$\mathbb{R}^3$ 内の非特異曲面の中心射影の普遍性
Versality of central projections of regular surfaces in $\mathbb{R}^3$

Shuhei Honda

Saitama University, M2, Fukui laboratory

January 21, 2021　研究集会「接触構造、特異点、微分方程式及びその周辺」

Introduction

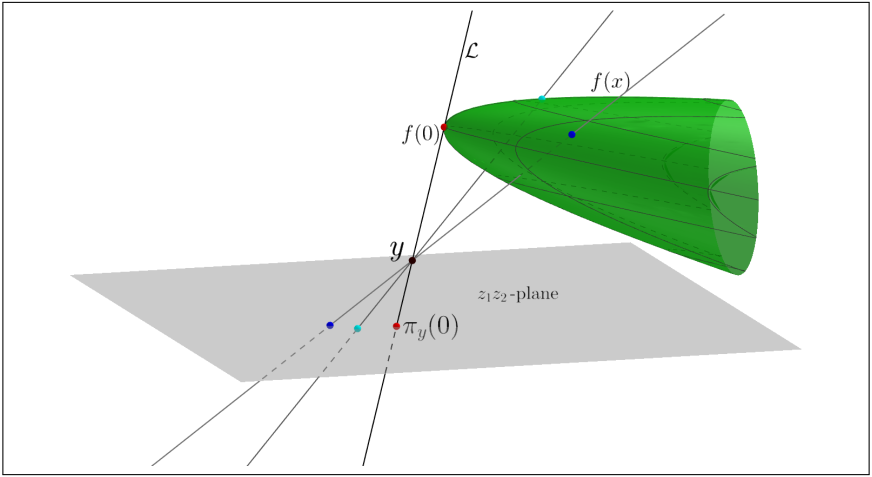


Figure: a central projection of a regular surface

Introduction

Let $y \in \mathbb{R}^3$ called a **viewpoint**. And we call the line $\mathcal{L}$ which is through viewpoint and $f(x)$ the **viewline** to $f(x)$.
A **central projection** of a regular surface $f(x)$ from y to a $z_1 z_2$-plane is defined as the following map

$$\pi : (\mathbb{R}^2, 0) \times (\mathbb{R}^3, y) \longrightarrow (\mathbb{R}^2, \pi(0, y)),\ (x, y) \mapsto \pi_y(x)$$

where

$$\pi_y(x) := \begin{pmatrix} \langle t(x,y)f(x) + (1 - t(x,y))y, \boldsymbol{e}_1 \rangle \\ \langle t(x,y)f(x) + (1 - t(x,y))y, \boldsymbol{e}_2 \rangle \end{pmatrix},$$

$$t(x,y) := \frac{\langle y, \boldsymbol{e}_3 \rangle}{\langle y - f(x), \boldsymbol{e}_3 \rangle} \quad \text{and} \quad \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \ : \text{canonical basis of } \mathbb{R}^3.$$

We regard π as the **central projection unfoldings** with parameters y.

Preliminary (differential geometry 1/2)

Let consider a parametrized regular surface

$$f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, f(0,0)) : (x_1, x_2) \longmapsto f(0,0) + x_1 \boldsymbol{u} + x_2 \boldsymbol{v} + Q(x) \boldsymbol{w}$$

where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$: orthonormal frame of $\mathbb{R}^3$ and

$$Q(x) := \sum_{k \geq 2} H_k(x_1, x_2), \; H_k(x_1, x_2) := \sum_{i+j=k} \frac{1}{i!\,j!} a_{ij} \, {x_1}^i {x_2}^j.$$

We set the first and second fundamental quantities of the regular surface f

$$E := \langle f_{x_1}, f_{x_1} \rangle, \quad F := \langle f_{x_1}, f_{x_2} \rangle, \quad G := \langle f_{x_2}, f_{x_2} \rangle,$$
$$L := \langle f_{x_1 x_1}, \mathbf{n} \rangle, \quad M := \langle f_{x_1 x_2}, \mathbf{n} \rangle, \quad N := \langle f_{x_2 x_2}, \mathbf{n} \rangle$$

where $\mathbf{n}$ is the unit normal vector $\frac{f_{x_1} \times f_{x_2}}{|f_{x_1} \times f_{x_2}|}$ and the **Gauss curvature**

$$K := \frac{LN - M^2}{EG - F^2}.$$

- If $K > 0$ at x, we call a point $f(x)$ **elliptic point**.
- If $K = 0$ at x, we call a point $f(x)$ **parabolic point**.
- If $K < 0$ at x, we call a point $f(x)$ **hyperbolic point**.

Preliminary (differential geometry 2/2)

We set the second fundamental quantities of f

$$L := \langle f_{x_1 x_1}, \mathbf{n} \rangle, \ \ M := \langle f_{x_1 x_2}, \mathbf{n} \rangle, \ \ N := \langle f_{x_2 x_2}, \mathbf{n} \rangle.$$

Definition

(dx_1, dx_2) is an **asymptotic direction** of f at x_0 if the second fundamental form

$$II := L\, dx_1^2 + 2M\, dx_1 dx_2 + N\, dx_2^2$$

vanishes at x_0.

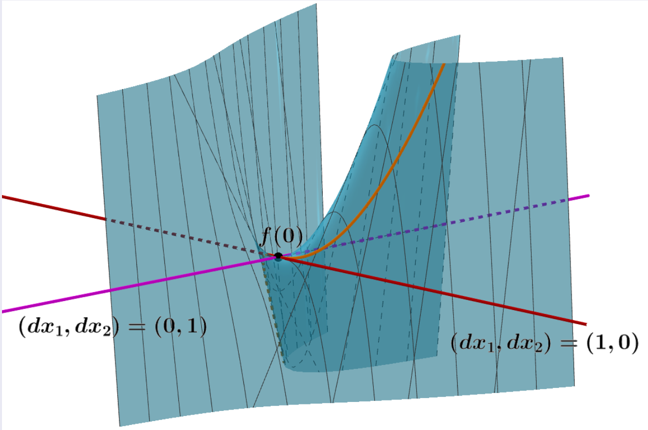
Definition

We call a line in $\mathbb{R}^3$ which is generated by an asymptotic direction of f at x_0 an **asymptotic direction line** of f at x_0.

Example (asymptotic line and asymptotic direction line)

Let $f(x) = (x_1, x_2, x_1 x_2 + \frac{x_1^3}{6})$. The origin is a hyperbolic point.

- Asymptotic lines are curves in f (orange curve and purple line).
- Asymptotic direction lines are curves in $\mathbb{R}^3$ (red and purple lines).



Preliminary (Singularity theory 1/2)

Let $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, f(0))$ be smooth map germs.
We set $Jf(x)$ a Jacobi matrix of f. Then, the origin is

- regular point $\Leftrightarrow$ $rank\, Jf(0) = min\,(m, n)$,
- singular point $\Leftrightarrow$ $rank\, Jf(0) < min\,(m, n)$.

Definition ($\mathcal{A}$-equivalence)

Let $f_i : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, f_i(0))$ $(i = 1, 2)$ denote smooth map germs.
If there exist diffeomorphism germs φ and ψ so that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^m, 0 & \xrightarrow{f_1} & \mathbb{R}^n, f_1(0) \\ \varphi \downarrow & & \downarrow \psi \\ \mathbb{R}^m, 0 & \xrightarrow{f_2} & \mathbb{R}^n, f_2(0) \end{array},$$

f_1 and f_2 are $\mathcal{A}$-**equivalent** ($f_1 \sim_{\mathcal{A}} f_2$).

Preliminary (Singularity theory 2/2)

Let $T\mathcal{A}_e(f)$ denote **tangent space of** f

$$T\mathcal{A}_e(f) := tf(\theta_m) + \omega f(\theta_n) \subset \theta(f)$$

with $\quad tf : \theta_m \to \theta(f) : \xi \mapsto df \circ \xi \quad$ and $\quad \omega f : \theta_n \to \theta(f) : \eta \mapsto \eta \circ f.$

We define the $\mathcal{A}_e$-**codimension** of f by

$$cod(\mathcal{A}_e, f) := \dim_{\mathbb{R}} \frac{\theta(f)}{T\mathcal{A}_e(f)}.$$

Definition ($\mathcal{A}$- (infinitesimal) versal unfolding)

Let $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of f with parameters y. Then, F is $\mathcal{A}$- **(infinitesimal) versal unfolding** if

$$T\mathcal{A}_e(f) + \sum_{i=1}^{k} \mathbb{R} \frac{\partial F}{\partial y_i}(x, 0) = \theta(f).$$

Criteria of singularity of a central projection

We consider **criteria of singularity type of π_y whose $\mathcal{A}_e$-codimension** ≤ 3 and **versality** of each of them.

Theorem

(1) *Let $\lambda(x_1, x_2)$ be Jacobian of $\pi_y := \pi(x, y)$.*
Then, the following two condition (i), (ii) are equivalent.
  (i) $\lambda(0, 0) = 0$
  (ii) *the viewline is contained in the tangent space of f at the origin.*

(2) $(x_1, x_2) = (0, 0)$ *is a singular point of π_y,*
  π_y *at* 0 *is $\mathcal{A}$-equivalent to fold* (x_1, x_2^2)
  $\Leftrightarrow$ *the viewline is not an asymptotic direction line.*

Singularity type of a central projection at elliptic point

Lemma

We assume that viewline from viewpoint y is contained in tangent space of f at 0.
Then, if f is elliptic at 0, a central projection π has fold singularity at 0 for any viewpoint y.

$(\because)$ f does not have any asymptotic direction at elliptic point.

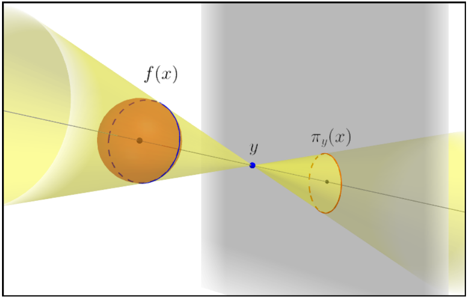


Figure: a central projection of a sphere ([7])

Singularity type of a central projection at hyperbolic point

We assume that $a_{20} = 0$, $a_{11} \neq 0$ and $y - f(0) = p_1 f_{x_1}(0)$ i.e. f is hyperbolic at 0 and $\mathcal{L}$ is an asymptotic direction line of f at 0.
The **criteria of singularity types of π_y whose $\mathcal{A}_e$-codimension ≤ 3** ([2], [4]) and **versality** of each of them are the following table in this case.

type	c	d	l	position of y	criteria of versality
cusp $(x_1, x_1x_2 + x_2^3)$	0	3	2		always
swallowtail $(x_1, x_1x_2 + x_2^4)$	1	4	3		always
butterfly $(x_1, x_1x_2 + x_2^5 \pm x_2^7)$	2	7	4	not h-focal ([2])	the flecnodal curve is not singular
elder butterfly $(x_1, x_1x_2 + x_2^5)$	3	7	4	h-focal ([2])	the flecnodal curve is not singular and y is not in a special position.
unimodal $(x_1, x_1x_2 + x_2^6 \pm x_2^8 + \alpha x_2^9)$	3	8	5	not one of two special positions	not versal

c : $\mathcal{A}_e$-codimension, d : $\mathcal{A}$-determinacy order, l : order of contact with $\mathcal{L}$ for f at 0 in criteria.

Singularity type of a central projection at hyperbolic point

We assume that $a_{20}=0$, $a_{11}\neq 0$ and $y-f(0)=p_1 f_{x_1}(0)$ i.e. f is hyperbolic at 0 and $\mathcal{L}$ is an asymptotic direction line of f at 0. Then, y is h-**focal** $\Leftrightarrow$ $(48a_{50}\,a_{70}\,{a_{11}}^2-35{a_{60}}^2\,{a_{11}}^2+42a_{21}\,a_{50}\,a_{60}\,a_{11}-1680a_{31}\,{a_{50}}^2\,a_{11}+2205{a_{21}}^2\,{a_{50}}^2){p_1}^2+(-84a_{50}\,a_{60}\,{a_{11}}^2+252a_{21}\,{a_{50}}^2\,a_{11})\,p_1+756{a_{50}}^2\,{a_{11}}^2=0.$

Goose series singularities of a central projection at parabolic point

We assume that $a_{20} = a_{11} = 0$, $a_{02} \neq 0$ and $y - f(0) = p_1 f_{x_1}(0)$
i.e. f is parabolic at 0 and $\mathcal{L}$ is an asymptotic direction line of f at 0.

type	c	d	l	additional criteria	criteria of versality
lips $(x_1, x_2^3 + x_1^2 x_2)$ or beaks $(x_1, x_2^3 - x_1^2 x_2)$	1	3	2	$G \circ \gamma$ is 1-st order contact with S and p_1 is more (resp. less) than the curvature radius of $G \circ \gamma$	always
goose $(x_1, x_2^3 + x_1^3 x_2)$	2	4	2	$G \circ \gamma$ is 2-nd order contact with S	f is not flat umbilic i.e. $a_{02} \neq 0$
ugly goose $(x_1, x_2^3 \pm x_1^4 x_2)$	3	5	2	$G \circ \gamma$ is 3-rd order contact with S	f is not flat umbilic i.e. $a_{02} \neq 0$
type 16 $(x_1, x_2^4 + x_1^2 x_2)$	3	5	3	$f(0)$ is 1-st or more order red subparabolic of f and y is not in a special position	not versal

$G : (\mathbb{R}^2, 0) \to (S^2, G(0))$, $G(x_1, x_2) := \frac{f_{x_1}(x_1,x_2) \times f_{x_2}(x_1,x_2)}{\| f_{x_1}(x_1,x_2) \times f_{x_2}(x_1,x_2) \|}$: the **Gauss map** of f.
γ : the **parabolic curve** of f.
S : **the characteristic surface** defined by

$$p_1 a_{02} z_1 + R_1 z_2^3/6 + R_2 z_2^4/24 - z_3 + 1 = 0.$$

Goose series singularities of a central projection at parabolic point

S : **the characterisitic surface** defined by

$$p_1\, a_{02}\, z_1 + R_1\, z_2^3/6 + R_2\, z_2^4/24 - z_3 + 1 = 0$$

where

$$R_1 := \frac{a_{03}\, {a_{30}}^2 - {a_{21}}^3}{(a_{12}\, a_{30} - {a_{21}}^2)^2\, {p_1}^2}$$

and

$$R_2 := (3{a_{30}}^4\, a_{02}^4 + S_1\, a_{02} + S_0)/{a_{30}}^4\, {a_{02}}^4,$$

$S_1 := 8a_{21}({a_{21}}^3\, a_{40} - 3{a_{21}}^2\, a_{30}\, a_{31} - a_{13}\, {a_{30}}^3 + 3a_{21}\, a_{22}\, {a_{30}}^2) - ({a_{21}}^4\, a_{40} - 4{a_{21}}^3\, a_{30}\, a_{31} + a_{04}\, {a_{30}}^4 - 4a_{13}\, a_{21}\, {a_{30}}^3 + 6{a_{21}}^2\, a_{22}\, {a_{30}}^2)$,
$S_0 := 3{a_{30}}^2\, (a_{12}\, a_{30} - {a_{21}}^2)^2 + 3{a_{03}}^2\, {a_{30}}^4 + 8a_{03}\, a_{12}\, a_{21}\, {a_{30}}^3 - 14a_{03}\, {a_{21}}^3\, {a_{30}}^2 - 36{a_{12}}^2\, {a_{21}}^2\, {a_{30}}^2 + 64a_{12}\, {a_{21}}^4\, a_{30} - 25{a_{21}}^6$.

Goose series singularities of a central projection at parabolic point

If $\mathcal{L}$ is the 2-nd order contact with f at the origin ($\Leftrightarrow a_{30} \neq 0$) and f is not flat umbilic ($\Leftrightarrow a_{02} \neq 0$) ,

- $G \circ \gamma$ is 1-st order contact with $S \Leftrightarrow \frac{k_2}{p_1} \neq \frac{1}{a_{30}} \begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{vmatrix}$.
- $G \circ \gamma$ is 2-nd order contact with $S \Leftrightarrow$

$$\frac{k_2}{p_1} = \frac{1}{a_{30}} \begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{vmatrix} \quad \text{and} \quad H_3(-a_{21}, a_{30}) \neq \frac{1}{2} H_{4x_1}(-a_{21}, a_{30})\, p_1 .$$

- $G \circ \gamma$ is 3-rd order contact with $S \Leftrightarrow$
$\frac{k_2}{p_1} = \frac{1}{a_{30}} \begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{vmatrix}$, $H_3(-a_{21}, a_{30}) = \frac{1}{2} H_{4x_1}(-a_{21}, a_{30})\, p_1$ and

$$a_{30}(H_{5x_1}(-a_{21}, a_{30})p_1 - 3H_4(-a_{21}, a_{30}))p_1 \neq \frac{1}{2}(H_{4x_1x_1}(-a_{21}, a_{30})p_1 - 2H_{3x_1}(-a_{21}, a_{30}))^2 .$$

Gulls series singularities of a central projection at parabolic point

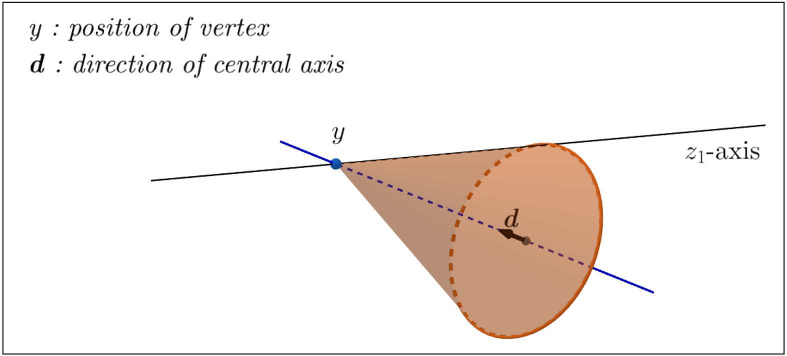
We can assume that $a_{20} = a_{11} = 0$, $a_{02}\, a_{40} - 3a_{21}^2 \neq 0$ and $y - f(0) = p_1\, f_{x_1}(0)$.

type	c	d	l	additional criteria	criteria of versality
gulls $(x_1, x_1x_2^2 + x_2^4 + x_2^5)$	2	5	3	$f(0)$ is not red subparabolic of f and A_4-contact with $C_{y,d,\theta}$	f is the first order blue ridge $\Leftrightarrow a_{02}\, a_{40} - 3a_{21}^2 \neq 0$
ugly gulls $(x_1, x_1x_2^2 + x_2^4 + x_2^7)$	3	7	3	$f(0)$ is not red subparabolic of f and "A_6-contact with $C_{y,d,\theta}$" or "A_5-contact with $C_{y,d,\theta}$ and y is not ug-focal"	f is the first order blue ridge $\Leftrightarrow a_{02}\, a_{40} - 3a_{21}^2 \neq 0$
type 12 $(x_1, x_1x_2^2 + x_2^5 + x_2^6)$	3	6	4	$f(0)$ is not red subparabolic of f and y is not one of two special positions	not versal

$C_{y,d,\theta}$: a **cone** whose vertex is y in $\mathbb{R}^3$, direction of the central axis is d in S^2 and ungle is θ in $(0, 1)$.

Gulls series singularities of a central projection at parabolic point

$C_{y,\boldsymbol{d},\theta}$: a **cone** whose vertex is y in $\mathbb{R}^3$, direction of the central axis is $d := (d_1, d_2, d_3)$ in S^2 and ungle is θ in $(0, 1)$.



We assume that $\mathcal{L}$ is the 3-rd order contact with f ($\Leftrightarrow a_{30} = 0$, $a_{40} \neq 0$) and f is not red subparabolic ($\Leftrightarrow a_{21} \neq 0$).

Gulls series singularities of a central projection at parabolic point

If f is the first order blue ridge ($a_{02}\,a_{40} - 3a_{21}^2 \neq 0$), $d_2 = 0$ and $(a_{02}\,a_{40} - 3{a_{21}}^2)\,d_3\,y_1 + a_{40}\,d_1 = 0$, then f is

- A_4-contact with $C_{y,d,\theta} \Leftrightarrow$

$$AC_4 := (3{a_{21}}^2\,a_{50}+5a_{12}\,{a_{40}}^2-10a_{21}\,a_{31}\,a_{40})\,p_1-5a_{40}\,(a_{40}\,k_2-3{a_{21}}^2) \neq 0.$$

- A_6-contact with $C_{y,d,\theta}$
  $\Leftrightarrow$ A_5-contact with $C_{y,d,\theta}$ and y is not ug-focal
  $\Leftrightarrow$ $AC_4 = 0$ and

$$AC_6 - 70\,a_{40}\,AC_5 = \tilde{A}_2 p_1^2 + \tilde{A}_1 p_1 + \tilde{A}_0 \neq 0.$$

where
AC_k : non degenerate condition of A_k-contact and
$\tilde{A}_2$, $\tilde{A}_1$ and $\tilde{A}_0$ are expressed as using coefficients of f whose degrees are 7 or less.

Summary : versality of a central projection

Concerning the versality of singularity of π_y, we get the following results.

type	$\mathcal{A}_e$-cod.	$\mathcal{A}$-det.	versality
swallowtail	1	4	always
butterfly	2	7	the flecnodal curve is not singular
elder butterfly	3	7	the flecnodal curve is not singular and y is not in a special position
unimodal	3	8	not versal
lips	1	3	always
beaks	1	3	always
goose	2	4	f is not flat umbilic
ugly goose	3	5	
gulls	2	5	f is the first order blue ridge
ugly gulls	3	7	
$(x_1, x_1x_2^2 + x_2^3 + x_2^5 + x_2^6)$	3	6	not versal
$(x_1, x_2^4 + x_1^2x_2)$	3	5	not versal

References

[1] C. T. C. Wall, Geometric properties of generic differentiable manifolds, Lecture Notes in Mathematics 597 (Springer, Berlin, 1977), 707-774.

[2] O. A. Platnova, Projections of smooth surfaces, Journal of Soviet Mathematics, Volume 35 (1986), 2796–2808.

[3] K. Saji, Criteria for singularities of smooth maps from the plane into the plane and their applications, Hiroshima Math. J. Volume 40, Number 2 (2010), 229-239.

[4] Yutaro Kabata, Recognition of plane-to-plane map-germs, Topology and its Applications 202 (2016) 216–238.

[5] H. Sano, Y. Kabata, J. L. Deolindo Silva and T. Ohmoto, Classification of jets of surfaces in projective 3-space via central projection, Bulletin of the Brazilian Mathematical Society, New Series 48 (2017), 623–639.

[6] S. Izumiya, M. C. R. Fuster, M. A. S. Ruas, F. Tari, Differential geometry from a singularity theory viewpoint, J. Geom. Symmetry Phys., Volume 41 (2016), 101-103.

[7] GeoGebra, Central projection of sphere, `https://www.geogebra.org/m/bUCVFJ64`

Appendix (definition of $\mathcal{A}$-determinacy)

Definition (finitely $\mathcal{A}$-determined)

A germ f is said to be k-$\mathcal{A}$-determined if any g with $j^k g = j^k f$ is $\mathcal{A}$-equivalent to f.The least integer k with this property is called the degree of determinacy of f. A finitely $\mathcal{A}$-determined germ is a k-$\mathcal{A}$-determined germ for some integer k.

Appendix (definition of contact)

Definition

Let $\alpha(t) := (x_1(t), x_2(t))$ be a regular plane curve and let β another plane curve given as the zero set of a smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$. We say that the curve α has $(k+1)$-**point contact** (k-th contact) at t_0 with the curve β if t_0 is a zero of order k of the function $g(t) = \Phi(\alpha(t)) = \Phi(x_1(t), x_2(t))$, that is,

$$g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0 \text{ and } g^{(k+1)}(t_0) \neq 0$$

where $g^{(i)}$ denotes the i^{th}-derivative of the function g.

The curve α has $(k+1)$-point contact (k-th order contact) at t_0 with β if and only if the function g has an A_k-singularity $x_1^{k+1} \pm x_2^2$ at t_0.

Appendix (*ug*-focal)

A_6-contact with $C_{y,d,\theta}$
$\Leftrightarrow$ A_5-contact with $C_{y,d,\theta}$ and y is not *ug*-focal
$\Leftrightarrow$ $AC_4 = 0$ and

$$AC_6 - 70\, a_{40}\, AC_5 = \tilde{A}_2 p_1^2 + \tilde{A}_1 p_1 + \tilde{A}_0 \neq 0.$$

where
$\tilde{A}_2 =$
$225 a_{21}{}^3\, a_{40}{}^2\, a_{70} + (1575 a_{21}{}^2\, a_{31}\, a_{40}{}^2 - 945 a_{21}{}^3\, a_{40}\, a_{50})\, a_{60} - 1575 a_{21}{}^2\, a_{40}{}^3\, a_{51} + 756 a_{21}{}^3\, a_{50}{}^3 - 3150 a_{21}{}^2\, a_{31}\, a_{40}\, a_{50}{}^2 + (3150 a_{21}{}^2\, a_{40}{}^2\, a_{41} - 1575 a_{21}\, a_{22}\, a_{40}{}^3 + 4200 a_{21}\, a_{31}{}^2\, a_{40}{}^2) a_{50} - 5250 a_{21}\, a_{31}\, a_{40}{}^3\, a_{41} - 875 a_{13}\, a_{40}{}^5 + (2625 a_{21}\, a_{32} + 2625 a_{22}\, a_{31})\, a_{40}{}^4 - 1750 a_{31}{}^3\, a_{40}{}^3$,
$\tilde{A}_1 = -70 a_{40}\, (45 a_{21}{}^3\, a_{40}\, a_{60} - 81 a_{21}{}^3\, a_{50}{}^2 - 45 a_{12}\, a_{21}\, a_{40}{}^2\, a_{50} + 315 a_{21}{}^2\, a_{31}\, a_{40}\, a_{50} - 225 a_{21}{}^2\, a_{40}{}^2\, a_{41} - 25 a_{03}\, a_{40}{}^4 + 75 a_{12}\, a_{31}\, a_{40}{}^3 + 225 a_{21}\, a_{22}\, a_{40}{}^3 - 300 a_{21}\, a_{31}{}^2\, a_{40}{}^2)$
and
$\tilde{A}_0 = 3150 a_{21}\, a_{40}{}^2\, (3 a_{21}{}^2\, a_{50} + 5 a_{12}\, a_{40}{}^2 - 10 a_{21}\, a_{31}\, a_{40})$.

Example (contact with cones at parabolic point)

Let $f(x) = (x_1, x_2, \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 x_2 + \frac{1}{24}x_1^4)$.