# Versality of the folding family

Atsuki Hiramatsu

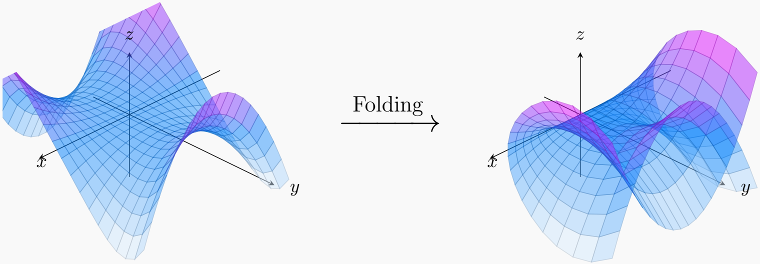
Extension of the Singularity theory

Saitama university

# 1. Introduction -What is the Folding Family?-

- $S \subset \mathbb{R}^3$ : A smooth surface defined by $z = f(x, y)$
- $F$ is "Folding map" with reflection plane $\Pi : y = 0$, that is,

$$F: \quad \begin{array}{ccccc} \mathbb{R}^2, \mathbf{0} & \xrightarrow{\text{graph}} & \mathbb{R}^3, \mathbf{0} & \xrightarrow{\text{folding}} & \mathbb{R}^3, \mathbf{0} \\ \cup\!\!\!\!\!\!\!\!\! & & \cup & & \cup \\ (x, y) & \longmapsto & (x, y, f(x, y)) & \longmapsto & (x, y^2, f(x, y)) \end{array}$$



# 1. Introduction -What is the Folding Family?-

- When we move $\Pi$ by Euclidean motion $A$, we obtain "Motion unfolding" of $F$:

$$\begin{array}{cccc} M: & \mathbb{R}^2 \times Euc/H & \to & \mathbb{R}^3 \\ & \Psi & & \Psi \\ & (p, A) & \longmapsto & A^{-1} \circ F \circ A(p) \end{array}$$

Here, $H$ is subgroup preserving $y = 0$.
This is the folding family due to Bruce and Wilkinson.

- Restricting to rotations, we obtain "Rotation unfolding" of $F$:

$$R : \mathbb{R}^2 \times S^2 \to \mathbb{R}^3.$$

# 1. Introduction -Infinitesimal Reflectional Symmetry-

- Bruce and Wilkinson are motivated by infinitesimal reflectional symmetry.

$\rightarrow$ "Infinitesimal" reflectional symmetry implies $f_o(x,y) = \frac{f(x,y)-f(x,-y)}{2}$ is closed to 0 near the origin.

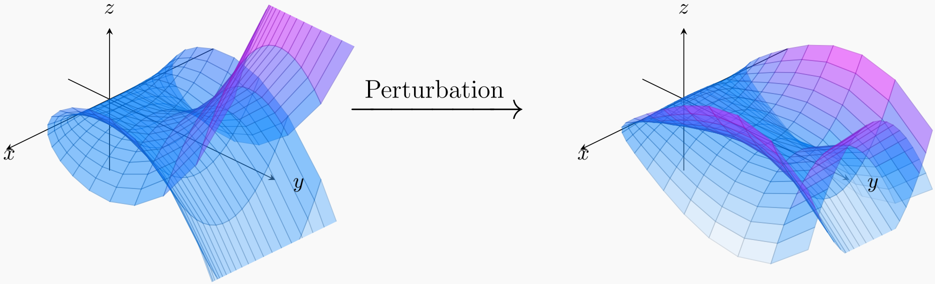
- Formulations are as follows:

(1) $f_o(x_0,y_0) = \frac{\partial}{\partial x} f_o(x_0,y_0) = \frac{\partial}{\partial y} f_o(x_0,y_0) = 0$

$\Rightarrow (x_0,y_0)$ is self-tangency point of folding map.

(2) If $F \overset{\mathcal{A}}{\sim} (x,y) \mapsto (x, y^2, y^5 - x^2 y)$, $B_2$-singularity, there is a perturbation of $F$ with a self-tangency point.

# 1. Introduction -Self Tangency-



# 1. Introduction -Bruce & Wilkinson's Paper (The folding family and focal )-

- For a residual set of embeddings $g : M \to \mathbb{R}^3$, the folding maps have $\mathcal{A}$-equivalent to the following.

| | $\mathscr{A}_e$ codimension | Name | C |
|---|---|---|---|
| $f(x,y) = (x,y,0)$ | 0 | Immersion | 0 |
| $f(x,y) = (x,y^2,xy)$ | 0 | Cross-cap | 1 |
| $f(x,y) = (x,y^2,x^2y \pm y^{2k+1})$, $1 \leq k \leq 3$ | k | $B_k^{\pm}$ | 2 |
| $f(x,y) = (x,y^2,y^3 \pm x^{k+1}y)$, $2 \leq k \leq 3$ | k | $S_k^{\pm}$ | k+1 |
| $f(x,y) = (x,y^2,xy^3 \pm x^k y)$, $k=3$ | k | $C_k^{\pm}$ | k |

Moreover these singularities are versally unfolded, by the family $F_g$.

# 1. Introduction -Bifurcation set due to Bruce & Wilkinson-

Since the family $F_g$ is an $\mathcal{A}$-versal unfolding of each of its singularities one can deduce local models for this dual set $\mathcal{B}(F_g)$. Ignoring multigerm singularities we have six such models, given in Figure 2. The bifurcation

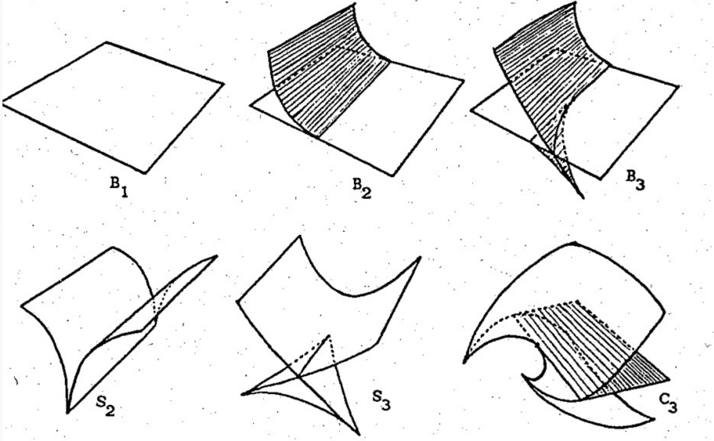
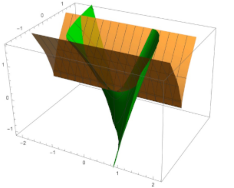
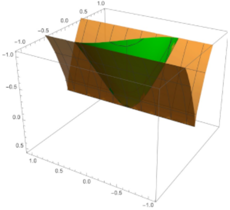



Fig. 2. Bifurcation Sets





# 1. Introduction -Geometry for the focal set due to Bruce & Wilkinson-

| | |
|---|---|
| $S_1 = B_1$ | general smooth point of focal set |
| $S_2$ | parabolic smooth point of focal set |
| $S_3$ | cusp of gauss at smooth point of focal set |
| $B_2$ | general cusp point of focal set |
| $B_3$ | (cusp) point of focal set in closure of parabolic curve on symmetry set |
| $C_3$ | intersection point of cuspidal edge and parabolic curve on focal set. |

| On the surface | $\longleftrightarrow$ | On the focal set |
|---|---|---|
| The "ridge" point | $\leftrightarrow$ | The cusp point |
| The "subparabolic" point | $\leftrightarrow$ | The parabolic point |

## Main Theorem for Rotation unfolding

We assume that

$$f(x,y) = \frac{1}{2}(k_1x^2 + k_2y^2) + \sum_{i+j\geq 3}^{m} \frac{1}{i!j!}a_{ij}x^iy^j + O(m+1)$$

where $m$ is an integer $\geq 3$.

(1) If $F \sim_{\mathcal{A}} S_1$, the rotation unfolding $R$ is always versal.
(2) If $F \sim_{\mathcal{A}} S_2$, $R$ is versal if and only if the origin is not umbilic.
(3) If $F \sim_{\mathcal{A}} B_2$, $R$ is versal if and only if the origin is not umbilic and ridge line transverse to the reflection plane $\Pi$: $y = 0$, or $D_4$ type umbilic.

## Main Theorem for Motion unfolding

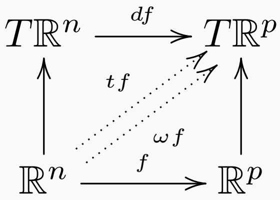
If the folding map $F$ is equivalent to the following singularities,
the motion unfolding M is versal.

| The sing. | The condition of versality for non-umbilic |
|---|---|
| $S_1$ | Always. |
| $S_2$ | Always |
| $S_3$ | The $v_2$-subparabolic line is non-singular. |
| $B_2$ | The $v_2$-ridge line is non-singular. |
| $B_3$ | 6-jet conditon. |
| $C_3$ | The $v_2$-ridge and $v_2$-subpara lines meet transversely. |

# 2. Preliminaries -Extended Tangent Space-

Let $g : \mathbb{R}^n, \mathbf{0} \to \mathbb{R}^p, \mathbf{0}$ be a $C^\infty$-germ.

- "$\theta_k$" is the set of germs of $C^\infty$-sections $\mathbb{R}^k, \mathbf{0} \to T\mathbb{R}^k$
- "$\theta(g)$" is the set of vector fields along $f$.
- "$tg$" : $\theta_n \mapsto \theta(f)$ is defined by $\xi \mapsto df \circ \xi$.
- "$\omega g$" : $\theta_p \mapsto \theta(f)$ is defined by $\eta \mapsto \eta \circ f$.
- "$T\mathcal{A}_e g$" $:= tg(\theta_n) + \omega g(\theta_p)$.



- An unfolding $G$ of $g$ is "versal" $\overset{\text{def}}{\Longleftrightarrow}$
  An unfolding which contains all other unfoldings of $g$ up to parameterized equivalence.

**Thm.2.1**

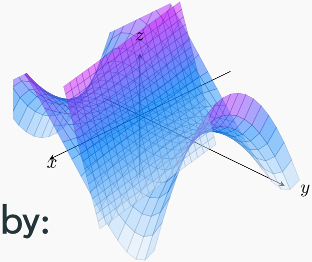
The unfolding $G$ of $g$ is versal if and only if $T\mathcal{A}_e g + V_G = \theta(g)$.
Here, $V_G := \langle \frac{\partial G}{\partial u_1}|_{\mathbb{R}^n \times \mathbf{0}}, \ldots, \frac{\partial F}{\partial u_r}|_{\mathbb{R}^n \times \mathbf{0}} \rangle_{\mathbb{R}} \quad (u_1, \ldots u_r \in \mathbb{R}^r)$.

# 2. Preliminaries -Explicit form of Rotation Unfolding-

- Let $\Pi_{\boldsymbol{v}}$ be a plane through $(0,0,0)$ with a normal vector $\boldsymbol{v} \in S^2$.
- $\boldsymbol{\nu} = (0,0,1)$ is a normal vector of the surface $M$ at $\mathbf{0}$.

We consider an orthonormal frame:

$$\boldsymbol{\nu} \times \boldsymbol{v},\ \boldsymbol{v},\ (\boldsymbol{v} \times \boldsymbol{\nu}) \times \boldsymbol{v}.$$




Then the folding map for $\boldsymbol{v}$-direction is given by:

$$s\boldsymbol{v} \times \boldsymbol{\nu} + t\boldsymbol{v} + r(\boldsymbol{v} \times \boldsymbol{\nu}) \times \boldsymbol{v} \longmapsto s\boldsymbol{v} \times \boldsymbol{\nu} + t^2\boldsymbol{v} + r(\boldsymbol{v} \times \boldsymbol{\nu}) \times \boldsymbol{v}.$$

# 3. Versality -The Rotation Unfolding in the case of $S_1$-

| | $b_{00}$ | $b_{10}$ | $b_{01}$ | $b_{11}$ | $b_{21}$ | $b_{03}$ | $c_{00}$ | $c_{10}$ | $c_{01}$ | $c_{20}$ | $c_{11}$ | $c_{02}$ | $q_1$ | $q_2$ | $q_3$ | $r_1$ | $r_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y\boldsymbol{e}_1$ | | | $\boxed{1}$ | | | | | | | | | | | | | $-1$ | |
| $xy\boldsymbol{e}_1$ | | | | $\boxed{1}$ | | | | | | | | | | | | | |
| $x^2y\boldsymbol{e}_1$ | | | | | $\boxed{1}$ | | | | | | | | $\frac{a_{21}}{2}$ | | | | |
| $y^3\boldsymbol{e}_1$ | | | | | | $\boxed{1}$ | | | | | | | $\frac{a_{03}}{6}$ | | | | |
| $y\boldsymbol{e}_2$ | | | | | | | $\boxed{2}$ | | | | | | | | | | |
| $xy\boldsymbol{e}_2$ | | | | | | | | $\boxed{2}$ | | | | | | | | $2$ | |
| $x^2y\boldsymbol{e}_2$ | | | | | | | | | | | $\boxed{2}$ | | | $\frac{a_{21}}{2}$ | | | $k_1$ |
| $y^3\boldsymbol{e}_2$ | | | | | | | | | | | | $2$ | | $\boxed{\frac{a_{03}}{6}}$ | | | $k_2$ |
| $y\boldsymbol{e}_3$ | | | | | | | $k_2$ | | | | | | | | | | $\boxed{-1}$ |
| $xy\boldsymbol{e}_3$ | $\boxed{a_{21}}$ | | $k_1$ | | | | $a_{12}$ | $k_2$ | | | | | | | | | |
| $x^2y\boldsymbol{e}_3$ | $\frac{a_{31}}{2}$ | $\boxed{a_{21}}$ | $\frac{a_{30}}{2}$ | $k_1$ | | | $\frac{a_{22}}{2}$ | $a_{12}$ | $\frac{a_{21}}{2}$ | $k_2$ | | | | | $\frac{a_{21}}{2}$ | | |
| $y^3\boldsymbol{e}_3$ | $\frac{a_{13}}{6}$ | | $\frac{a_{12}}{2}$ | | | | $\frac{a_{04}}{6}$ | | $\boxed{\frac{a_{03}}{6}}$ | | | $k_2$ | | | $\frac{a_{03}}{2}$ | | |

# 3. Versality -The Rotation Unfolding-

| Singularity | The condition of versality |
|---|---|
| $S_1$ | Always. |
| $S_2$ | $k_1 - k_2 \neq 0$. $\Leftrightarrow$ the umbilic. |
| $B_2$ | $3a_{21}a_{12} - a_{13}(k_1 - k_2) \neq 0$ |
| | $\Leftarrow$ the $v_2$-ridge line is transverse to $\Pi$. |

The ridge line is expressed by:

$$0 = a_{03} + \frac{1}{k_2 - k_1}\{3a_{21}a_{12} + a_{13}(k_2 - k_1)\}u + \frac{1}{k_2 - k_1}\{3a_{12}^2 + (a_{04} - 3k_2^3)(k_2 - k_1)\}v + O(2)$$

# 3. Versality -The Motion Unfolding-

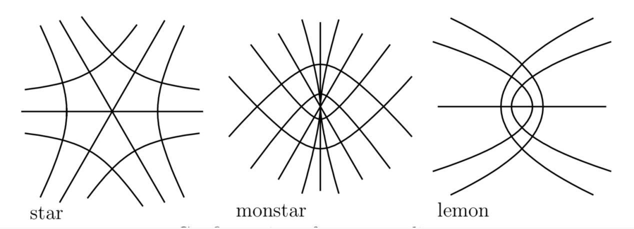
| Singularity | The condition for non umbilic |
|---|---|
| $S_1$ | Always. |
| $S_2$ | Always. |
| $S_3$ | The $v_2$-subparabolic line is non-singular. |
| $B_2$ | The $v_2$-ridge line is non-singular. |
| $B_3$ | 6-jet conditon. |
| $C_3$ | The $v_2$-ridge and subpara line meet transversely. |

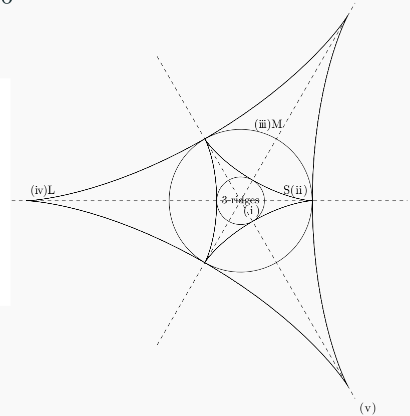
$B_3$ condition: full-rank of the below matrix $\mathcal{B}_3$:

$$\begin{pmatrix} a_{12} + \frac{a_{13}(k_2 - k_1)}{3a_{21}} & \frac{a_{14}}{2} + \frac{a_{15}}{10a_{21}}(k2 - k1) + \frac{a_{13}}{3a_{21}}(a_{04} - 3a_{22} + \frac{a_{23}(k1 - k2)}{a_{21}}) + \frac{a_{13}^2}{6a_{21}^2}(a_{30} - 2a_{12} + \frac{a_{31}}{a_{21}}(k_2 - k_1)) \\ a_{04} - 3k_2^3 - \frac{a_{12}a_{13}}{a_{21}} & \frac{3a06}{10} - \frac{9a_{04}k_2^2}{2} - \frac{3a_{31}a_{13}^2}{a_{21}^2} + \frac{a_{13}}{a_{21}}(-a_{14} + 6a_{12}k_2^2 + \frac{a_{12}a_{23}}{a_{21}} + \frac{a_{13}^2}{a_{21}^2}(a_{22} - k_1k_2^2 - \frac{a_{12}a_{31}}{a_{21}})) \end{pmatrix}$$

# 4. Umbilic -Geometry of the umbilic-

We express $M$ using complex coordinates $z = x + iy$,
$f(z) = \frac{k}{2} z\bar{z} + c(z) + O(4)$, where $c(z) = \frac{1}{6}\mathrm{Re}(\alpha z^3 + 3\beta z^2 \overline{z})$.



# 4. Umbilic -Versality for Rotation Unfolding-

- We chose $w \in \mathbb{C}$ s.t. $|w| = 1$, and that

$$c(wz) = \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3).$$

Then,
$$a_{30} = c(wz)\Big|_{z=1}, \quad a_{21} = \frac{\partial c(wz)}{\partial y}\Big|_{z=1},$$
$$a_{12} = \frac{\partial c(wz)}{\partial x}\Big|_{z=i}, \quad a_{03} = c(wz)\Big|_{z=i}.$$

- If $\sim_{\mathcal{A}} B_2$, i.e., $a_{21} \neq 0, a_{03} = 0$, then $R$ is versal

$$\Leftrightarrow 3a_{21}a_{12} - a_{13}(k_1 - k_2) \neq 0 \Leftrightarrow a_{12} \neq 0$$

# 4. Umbilic -Versality for Motion Unfolding-

| Sing. | Condition for sing. | Condition for versality |
|---|---|---|
| $S_1$ | $a_{21} \neq 0,\ a_{03} \neq 0$ | always |
| $S_2$ | $a_{21} = 0,\ a_{03} \neq 0,\ a_{31} \neq 0$ | $a_{12} \neq 0$ |
| $S_3$ | $a_{21} = 0,\ a_{03} \neq 0,\ a_{31} = 0,\ a_{41} = 0$ | $a_{12}(2a_{12} - a_{30}) \neq 0$ |
| $B_2$ | $a_{21} \neq 0,\ a_{03} = 0, B_2 \neq 0$ | $a_{12} \neq 0$ or $a_{13} \neq 0$ |
| $B_3$ | $a_{21} \neq 0,\ a_{03} = 0, B_2 = 0, B_3 \neq 0$ | $\mathcal{B}_3 \neq 0$. |
| $C_3$ | $a_{21} = 0,\ a_{03} = 0,\ a_{13} \neq 0,\ a_{13} \neq 0$ | $\mathcal{C} \neq 0$ |

**Here,** $B_2 = \frac{a_{05}}{5} - \frac{a_{13}^2}{3a_{21}}, B_3 = \frac{a_{07}}{7} - \frac{a_{15}a_{13}}{a_{21}} + \frac{5a_{23}a_{13}^2}{a_{21}^2} - \frac{5a_{31}a_{13}^3}{a_{21}^3}$.

$\mathcal{C} = a_{12}(3a_{31}a_{12} + a_{13}(2a_{12} - a_{30}))$

# Thank you for listening.

arXiv QR code

# Reference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