toric manifold). We assume that (Σ, β) is a mixed fan. We define U_Σ by

$$U_\Sigma = \bigcup_{\sigma \in \Sigma(n)} U_\sigma, \text{ where } U_\sigma = \Big\{ (u_\tau)_{\tau \in \Sigma(1)} \in \mathbb{C}^{\Sigma(1)} : \prod_{\tau \notin \sigma(1)} u_\tau \neq 0 \Big\}.$$

We remark that $(z_\tau)_{\tau \in \Sigma(1)} \in G$ acts on U_σ by

$$(z_\tau)_{\tau \in \Sigma(1)} : U_\sigma \longrightarrow U_\sigma, \ (u_\tau)_{\tau \in \sigma(1)} \longmapsto (z_\tau u_\tau)_{\tau \in \sigma(1)},$$

and thus on U_Σ. We define the **mixed toric manifold** $M_{\Sigma,\beta}$ by

$$M_{\Sigma,\beta} = U_\Sigma / G, \text{ and } M_\sigma = U_\sigma / G.$$

Set $V_\sigma = \{(u_\tau)_{\tau \in \Sigma(1)} : u_{\tau'} = 1, \ \tau' \notin \sigma(1)\}$. We easily have the following:

$$(u_\tau)_{\tau \in \Sigma(1)} \in U_\sigma \iff \exists (z_\tau)_{\tau \in \Sigma(1)} \in G \text{ such that } (z_\tau u_\tau)_{\tau \in \Sigma(1)} \in V_\sigma.$$

We thus conclude that the composition $V_\sigma \subset U_\sigma \longrightarrow U_\sigma / G = M_\sigma$ is a semi-algebraic homeomorphism. We consider this map as a semi-algebraic chart of a mixed toric manifold $M_{\Sigma,\beta}$, identifying V_σ with $\mathbb{C}^{\sigma(1)}$.

It is interesting to ask when $M_{\Sigma,\beta}$ is a real algebraic manifold.

Proposition 1.11. *Let (Σ, β) denote a mixed fan. Then $M_{\Sigma,\beta}$ is a real algebraic manifold if*

$$\boldsymbol{a}^\tau \equiv \boldsymbol{b}^\tau \mod 2 \quad \textit{for} \quad \tau \in \Sigma(1).$$

Proof. By supposition, we have

$$(\boldsymbol{a}^\tau)_{\tau \in \sigma(1)} \equiv (\boldsymbol{b}^\tau)_{\tau \in \sigma(1)} \mod 2 \quad \text{for } \sigma \in \Sigma(n).$$

By the condition (ii) in Definition 1.2, $\det(\boldsymbol{b}^\tau)_{\tau \in \sigma(1)} \equiv 1 \mod 2$, and $\det(\boldsymbol{a}^\tau)_{\tau \in \sigma(1)} \equiv 1 \mod 2$. Moreover, $(\boldsymbol{b}^\tau)^{-1}_{\tau \in \sigma(1)} (\boldsymbol{a}^\tau)_{\tau \in \sigma(1)}$ is the identity matrix modulo 2. Thus the lemma below implies the result. □

Lemma 1.12. *Assume that (Σ, β) is a mixed fan. For $\sigma, \sigma' \in \Sigma(n)$, we have two charts $V_\sigma \longrightarrow M_{\Sigma,\beta}$ and $V_{\sigma'} \longrightarrow M_{\Sigma,\beta}$. The patch $V_\sigma \longrightarrow V_{\sigma'}$ is a semi-algebraic map (defined except a thin set). This is given by mixed Laurent monomials if*

$$(\boldsymbol{a}^{\tau'})_{\tau' \in \sigma'(1)} (\boldsymbol{b}^{\tau'})^{-1}_{\tau' \in \sigma'(1)} \equiv (\boldsymbol{a}^\tau)_{\tau \in \sigma(1)} (\boldsymbol{b}^\tau)^{-1}_{\tau \in \sigma(1)} \text{ mod } 2. \tag{1.13}$$

Proof. We consider two semi-algebraic maps (possibly with indeterminacy sets)

$$V_\sigma \longrightarrow \mathbb{C}^n, \qquad (v_\tau)_{\tau \in \sigma(1)} \longmapsto \left(\prod_{\tau \in \sigma(1)} |v_\tau|^{a_i^\tau} \Big(\frac{v_\tau}{|v_\tau|} \Big)^{b_i^\tau} \right)_{i=1,\ldots,n}, \tag{1.14}$$

$$V_{\sigma'} \longrightarrow \mathbb{C}^n, \qquad (v'_{\tau'})_{\tau' \in \sigma'(1)} \longmapsto \left(\prod_{\tau' \in \sigma'(1)} |v'_\tau|^{a_i^{\tau'}} \Big(\frac{v'_{\tau'}}{|v'_{\tau'}|} \Big)^{b_i^{\tau'}} \right)_{i=1,\ldots,n}.$$

For $\tau' \in \sigma'(1)$, we assume that

$$\boldsymbol{a}^\tau = \sum_{\tau \in \sigma} p^\tau_{\tau'} \boldsymbol{a}^{\tau'}, \quad \boldsymbol{b}^\tau = \sum_{\tau \in \sigma} q^\tau_{\tau'} \boldsymbol{b}^{\tau'}, \tag{1.15}$$

that is,

$$(\boldsymbol{a}^{\tau'})_{\tau'\in\sigma'(1)}(p^{\tau}_{\tau'}) = (\boldsymbol{a}^{\tau})_{\tau\in\sigma(1)}, \quad (\boldsymbol{b}^{\tau'})_{\tau'\in\sigma'(1)}(q^{\tau}_{\tau'}) = (\boldsymbol{b}^{\tau})_{\tau\in\sigma(1)}. \tag{1.16}$$

We thus have

$$|v'_{\tau'}| = \prod_{\tau\in\sigma(1)} |v_\tau|^{p^{\tau}_{\tau'}}, \qquad \frac{v'_{\tau'}}{|v'_{\tau'}|} = \prod_{\tau\in\sigma(1)} \Big(\frac{v_\tau}{|v_\tau|}\Big)^{q^{\tau}_{\tau'}},$$

and

$$v'_{\tau'} = \prod_{\tau\in\sigma(1)} v_\tau^{q^{\tau}_{\tau'}} (v_\tau \bar{v}_\tau)^{\frac{p^{\tau}_{\tau'}-q^{\tau}_{\tau'}}{2}} = \prod_{\tau\in\sigma(1)} v_\tau^{\frac{p^{\tau}_{\tau'}+q^{\tau}_{\tau'}}{2}} \bar{v}_\tau^{\frac{p^{\tau}_{\tau'}-q^{\tau}_{\tau'}}{2}}.$$

So $v'_{\tau'}$ and $\bar{v}'_{\tau'}$ are mixed Laurent monomials in v_τ and $\overline{v_\tau}$ if

$$(\boldsymbol{a}^{\tau'})^{-1}_{\tau'\in\sigma'(1)}(\boldsymbol{a}^{\tau})_{\tau\in\sigma(1)} \equiv (\boldsymbol{b}^{\tau'})^{-1}_{\tau'\in\sigma'(1)}(\boldsymbol{b}^{\tau})_{\tau\in\sigma(1)} \mod 2.$$

Let $(\boldsymbol{a}^{\tau'})^{*}_{\tau'\in\sigma'(1)}$ denote the adjugate matrix of $(\boldsymbol{a}^{\tau'})_{\tau'\in\sigma'(1)}$ and $\delta_\sigma = \det(\boldsymbol{a}^{\tau})_{\tau\in\sigma(1)}$. Then this condition is equivalent that

$$(\boldsymbol{a}^{\tau'})^{*}_{\tau'\in\sigma'(1)}(\boldsymbol{a}^{\tau})_{\tau\in\sigma(1)} \equiv \delta_{\sigma'}(\boldsymbol{b}^{\tau'})^{-1}_{\tau'\in\sigma'(1)}(\boldsymbol{b}^{\tau})_{\tau\in\sigma(1)} \mod 2\delta_{\sigma'}.$$

Multiplying $(\boldsymbol{a}^{\tau'})_{\tau'\in\sigma'(1)}$ from the left, and $(\boldsymbol{b}^{\tau})_{\tau\in\sigma(1)}$ from the right, we obtain that

$$\delta_{\sigma'}(\boldsymbol{a}^{\tau})_{\tau\in\sigma(1)}(\boldsymbol{b}^{\tau})^{-1}_{\tau\in\sigma(1)} \equiv \delta_{\sigma'}(\boldsymbol{a}^{\tau'})_{\tau'\in\sigma'(1)}(\boldsymbol{b}^{\tau'})^{-1}_{\tau'\in\sigma'(1)} \mod 2\delta_{\sigma'},$$

which is equivalent to (1.13). □

Remark 1.17. Let (Σ,β) is a mixed fan and $\beta(\tau) = (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$ for $\tau \in \Sigma(1)$. Let S and T be integer $n \times n$ matrices with $\det S = \det T = 1$. If we set $\Sigma' = \{S\sigma : \sigma \in \Sigma\}$ and $\beta'(S\tau) = (S\boldsymbol{a}^\tau, T\boldsymbol{b}^\tau)$ for $\tau \in \Sigma(1)$, then $M_{\Sigma',\beta'}$ is isomorphic to $M_{\Sigma,\beta}$, because of the proof above.

Example 1.18. Let (Σ,β) is a mixed fan so that Σ is a subdivision of $\mathbb{R}^2$.

Assume that the cardinal of $\Sigma(1)$ is three, and $\boldsymbol{b}^1 = \boldsymbol{e}_1$, $\boldsymbol{b}^2 = \boldsymbol{e}_2$ and $\boldsymbol{b}^3 = \pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2$. Let $M_{\pm\pm}$ denote the mixed toric manifold $M_{\Sigma,\beta}$. They are real algebraic manifolds. We call them mixed complex projective planes if Σ is nonsingular, since M_{--} is the complex projective plane (see [5, 0.3]).

Assume that the cardinal of $\Sigma(1)$ is four, and $\boldsymbol{b}^1 = \boldsymbol{e}_1$, $\boldsymbol{b}^2 = \boldsymbol{e}_2$, $\boldsymbol{b}^3 = \epsilon_1\boldsymbol{e}_1 + s_1\boldsymbol{e}_2$ and $\boldsymbol{b}^4 = s_2\boldsymbol{e}_1 + \epsilon_2\boldsymbol{e}_2$, $\epsilon_i = \pm 1$, $s_i \in \mathbb{Z}$ so that $s_1 s_2 + \epsilon_1\epsilon_2 = \pm 1$. Let M denote the mixed toric manifold $M_{\Sigma,\beta}$. We call them mixed Hirzeburch surfaces if Σ is nonsingular, since M is the Hirzebruch surface when $\boldsymbol{a}^i = \boldsymbol{b}^i$, $i = 1,2,3,4$.

Remark 1.19 (Orientation of $M_{\Sigma,\beta}$). The complex structure of $\mathbb{C}^n$ defines a natural orientation of $(\mathbb{C}^*)^n$. We consider an orientation of $M_{\Sigma,\beta}$ induced by the map defined by (1.14). The sign of the jacobian of the map defined by (1.14) is the sign of

$$\det(\boldsymbol{a}^\tau)_{\tau\in\sigma(1)} \det(\boldsymbol{b}^\tau)_{\tau\in\sigma(1)},$$

which we refer as ε_σ. For $\sigma \in \Sigma(n)$, $\{\boldsymbol{a}^\tau\}_{\tau\in\sigma(1)}$ generates the n-cone σ, and, changing the order of $\boldsymbol{a}^\tau$ if necessary, we can assume that $\det(\boldsymbol{a}^\tau)_{\tau\in\sigma(1)}$ is always positive in the calculation. In this case, ε_σ is the sign of the jacobian of the map defined by (1.14).

Remark 1.20 (**Real oriented blow up**, cf. [1, §2])**.** Let $E = \bigcup_{j=1}^{k} C_j$ denote the critical set of $\pi = \pi_{\Sigma,\beta}$ where C_j, $j = 1, \dots, k$, denote the semi-algebraically nonsingular submanifolds, which form a semi-algebarically normal crossing set. Let $\rho_j : Z_j \longrightarrow M_{\Sigma,\beta}$ denote real orientaed blow up along C_j, that is, a map described by $(\widehat{\mathbb{C}} \times \mathbb{C}^{n-1}, S^1 \times \mathbf{0}) \longrightarrow (\mathbb{C}^n, \mathbf{0})$ locally where $C_j = \{0\} \times \mathbb{C}^{n-1}$ locally. The map $\widehat{M_{\Sigma,\beta}} \longrightarrow M_{\Sigma,\beta}$ is the fiber product of $Z_j \longrightarrow M_{\Sigma,\beta}$, $j = 1, \dots, k$. We denote by $\hat{\pi}_{\Sigma,\beta}$ the composition: $\widehat{M_{\Sigma,\beta}} \longrightarrow M_{\Sigma,\beta} \xrightarrow{\pi} \mathbb{C}^n$.

1.3 Mixed toric modification

Definition 1.21 (**Mixed toric modification**)**.** We assume that
(i) (Σ, β) is a mixed fan. Set $\beta(\tau) = (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$ for $\tau \in \Sigma(1)$.
(ii) A fan Σ is a subdivision of $\mathbb{R}^n_{\geq}$. In particular, each a_j^τ, $\tau \in \Sigma(1)$, $j = 1, \dots, n$, is non-negative.
(iii) For any $\tau \in \Sigma(1)$ and $j = 1, \dots, n$, $a_j^\tau = 0$ implies $b_j^\tau = 0$.
Then the map $\lambda_{\Sigma,\beta}$ (see (1.5)) extends to the map

$$\tilde{\pi}_{\Sigma,\beta} : U_\Sigma \longrightarrow \mathbb{C}^n, \quad (u_\tau)_{\tau \in \Sigma(1)} \longmapsto \left(\prod_{\tau \in \Sigma(1)} |u_\tau|^{a_i^\tau} \left(\frac{u_\tau}{|u_\tau|} \right)^{b_i^\tau} \right)_{i=1,\dots,n}.$$

Since this map is G-invariant, $\tilde{\pi}_{\Sigma,\beta}$ induces the natural map

$$\pi = \pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n, \tag{1.22}$$

which we call the **mixed toric modification** defined by the mixed fan (Σ, β).

We always assume that conditions (i), (ii), and (iii) for (Σ, β) when discussing mixed toric modifications (1.22).

Lemma 1.23. *The map* (1.22) *is proper.*

Proof. Take a sequence $\{\boldsymbol{y}^{(m)}\}_{m=1,2,\dots}$ in $M_{\Sigma,\beta}$ so that $\{\pi_{\Sigma,\beta}(\boldsymbol{y}^{(m)})\}_{m=1,2,\dots}$ is convergent. We are going to prove that there is a convergent subsequence of $\{\boldsymbol{y}^{(m)}\}_{m=1,2,\dots}$. Since $M_{\Sigma,\beta}$ is a finite union of V_σ ($\sigma \in \Sigma(n)$) there is an n-cone $\sigma \in \Sigma(n)$ so that infinitely many $\{\boldsymbol{y}^{(m)}\}$ are in V_σ. We write this sequence as $\{\boldsymbol{v}^{(m)}\}_{m=1,2,\dots}$. We take a sequence $\{\hat{\boldsymbol{v}}^{(m)}\}_{m=1,2,\dots}$ in $(\mathbb{C}^*)^{\sigma(1)}$ so that

$$|\boldsymbol{v}^{(m)} - \hat{\boldsymbol{v}}^{(m)}| \to 0 \quad (m \to \infty)$$

It is enough to show that $\{\hat{\boldsymbol{v}}^{(m)}\}_{m=1,2,\dots}$ has a convergent subsequence. Thus we can assume that $\boldsymbol{v}^{(m)} \in (\mathbb{C}^*)^{\sigma(1)}$. Since $\pi_\sigma(\boldsymbol{v}^{(m)})$ is convergent, there is a positive constant L so that

$$\log |x_i^{(m)}| \leq L \quad \text{where} \quad |x_i^{(m)}| = \prod_{\tau \in \sigma(1)} |v_\tau^{(m)}|^{a_i^\tau}.$$

Since

$$(L - \log |x_i^{(m)}|)_{i=1,\dots,n} \in \mathbb{R}^n_{\geq} = \bigcup_{\sigma \in \Sigma(n)} \sigma,$$

we can assume that, taking a subsequence if necessary, there is $\sigma \in \Sigma(n)$ so that

$$(L - \log |x_i^{(m)}|)_{i=1,\dots,n} \in \sigma. \tag{1.24}$$

Taking constants L_τ ($\tau \in \sigma(1)$) so that $L = \sum_{\tau\in\sigma(1)} \boldsymbol{a}^\tau L_\tau$,

$$(L - \log|x_i^{(m)}|)_{i=1,\dots,n} = (\boldsymbol{a}^\tau)_{\tau\in\sigma(1)}(L_\tau - \log|v_\tau^{(m)}|)_{\tau\in\sigma(1)}$$

By (1.24), we obtain that $L_\tau - \log|v_\tau^{(m)}| \geq 0$. Thus $\{\boldsymbol{v}^{(m)}\}_{m=1,2,\dots}$ is bounded, and $\{\boldsymbol{v}^{(m)}\}_{m=1,2,\dots}$ has a convergent subsequence. □

Remark 1.25. In general, the map $\pi_{\Sigma,\beta}$ is semi-algebraic. Since

$$\tilde{\pi}_{\Sigma,\beta}((u_\tau)_{\tau\in\Sigma(1)}) = \left(\prod_{\tau\in\Sigma(1)} u_\tau^{\frac{a_i^\tau+b_i^\tau}{2}} \overline{u_\tau}^{\frac{a_i^\tau-b_i^\tau}{2}}\right)_{i=1,\dots,n},$$

the map $\pi_{\Sigma,\beta}$ is expressed by mixed monomials if $|b_i^\tau| \leq a_i^\tau$ for $\tau \in \Sigma(1)$, $i = 1,\dots,n$. The map $\pi_{\Sigma,\beta}$ is real algebraic if we further assume that $\boldsymbol{a}^\tau \equiv \boldsymbol{b}^\tau \bmod 2$ for $\tau \in \Sigma(1)$ (cf. Proposition 1.11).

Definition 1.26 (E_σ and E_σ^*)**.** For $\tau \in \Sigma(1)$, we define E_τ as the image of the set defined by $u_\tau = 0$ by the map $\tilde{\pi}_{\Sigma,\beta} : U_\Sigma \longrightarrow M_{\Sigma,\beta}$. We call E_τ the **mixed divisor corresponding to** $\tau \in \Sigma(1)$**.** It is clear that $v_\tau = 0$ define $V_\sigma \cap E_\tau$ under the identification of $\mathbb{C}^{\sigma(1)}$ with V_σ. For $\sigma \in \Sigma$, we set

$$E_\sigma = \bigcap_{\tau\in\Sigma(1)} E_\tau, \text{ and } E_\sigma^* = E_\sigma \setminus \bigcup_{\sigma'\in\Sigma:\tau\not\subset\sigma'} E_{\sigma'}. \tag{1.27}$$

Since the natural map $V_\sigma \longrightarrow \mathbb{C}^n$, $\sigma \in \Sigma(n)$, is expressed by

$$|x_i| = \prod_{\tau\in\sigma(1)} |v_\tau|^{a_i^\tau}, \quad \frac{x_i}{|x_i|} = \prod_{\tau\in\sigma(1)} \left(\frac{v_\tau}{|v_\tau|}\right)^{b_i^\tau}, \tag{1.28}$$

we have that the restriction of the map $V_\sigma \longrightarrow \mathbb{C}^n$ to the set defined by $\prod_{\tau\in\sigma(1)} v_\tau \neq 0$ is an isomorphism onto $(\mathbb{C}^*)^n$, and

$$\pi_{\Sigma,\beta}(E_\tau) = \{x \in \mathbb{C}^n : x_i = 0 \text{ for } i \text{ with } a_i^\tau > 0\}. \tag{1.29}$$

Remark 1.30 (**Moment maps**)**.** We consider a moment map associated to a finite subset Γ of $\mathbb{Z}^n$. This is a completion of the map

$$\mu : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad \boldsymbol{x} \longmapsto \mu(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{\nu}\in\Gamma} |\boldsymbol{x}^{\boldsymbol{\nu}}|\boldsymbol{\nu}}{\sum_{\boldsymbol{\nu}\in\Gamma} |\boldsymbol{x}^{\boldsymbol{\nu}}|}. \tag{1.31}$$

Let Δ denote the convex hull of Γ. For a vector $\boldsymbol{a}$, we define $\Delta(\boldsymbol{a})$ by

$$\Delta(\boldsymbol{a}) = \{\boldsymbol{\nu} \in \Delta : \langle \boldsymbol{a}, \boldsymbol{\nu}\rangle = m_\Delta(\boldsymbol{a})\}$$

where $m_\Delta(\boldsymbol{a}) = \min\{\langle \boldsymbol{a}, \boldsymbol{\nu}\rangle : \boldsymbol{\nu} \in \Delta\}$. Similarly for a cone σ, we define $\Delta(\sigma)$ by

$$\Delta(\sigma) = \{\boldsymbol{\nu} \in \Delta : \langle \boldsymbol{a}, \boldsymbol{\nu}\rangle = m_\Delta(\boldsymbol{a}) \ \forall \boldsymbol{a} \in \sigma\}.$$

We define an equivalence relation, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, by

$$\boldsymbol{a} \sim \boldsymbol{b} \iff \Delta(\boldsymbol{a}) = \Delta(\boldsymbol{b}).$$

Their equivalence classes are identified with cones in $\mathbb{R}^n$, which form the **dual fan** of Δ. Let Σ be a simplicial subdivision of the dual fan of Δ.

For any $\boldsymbol{\nu}^0 \in \mathbb{Z}^n$,

$$\mu(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{\nu}\in\Gamma} |\boldsymbol{x}^{\boldsymbol{\nu}-\boldsymbol{\nu}^0}|\boldsymbol{\nu}}{\sum_{\boldsymbol{\nu}\in\Gamma} |\boldsymbol{x}^{\boldsymbol{\nu}-\boldsymbol{\nu}^0}|}.$$

Let σ' be a face of σ, $\sigma \in \Sigma(n)$. When $\boldsymbol{\nu}^0$ is a point in the relative interior of $\Delta(\sigma')$, by (1.28), we can identify this with

$$\frac{\sum_{\boldsymbol{\nu}\in\Gamma} \prod_{\tau\in\sigma(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu}-\boldsymbol{\nu}^0\rangle}\boldsymbol{\nu}}{\sum_{\boldsymbol{\nu}\in\Gamma} \prod_{\tau\in\sigma(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu}-\boldsymbol{\nu}^0\rangle}}. \tag{1.32}$$

When $v_\tau \to 0$ for $\tau \in \sigma'(1)$, we have

$$(1.32) \longrightarrow \frac{\sum_{\boldsymbol{\nu}\in\Gamma\cap\Delta(\sigma)} \prod_{\tau\notin\sigma(1)\setminus\sigma'(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu}-\boldsymbol{\nu}^0\rangle}\boldsymbol{\nu}}{\sum_{\boldsymbol{\nu}\in\Gamma\cap\Delta(\sigma)} \prod_{\tau\notin\sigma(1)\setminus\sigma'(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu}-\boldsymbol{\nu}^0\rangle}}.$$

This implies that the map (1.31) extend to $\mu_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{R}^n$ and $\mu_{\Sigma,\beta}(E_{\sigma'}) \subset \Delta(\sigma')$.

If the dual fan of Δ is nonsingular, the fiber $\mu_{\Sigma,\beta}^{-1}(\boldsymbol{p})$ is isomorphic to $(S^1)^{n-d}$ where $d = \dim \Delta(\sigma)$, for $\boldsymbol{p} \in \Delta(\sigma)$.

Let us consider a mixed fan (Σ, β) with the following property:

- The fan Σ is a part of the dual fan of Δ.

Then we have the moment map

$$\mu_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \Delta.$$

For a mixed toric modification $\pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n$, then $\mu_{\Sigma,\beta}(\pi_{\Sigma,\beta}^{-1}(\mathbf{0}))$ is the union of faces of Δ which are facing to the origin.

2 Mixed toric modifications for mixed polynomials

2.1 Newton polyhedron of f and construction of a mixed fan (Σ, β)

We consider the Newton polyhedron $\Gamma_+(f)$ defined by (0.6). For $\boldsymbol{a} \in \mathbb{R}^n$, set

$$\begin{aligned}\ell(\boldsymbol{a}) &= \min\{\langle \boldsymbol{a}, \boldsymbol{\nu}\rangle : \boldsymbol{\nu} \in \Gamma_+(f)\}, \text{ and}\\ \gamma(\boldsymbol{a}) &= \{\boldsymbol{\nu} \in \Gamma_+(f) : \langle \boldsymbol{a}, \boldsymbol{\nu}\rangle = \ell(\boldsymbol{a})\}.\end{aligned}$$

For $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n_{\geq}$, we introduce an equivalence relation $\sim$ by

$$\boldsymbol{a} \sim \boldsymbol{b} \iff \gamma(\boldsymbol{a}) = \gamma(\boldsymbol{b}).$$

The set of closures of equivalence classes gives a polyhedral cone subdivision of $\mathbb{R}^n_{\geq}$. We denote it as $\Gamma^*(f)$ and we call it the **dual Newton diagram**.

Define $\mathrm{LE}_\gamma(f)$ by

$$\mathrm{LE}_\gamma(f) = \text{convex hull of } \{\boldsymbol{\nu} - \bar{\boldsymbol{\nu}} : c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} \neq 0,\ \boldsymbol{\nu} + \bar{\boldsymbol{\nu}} \in \gamma\}. \tag{2.1}$$

We call $(\Gamma_+(f); \mathrm{LE}_\gamma(f), \gamma \in \mathcal{F}_0(\Gamma_+(f))$ **mixed Newton polyhedrons** of f where $\mathcal{F}_0(\Gamma_+(f))$ denotes the set of compact faces of $\Gamma_+(f)$.

For a mixed Newton non-degenerate mixed polynomial, we construct a mixed fan (Σ, β) as follows: We first take a simplicial subdivision Σ_0 of $\Gamma^*(f)$. Let $\boldsymbol{a}^\tau$ denote the primitive vector which generates τ for $\tau \in \Sigma_0(1)$. Set

$$m(\boldsymbol{b}^\tau) = \min\{\langle \boldsymbol{b}^\tau, \boldsymbol{\nu}\rangle : \boldsymbol{\nu} \in \mathrm{LE}_{\gamma(\boldsymbol{a}^\tau)}(f)\}.$$

We can assume that $m(\boldsymbol{b}^\tau) \geq 0$, $\tau \in \Sigma(1)$, changing the sign of $\boldsymbol{b}^\tau$, if necessary. If we assume that Condition (b) of Definition 0.7, we can choose $\boldsymbol{b}^\tau$, $\tau \in \Sigma_0(1)$, which satisfies the flollowing properties.
(a1) $\mathrm{LE}_{\gamma(\boldsymbol{a}^\tau)}(f) = \{\boldsymbol{\nu} \in \mathrm{LE}_{\gamma(\boldsymbol{a}^\tau)}(f) : \langle \boldsymbol{b}^\tau, \boldsymbol{\nu}\rangle = m(\boldsymbol{b}^\tau)\}$ for $\tau \in \Sigma_0(1)$.
(a2) $\boldsymbol{b}^\tau$, $\tau \in \sigma(1)$, are linearly independent for $\sigma \in \Sigma_0(n)$.
By Lemma 1.3, there is a mixed fan (Σ, β) with the following properties.

- Σ is a simplicial subdivision of Σ_0.
- $\beta(\tau) = (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$ for $\tau \in \Sigma_0(1)$.

By condition (b) of Definition 0.7, f_γ is a mixed weighted homogeneous with respect to the weight $(\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$ and degrees $(\ell(\boldsymbol{a}^\tau), m(\boldsymbol{b}^\tau))$ where $\tau \in \Sigma(1)$ with $\gamma(\boldsymbol{a}^\tau) = \gamma$. If Condition (b) of Definition 0.7 does not hold, f_γ is no longer mixed weighted homogeneous, but we can choose $\boldsymbol{b}^\tau$ with Condition (a2) above, and we are able to construct a mixed fan (Σ, β) similarly.

Lemma 2.2. $\pi_{\Sigma,\beta}(E_\tau) \subset f^{-1}(0)$, *whenever* $\ell(\boldsymbol{a}^\tau) \neq 0$ *for* $\tau \in \Sigma(1)$,

Proof. Set $I(\tau) = \{i = 0, a_i^\tau > 0\}$. We have

$$\langle \boldsymbol{\nu} + \bar{\boldsymbol{\nu}}, \boldsymbol{a}^\tau\rangle = \sum_{i \in I(\tau)} (\nu_i + \bar{\nu}_i) a_i^\tau < \ell(\boldsymbol{a}^\tau) \implies c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} = 0.$$

This implies $f|_{\{x_i = 0:\, i \in I(\tau)\}} = 0$ by (1.29). □

We often refer $\ell(\boldsymbol{a}^\tau)$ and $m(\boldsymbol{b}^\tau)$ as ℓ_τ and m_τ, respectively, for $\tau \in \Sigma(1)$, when discussing one fixed mixed fan (Σ, β)

2.2 Remark on the singular set of f

We say that $\Gamma_+(f)$ is **convenient** if $\Gamma_+(f)$ intersects with each coordinate axis. In this subsection, we show that the zero of a mixed Newton non-degenerate polynomial f has isolated singularity whenever f is convenient. We also show several conclusions for non-convenient cases from the related discussion of its proof.

For $I \subset \{1, \ldots, n\}$, we set

$$\mathbb{C}^I = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i = 0, i \notin I\}$$

and identify $\mathbb{C}^I \times \mathbb{C}^{I^c}$ with $\mathbb{C}^n$, $I^c = \{1, \ldots, n\} \setminus I$, without notice.

Notation 2.3 ($\mathcal{A}_{\boldsymbol{p}}$)**.** For $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_{\geq}$, $\mathcal{A}_{\boldsymbol{p}}$ denotes the set of real-analytic arcs

$$\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t)), \quad x_i(t) = \begin{cases} \alpha_i t^{p_i} + \text{h.o.t.}, \ \alpha_i \neq 0, & i \in I(\boldsymbol{p}), \\ 0, & i \notin I(\boldsymbol{p}), \end{cases} \tag{2.4}$$

where $I(\boldsymbol{p}) = \{i \in \{1, \ldots, n\} : p_i > 0\}$.

Proposition 2.5. *If there is a real analytic arc $\boldsymbol{x}(t) \in \mathcal{A}_{\boldsymbol{p}}$ in $\Sigma(f)$ (resp. $\Sigma(f) \cap f^{-1}(0)$), then the set*

$$\widetilde{\Sigma}_{\boldsymbol{p}}(f) = \{\boldsymbol{x} \in (\mathbb{C}^*)^{I(\boldsymbol{p})} : \operatorname{rank} \tilde{J}_{\boldsymbol{p}} f(\boldsymbol{x}) < 2\}$$

(resp. $\widetilde{\Sigma}_{\boldsymbol{p}}(f) \cap f^{-1}_{\gamma(\boldsymbol{p})}(0)$) is not empty, where, setting $I = I(\boldsymbol{p})$,

$$\tilde{J}_{\boldsymbol{p}} f = \begin{pmatrix} x_i \partial_{x_i} (f|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & \overline{x_i} \partial_{\overline{x_i}} (f|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & ((\partial_{x_j} f)|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & ((\partial_{\overline{x_j}} f)|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} \\ x_i \partial_{x_i} (\overline{f}|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & \overline{x_i} \partial_{\overline{x_i}} (\overline{f}|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & ((\partial_{x_j} \overline{f})|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} & ((\partial_{\overline{x_j}} \overline{f})|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} \end{pmatrix}_{i \in I;\ j \notin I}.$$

Corollary 2.6. *If a mixed polynomial f is non-degenerate, then*

$$\Sigma(f) \cap f^{-1}(0) \subset \bigcup_{I \in \mathcal{I}_0(f)} \mathbb{C}^I \quad \textit{near } \mathbf{0}$$

where $\mathcal{I}_0(f)$ denotes the set of subsets I of $\{1, \ldots, n\}$ so that $f|_{\mathbb{C}^I}$ is identically zero, that is,

$$\mathcal{I}_0(f) = \{I \subset \{1, \ldots, n\} : \mathbb{R}^I \cap \Gamma_+(f) = \varnothing\}.$$

In particular, if f is convenient, then the origin $\mathbf{0}$ is an isolated point of $\Sigma(f) \cap f^{-1}(0)$.

Proof of Proposition 2.5. Choose a real-analytic arc $\boldsymbol{x}(t) \in \mathcal{A}_{\boldsymbol{p}}$, so that $\boldsymbol{x}(t) \in \Sigma(f)$. By definition, we have

$$\operatorname{rank} \begin{pmatrix} \partial_{x_1} f & \partial_{\overline{x_1}} f & \cdots & \partial_{x_n} f & \partial_{\overline{x_n}} f \\ \partial_{x_1} \overline{f} & \partial_{\overline{x_1}} \overline{f} & \cdots & \partial_{x_n} \overline{f} & \partial_{\overline{x_n}} \overline{f} \end{pmatrix} (\boldsymbol{x}(t)) < 2,$$

and we obtain 2×2 minors of $\tilde{J}_{\boldsymbol{p}} f$ are zero along $\boldsymbol{\alpha} t^{\boldsymbol{p}} = (\alpha_i t^{p_i})_{i \in I} \in \mathbb{C}^I$, $I = I(\boldsymbol{p})$, and a point of $\widetilde{\Sigma}_{\boldsymbol{p}}(f)$ is attained by $\boldsymbol{\alpha} = (\alpha_i)_{i \in I}$.

If we assume moreover $f(\boldsymbol{x}(t)) = 0$, then $(f|_{\mathbb{C}^I})_{\gamma(\boldsymbol{p})} (\boldsymbol{\alpha} t^{\boldsymbol{p}})_{i \in I} = 0$ and a point of $\widetilde{\Sigma}_{\boldsymbol{p}}(f) \cap (f|_{\mathbb{C}^I})^{-1}_{\gamma(\boldsymbol{p})}(0)$ is attained by $\boldsymbol{\alpha} = (\alpha_i)_{i \in I}$. □

Proof of Corollary 2.6. Remark that $\widetilde{\Sigma}_{\boldsymbol{p}}(f) \subset \Sigma((f|_{(\mathbb{C}^*)^{I(\boldsymbol{p})}})_{\gamma(\boldsymbol{p})})$. For $\boldsymbol{p}$ with $I(\boldsymbol{p}) \notin \mathcal{I}_0(f)$, $f|_{\mathbb{C}^{I(\boldsymbol{p})}}$ is not identically zero and non-degeneracy implies that $\Sigma((f|_{(\mathbb{C}^*)^{I(\boldsymbol{p})}})_{\gamma(\boldsymbol{p})}) \cap f^{-1}_{\gamma(\boldsymbol{p})}(0)$ is empty. □

2.3 Strict transform of f via a mixed toric modification

Let us consider a mixed fan (Σ, β), so that Σ is a subdivision of the dual Newton diagram $\Gamma^*(f)$.

For $\sigma \in \Sigma(n)$, we have

$$f \circ \pi_{\Sigma,\beta}|_{V_\sigma} (v_\tau)_{\tau \in \sigma(1)} = \sum_{\boldsymbol{\nu}, \overline{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \overline{\boldsymbol{\nu}}} \prod_{\tau \in \sigma(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu} + \overline{\boldsymbol{\nu}} \rangle} e^{(\sum_{\tau \in \sigma(1)} \langle \boldsymbol{b}^\tau, \boldsymbol{\nu} - \overline{\boldsymbol{\nu}} \rangle \arg v_\tau) \mathbf{i}}.$$

If we set

$$f \circ \pi_{\Sigma,\beta} (v_\tau)_{\tau \in \sigma(1)} = \hat{f} \prod_{\tau \in \sigma(1)} |v_\tau|^{\ell(\boldsymbol{a}^\tau)} (v_\tau / |v_\tau|)^{m(\tau)}, \tag{2.7}$$

where $m(\tau) = \min\{\langle \boldsymbol{\nu}, \boldsymbol{b}^\tau \rangle : \boldsymbol{\nu} \in \mathrm{LE}_{\gamma(\boldsymbol{a}^\tau)}(f)\}$, we have

$$\hat{f} = \sum_{\boldsymbol{\nu}, \overline{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \overline{\boldsymbol{\nu}}} \prod_{\tau \in \sigma(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu} + \overline{\boldsymbol{\nu}} \rangle - \ell(\boldsymbol{a}^\tau)} e^{(\sum_{\tau \in \sigma(1)} (\langle \boldsymbol{b}^\tau, \boldsymbol{\nu} - \overline{\boldsymbol{\nu}} \rangle - m(\tau)) \arg v_\tau) \mathbf{i}} \tag{2.8}$$

and we thus have

$$\hat{f}|_{E_{\sigma'}} = \sum_{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\in\gamma} c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} \prod_{\tau\in\sigma(1)\setminus\sigma'(1)} |v_\tau|^{\langle \boldsymbol{a}^\tau, \boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\rangle - \ell(\boldsymbol{a}^\tau)} e^{((\langle \boldsymbol{b}^\tau, \boldsymbol{\nu}-\bar{\boldsymbol{\nu}}\rangle - m(\tau))\arg v_\tau)\mathbf{i}} = f'_\gamma \tag{2.9}$$

where $\gamma = \bigcap_{\tau\in\sigma'(1)}\gamma(\boldsymbol{a}^\tau)$. Remark that $\hat{f}$ is semi-algebraic. Setting $e_\tau = e^{(\arg v_\tau)\mathbf{i}}$, we have

$$\begin{aligned}\hat{f}_\gamma &= \sum_{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\in\gamma} c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} \prod_{\tau\in\sigma'(1)} e_\tau^{\langle \boldsymbol{b}^\tau, \boldsymbol{\nu}-\bar{\boldsymbol{\nu}}\rangle - m(\tau)} \prod_{\tau'\in\sigma(1)\setminus\sigma'(1)} |v_{\tau'}|^{\langle \boldsymbol{a}^{\tau'}, \boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\rangle - \ell(\boldsymbol{a}^{\tau'})} e_{\tau'}^{\langle \boldsymbol{b}^{\tau'}, \boldsymbol{\nu}-\bar{\boldsymbol{\nu}}\rangle - m(\tau')},\\ \overline{\hat{f}_\gamma} &= \sum_{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\in\gamma} \overline{c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}}} \prod_{\tau\in\sigma'(1)} e_\tau^{\langle \boldsymbol{b}^\tau, \bar{\boldsymbol{\nu}}-\boldsymbol{\nu}\rangle + m(\tau)} \prod_{\tau'\in\sigma(1)\setminus\sigma'(1)} |v_{\tau'}|^{\langle \boldsymbol{a}^{\tau'}, \boldsymbol{\nu}+\bar{\boldsymbol{\nu}}\rangle - \ell(\boldsymbol{a}^{\tau'})} e_{\tau'}^{\langle \boldsymbol{v}^{\tau'}, \bar{\boldsymbol{\nu}}-\boldsymbol{\nu}\rangle + m(\tau')}.\end{aligned} \tag{2.10}$$

Remark that

$$\hat{f} = \hat{f}_\gamma + O(|v_\tau| : \tau\in\sigma'(1)).$$

and the equation $\hat{f} = 0$ defines the strict transform $\widehat{Z}$ of f by $\hat{\pi}_{\Sigma,\beta}$. The equation $\hat{f}_\gamma$ depends on $|v_\tau|$, $\arg v_\tau$, $\tau\in\sigma(1)\setminus\sigma'(1)$, and the arguments $\arg v_\tau$, $\tau\in\sigma'(1)$, in general. Eliminating $\arg v_\tau$, $\tau\in\sigma'(1)$, from the system $\hat{f}_\gamma = \overline{\hat{f}_\gamma} = 0$, we obtain the equations defining the intersection of $E_{\sigma'}$ with the strict transform in terms of coordinate system $(v_\tau)_{\tau\notin\sigma('1)}$ of $E_{\sigma'}$. When we assume Condition (b) in Definition 0.7, the equation $\hat{f}_\gamma = 0$ depends on $|v_\tau|$, $\arg v_\tau$, $\tau\in\sigma(1)\setminus\sigma'(1)$, and $\hat{f}$ and $\overline{\hat{f}}$ have continuous extensions on V_σ. We denote these extensons by f' and $\overline{f'}$ respectively.

2.4 Mixed weighted homogeneous polynomials

We consider a mixed weighted homogeneous polynomial defined in Definition 0.2. When the mixed weighted homogeneous polynomial f does not have a constant term, we always have $\ell > 0$. Moreover, we can always assume that $m \ge 0$, changing the sign of $\boldsymbol{b}$, if necessary.

For a mixed weighted homogeneous polynomial f, we have the following:

$$\begin{pmatrix} x_1\partial_{x_1}f & \overline{x_1}\partial_{\overline{x_1}}f & \cdots & x_n\partial_{x_n}f & \overline{x_n}\partial_{\overline{x_n}}f \\ x_1\partial_{x_1}\overline{f} & \overline{x_1}\partial_{\overline{x_1}}\overline{f} & \cdots & x_n\partial_{x_n}\overline{f} & \overline{x_n}\partial_{\overline{x_n}}\overline{f}\end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_1 & -b_1 \\ \vdots & \vdots \\ a_n & b_n \\ a_n & -b_n \end{pmatrix} = \begin{pmatrix} \ell f & mf \\ \ell\overline{f} & -m\overline{f}\end{pmatrix}. \tag{2.11}$$

Taking the determinants of the both hand side of (2.11), Cauchy-Binet formula implies $\Sigma(f)\subset f^{-1}(0)$ whenever $m\neq 0$. When $m=0$, $\Sigma(f)\subset f^{-1}(0)$ does not hold in general, as the following example shows.

Example 2.12. Set $f = |x_1|^2 + x_2^2$. This is mixed weighted homogeneous with respect to the weights $((1,1),(1,0))$ and degree $(2,0)$, and $\Sigma(f) = \{x_2 = 0\}\not\subset f^{-1}(0)$.

The condition (b) in Defiinition 0.7 implies that f_γ is mixed weighted homogeneous with respect to the weights $(\boldsymbol{a}^\tau, \boldsymbol{b}^\tau)$, $\tau\in\sigma(1)$, when $\gamma = \bigcap_{\tau\in\sigma(1)}\gamma(\boldsymbol{a}^\tau)$ is a compact face of $\Gamma_+(f)$, $\sigma\in\Sigma$.

Example 2.13. A mixed monomial $f = \boldsymbol{x}^{\boldsymbol{\nu}}\bar{\boldsymbol{x}}^{\bar{\boldsymbol{\nu}}}$, $\boldsymbol{\nu}\neq\bar{\boldsymbol{\nu}}$, is mixed weighted homogeneous with respect to any weights $(\boldsymbol{a},\boldsymbol{b})$. We obtain $\Sigma(f)\subset f^{-1}(0)$, since there is a weight $\boldsymbol{b}$ so that $m = \langle\boldsymbol{\nu}-\bar{\boldsymbol{\nu}},\boldsymbol{b}\rangle\neq 0$.

3 Semi-algebraic analogue of resolution of singularities

3.1 Normal crossing property

We introduce semi-algebraic analogue of normal crossing varieties as mixed version of normal crossing varieties.

Definition 3.1. We say a subset Z of $\mathbb{C}^n$ is of **semi-algebraically normal crossing** at $z \in Z$ if there is a semi-algebraic coordinate system (U, φ), U an open neighborhood of z, and a semi-algebraic homeomorphism $\varphi : U \longrightarrow \varphi(U) \subset \mathbb{C}^n$ centred at z, so that $Z \cap U$ is the inverse image of zero of a pure monomial by φ.

Theorem 3.2. *Let f be a mixed polynomial, which is mixed Newton non-degenerate in the sense of Definition 0.7, and let (Σ, β) denote a mixed fan constructed in §2.1. Then, for the mixed toric modification $\pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n$, the subset $(f \circ \pi_{\Sigma,\beta})^{-1}(0)$ in V_σ is of semi-algebraically normal crossing near $\pi^{-1}(0)$.*

To conclude an analogy of Milnor fibration for mixed polynomial, we need to assume that $\Sigma(f) \subset f^{-1}(0)$. The next theorem shows when this is the case.

Theorem 3.3. *For a mixed polynomial f, which is mixed Newton non-degenerate in the sense of Definition 0.7, we take (Σ, β) as in Theorem 3.2. Then $\Sigma(f) \subset f^{-1}(0)$ near the origin $\mathbf{0}$, whenever one of the following conditions satisfied.*
(i) $m_\tau \neq 0$ for all $\tau \in \Sigma(1)$.
(ii) If there is $\sigma \in \Sigma$ with $m_\tau = 0$ for $\tau \in \sigma(1)$, then $\Sigma(f_\gamma) \subset \{x_1 \cdots x_n f_\gamma = 0\}$ where $\gamma = \bigcap_{\tau \in \sigma(1)} \gamma(\boldsymbol{a}^\tau)$.

Remark 3.4. Under the assumptions of Theorem 3.3, we often say that the mixed toric modification $\pi = \pi_{\Sigma,\beta}$ provides a semi-algebraic analogue of resolution of singularity of f, since the zero of $f \circ \pi$ is of semi-algebraically normal crossing and $\Sigma(f \circ \pi) \subset (f \circ \pi)^{-1}(0)$.

Remark 3.5. We use the notation in §2.3. As we see in the proof below, non-degeneracy implies the strict transform of f by $\hat{\pi}$ intersects with the set defined by $|v_\tau| = 0$, $\tau \in \sigma'(1)$, transversely, when $|v_{\tau'}| \neq 0$, $\tau' \in \sigma(1) \setminus \sigma'(1)$. This fact is proved by Oka [12, Theorem 24] when π is a certain toric modification.

Oka ([14, Theorem 9.19]) also proved that π provides "a resolution of singularity" when $\boldsymbol{a}^\tau = \boldsymbol{b}^\tau$ for $\tau \in \Sigma(1)$ with the condition (i) in Theorem 3.3. Dropping this condition, Saito and Takashimizu ([16, §8]) discuss the case that a toric modification provides an analogy of resolution.

Proof of Theorem 3.2. We take a mixed fan (Σ, β) constructed in §2.1. The condition (a) of Definition 0.7 implies that

$$(3.6) \qquad x_1 \cdots x_n \neq 0,\ f_\gamma(x) = 0 \implies \operatorname{rank} \begin{pmatrix} \partial_{x_i} f_\gamma(x) & \partial_{\overline{x_i}} f_\gamma(x) \\ \partial_{x_i} \overline{f_\gamma}(x) & \partial_{\overline{x_i}} \overline{f_\gamma}(x) \end{pmatrix}_{i=1,\dots,n} = 2.$$

This is clearly equivalent to the following condition:

$$(3.7) \qquad x_1 \cdots x_n \neq 0,\ f_\gamma(x) = 0 \implies \operatorname{rank} \begin{pmatrix} x_i \partial_{x_i} f_\gamma(x) & \overline{x_i} \partial_{\overline{x_i}} f_\gamma(x) \\ x_i \partial_{x_i} \overline{f_\gamma}(x) & \overline{x_i} \partial_{\overline{x_i}} \overline{f_\gamma}(x) \end{pmatrix}_{i=1,\dots,n} = 2.$$

Since $|x_i|\partial_{|x_i|} = x_i\partial_{x_i} + \overline{x_i}\partial_{\overline{x_i}}$, $\partial_{\arg x_i} = \mathbb{i}(x_i\partial_{x_i} - \overline{x_i}\partial_{\overline{x_i}})$, we have

$$\begin{pmatrix} |x_i|\partial_{|x_i|}f & \partial_{\arg x_i}f \\ |x_i|\partial_{|x_i|}\overline{f} & \partial_{\arg x_i}\overline{f} \end{pmatrix} = \begin{pmatrix} x_i\partial_{x_i}f & \overline{x_i}\partial_{\overline{x_i}}f \\ x_i\partial_{x_i}\overline{f} & \overline{x_i}\partial_{\overline{x_i}}\overline{f} \end{pmatrix} \begin{pmatrix} 1 & \mathbb{i} \\ 1 & -\mathbb{i} \end{pmatrix}$$

and condition (3.7) is equivalent to the following condition:

$$x_1\cdots x_n \neq 0,\ f_\gamma(x) = 0 \implies \operatorname{rank}\begin{pmatrix} |x_i|\partial_{|x_i|}f_\gamma(x) & \partial_{\arg x_i}f_\gamma(x) \\ |x_i|\partial_{|x_i|}\overline{f_\gamma}(x) & \partial_{\arg x_i}\overline{f_\gamma}(x) \end{pmatrix} = 2. \tag{3.8}$$

We take $\sigma \in \Sigma(n)$ and discuss the behavior of the pull back $f\circ\pi_\sigma$ where $\pi_\sigma = \pi_{\Sigma,\beta}|_{V_\sigma}$ near $\pi_\sigma^{-1}(0)$. By (3.22), we have

$$\begin{pmatrix} |v_\tau|\partial_{|v_\tau|}f \\ |v_\tau|\partial_{|v_\tau|}\overline{f} \end{pmatrix}_{\tau\in\sigma(1)} = \begin{pmatrix} |x_1|\partial_{|x_1|}f & \cdots & |x_n|\partial_{|x_n|}f \\ |x_1|\partial_{|x_1|}\overline{f} & \cdots & |x_n|\partial_{|x_n|}\overline{f} \end{pmatrix} (\boldsymbol{a}^\tau)_{\tau\in\sigma(1)} \quad \text{and}$$

$$\begin{pmatrix} \partial_{\arg v_\tau}f \\ \partial_{\arg v_\tau}\overline{f} \end{pmatrix}_{\tau\in\sigma(1)} = \begin{pmatrix} \partial_{\arg x_1}f & \cdots & \partial_{\arg x_n}f \\ \partial_{\arg x_1}\overline{f} & \cdots & \partial_{\arg x_n}\overline{f} \end{pmatrix} (\boldsymbol{b}^\tau)_{\tau\in\sigma(1)}.$$

We thus obtain

$$\prod_{\tau\in\sigma(1)} v_\tau \neq 0,\ f_\gamma\circ\pi_\sigma(v_\tau) = 0 \implies \operatorname{rank}\begin{pmatrix} |v_\tau|\partial_{|v_\tau|}f_\gamma\circ\pi_\sigma & \partial_{\arg v_\tau}f_\gamma\circ\pi_\sigma \\ |v_\tau|\partial_{|v_\tau|}\overline{f_\gamma}\circ\pi_\sigma & \partial_{\arg v_\tau}\overline{f_\gamma}\circ\pi_\sigma \end{pmatrix}_{\tau\in\sigma(1)}(v) = 2. \tag{3.9}$$

Since $f_\gamma = f'_\gamma \prod_{\tau\in\sigma(1)} |v_\tau|^{\ell_\tau} e^{(m_\tau \arg v_\tau)\mathbb{i}}$, $\ell_\tau = \ell(\boldsymbol{a}^\tau)$, $m_\tau = m(\boldsymbol{b}^\tau)$, we have

$$|v_\tau|\partial_{|v_\tau|}f_\gamma = (|v_\tau|\partial_{|v_\tau|}f'_\gamma + \ell_\tau f'_\gamma)\prod_{\tau\in\sigma(1)}|v_\tau|^{\ell_\tau}e^{(m_\tau \arg v_\tau)\mathbb{i}}, \tag{3.10}$$

$$|v_\tau|\partial_{|v_\tau|}\overline{f_\gamma} = (|v_\tau|\partial_{|v_\tau|}\overline{f'_\gamma} + \ell_\tau \overline{f'_\gamma})\prod_{\tau\in\sigma(1)}|v_\tau|^{\ell_\tau}e^{(m_\tau \arg v_\tau)\mathbb{i}},$$

$$\partial_{\arg v_\tau}f_\gamma = (\partial_{\arg v_\tau}f'_\gamma + m_\tau f'_\gamma\mathbb{i})\prod_{\tau\in\sigma(1)}|v_\tau|^{\ell_\tau}e^{(m_\tau \arg v_\tau)\mathbb{i}}, \tag{3.11}$$

$$\partial_{\arg v_\tau}\overline{f_\gamma} = (\partial_{\arg v_\tau}\overline{f'_\gamma} - m_\tau \overline{f'_\gamma}\mathbb{i})\prod_{\tau\in\sigma(1)}|v_\tau|^{\ell_\tau}e^{(m_\tau \arg v_\tau)\mathbb{i}}.$$

The function f'_γ does not depend on $|v_\tau|$. The condition (b) of Definition 0.7 and (2.10) implies that f'_γ does not depend on $\arg v_\tau$, $\tau \in \sigma(1)$. We thus obtain that

$$\prod_{\tau\notin\sigma'(1)} v_\tau \neq 0, \quad f'_\gamma(v) = 0 \implies \operatorname{rank}\begin{pmatrix} |v_\tau|\partial_{|v_\tau|}f'_\gamma & \partial_{\arg v_\tau}f'_\gamma \\ |v_\tau|\partial_{|v_\tau|}\overline{f'_\gamma} & \partial_{\arg v_\tau}\overline{f'_\gamma} \end{pmatrix}_{\tau\notin\sigma'(1)}(v) = 2.$$

Thus we have the following condition:

(3.12)

$$\prod_{\tau\in\sigma(1)} v_\tau \neq 0,\ f_\gamma\circ\pi_\sigma(v) = 0 \implies \operatorname{rank}\begin{pmatrix} |v_\tau|\partial_{|v_\tau|}f'_\gamma\circ\pi_\sigma & \partial_{\arg v_\tau}f'_\gamma\circ\pi_\sigma \\ |v_\tau|\partial_{|v_\tau|}\overline{f'_\gamma}\circ\pi_\sigma & \partial_{\arg v_\tau}\overline{f'_\gamma}\circ\pi_\sigma \end{pmatrix}_{\tau\in\sigma(1)\setminus\sigma'(1)}(v) = 2.$$

We can rewrite this condition as follows:

$$\prod_{\tau\in\sigma(1)} v_\tau \neq 0,\ f_\gamma\circ\pi_\sigma(v) = 0 \implies \operatorname{rank}\begin{pmatrix} v_\tau\partial_{v_\tau}f'_\gamma & \overline{v_\tau}\partial_{\overline{v_\tau}}f'_\gamma \\ v_\tau\partial_{v_\tau}\overline{f'_\gamma} & \overline{v_\tau}\partial_{\overline{v_\tau}}\overline{f'_\gamma} \end{pmatrix}_{\tau\in\sigma(1)\setminus\sigma'(1)}(v) = 2, \tag{3.13}$$

since $|v_\tau|\partial_{|v_\tau|} = v_\tau\partial_{v_\tau} + \overline{v_\tau}\partial_{\overline{v_\tau}}$, $\partial_{\arg v_\tau} = \mathbb{i}(v_\tau\partial_{v_\tau} - \overline{v_\tau}\partial_{\overline{v_\tau}})$. This implies that the set defined by $f' = 0$ intersects $E_{\sigma'} \setminus \bigcup_{\tau\subsetneq\sigma'} E_\tau$ transversely. $\square$

Remark 3.14. From the proof above, we see that f' can be a part of coordinate system at a point of zero of f' in $\pi^{-1}(0)$.

Proof of Theorem 3.3. We continue the notation of the proof of Theorem 3.2. Assume that there is a real-analytic arc $\boldsymbol{x}(t) \in \mathcal{A}_{\boldsymbol{p}}$ with $\boldsymbol{x}(t) \in \Sigma(f)$. We are going to show that $\boldsymbol{x}(t) \in f^{-1}(0)$ for $0 < t \ll 1$.

We take a lift $\tilde{\boldsymbol{x}}(t)$ of $\boldsymbol{x}(t)$ by π. Clearly we have $\tilde{\boldsymbol{x}}(t) \in \Sigma(f \circ \pi)$. If $\tilde{\boldsymbol{x}}(t) \in E_\tau$ with $\ell_\tau \neq 0$, Lemma 2.2 implies $\boldsymbol{x}(t) \in f^{-1}(0)$.

We assume that $\tilde{\boldsymbol{x}}(t) \notin E_\tau$ with $\ell_\tau \neq 0$. We can also assume that $\tilde{\boldsymbol{x}}(t) \in V_\sigma$ for some $\sigma \in \Sigma(n)$. We also assume that $\tilde{\boldsymbol{x}}(0) \in E_\tau$. Then $\tilde{\boldsymbol{x}}(0)$ represents a point in $(\mathbb{C}^*)^n$ so that the Jacobi matrix of f_γ is not of full rank. If $m_\tau \neq 0$, then we deduce $f_\gamma(\tilde{\boldsymbol{x}}(0)) = 0$. By Remark 3.14, one can take a local coordinate system at $\tilde{\boldsymbol{x}}(0)$ so that $f \circ \pi$ is expressed as Example 2.13, and we conclude that $\tilde{\boldsymbol{x}}(t) \in (f \circ \pi)^{-1}(0)$.

If $m_\tau = 0$, then we deduce $f_\gamma(\tilde{\boldsymbol{x}}(0)) = 0$ by condition (ii). Again, by Remark 3.14, one can take a local coordinate system at $\tilde{\boldsymbol{x}}(0)$ so that $f \circ \pi$ is expressed as Example 2.13, and we conclude that $\tilde{\boldsymbol{x}}(t) \in (f \circ \pi)^{-1}(0)$. □

3.2 Monodromy zeta function

The zeta function of a transformation $h : Y \longrightarrow Y$ of a topological space Y is the rational function

$$\zeta_h(t) = \prod_{i \geq 0} \Delta_i(h)^{(-1)^i}, \quad \Delta_i(h) = \det\{1 - t h_* : H_i(Y) \longrightarrow H_i(Y)\},$$

where $H_i(Y)$ is the i-th homology group of chain complex with closed support whose coefficient is $\mathbb{C}$.

Let $f : (\mathbb{C}^n, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function germ. We have a locally trivial fibration

$$f^{-1}(S^1_r) \cap B^{2n}_\varepsilon \longrightarrow S^1_r, \ \boldsymbol{x} \longmapsto f(\boldsymbol{x}), \quad S^1_r = \{z \in \mathbb{C} : |z| = r\}, \tag{3.15}$$

for $0 < r \ll \varepsilon \ll 1$, since $\Sigma(f) \subset f^{-1}(0)$ near 0. We consider the zeta function $\zeta_h(t)$ of a monodromy transformation

$$h : F(r, 0) \longrightarrow F(r, 2\pi) = F(r, 0)$$

where $F(r, \theta) = f^{-1}(re^{\theta \mathbf{i}}) \cap B^{2n}_\varepsilon$. This monodromy transformation h is obtained by integrateing a vector field ξ which is a lift of $\partial_{\arg z}$ by (3.15). Remark that the zeta function $\zeta_h(t)$ is determined by f. In fact, if two lifts ξ_0 and ξ_1 of the vector field $\partial_{\arg z}$ generate two monodromy transformations h_0 and h_1, then they are connected by a homotopy, which is generated by $\xi_t = (1-t)\xi_0 + t\xi_1$ $(0 \leq t \leq 1)$.

Let $\pi : M \longrightarrow \mathbb{C}^n$ be a proper modification such that at any point of $\pi^{-1}(0)$ the divisor $X_0 = \pi^{-1}(f^{-1}(0))$ is of normal crossing. For $m \in \mathbb{Z}$, $m \geq 1$, we put

$$S_m = \left\{ s \in \pi^{-1}(0) : \begin{array}{c} \text{the equation of } X_0 \text{ at } s \text{ is of the form } z_1^m \text{ for a} \\ \text{local coordinate system } (z_1, \ldots, z_n) \text{ of } M \text{ at } s \end{array} \right\}.$$

Theorem 3.16 ([1, THÉORÈME 3])**.** *The monodromy zeta function $\zeta(t)$ and the Euler characteristics of Milnor fiber $F = f^{-1}(r) \cap B^{2n}_\varepsilon$, $0 < r \ll \varepsilon \ll 1$, are given by*

$$\zeta(t) = \prod_{m \geq 1} (1 - t^m)^{\chi(S_m)}, \quad \chi(F) = \sum_{m \geq 1} m \chi(S_m).$$

Remark 3.17. Remark that A'Campo stated the formula with opposite sign in the exponents ([1, THÉORÈME 3]), since he defined the monodromy zeta function with the opposite sign in the exponents in the first paragraph loc.cite..

Remark 3.18. We quickly recall A'Campo's construction of geometric monodromy ([1, §2]). Let $X = (f\circ\pi)^{-1}(D_r) \cap B^{2n}_{\varepsilon}$ for $0 < r \ll \varepsilon \ll 1$, where $D_r = \{w \in \mathbb{C} : |w| < r\}$. Let $C_1, \ldots, C_s$ be nonsingular components of $X_0 = (f\circ\pi)^{-1}(0)$. Let Δ_x be a simplex spanned by the pre-image of $x \in X_0$ by the natural map: $\bigsqcup_{i=1}^{s} C_i \longrightarrow X_0$. Define $\widehat{X}_0$ by

$$\widehat{X}_0 = \Big(\bigsqcup_{i=1}^{s} C_i\Big) \cup \Big(\bigsqcup_{x\in X_0} \Delta_x\Big).$$

Let $\rho : \widehat{X} \longrightarrow X$ denote the fiber product of $\rho_j : \widehat{X}_j \longrightarrow X$, the real oriented blow-up of X with center C_j for $j = 1, \ldots, s$. Remark that $\widehat{X}$ is a manifold with corner and $\partial\widehat{X} = \rho^{-1}(X_0)$. Let Y denote the fiber product of the natural map $\widehat{X}_0 \longrightarrow X_0$ and $\rho : \partial\widehat{X} \longrightarrow X_0$. We remark that the inverse image of $x \in X_0$ is $(S^1)^{k+1}$ where k is dimension of Δ_x. Let m_i denote the multiplicity of f along C_i. Then the geometric monodromy $h : Y \longrightarrow Y$ is described as follows:

$$h(a, \theta_{i_0}, \ldots, \theta_{i_k}) = (a, \theta_{i_0} + \tfrac{2\pi a_{i_0}}{m_{i_0}}, \ldots, \theta_{i_k} + \tfrac{2\pi a_{i_k}}{m_{i_k}}), \quad a \in \Delta_x,\ (\theta_{i_0}, \ldots, \theta_{i_k}) \in \rho^{-1}(x).$$

where $a_{i_0}, \ldots, a_{i_k}$ denote the barycentric coordinate of $a \in \Delta_x$. Remark that, when $k = 0$, we have $h(x, \theta_{i_0}) = (x, \theta_{i_0} + \frac{2\pi}{m_{i_0}})$.

Now we consider a map $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by a mixed polynomial with $\Sigma(f) \subset f^{-1}(0)$. Then we have an analogy of a locally trivial fibration (3.15) and we can consider a monodromy transformation as holomorphic case. Let $\pi : M \longrightarrow \mathbb{C}^n$ be a proper modification which provides semi-algebraic analogue of resolution of singularities of f. For $m \in \mathbb{Z}$, $m \geq 1$, we put

$$S_m = \left\{ s \in \pi^{-1}(0) : \begin{array}{c} \text{the equation of } X_0 \text{ at } s \text{ is of the form } u{z_1}^{\alpha}\overline{z_1}^{\beta} \text{ for a} \\ \text{local coordinate system } (z_1, \ldots, z_n) \text{ of } M \text{ at } s \text{ with} \\ m = |\alpha - \beta| \text{ where } u \text{ is an invertible function near } s. \end{array} \right\}.$$

Under the notation above, we obtain the following

Theorem 3.19. *If S_0 is empty, then the monodromy zeta function $\zeta(t)$ is given by*

$$\zeta(t) = \prod_{m\geq 1} (1 - t^m)^{\chi(S_m)}. \tag{3.20}$$

In particular, we have the following formula for Euler characteristics for local nearby fiber:

$$\chi(F(r, \theta)) = \lim_{r\to 0} \chi(F(r, \theta)) = \sum_{m\geq 1} m\chi(S_m) \tag{3.21}$$

where $F(r, \theta) = f^{-1}(re^{\theta\mathrm{i}}) \cap B^{2n}_{\varepsilon}$ for $0 < r \ll \varepsilon \ll 1$.

3.3 Intersection numbers among components of the exceptional set

Let (Σ, β) be a mixed fan. Using the orientation introduced by Remark 1.19, we can discuss intersection numbers of submanifolds of $M_{\Sigma,\beta}$. Let E_τ be the mixed divisor corresponding to $\tau \in \Sigma(1)$.

Denoting the coordinate of $V_\sigma = \mathbb{C}^{\sigma(1)}$ by $(v_\tau)_{\tau\in\sigma(1)}$, we have

$$d\pi_\sigma(|v_\tau|\partial_{|v_\tau|}) = \sum_{i=1}^n a_i^\tau |x_i|\partial_{|x_i|}, \quad d\pi_\sigma(\partial_{\arg v_\tau}) = \sum_{i=1}^n b_i^\tau \partial_{\arg x_i} \tag{3.22}$$

and they define co-orientations of E_τ, $\tau \in \Sigma(1)$. Using these co-orientations, we can compute their intersection numbers. Actually the intersection number of $E_{\tau_1}, \ldots, E_{\tau_n}$ is

$$E_{\tau_1}\cdots E_{\tau_n} = \begin{cases} \varepsilon_\sigma, & \tau_1,\ldots,\tau_n \text{ generate an } n\text{-cone } \sigma\in\Sigma(n), \\ 0, & \text{otherwise,} \end{cases}$$

whenever $\tau_1, \ldots, \tau_n$ are all distinct. Here ε_σ is defined in Remark 1.19.

Define D_i by

$$D_i = \sum_{\tau\in\Sigma(1)} b_i^\tau E_\tau. \tag{3.23}$$

Lemma 3.24. *We have* $\sum_{\tau\in\Sigma(1)} b_i^\tau E_\tau \cdot E_{\tau_2}\cdots E_{\tau_n} = 0$.

Proof. We consider the map $\pi_\sigma = \pi_{\Sigma,\beta}|_{V_\sigma} : V_\sigma \longrightarrow \mathbb{C}^n$. Since

$$\pi_\sigma^* x_i = \prod_{\tau\in\sigma(1):b_i^\tau\neq 0} \left(|v_\tau|^{\frac{a_i^\tau}{b_i^\tau}}(v_\tau/|v_\tau|)\right)^{b_i^\tau} \prod_{\tau\in\sigma(1):b_i^\tau=0} |v_\tau|^{a_i^\tau},$$

we have

$$\frac{\pi_\sigma^* x_i}{\rho_i} \sim \prod_{\tau\in\sigma(1):b_i^\tau\neq 0} w_\tau^{b_i^\tau}, \quad w_\tau = |v_\tau|^{\frac{a_i^\tau}{b_i^\tau}}(v_\tau/|v_\tau|) \tag{3.25}$$

where $\rho_i : M \longrightarrow \mathbb{R}$ be a continuous function defined by

$$\rho_i(z) = \prod_{\tau\in\Sigma(1):b_i^\tau=0} \operatorname{dist}(z, E_\tau)^{a_i^\tau}$$

and $\operatorname{dist}(z, E_\tau)$ is the distance between z and E_τ. Here $p \sim q$ means $C_1|p| \le |q| \le C_2|p|$ for some positive constants C_1, C_2. Let $A_{\varepsilon,\delta}$ denote the locus defined by

$$\frac{\pi^* x_i}{\varepsilon + \rho_i} = \delta, \quad \varepsilon \ge 0,\ \delta \ge 0,$$

which defines D_i when $\varepsilon = \delta = 0$ by (3.25). Thus we obtain

$$0 = A_{\varepsilon,\delta}\cdot E_{\tau_2}\cdots E_{\tau_n} = D_i \cdot E_{\tau_2}\cdots E_{\tau_n} = \sum_{\tau\in\Sigma(1)} b_i^\tau E_\tau \cdot E_{\tau_2}\cdots E_{\tau_n}. \qquad \square$$

This lemma often determines intersection numbers $E_{\tau_1}\cdots E_{\tau_n}$.

Proposition 3.26. *Assume that* $n = 2$*, and* (Σ, β) *is a mixed fan. Suppose that* $\Sigma(1) = \{\tau_0, \tau_1, \dots, \tau_{k+1}\}$ *and* $\beta(\tau_i) = (\boldsymbol{a}^i, \boldsymbol{b}^i)$*,* $i = 0, 1, \dots, k+1$*. We assume that* $\boldsymbol{a}^0 = \boldsymbol{e}_1$*,* $\boldsymbol{a}^{k+1} = \boldsymbol{e}_2$*, and* $\det(\boldsymbol{a}^i, \boldsymbol{a}^{i+1}) > 0$*,* $i = 0, \dots, k$*. We set* $\varepsilon_j = \det(\boldsymbol{b}^{j-1}\ \boldsymbol{b}^j)$*. Then we have*

$$E_i \cdot E_j = \begin{cases} 0 & (j - i \geq 2) \\ \varepsilon_j & (j = i + 1) \\ q_j & (i = j) \end{cases}$$

where q_j *is defined by* $0 = b_i^j q_j + b_i^{j-1}\varepsilon_j + b_i^{j+1}\varepsilon_{j+1}$*,* $i = 1, 2$*.*

Proof. By (3.23), we have

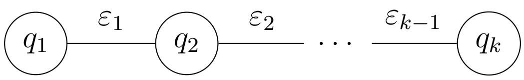
$$(\pi^*_{\Sigma,\beta} x_i)_0 = \sum_{s=0}^{k+1} b_i^s E_s.$$

Then we conclude that, for $i = 1, 2$,

$$0 = (\pi^*_{\Sigma,\beta} x_i)_0 \cdot E_j = \sum_{s=0}^{k+1} b_i^s E_s \cdot E_j = b_i^{j-1}(E_{j-1} \cdot E_j) + b_i^j (E_j \cdot E_j) + b_i^{j+1}(E_{j+1} \cdot E_j).$$

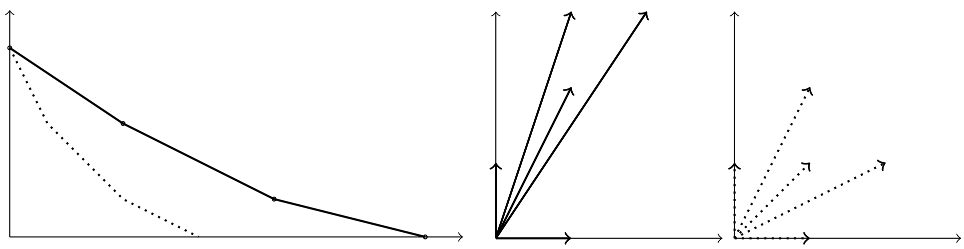
We complete the proof, since $E_{j-1} \cdot E_j = \det(\boldsymbol{b}^{j-1}\ \boldsymbol{b}^j)$ and $E_{j+1} \cdot E_j = \det(\boldsymbol{b}^j\ \boldsymbol{b}^{j+1})$. □

Remark 3.27 (Dual graph of exceptional sets)**.** We can consider a dual weighted graph associated to a finite collection of connected oriented surfaces E_i in an oriented compact 4-manifold with boundary, which form a variety with sub-algebraically normal crossing singularities. The intersection relations give rise a dual weighted graph where each vertex corresponds to a surface E_i. The vertex corresponding to E_i is labelled by self-intersection number of E_i. We connect two vertices by a solid edge (resp. dashed edge) if the corresponding surfaces intersect with intersection number 1 (resp. -1). In general, we connect two vertices by edge if the corresponding surfaces E_i and E_j intersect, labeling the intersection number $E_i \cdot E_j$.



Since the dual graph is a tree, we can choose orientations of E_j so that the numbers ε_j are 1. It is also useful to add the information of the strict transform to the dual graph. We associate a vertex • to each component C of the strict transform and connect the vertex corresponding to E_i when C and E_i intersect. In this case, $\boldsymbol{a}^i$ supports a 1-dimensional face of $\Gamma_+(f)$, since the intersection corresponds to the solution of $f_\gamma = 0$.

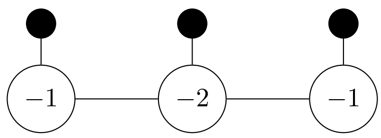
Example 3.28. Set $f = x_2^5 + x_1^2\overline{x_1}x_2^3 + x_1^5\overline{x_1}^2 x_2 + x_1^8\overline{x_1}^3$. We show the boundary of $\Gamma_+(f)$ (solid lines) and $\mathrm{LE}_\gamma(f)$'s (dotted lines) in the left figure below.



We consider the mixed fan from the data

$$(\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2\ \boldsymbol{a}^3\ \boldsymbol{a}^4) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 2 & 4 & 1 \end{pmatrix},\ (\boldsymbol{b}^0\ \boldsymbol{b}^1\ \boldsymbol{b}^2\ \boldsymbol{b}^3\ \boldsymbol{b}^4) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix},$$

which are shown in the middle and right figures above. The corresponding $\ell = (\ell_\tau)_{\tau\in\Sigma(1)}$ and $m = (m_\tau)_{\tau\in\Sigma(1)}$ are given by $\ell = (0\ 15\ 9\ 11\ 0)$, $m = (0\ 5\ 4\ 5\ 0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 1, 1, 1)$. Thus we obtain the dual graph

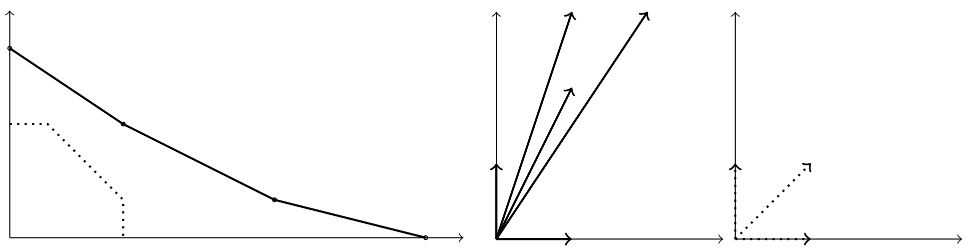


since $\boldsymbol{b}^1 = \boldsymbol{b}^0 + \boldsymbol{b}^2$, $2\boldsymbol{b}^2 = \boldsymbol{b}^1 + \boldsymbol{b}^3$ and $\boldsymbol{b}^3 = \boldsymbol{b}^2 + \boldsymbol{b}^4$. Remark that the intersection numbers of a component C_i of the strict transform and a component E_j of the exceptional set depend on the given orientation of the components of the strict transform. For simplicity, we assume that $E_i \cdot C_j \geq 0$.

Since $\chi(S_4) = 0$ and $\chi(S_5) = -1$, we obtain

$$\zeta(t) = (1 - t^5)^{-1}.$$

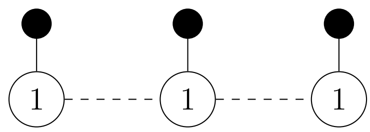
Example 3.29. Set $f = x_2^3\overline{x_2} + x_1^2\overline{x_1}x_2^3 + x_1^5\overline{x_1}^2x_2 + x_1^7\overline{x_1}^4$. We show the boundary of $\Gamma_+(f)$ (solid lines) and $\mathrm{LE}_\gamma(f)$'s (dotted lines) in the left figure below.



We consider the mixed fan from the data

$$(\boldsymbol{a}^0\ \boldsymbol{a}^1\ \boldsymbol{a}^2\ \boldsymbol{a}^3\ \boldsymbol{a}^4) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 2 & 4 & 1 \end{pmatrix},\ (\boldsymbol{b}^0\ \boldsymbol{b}^1\ \boldsymbol{b}^2\ \boldsymbol{b}^3\ \boldsymbol{b}^4) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

which are shown in the middle and righte figures above. The corresponding $\ell = (\ell_\tau)_{\tau\in\Sigma(1)}$ and $m = (m_\tau)_{\tau\in\Sigma(1)}$ are given by $\ell = (0\ 15\ 9\ 11\ 0)$, $m = (0\ 3\ 4\ 3\ 0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, -1, -1, 1)$. Thus we obtain the dual graph



since $\boldsymbol{b}^1 = -\boldsymbol{b}^0 + \boldsymbol{b}^2$, $\boldsymbol{b}^2 = \boldsymbol{b}^1 + \boldsymbol{b}^3$, $\boldsymbol{b}^3 = \boldsymbol{b}^2 - \boldsymbol{b}^4$.

Since $\chi(S_4) = 0$ and $\chi(S_3) = -1$, we obtain

$$\zeta(t) = (1 - t^3)^{-1}.$$

As Remark 3.27, if $n=3$, we can consider the dual graph for intersection numbers of the exceptional mixed divisors E_i in the strict transform Z. In this case, it is convenient to add the information of the genus of E_i.

If $\dim \gamma(\boldsymbol{a}^\tau) = 0$, then $f_{\gamma(\boldsymbol{a}^\tau)} = 0$ defines the empty set and $Z \cdot E_\tau \cdot E_\tau = 0$. If $\dim \gamma(\boldsymbol{a}^\tau) = 1$, then $f_{\gamma(\boldsymbol{a}^\tau)} = 0$ defines $\mathbb{C}^* \times S$ in $(\mathbb{C}^*)^2$ where S is a finite set. Since S can move a nearby finite set by a homotopy and $Z \cdot E_\tau \cdot E_\tau = 0$.

Since $(\pi^*_{\Sigma,\beta} x_i)_0 = \sum_{\tau \in \Sigma(1)} b_i^\tau E_\tau$, we have

$$\begin{aligned} 0 =& Z \cdot (\pi^*_{\Sigma,\beta} x_i)_0 \cdot E_\tau \\ =& Z \cdot (b_i^\tau E_\tau + \sum_{\tau' \neq \tau} b_i^{\tau'} E_{\tau'}) \cdot E_\tau \\ =& b_i^\tau Z \cdot E_\tau \cdot E_\tau + \sum_{\tau' \neq \tau} b_i^{\tau'} N(f_{\gamma(\tau') \cap \gamma(\tau)}). \end{aligned}$$

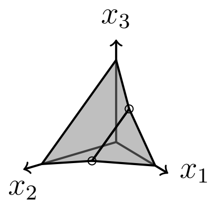
Example 3.30. Set $f = x_1 \overline{x_1} + x_2^2 + x_3^2$. We consider a simplicial fan Σ so that

$\Sigma(3) = \{\langle \boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^5 \rangle_{\mathbb{R}_\geq}, \langle \boldsymbol{a}^2, \boldsymbol{a}^4, \boldsymbol{a}^5 \rangle_{\mathbb{R}_\geq}, \langle \boldsymbol{a}^1, \boldsymbol{a}^3, \boldsymbol{a}^5 \rangle_{\mathbb{R}_\geq}, \langle \boldsymbol{a}^3, \boldsymbol{a}^4, \boldsymbol{a}^5 \rangle_{\mathbb{R}_\geq}, \langle \boldsymbol{a}^2, \boldsymbol{a}^3, \boldsymbol{a}^4 \rangle_{\mathbb{R}_\geq}\}$, where

$$\Sigma(1) = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \dots, \boldsymbol{a}^5\}, \quad (\boldsymbol{a}^i)_{i=1,\dots,5} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

We consider the mixed fan (Σ, β) with the following data

$$(\boldsymbol{b}^i)_{i=1,\dots,5} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$



Set $(\ell_i) = \begin{pmatrix} 0 & 0 & 0 & 2 & 4 \end{pmatrix}$, $(m_i) = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

Taking the intersection product of

$$(\pi^* x_1)_0 = E_1 + E_4 + E_5, \qquad (\pi^* x_2)_0 = E_2 + E_5, \qquad (\pi^* x_3)_0 = E_3 + E_5, \tag{3.31}$$

with E_4 and E_5, we obtain

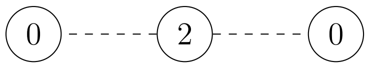
$$\begin{aligned} 0 =& e_{1,4} + e_{4,4} + e_{4,5}, & 0 =& e_{2,4} + e_{4,5}, & 0 =& e_{3,4} + e_{4,5}, \\ 0 =& e_{1,5} + e_{4,5} + e_{5,5}, & 0 =& e_{2,5} + e_{5,5}, & 0 =& e_{3,5} + e_{5,5}. \end{aligned}$$

where $e_{i,j} = Z \cdot E_i \cdot E_j$. Since $e_{2,5} = e_{3,5} = 0$, we have $e_{5,5} = 0$. We also have

$$\frac{|\boldsymbol{b}^1\ \boldsymbol{b}^2\ \boldsymbol{b}^5|}{|\boldsymbol{a}^1\ \boldsymbol{a}^2\ \boldsymbol{a}^5|} = 1, \quad \frac{|\boldsymbol{b}^2\ \boldsymbol{b}^4\ \boldsymbol{b}^5|}{|\boldsymbol{a}^2\ \boldsymbol{a}^4\ \boldsymbol{a}^5|} = -1,$$

$$e_{1,5} = -e_{4,5} = N(x_2^2 + x_3^2) = 2.$$

Since $e_{1,4} = 0$, $e_{4,4} = -e_{4,5} = 2$. The dual graph of the exceptional set is as follows:



Taking the intersection products of (3.31) with $E_4 \cdot E_4$, $E_4 \cdot E_5$, and $E_5 \cdot E_5$, we obtain

$$\begin{aligned} 0 &= e_{1,4,4} + e_{4,4,4} + e_{4,4,5} & 0 &= e_{2,4,4} + e_{4,4,5} & 0 &= e_{3,4,4} + e_{4,5,5} \\ 0 &= e_{1,4,5} + e_{4,4,5} + e_{4,5,5} & 0 &= e_{2,4,5} + e_{4,5,5} & 0 &= e_{3,4,5} + e_{4,5,5} \\ 0 &= e_{1,5,5} + e_{4,5,5} + e_{5,5,5} & 0 &= e_{2,5,5} + e_{5,5,5} & 0 &= e_{3,5,5} + e_{5,5,5} \end{aligned}$$

where $e_{i,j,k} = E_i \cdot E_j \cdot E_k$. Since $E_1 \cap E_4 = \varnothing$, we have $e_{1,4,4} = e_{1,4,5} = 0$. Since $e_{2,5,5} = e_{3,5,5} = 0$, we have $e_{5,5,5} = 0$. Since $e_{2,4,5} = e_{3,4,5} = -1$, $e_{4,5,5} = 1$ and $e_{4,4,5} = -1$. We also conclude that $e_{4,4,4} = 1$.

Remark 3.32 (Links of singularities)**.** In general, the link of an isolated singularity of the zero set of a holomorphic function is a graph manifold, which is a manifold obtained by plumbing of several S^1-bundle on the irreducible components of the exceptional set in the strict transform of its resolution of singularity. The plumbing data are given by resolution graphs.

The situation is analogous for the link of zero of a mixed polynomial f which is mixed Newton non-degenerate. Let $K(f)$ denote the link of f, that is,

$$K(f) = f^{-1}(0) \cap S_\varepsilon^{2n-1}, \quad S_\varepsilon^{2n-1} = \{\boldsymbol{x} \in \mathbb{C}^n : |\boldsymbol{x}| = \varepsilon\} \quad \text{for } 0 < \varepsilon \ll 1.$$

Assume that f is mixed Newton non-degenerate and (Σ, β) denote a mixed fan constructed in §2.1. We take a polygon Δ so that Σ is a part of the dual fan of Δ. Then we have maps

$$K(f) \longrightarrow \pi_{\Sigma,\beta}^{-1}(\mathbf{0}) \cap Z \overset{\mu_{\Sigma,\beta}}{\longrightarrow} \mu_{\Sigma,\beta}(\pi_{\Sigma,\beta}^{-1}(\mathbf{0})) \subset \Delta \subset \mathbb{R}^n$$

where the first arrow is determined by pluming and Z denotes the strict transform of $f^{-1}(0)$ and $\mu_{\Sigma,\beta}$ is the moment map (see Remark 1.30).

4 Remarks on non-degenerate case

4.1 Topological trivialities induced by real-analytic isomorphisms of resolution spaces

We consider a family of mixed polynomials, which are simultaneously non-degenerate, and show a topological triviality theorem for them which comes down from real-analytic isomorphisms of a resolution space if they have the same Newton polyhedron which intersects each coordinate axis.

Theorem 4.1. *Let I be an interval which contains 0. Let $f_t : \mathbb{C}^n \longrightarrow \mathbb{C}$, $t \in I$, be a real-analytic family of mixed polynomials with $\Gamma_+(f_t) = \Gamma_+(f_0)$ with $\Sigma(f_t) \cap f_t^{-1}(0) = \{\mathbf{0}\}$, $t \in I$, near $\mathbf{0}$. Let $\pi = \pi_{\Sigma,\beta} : M \longrightarrow \mathbb{C}^n$ denote the mixed toric modification associated to a mixed fan (Σ, β) constructed in §2.1 and $\hat{\pi} = \hat{\pi}_{\Sigma,\beta} : \widehat{M} \longrightarrow \mathbb{C}^n$ denote its real oriented blow-up (Remark 1.20). If f_t, $t \in I$, are mixed Newton non-degenerate (resp. non-degenerate) simultaneously, then there exists a family of real-analytic isomorphisms*

$$h'_t : (M, \pi^{-1}(\mathbf{0})) \longrightarrow (M, \pi^{-1}(\mathbf{0})) \quad (\textit{resp. } \hat{h}_t : (\widehat{M}, \pi^{-1}(\mathbf{0})) \longrightarrow (\widehat{M}, \pi^{-1}(\mathbf{0}))),$$

so that

$$h'_t((\pi \circ f_0)^{-1}(0)) = (\pi \circ f_t)^{-1}(0) \quad (\textit{resp. } \hat{h}_t((\hat{\pi} \circ f_0)^{-1}(0)) = (\hat{\pi} \circ f_t)^{-1}(0)).$$

Moreover, if f_t, $t \in I$, are convenient, h'_t (resp. $\hat{h}_t$) induces a family of homeomorphism-germs $h_t : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$, that is, $h_t \circ \pi = \pi \circ h'_t$ (resp. $h_t \circ \hat{\pi} = \hat{\pi} \circ \hat{h}_t$), so that

$$(h_t(f_0^{-1}(0)), \mathbf{0}) = (f_t^{-1}(0), \mathbf{0}), \quad t \in I.$$

We can take such h_t, $t \in I$, so that $f_t \circ h_t(x) = f_0(x)$ whenever $\Sigma(f_t) = \{\mathbf{0}\}$ near $\mathbf{0}$.

Proof. The proof is an adaption of the discussion appeared in [6] and we present here briefly. Assume that f_t, $t \in I$, are mixed Newton non-degenerate (resp. non-degenerate) simultaneously, and set $F(x,t) = f_t(x)$. Consider the vector field

$$\begin{aligned}
&\tilde{\boldsymbol{\xi}} = \det(PV), \text{ where } P = \begin{pmatrix} J & J_t \\ 0 & 1 \end{pmatrix}, \ V = \begin{pmatrix} {}^tJ & {}^t\boldsymbol{v} \\ {}^tJ_t & \partial_t \end{pmatrix}, \\
&J = \begin{pmatrix} |x_1|\partial_{|x_1|} \operatorname{Re} F & \partial_{\arg x_1} \operatorname{Re} F & \cdots & |x_n|\partial_{|x_n|} \operatorname{Re} F & \partial_{\arg x_n} \operatorname{Re} F \\ |x_1|\partial_{|x_1|} \operatorname{Im} F & \partial_{\arg x_1} \operatorname{Im} F & \cdots & |x_n|\partial_{|x_n|} \operatorname{Im} F & \partial_{\arg x_n} \operatorname{Im} F \end{pmatrix}, \ J_t = \begin{pmatrix} \partial_t \operatorname{Re} F \\ \partial_t \operatorname{Im} F \end{pmatrix}, \\
&\boldsymbol{v} = \begin{pmatrix} |x_1|\partial_{|x_1|} & \partial_{\arg x_1} & \cdots & |x_n|\partial_{|x_n|} & \partial_{\arg x_n} \end{pmatrix}.
\end{aligned}$$

Remark that the elements of the last column of V are vectors. We have $\tilde{\boldsymbol{\xi}} \operatorname{Re} F = \tilde{\boldsymbol{\xi}} \operatorname{Im} F = 0$, since both $V \operatorname{Re} F$ and $V \operatorname{Im} F$ are not of full rank. Define

$$\boldsymbol{\xi} = \begin{cases} \frac{\tilde{\boldsymbol{\xi}}}{\det(J\,{}^tJ)} & \text{case (1): } \Sigma(f_t) = \{\mathbf{0}\} \text{ near } \mathbf{0}, \\ \frac{\tilde{\boldsymbol{\xi}} + |F|^4 \partial_t}{\det(J\,{}^tJ) + |F|^4} & \text{case (2): } \Sigma(f_t) \neq \{\mathbf{0}\}, \Sigma(f_t) \cap f^{-1}(0) = \{\mathbf{0}\} \text{ near } \mathbf{0}. \end{cases}$$

The coefficient of ∂_t in $\boldsymbol{\xi}$ is 1, since the coefficient of ∂_t in $\tilde{\boldsymbol{\xi}}$ is $\det(J{}^tJ)$. If $\Sigma(f_t) = \{\mathbf{0}\}$ near $\mathbf{0}$, then $\boldsymbol{\xi} \operatorname{Re} F = \boldsymbol{\xi} \operatorname{Im} F = 0$. If $\Sigma(f_t) \neq \{\mathbf{0}\}$, $\Sigma(f_t) \cap f^{-1}(0) = \{\mathbf{0}\}$, then $\boldsymbol{\xi} \operatorname{Re} F = \boldsymbol{\xi} \operatorname{Im} F = 0$ on the set defined by $F = 0$.

Since $J = \frac{1}{2} I J_c I_1$, where

$$J_c = \begin{pmatrix} x_1\partial_{x_1} F & \overline{x_1}\partial_{\overline{x_1}} F & \cdots & x_n\partial_{x_n} F & \overline{x_n}\partial_{\overline{x_n}} F \\ x_1\partial_{x_1} \overline{F} & \overline{x_1}\partial_{\overline{x_1}} \overline{F} & \cdots & x_n\partial_{x_n} \overline{F} & \overline{x_n}\partial_{\overline{x_n}} \overline{F} \end{pmatrix}, \ I = \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \ I_1 = \begin{pmatrix} {}^t\overline{I} & & \\ & \ddots & \\ & & {}^t\overline{I} \end{pmatrix},$$

we have

$$\begin{aligned}
\det(J\,{}^t\overline{J}) &= \tfrac{1}{4} \det(I J_c I_1 {}^tI_1 {}^t\overline{J_c} {}^tI) = 4^{n-1} \det(J_c {}^t\overline{J_c}) \\
&= 4^{n-1} \begin{vmatrix} 2\sum_{i=1}^n |x_i\partial_{x_i} F|^2 & \sum_{i=1}^n [(x_i\partial_{x_i} F)^2 + (\overline{x_i}\partial_{\overline{x_i}} F)^2] \\ \sum_{i=1}^n [(x_i\partial_{x_i} F)^2 + (\overline{x_i}\partial_{\overline{x_i}} F)^2] & 2\sum_{i=1}^n |x_i\partial_{x_i} F|^2 \end{vmatrix},
\end{aligned}$$

since

$$\det(\tfrac{1}{2} I) = \frac{\mathrm{i}}{2}, \quad \det(\tfrac{1}{2} {}^t\overline{I}) = -\frac{\mathrm{i}}{2}, \quad {}^t\overline{I}\, I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We first remark that $\boldsymbol{v}$ has a lift on M (resp. $\widehat{M}$) by (3.22). We denote them by $\boldsymbol{v}'$ (resp. $\widehat{\boldsymbol{v}}$). Then we see the pull backs of coefficients of $\boldsymbol{\xi}$ by π (resp. $\hat{\pi}$). We consider these pull backs on V_σ, $\sigma \in \Sigma(n)$. Define J' and J'_t (resp. $\widehat{J}$ and $\widehat{J}_t$) by

$$\begin{aligned}
J \circ \pi_\sigma &= J' \prod_{\tau \in \sigma(1)} |u_\tau|^{\ell(\boldsymbol{a}^\tau)}, & J_t \circ \pi_\sigma &= J'_t \prod_{\tau \in \sigma(1)} |u_\tau|^{\ell(\boldsymbol{a}^\tau)} \\
\Big(\text{resp. } J \circ \hat{\pi}_\sigma &= \widehat{J} \prod_{\tau \in \sigma(1)} |u_\tau|^{\ell(\boldsymbol{a}^\tau)}, & J_t \circ \hat{\pi}_\sigma &= \widehat{J}_t \prod_{\tau \in \sigma(1)} |u_\tau|^{\ell(\boldsymbol{a}^\tau)}\Big).
\end{aligned}$$

Remark that J' and J'_t (resp. $\widehat{J}$ and $\widehat{J}_t$) is locally well-defined functions. Setting

$$\tilde{\boldsymbol{\xi}}' = \det(P'V'), \text{ where } P' = \begin{pmatrix} J' & J'_t \\ 0 & 1 \end{pmatrix}, \ V' = \begin{pmatrix} {}^tJ' & {}^t\boldsymbol{v}' \\ {}^tJ'_t & \partial_t \end{pmatrix}$$

$$\left(\text{rssp. } \widehat{\tilde{\boldsymbol{\xi}}} = \det(\widehat{P}\widehat{V}), \text{ where } \widehat{P} = \begin{pmatrix} \widehat{J} & \widehat{J}_t \\ 0 & 1 \end{pmatrix}, \ \widehat{V} = \begin{pmatrix} {}^t\widehat{J} & {}^t\widehat{\boldsymbol{v}} \\ {}^t\widehat{J}_t & \partial_t \end{pmatrix}\right),$$

we obtain the lift $\boldsymbol{\xi}'$ (resp. $\widehat{\tilde{\boldsymbol{\xi}}}$) of $\boldsymbol{\xi}$ by π_σ (resp. $\hat{\pi}_\sigma$) as follows:

$$\boldsymbol{\xi}' = \begin{cases} \frac{\tilde{\boldsymbol{\xi}}'}{\det(J'{}^tJ')} & \text{case (1)} \\ \frac{\tilde{\boldsymbol{\xi}}'}{\det(J'{}^tJ')+(F')^4} & \text{case (2)} \end{cases} \quad \left(\text{resp. } \widehat{\boldsymbol{\xi}} = \begin{cases} \frac{\widehat{\tilde{\boldsymbol{\xi}}}}{\det(\widehat{J}{}^t\widehat{J}')} & \text{case (1)} \\ \frac{\widehat{\tilde{\boldsymbol{\xi}}}}{\det(\widehat{J}{}^t\widehat{J})+|\widehat{F}|^4} & \text{case (2)} \end{cases}\right),$$

since

$$\det(J{}^tJ)\circ\pi_\sigma = \det(J'{}^tJ') \prod_{\tau\in\sigma(1)} |u_\tau|^{2\ell(\boldsymbol{a}^\tau)}, \qquad F\circ\pi_\sigma = F' \prod_{\tau\in\sigma(1)} |u_\tau|^{2\ell(\boldsymbol{a}^\tau)}$$

$$\left(\text{resp. } \det(J{}^tJ)\circ\hat{\pi}_\sigma = \det(\widehat{J}{}^t\widehat{J}) \prod_{\tau\in\sigma(1)} |u_\tau|^{2\ell(\boldsymbol{a}^\tau)}, \qquad F\circ\hat{\pi}_\sigma = \widehat{F} \prod_{\tau\in\sigma(1)} |u_\tau|^{2\ell(\boldsymbol{a}^\tau)}\right).$$

Take a cone $\sigma' \in \Sigma$ so that $\pi(E_{\sigma'}) = \{\boldsymbol{0}\}$, $\sigma' \subset \sigma$. It is enough to show that

$$Q = \begin{cases} \det(J'\,{}^tJ')|_{E_{\sigma'}} & (\text{resp. } \det(\widehat{J}\,{}^t\widehat{J})|_{\widehat{E}_{\sigma'}}) & (\text{case (1)}) \\ (\det(J'\,{}^tJ') + |F'|^4)|_{E_{\sigma'}} & (\text{resp.} (\det(\widehat{J}\,{}^t\widehat{J}) + |\widehat{F}|^4)|_{\widehat{E}_{\sigma'}}) & (\text{case (2)}) \end{cases}$$

is nowhere zero. Remark that only the terms in $(f_t)_\gamma$, $\gamma = \bigcap_{\tau\in\sigma'(1)} \gamma(\boldsymbol{a}^\tau)$, contribute to the terms in $J'|_{E_{\sigma'}}$ $J'_t|_{E_{\sigma'}}$, $F'|_{E_{\sigma'}}$ (resp. $\widehat{J}|_{\widehat{E}_{\sigma'}}$, $\widehat{J}_t|_{\widehat{E}_{\sigma'}}$, $\widehat{F}|_{\widehat{E}_{\sigma'}}$), and so is Q. Since f_γ is mixed weighted homogeneous (resp. radially weighted homogeneous), if it is zero, it must be zero at some point in $(\mathbb{C}^*)^n$, which contradicts the non-degeneracy assumption. Thus the vector field $\boldsymbol{\xi}'$ (resp. $\widehat{\boldsymbol{\xi}}$) is well-defined and its flow provides the desired real-analytic isomorphisms. The last assertion is a consequence of the fact that $\boldsymbol{\xi}'$ (resp. $\widehat{\boldsymbol{\xi}}$) tangent to each E_τ (resp. $\widehat{E}_\tau$) for $\tau \in \Sigma(1)$. □

Remark that H_t (resp. $\widehat{H_t}$) induces a homeomorphism when $(f_t)_\gamma$ does not depend on t for each non-compact face γ which does not lie in the union of coordinate hyperplanes, since $\partial_t F'|_{E_\tau}$ is zero for $\tau \in \Sigma(1)$ with $\ell(\boldsymbol{a}^\tau) > 0$ and $\pi(E_\tau) \neq \{\boldsymbol{0}\}$.

Corollary 4.2. *Let f be a mixed polynomial. If f is non-degenerate and convenient, then the local topological type of $(\mathbb{C}^n, f^{-1}(0), \boldsymbol{0})$ is determined by f_Γ where Γ is the union of compact faces of $\Gamma_+(f)$.*

Proof. Apply the previous Theorem for $f_t(x) = f(x) + tg(x)$, $0 \le t \le 1$, so that $\Gamma_+(g)$ is in the interior of $\Gamma_+(f)$. □

4.2 Coordinate crossing property

In this subsection, we present an attempt to understand the situation that a mixed polynomial is not mixed Newton non-degenerate, but is non-degenerate.

We say that two subsets Z_1 and Z_2 of $\mathbb{R}^{2n}$ at $p \in Z_1 \cap Z_2$ are of **coordinate crossing** if there exists submersion germs $g_i : (\mathbb{R}^{2n}, p) \longrightarrow (\mathbb{R}^{k_i}, 0)$, $i = 1, 2$, so that $(Z_i, p) = (g_i^{-1}(0), p)$, $i = 1, 2$, and the Jacobi matrix of $G = (g_1, g_2) : (\mathbb{R}^{2n}, p) \longrightarrow (\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}, 0)$ is of rank k along $G^{-1}(0)$. Remark that $k \le k_1 + k_2$ and equality holds if and only if $g_1^{-1}(0)$ and $g_2^{-1}(0)$ intersects transversely.

We consider the real oriented blow up $\widehat{V}_\sigma \longrightarrow V_\sigma$ along the exceptional set $E_{\sigma'}$. The set $\widehat{V}_\sigma$ is $\widehat{\mathbb{C}}^{\sigma'(1)} \times \mathbb{C}^{\sigma(1)\setminus\sigma'(1)}$ where $\widehat{\mathbb{C}}^{\sigma'(1)} = \{(|v_{\tau'}|, \arg v_{\tau'}) \in \mathbb{R}_{\ge 0} \times S^1 : \tau' \in \sigma'(1)\}$. The condition (3.9) implies that the strict transform of f intersects the set defined by $|v_{\tau'}| = 0$, $\tau' \in \sigma'(1)$ transversely (Remark 3.5).

We consider how the strict transform Z of f by $\pi_{\Sigma,\beta}$ and the exceptional set E_τ, $\tau \in \Sigma(1)$ intersect in V_σ. By (1.28), the map $\pi_\sigma = \pi_{\Sigma,\beta}|_{V_\sigma}$ is expressed by

$$x_i = \prod_{\tau \in \sigma(1)} v_\tau^{\frac{a_i^\tau + b_i^\tau}{2}} \overline{v_\tau}^{\frac{a_i^\tau - b_i^\tau}{2}}, \quad i = 1, \dots, n, \tag{4.3}$$

and we obtain that

$$f \circ \pi_\sigma = \sum_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \prod_{\tau \in \sigma(1)} v_\tau^{\langle \frac{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{a}^\tau \rangle + \langle \frac{\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{b}^\tau \rangle} \overline{v_\tau}^{\langle \frac{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{a}^\tau \rangle - \langle \frac{\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{b}^\tau \rangle}.$$

Define $\tilde{f}$ and $\tilde{f}_\gamma$, $\gamma \subset \mathbb{R}^n$, by

$$f \circ \pi_\sigma = \tilde{f} \prod_{\tau \in \sigma(1)} v_\tau^{\frac{\ell(\boldsymbol{a}^\tau) + m(\tau)}{2}} \overline{v_\tau}^{\frac{\ell(\boldsymbol{a}^\tau) - \overline{m}(\tau)}{2}}, \qquad f_\gamma \circ \pi_\sigma = \tilde{f}_\gamma \prod_{\tau \in \sigma(1)} v_\tau^{\frac{\ell(\boldsymbol{a}^\tau) + m(\tau)}{2}} \overline{v_\tau}^{\frac{\ell(\boldsymbol{a}^\tau) - \overline{m}(\tau)}{2}},$$

where $\overline{m}(\tau) = \max\{\langle \boldsymbol{\nu}, \boldsymbol{b}^\tau \rangle : \boldsymbol{\nu} \in \mathrm{LE}_{\gamma(\boldsymbol{a}^\tau)}(f)\}$. We then have

$$\begin{aligned}
\tilde{f} &= \sum_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \prod_{\tau \in \sigma(1)} v_\tau^{\langle \frac{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{a}^\tau \rangle + \langle \frac{\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{b}^\tau \rangle - \frac{\ell(\boldsymbol{a}^\tau) + m(\tau)}{2}} \overline{v_\tau}^{\langle \frac{\boldsymbol{\nu}+\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{a}^\tau \rangle - \langle \frac{\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}}{2}, \boldsymbol{b}^\tau \rangle - \frac{\ell(\boldsymbol{a}^\tau) - \overline{m}(\tau)}{2}} \\
&= \sum_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu}, \bar{\boldsymbol{\nu}}} \prod_{\tau \in \sigma(1)} |v_\tau|^{\langle \boldsymbol{\nu}+\bar{\boldsymbol{\nu}}, \boldsymbol{a}^\tau \rangle - \ell(\boldsymbol{a}^\tau)} (v_\tau / |v_\tau|)^{\langle \boldsymbol{\nu}-\bar{\boldsymbol{\nu}}, \boldsymbol{b}^\tau \rangle - m(\tau)} \overline{v_\tau}^{\frac{\overline{m}(\tau) - m(\tau)}{2}}.
\end{aligned}$$

Then we have

$$\tilde{f} = f' \prod_{\tau \in \sigma(1)} \overline{v_\tau}^{\frac{\overline{m}(\tau) - m(\tau)}{2}}, \quad \text{and} \quad \tilde{f}_\gamma = f'_\gamma \prod_{\tau \in \sigma(1)} \overline{v_\tau}^{\frac{\overline{m}(\tau) - m(\tau)}{2}}.$$

When $\overline{m}(\tau) = m(\tau)$ for $\tau \in \Sigma(1)$, we have $\tilde{f} = f'$, and $\tilde{f}_\gamma = f'_\gamma$. This case is already treated in the previous section.

In this section, we consider the case $\overline{m}(\tau_0) - m(\tau_0) = 2$ assuming $n = 2$. We assume that (v_0, v_1) denotes the coordinate system of V_σ, $\sigma \in \Sigma(2)$, and $E_\tau = \{v_0 = 0\}$. We have

$$\tilde{f} = v_0 \tilde{\phi}_0 + \overline{v_0} \tilde{\phi}_1, \qquad \tilde{f}_{\gamma(\boldsymbol{a}^{\tau_0})} = v_0 \phi_0 + \overline{v_0} \phi_1$$

where $\tilde{\phi}_i = \phi_i + R_i$, $R_i = o(|v_0|)$, $i = 1, 2$. We set $\phi = \det \Phi$, and $\tilde{\phi} = \det \widetilde{\Phi}$ where

$$\Phi = \begin{pmatrix} \phi_0 & \phi_1 \\ \overline{\phi_1} & \overline{\phi_0} \end{pmatrix}, \quad \widetilde{\Phi} = \begin{pmatrix} \tilde{\phi}_0 & \tilde{\phi}_1 \\ \overline{\tilde{\phi}_1} & \overline{\tilde{\phi}_0} \end{pmatrix}.$$

Proposition 4.4. *The strict transform Z and the set $E_\tau \setminus \{v_1 = 0\}$, $\tau \in \sigma(1)$, intersect as coordinate crossing when* $\operatorname{rank}\Phi = 1$ *and $\{\tilde{\phi} = 0\}$ is nonsingular at $Z \cap E_\tau$.*

Proof. We first remark that

$$\begin{pmatrix} \operatorname{Re}\tilde{f} \\ \operatorname{Im}\tilde{f} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} \widetilde{\Phi} \begin{pmatrix} v_0 \\ \overline{v_0} \end{pmatrix}$$

and

$$\begin{pmatrix} \operatorname{Re} f' \\ \operatorname{Im} f' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} \widetilde{\Phi} \begin{pmatrix} e^{\arg v_0 \mathrm{i}} \\ e^{-\arg v_0 \mathrm{i}} \end{pmatrix}.$$

The system $f' = \overline{f'} = 0$ thus reduces to the system

$$\widetilde{\Phi} \begin{pmatrix} e^{\arg v_0 \mathrm{i}} \\ e^{-\arg v_0 \mathrm{i}} \end{pmatrix} = 0,$$

and we conclude that $\tilde{\phi} = 0$ on $Z \cap \{v_0 = 0\}$.

Since

$$\left.\frac{\partial(\operatorname{Re}\tilde{f}, \operatorname{Im}\tilde{f}, \tilde{\phi})}{\partial(\operatorname{Re} v_0, \operatorname{Im} v_0, \operatorname{Re} u_1, \operatorname{Im} u_1)}\right|_{v_0=0}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (\tilde{f}_\gamma)_{v_0} & (\tilde{f}_\gamma)_{\overline{v_0}} & (\tilde{f}_\gamma)_{u_1} & (\tilde{f}_\gamma)_{\overline{u_1}} \\ \overline{(\tilde{f}_\gamma)}_{v_0} & \overline{(\tilde{f}_\gamma)}_{\overline{v_0}} & \overline{(\tilde{f}_\gamma)}_{v_1} & \overline{(\tilde{f}_\gamma)}_{\overline{v_1}} \\ \phi_{v_0} & \phi_{\overline{v_0}} & \phi_{v_1} & \phi_{\overline{v_1}} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -\mathrm{i} & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & -\mathrm{i} \\ 0 & 0 & 1 & \mathrm{i} \end{pmatrix}$$

$$= \begin{pmatrix} \operatorname{Re}(\phi_0 + \phi_1) & \operatorname{Im}(\phi_1 - \phi_0) & 0 & 0 \\ \operatorname{Im}(\phi_0 + \phi_1) & \operatorname{Re}(\phi_1 - \phi_0) & 0 & 0 \\ \phi_{v_0} + \phi_{\overline{v_0}} & \mathrm{i}[\phi_{\overline{v_0}} - \phi_{v_0}] & \phi_{v_1} + \phi_{\overline{v_1}} & \mathrm{i}[\phi_{\overline{v_1}} - \phi_{v_1}] \end{pmatrix},$$

where $\gamma = \gamma(\boldsymbol{a}^\tau)$, we have the following:

- $\operatorname{Re} v_0$, $\operatorname{Re}\tilde{f}$, $\tilde{\phi}$ form a part of a coordinate system when $\operatorname{Im}(\phi_0 - \phi_1) \neq 0$.
- $\operatorname{Re} v_0$, $\operatorname{Im}\tilde{f}$, $\tilde{\phi}$ form a part of a coordinate system when $\operatorname{Re}(\phi_0 - \phi_1) \neq 0$.
- $\operatorname{Im} v_0$, $\operatorname{Re}\tilde{f}$, $\tilde{\phi}$ form a part of a coordinate system when $\operatorname{Re}(\phi_0 + \phi_1) \neq 0$.
- $\operatorname{Im} v_0$, $\operatorname{Im}\tilde{f}$, $\tilde{\phi}$ form a part of a coordinate system when $\operatorname{Im}(\phi_0 + \phi_1) \neq 0$.

Thus we complete the proof, because of the following lemma. □

Lemma 4.5. *Assume that* $\operatorname{rank}\Phi = 1$. *Let $\boldsymbol{p}$ is a point in $Z \cap \{v_0 = 0\}$.*
(i) *The strict transform Z near $\boldsymbol{p}$ is defined by one of the following ideals:*

$$\langle \operatorname{Re} f', \tilde{\phi} \rangle, \ \langle \operatorname{Im} f', \tilde{\phi} \rangle.$$

(ii) *The exceptional set $\{v_0 = 0\}$ near $\boldsymbol{p}$ is defined by one of the following ideals:*

$$\langle \operatorname{Re} v_0, \operatorname{Im}\tilde{f} \rangle, \ \langle \operatorname{Re} v_0, \operatorname{Re}\tilde{f} \rangle, \ \langle \operatorname{Im} v_0, \operatorname{Im}\tilde{f} \rangle, \ \langle \operatorname{Im} v_0, \operatorname{Re}\tilde{f} \rangle.$$

Proof. The item (i) is a consequence of the following implications.

- $\operatorname{Re} f' = \det\tilde{\phi} = 0$ implies $\operatorname{Im}\tilde{f} = 0$ for $|v_0| \ll 1$, $\tau' \in \sigma'(1)$.
- $\operatorname{Im} f' = \det\tilde{\phi} = 0$ implies $\operatorname{Re}\tilde{f} = 0$ for $|v_0| \ll 1$, $\tau' \in \sigma'(1)$.

$\operatorname{Re} f' = 0$ implies $\widetilde{\Phi} X = 0$ has only solution

$$X = c \begin{pmatrix} e^{(\arg v_0)\mathrm{i}} \\ e^{(-\arg u_0)\mathrm{i}} \end{pmatrix}, \quad c \in \mathbb{C},$$

and we obtain $\operatorname{Im} f' = 0$, which shows the first item. The second item is proved similarly.

Now we show the item (ii). Since $(\phi_0, \phi_1) \neq (0,0)$ on $Z \cap E_{\tau_0}$, we have $(\phi_0 + \phi_1, \phi_0 - \phi_1) \neq (0,0)$ on $Z \cap E_{\tau_0}$. Thus the lemma is a consequence of the following implications.

- $\operatorname{Re}(\phi_0 - \phi_1) \neq 0$, $\operatorname{Re} v_0 = 0$, $\operatorname{Im} \tilde{f} = 0 \Longrightarrow \operatorname{Im} v_0 = 0$.
- $\operatorname{Im}(\phi_0 - \phi_1) \neq 0$, $\operatorname{Re} v_0 = 0$, $\operatorname{Re} \tilde{f} = 0 \Longrightarrow \operatorname{Im} v_0 = 0$.
- $\operatorname{Re}(\phi_0 + \phi_1) \neq 0$, $\operatorname{Im} v_0 = 0$, $\operatorname{Re} \tilde{f} = 0 \Longrightarrow \operatorname{Re} v_0 = 0$.
- $\operatorname{Im}(\phi_0 + \phi_1) \neq 0$, $\operatorname{Im} v_0 = 0$, $\operatorname{Im} \tilde{f} = 0 \Longrightarrow \operatorname{Re} v_0 = 0$.

These implications follow from the following identity:

$$\begin{pmatrix} \operatorname{Re} \tilde{f} \\ \operatorname{Im} \tilde{f} \\ \operatorname{Re} v_0 \\ \operatorname{Im} v_0 \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(\tilde{\phi}_0 + \tilde{\phi}_1) & \operatorname{Im}(\tilde{\phi}_0 - \tilde{\phi}_1) \\ \operatorname{Im}(\tilde{\phi}_0 + \tilde{\phi}_1) & \operatorname{Re}(\tilde{\phi}_0 - \tilde{\phi}_1) \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \operatorname{Re} v_0 \\ \operatorname{Im} v_0 \end{pmatrix}. \qquad \square$$

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