Singularities of mixed polynomials with Newton polyhedra

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15:00–15:30, 18 September 2023

WORKSHOP on
Algebraic and Analytic Singularities
18–22 September 2023,
Warsaw University of Technology

Mixed polynomial

A Mixed polynomial is a linear combination

$$f = \sum_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} \mathbf{x}^{\boldsymbol{\nu}} \bar{\mathbf{x}}^{\bar{\boldsymbol{\nu}}}, \quad c_{\boldsymbol{\nu},\bar{\boldsymbol{\nu}}} \in \mathbb{C}$$

of mixed monomials

$$\mathbf{x}^{\boldsymbol{\nu}} \bar{\mathbf{x}}^{\bar{\boldsymbol{\nu}}} = x_1^{\nu_1} \cdots x_n^{\nu_n} \overline{x_1}^{\bar{\nu}_1} \cdots \overline{x_n}^{\bar{\nu}_n}$$

where

$$\mathbf{x} = (x_1, \ldots, x_n), \quad \bar{\mathbf{x}} = (\overline{x_1}, \ldots, \overline{x_n})$$

($\overline{x_i}$ is the complex conjugate of x_i) and

$$\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}_{\geq}^n, \quad \bar{\boldsymbol{\nu}} = (\bar{\nu}_1, \ldots, \bar{\nu}_n) \in \mathbb{Z}_{\geq}^n.$$

Maps defined by mixed polynomial

A mixed polynomial $f = \sum_{\nu,\bar{\nu}} c_{\nu,\bar{\nu}} x^{\nu} \bar{x}^{\bar{\nu}}$ defines a map

$$\mathbb{C}^n \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{\nu,\bar{\nu}} c_{\nu,\bar{\nu}} x^{\nu} \bar{x}^{\bar{\nu}}$$

which we also denote by f.

Pichon-Seade 2008: $f\bar{g}$ may admit Milnor fibration
M. Oka 2010-: topology of singularities of mixed polynomials using toric modifications

Motivation of today's talk

For polynomial-germ,
we consider Newton polyhedrons
and construct a toric modification using fan,
this provides a resolution of singularity
when the polynomial is non-degenerate with respect to its Newton polyhedron.

What is the mixed counter part of this theory?
Today we seek a mixed analogy of this theory.

Today's story (conclusion)

In a nutshell, for a mixed polynomial,
we consider a mixed Newton polyhedron
and then construct a mixed toric modification,
using mixed fan
it provides a mixed analogue of a resolution of singularity
under mixed Newton non-degeneracy condition.

Any real polynomial can be expressed as a mixed polynomial, since

$$\operatorname{Re} x_i = \frac{x_i + \overline{x_i}}{2}, \quad \operatorname{Im} x_i = \frac{x_i + \overline{x_i}}{2\mathbb{i}}$$

where $\mathbb{i}$ is the imaginary unit.

A pure polynomial is a linear combination

$$\sum_{\nu} c_{\nu} \mathbf{x}^{\nu}, \quad c_{\nu,\bar{\nu}} \in \mathbb{C},$$

of pure monomials

$$\mathbf{x}^{\nu} = x_1^{\nu_1} \cdots x_n^{\nu_n}$$

For a mixed polynomial $f = \sum_{\nu,\bar{\nu}} c_{\nu,\bar{\nu}} x^{\nu}\bar{x}^{\bar{\nu}}$ we define

$$\Gamma_+(f) = \mathrm{co}\{\boldsymbol{\nu} + \bar{\boldsymbol{\nu}} + \mathbb{R}^n_{\geq} : c_{\nu,\bar{\nu}} \neq 0\}.$$

For $a \in \mathbb{R}^n$, we define

$$\ell(a) = \min\{\langle a, \boldsymbol{\nu}\rangle : \boldsymbol{\nu} \in \Gamma_+(f)\},$$
$$\gamma(a) = \{\boldsymbol{\nu} \in \Gamma_+(f) : \langle a, \boldsymbol{\nu}\rangle = \ell(a)\}.$$

If f is a pure polynomial, that is, $\bar{\boldsymbol{\nu}} = 0$ for all $\bar{\boldsymbol{\nu}}$,
these are usual data for a Newton diagram.
If f is a mixed polynomial, $\Gamma_+(f)$ is

the absolute Newton polyhedrons of f.
(called by radial Newton polyhedrons by M.Oka)

We consider the dual Newton diagram

$$\Gamma^*(f) = \{\overline{[a]} : a \in \mathbb{R}^n_{\geq}\}$$

where $[a]$ is the equivalence class of a by the equivalence relation defined by

$$a \sim b \iff \gamma(a) = \gamma(b),$$

and, we take
a nonsingular fan Σ, which is a subdivisuon of $\Gamma^*(f)$.
Then we have a toric modification

$$\pi_\Sigma : M_\Sigma \longrightarrow \mathbb{C}^n.$$

Fan

Let Σ denote a fan, that is, a finite collection of rational polyhedral cones in $\mathbb{R}^n$ with the following properties.

- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of σ.

We assume that Σ is nonsingular, that is, each $\sigma \in \Sigma$ is generated by a part of $\mathbb{Z}$-basis of $\mathbb{Z}^n$.
For a fan Σ,

$$\Sigma(k) := \{\sigma \in \Sigma : \dim \sigma = k\}$$

For a cone σ,

$$\sigma(k) := \{\tau : k\text{-dim. face of } \sigma\}$$

Let $a^\tau = \begin{pmatrix} a_1^\tau \\ \vdots \\ a_n^\tau \end{pmatrix}$ be a primitive integral vector generating 1-cone $\tau \in \Sigma(1)$.

Let G be the kernel of the group morphism

$$\pi^* : (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow (\mathbb{C}^*)^n, \quad (z_\tau)_{\tau \in \Sigma(1)} \longmapsto \Big(\prod_{\tau \in \Sigma(1)} {z_\tau}^{a_i^\tau} \Big)_{i=1,\dots,n}.$$

For $\sigma \in \Sigma(n)$, we set

$$U_\sigma = \{(u_\tau)_{tau \in \Sigma(1)} \in \mathbb{C}^{\Sigma(1)} : \prod_{\tau' \notin \sigma(1)} u_{\tau'} \neq 0\}$$

An element $(z_\tau)_{\tau \in \Sigma(1)} \in G$ acts on U_σ, as

$$U_\sigma \longrightarrow U_\sigma, \quad (u_\tau)_{\tau \Sigma(1)} \mapsto (z_\tau u_\tau)_{\tau \Sigma(1)},$$

and thus on

$$U_\Sigma = \bigcup_{\sigma \in \Sigma(n)} U_\sigma.$$

Def of $M_\Sigma \to \mathbb{C}^n$

We define

$$M_\Sigma = U_\Sigma/G = \bigcup_{\sigma\in\Sigma(n)} M_\sigma, \quad M_\sigma = U_\sigma/G.$$

For $\sigma \in \Sigma(n)$, setting

$$V_\sigma = \{(u_\tau)_\tau \in U_\sigma : u_{\tau'} = 1, \tau' \notin \sigma(1)\},$$

we have an isomorphism:

$$\mathbb{C}^{\sigma(1)} = V_\sigma \longrightarrow M_\sigma$$

We have the following commutative diagram:

$$\begin{array}{ccc} U_\Sigma \supset (\mathbb{C}^*)^{\Sigma(1)} & \xrightarrow{\pi^*} & (\mathbb{C}^*)^n \\ \big\downarrow /G & & \cap \\ M_\Sigma & \xrightarrow{\pi} & \mathbb{C}^n \end{array}$$

Resolution for pure poly.

For pure poly. $f = \sum_\nu c_\nu x^\nu$, set

$$f_\gamma = \sum_{\nu \in \gamma} c_\nu x^\nu .$$

Remark that f_γ is a weighted homogeneous when γ is a face of $\Gamma_+(f)$.
If f is non-degenerate, that is,

$$\Sigma(f_\gamma) \subset \{x_1 \cdots x_n = 0\}$$

for any compact face γ of $\Gamma_+(f)$, then

$$\pi : M_\Sigma \longrightarrow \mathbb{C}^n \text{ provides a resolution of } f.$$

Mixed weighted homogeneous polynomial

We say that a mixed polynomial f is

a mixed weighted homogeneous polynomial
of weight (a, b) with degree (ℓ, m),
if f is a $\mathbb{C}$-linear combination of $x^{\nu}\bar{x}^{\bar{\nu}}$ with

$$\langle \boldsymbol{\nu} + \bar{\boldsymbol{\nu}}, \boldsymbol{a} \rangle = \ell, \ \langle \boldsymbol{\nu} - \bar{\boldsymbol{\nu}}, \boldsymbol{b} \rangle = m. \tag{1}$$

Property of mixed w.h.poly.

If f is mixed w.h, we have

$$m \neq 0 \Longrightarrow \Sigma(f) \subset f^{-1}(0),$$

applying Cauchy-Binet formula for

$$\begin{pmatrix} x_1\partial_{x_1} f & \overline{x_1}\partial_{\overline{x_1}} f & \cdots & x_n\partial_{x_n} f & \overline{x_n}\partial_{\overline{x_n}} f \\ x_1\partial_{x_1} \bar{f} & \overline{x_1}\partial_{\overline{x_1}} \bar{f} & \cdots & x_n\partial_{x_n} \bar{f} & \overline{x_n}\partial_{\overline{x_n}} \bar{f} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_1 & -b_1 \\ \vdots & \vdots \\ a_n & b_n \\ a_n & -b_n \end{pmatrix} = \begin{pmatrix} \ell f & mf \\ \ell \bar{f} & -m\bar{f} \end{pmatrix}.$$

Newton diagram for mixed poly.

If f is a mixed polynomial, we have more "Newton diagram" (i.e., mixed Newton diagram).
For $\gamma = \gamma(a)$, we set

$$\mathrm{LE}_\gamma(f) = \mathrm{co}\{\boldsymbol{\nu} - \bar{\boldsymbol{\nu}} : \boldsymbol{\nu} + \bar{\boldsymbol{\nu}} \in \gamma, c_{\nu,\bar{\nu}} \neq 0\}$$

We consider

$$\Gamma_+(f), \mathrm{LE}_\gamma(f) \ (\gamma : \text{compact faces of } \Gamma_+(f))$$

as a counter part of Newton polyhedra of f.

Mixed Newton non-deg

We say that a mixed polynomial f is mixed Newton non-degenerate if the following conditions hold.

(a) For each compact face γ of $\Gamma_+(f)$, we have

$$\Sigma(f_\gamma) \cap f_\gamma^{-1}(0) \subset \{x_1 \cdots x_n = 0\} \tag{2}$$

where $f_\gamma = \sum_{\nu+\bar{\nu}\in\gamma} c_{\nu,\bar{\nu}} x^\nu \bar{x}^{\bar{\nu}}$.

(b) The polynomial f_γ is mixed w.h. for each compact face γ of $\Gamma_+(f)$.

As Oka, we say that a mixed polynomial f is non-degenerate if the condition (a) above holds.

Mixed fan

Let Σ be a simplicial fan (i.e., each $\sigma \in \Sigma$ is simplicial), and let β be a map

$$\Sigma(1) \longrightarrow \mathbb{Z}^n \times \mathbb{Z}^n, \quad \tau \longmapsto (\boldsymbol{a}^\tau, \boldsymbol{b}^\tau).$$

We say that (Σ, β) is a mixed fan, if the following conditions hold.

(i) $\{\boldsymbol{a}^\tau : \tau \in \Sigma(1)\}$: a^τ a primitive generator of $\tau \in \Sigma(1)$.

(ii) $\{\boldsymbol{b}^\tau : \tau \in \sigma(1)\}$ forms a part of $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for all $\sigma \in \Sigma$.

If $\boldsymbol{a}^\tau = \boldsymbol{b}^\tau$ for $\tau \in \Sigma(1)$, the mixed fan (Σ, β) is a nonsingular fan mentioning primitive generators $\boldsymbol{a}^\tau$ for $\tau \in \Sigma(1)$.

Let us define the group G as the kernel of the group morphism:

$$\begin{aligned} \pi^*_{\Sigma,\beta} : (\mathbb{C}^*)^{\Sigma(1)} &\longrightarrow (\mathbb{C}^*)^n, \\ (z_\tau)_{\tau\in\Sigma(1)} &\longmapsto \left(\prod_{\tau\in\Sigma(1)} |z_\tau|^{a_i^\tau} \left(\frac{z_\tau}{|z_\tau|}\right)^{b_i^\tau} \right)_{i=1,\dots,n}. \end{aligned}$$

We clearly have

$$(z_\tau)_{\tau\in\Sigma(1)} \in G \iff \begin{cases} \sum_{\tau\in\Sigma(1)} a_i^\tau \log|z_\tau| = 0, \\ \sum_{\tau\in\Sigma(1)} b_i^\tau \arg z_\tau \equiv 0 \pmod{2\pi}. \end{cases}$$

For $\sigma \in \Sigma(n)$, these equations can be written as

$$\begin{aligned} \sum_{\tau\in\sigma(1)} a_i^\tau \log|z_\tau| &= -\sum_{\tau'\notin\sigma(1)} a_i^{\tau'} \log|z_{\tau'}|, \\ \sum_{\tau\in\sigma(1)} b_i^\tau \arg z_\tau &\equiv -\sum_{\tau'\notin\sigma(1)} b_i^{\tau'} \arg z_{\tau'} \pmod{2\pi}, \end{aligned}$$

and we conclude that an element $(z_\tau)_{\tau\in\sigma(1)} \in G$ is determined by $(z_{\tau'})_{\tau'\notin\sigma(1)}$.

We assume that (Σ, β) is a mixed fan. We define U_Σ by

$$U_\Sigma = \bigcup_{\sigma \in \Sigma(n)} U_\sigma, \;\; U_\sigma = \left\{ (u_\tau)_{\tau \in \Sigma(1)} \in \mathbb{C}^{\Sigma(1)} : \prod_{\tau \notin \sigma(1)} u_\tau \neq 0 \right\}.$$

We remark that $(z_\tau)_{\tau \in \Sigma(1)} \in G$ acts on U_σ by

$$(z_\tau)_{\tau \in \Sigma(1)} : U_\sigma \longrightarrow U_\sigma, \;\; (u_\tau)_{\tau \in \sigma(1)} \longmapsto (z_\tau u_\tau)_{\tau \in \sigma(1)},$$

and thus on U_Σ.
We define the mixed toric manifold $M_{\Sigma,\beta}$ by

$$M_{\Sigma,\beta} = U_\Sigma / G = \bigcup_{\sigma \in \Sigma(n)} M_\sigma, \text{ and } M_\sigma = U_\sigma / G.$$

Chart for $M_{\Sigma,\beta}$

Set

$$V_\sigma = \{(u_\tau)_{\tau\in\Sigma(1)} : u_{\tau'} = 1,\ \tau' \notin \sigma(1)\}.$$

We conclude that the composition

$$V_\sigma \subset U_\sigma \longrightarrow U_\sigma/G = M_\sigma$$

is a semi-algebraic homeomorphism. We consider this map as a semi-algebraic chart of a mixed toric manifold $M_{\Sigma,\beta}$, identifying V_σ with $\mathbb{C}^{\sigma(1)}$.

Algbraicity of $M_{\Sigma,\beta}$

Prop.
Let (Σ, β) denote a mixed fan. Then $M_{\Sigma,\beta}$ is a real algebraic manifold if

$$\boldsymbol{a}^{\tau} \equiv \boldsymbol{b}^{\tau} \mod 2 \quad \text{for} \quad \tau \in \Sigma(1).$$

Mixed toric modification (I)

We assume that

(i) (Σ, β) is a mixed fan. Set $\beta(\tau) = (a^\tau, b^\tau)$ for $\tau \in \Sigma(1)$.

(ii) A fan Σ is a subdivision of $\mathbb{R}^n_{\geq}$. In particular, each a^τ_j, $\tau \in \Sigma(1)$, $j = 1, \ldots, n$, is non-negative.

(iii) For any $\tau \in \Sigma(1)$ and $j = 1, \ldots, n$, $a^\tau_j = 0$ implies $b^\tau_j = 0$.

Then the map $\pi^*_{\Sigma,\beta}$ extends to the map

$$\tilde{\pi}_{\Sigma,\beta} : U_\Sigma \longrightarrow \mathbb{C}^n,$$

$$(u_\tau)_{\tau\in\Sigma(1)} \longmapsto \left(\prod_{\tau\in\Sigma(1)} |u_\tau|^{a^\tau_i} \left(\frac{u_\tau}{|u_\tau|} \right)^{b^\tau_i} \right)_{i=1,\ldots,n}.$$

Mixed toric modification (2)

Since this map is G-invariant, $\tilde{\pi}_{\Sigma,\beta}$ induces the natural map

$$\pi = \pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n, \tag{3}$$

which we call the mixed toric modification defined by the mixed fan (Σ, β).

$$\begin{array}{ccc} U_\Sigma \supset (\mathbb{C}^*)^{\Sigma(1)} & \xrightarrow{\pi^*} & (\mathbb{C}^*)^n \\ \big\downarrow /G & & \cap \\ M_{\Sigma,\beta} & \xrightarrow{\pi} & \mathbb{C}^n \end{array}$$

Thm: The map $\pi = \pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n$ is proper.

We consider the Newton polyhedron $\Gamma_+(f)$.
For $a \in \mathbb{R}^n$, set

$$\ell(a) = \min\{\langle a, \nu\rangle : \nu \in \Gamma_+(f)\}, \text{ and}$$
$$\gamma(a) = \{\nu \in \Gamma_+(f) : \langle a, \nu\rangle = \ell(a)\}.$$

Define $\mathrm{LE}_\gamma(f)$ by

$$\mathrm{LE}_\gamma(f) = \mathrm{co}\{\nu - \bar{\nu} : c_{\nu,\bar{\nu}} \neq 0, \ \nu + \bar{\nu} \in \gamma\}. \tag{4}$$

Let a^τ denote the primitive vector which generates τ for $\tau \in \Sigma_0(1)$. Set

$$m(b^\tau) = \min\{\langle b^\tau, \nu\rangle : \nu \in \mathrm{LE}_{\gamma(a^\tau)}(f)\}.$$

We can assume that $m(b^\tau) \geq 0$, $\tau \in \Sigma(1)$, changing the sign of b^τ, if necessary.

Mixed analogy of normal crossing

We say a subset Z of $\mathbb{C}^n$ is of semi-algebraically normal crossing at $z \in Z$
if there is a semi-algebraic coordinate system (U, φ), U an open neighborhood of z, and a semi-algebraic homeomorphism

$$\varphi : U \longrightarrow \varphi(U) \subset \mathbb{C}^n \text{ centred at } z,$$

so that $Z \cap U$ is the inverse image of zero of a pure monomial by φ.

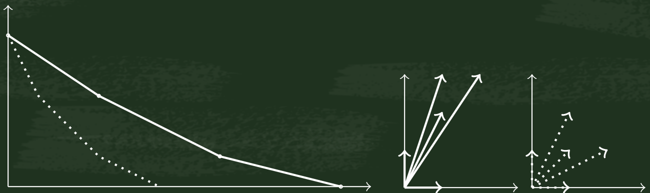
Theorem

Let f be a mixed polynomial,
which is mixed Newton non-degenerate,
and let (Σ, β) denote a mixed fan as above.
Then, for a mixed toric modification

$$\pi_{\Sigma,\beta} : M_{\Sigma,\beta} \longrightarrow \mathbb{C}^n,$$

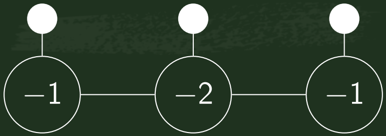
the subset $(f \circ \pi_{\Sigma,\beta})^{-1}(0)$ in V_σ is of semi-algebraically normal crossing near $\pi^{-1}(0)$.

Example $f = x_2^5 + x_1^2\overline{x_1}x_2^3 + x_1^5\overline{x_1}^2x_2 + x_1^8\overline{x_1}^3$.

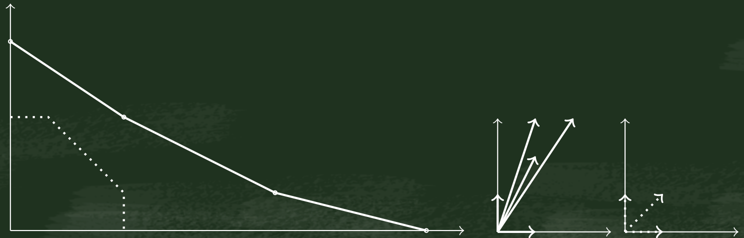


$$(\boldsymbol{a}^\tau) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 2 & 4 & 1 \end{pmatrix},\ (\boldsymbol{b}^\tau) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix},$$

$\ell = (0\ 15\ 9\ 11\ 0)$, $m = (0\ 5\ 4\ 5\ 0)$

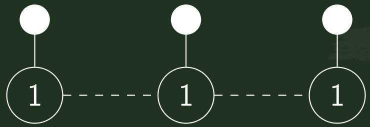


Example $f = x_2^3\overline{x_2} + x_1^2\overline{x_1}x_2^3 + x_1^5\overline{x_1}^2x_2 + x_1^7\overline{x_1}^4$.



$$(\boldsymbol{a}^{\tau}) = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 2 & 4 & 1 \end{pmatrix}, \; (\boldsymbol{b}^{\tau}) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$\ell = (0\ 15\ 9\ 11\ 0)$, $m = (0\ 3\ 4\ 3\ 0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, -1, -1, 1)$. Thus we obtain the dual graph



since $\boldsymbol{b}^1 = -\boldsymbol{b}^0 + \boldsymbol{b}^2$, $\boldsymbol{b}^2 = \boldsymbol{b}^1 + \boldsymbol{b}^3$, $\boldsymbol{b}^3 = \boldsymbol{b}^2 - \boldsymbol{b}^4$.

Thank you very much for
your attention!
Dziękujemy Bardzo za uwagę!

Congratulation on 70th
Birthday, Stanislaw!
Wish good health
and good maths!