Pseudospheres from singularity theory view point with a classification of 2-soliton surfaces
(j/w with Yutaro Kabata)

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Surfaces in $\mathbb{R}^3$

$\varphi : \mathbb{R}^2 \longrightarrow M = \varphi(\mathbb{R}^2) \subset \mathbb{R}^3$, C^∞

$$E = \langle \varphi_u, \varphi_u \rangle,\ \ F = \langle \varphi_u, \varphi_v \rangle,\ \ G = \langle \varphi_v, \varphi_v \rangle$$

$$L = \langle \varphi_{uu}, \boldsymbol{\nu} \rangle,\ \ M = \langle \varphi_{uv}, \boldsymbol{\nu} \rangle,\ \ N = \langle \varphi_{vv}, \boldsymbol{\nu} \rangle$$

where ν is a unit normal.
The first fundamental form

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

The second fundamental form

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

Chebyshev' net

A pseudosphere is a surface with constant negative Gauss curvatures. We can assume that they have Gauss curvature -1 up to similarity transformations.
For a surface with $K = -1$, we can take the asymptotic coordinate (u, v) with the following fundamental forms:

$$\begin{aligned} I &= du^2 + 2\cos\phi \, du \, dv + dv^2 \\ II &= 2\sin\phi \, du \, dv \end{aligned}$$

where ϕ is the asymptotic angle.
Gauss-Coddazi equation becomes sine-Gordon equation:

$$\phi_{uv} = \sin\phi$$

Curvature coordinate

The curvature coordinate is given by

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

The fundamental forms are

$$I = \cos^2 \tfrac{\phi}{2}\, dx^2 + \sin^2 \frac{\phi}{2}\, dy^2$$

$$II = \frac{1}{2} \sin\phi \,(dx^2 - dy^2)$$

The principal curvatures are

$$\tan\frac{\phi}{2}, \text{ and } -\cot\frac{\phi}{2}$$

Ridge and flecnodal

Let v_i denote a principal vector of a surface and let κ_i denote the corresponding principal curvature of a surface.

A point P on a surface is v_i-ridge if $v_i\kappa_i(P) = 0$.

A point P on a surface is flecnodal if there is a line with at least 4 point contact with the surface at P.

1. The level sets of ϕ, κ_1 and κ_2 containing P are equal.
2. The differentials of the principal curvatures are given as follows:

$$\partial_x\kappa_1 = \frac{\phi_x}{1+\cos\phi}, \qquad \partial_y\kappa_1 = \frac{\phi_y}{1+\cos\phi},$$
$$\partial_x\kappa_2 = \frac{\phi_x}{-1+\cos\phi}, \qquad \partial_y\kappa_2 = \frac{\phi_y}{-1+\cos\phi}.$$

So ∂_x-ridge (resp. ∂_y-ridge) is given by $\phi_x = 0$ (resp. $\phi_y = 0$). (A level of ϕ has a horizontal (or vertical) tangent.)

Flecnodal point on pseudosphere is given by $\phi_u\phi_v = 0$. (i.e., A level of ϕ has a diagonal (or anti-diagonal) tangent.)

Backlünd transformation

We say $\tilde{\phi}$ is Backlünd transformation of ϕ if

$$\left(\frac{\tilde{\phi}+\phi}{2}\right)_u = \lambda \sin\frac{\tilde{\phi}-\phi}{2}, \quad \left(\frac{\tilde{\phi}-\phi}{2}\right)_v = \lambda^{-1} \sin\frac{\tilde{\phi}+\phi}{2}. \quad (1)$$

where $\lambda = \tan\theta/2$. θ is in the next sheet.
If ϕ is a solution of sine-Gordon equation, so is $\tilde{\phi}$.

$$\{\text{sol. of sine-Gordon}\} \xrightarrow{BT} \{\text{sol. of sine-Gordon}\}$$

Geometric BT

We say

$$M \longrightarrow \widetilde{M}, \quad p \longmapsto \tilde{p},$$

is gemetric BT, if

- The line $\overline{p\tilde{p}}$ is in T_pM and also in $T_{\tilde{p}}\widetilde{M}$.
- $d(p, \tilde{p})$ is constant $(= r)$.
- the unit normals $\boldsymbol{\nu}_p$ and $\tilde{\boldsymbol{\nu}}_{\tilde{p}}$ has a constant angle θ, that is $\langle \boldsymbol{\nu}_p, \tilde{\boldsymbol{\nu}}_{\tilde{p}} \rangle = \cos\theta$.

Geometric BT between $K = -1$ surfaces is given by

$$\tilde{\varphi} = \varphi + r\left(\frac{\cos\tilde{\phi}/2}{\cos\phi/2}\varphi_x + \frac{\sin\tilde{\phi}/2}{\sin\phi/2}\varphi_y\right), \quad r = \sin\theta$$

and it preserves Chebyshev's nets.

Bianchi's permutability

If ϕ_i $(i = 1, 2)$ satisfies

$$\left(\frac{\phi_i + \phi}{2}\right)_u = \lambda_i \sin \frac{\phi_i - \phi}{2}, \quad \left(\frac{\phi_i - \phi}{2}\right)_v = \lambda_i^{-1} \sin \frac{\phi_i + \phi}{2},$$

and $\tilde{\phi}$ satisfies

$$(\lambda_2 - \lambda_1) \tan \frac{\tilde{\phi} - \phi}{4} = (\lambda_2 + \lambda_1) \tan \frac{\phi_2 - \phi_1}{4},$$

then

$$\left(\frac{\tilde{\phi} + \phi_1}{2}\right)_u = \lambda_2 \sin \frac{\tilde{\phi} - \phi_1}{2}, \quad \left(\frac{\tilde{\phi} - \phi_1}{2}\right)_v = \lambda_2^{-1} \sin \frac{\tilde{\phi} + \phi_1}{2}.$$

$$\left(\frac{\tilde{\phi} + \phi_2}{2}\right)_u = \lambda_1 \sin \frac{\tilde{\phi} - \phi_2}{2}, \quad \left(\frac{\tilde{\phi} - \phi_2}{2}\right)_v = \lambda_1^{-1} \sin \frac{\tilde{\phi} + \phi_2}{2}.$$

Soliton

0-soliton $\xrightarrow{\text{BT}}$ 1-soliton $\xrightarrow{\text{BT}}$ 2-soliton

$\phi = 0$

$$\phi_\lambda = 4\tan^{-1}(\lambda u + \lambda^{-1} v) \qquad \xi_i = \lambda_i u + \lambda_i^{-1} v$$

$$\phi_{\lambda_1,\lambda_2} = 4\tan^{-1}\left(\frac{\lambda_1+\lambda_2}{\lambda_2-\lambda_1} \cdot \frac{\sinh\frac{\xi_1-\xi_2}{2}}{\cosh\frac{\xi_1+\xi_2}{2}}\right)$$

line

Beltrami's pseudosphere

Dini's pseudosphere

2-soliton surfaces

Singular locus of φ

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be a Chevyshev net for a pseudosphere with $K = -1$. Let ϕ denote the asymptotic angle. Then

$$\begin{aligned} I &= du^2 + 2\cos\phi\, du\, dv + dv^2 \\ II &= 2\sin\phi\, du\, dv \end{aligned}$$

Remark that the singular locus of φ is defined by

$$\Sigma : \sin\phi = 0, \quad \text{i.e.,} \quad \phi = k\pi, \ \ k \in \mathbb{Z}.$$

For 2-soliton surface, we have $k = 0, \pm 1$.

Criteria of singularities

Let C denote the curvature line through P whose principal direction is null direction at P.

1. Assume that ϕ is nonsingular at P, i.e., the singular locus of φ is nonsingular at P.
   1.1 φ is cuspidal edge at P if and only if Σ and C intersect transversely at P.
   1.2 φ is swallowtail at P if and only if Σ has 2-point contact with C at P.
2. Assume that ϕ has a Morse singularity at P.
   2.1 φ is cuspidal beaks at P if and only if the Hessian of ϕ is positive.
   2.2 φ is cuspidal lips at P if and only if the Hessian of ϕ is negative.

Flecnodal and ridge on a 2-soliton surface

On pseudospheres, we have
∂_u-flecnodal line ($\phi_u = 0$),
∂_v-flecnodal line ($\phi_v = 0$),
∂_x-ridge line ($\phi_x = 0$),
∂_y-ridge line ($\phi_y = 0$)
and, on 2-soliton surfaces, they are

$$\frac{\cosh \xi_2}{\cosh \xi_1} = \frac{\lambda_2}{\lambda_1}, \ \frac{\lambda_1}{\lambda_2}, \ \frac{\lambda_2 + \lambda_2^{-1}}{\lambda_1 + \lambda_1^{-1}}, \ \frac{\lambda_2 - \lambda_2^{-1}}{\lambda_1 - \lambda_1^{-1}}, \text{ respectively.}$$

Here $\xi_i = \lambda_i u + v/\lambda_i$, $i = 1, 2$.

When $\lambda_2 \to \lambda_1 = \lambda$,

$$\phi_{\lambda,\lambda} = \lim_{\lambda' \to \lambda} \phi_{\lambda,\lambda'} = 4\tan^{-1}\frac{-\eta}{\cosh\xi},$$

where $\xi = \lambda u + \lambda^{-1} v + c$ and $\eta = \lambda u - \lambda^{-1} v$.
The ∂_u-flecnodal, ∂_v-flecnodal, ∂_x-ridge and ∂_y-ridge are defined by

$$\eta \tanh\xi = 1, \quad -1, \quad \frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}}, \quad \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}}, \quad \text{respectively.}$$

Classification of 2-soliton

The result in this section should compare the classification of 2-soliton surfaces (Popov). They show four types for generic 2-soliton surfaces. The correspondence between their classification and our results is summarized as follows:

Type	$\lambda_1\lambda_2$	μ	flecnodal	∂_x-ridge	∂_y-ridge
1	$+$	$+$	exist	exist	exist
2	$+$	$-$	exist	exist	not exist
3	$-$	$+$	not exists	not exist	not exist
4	$-$	$-$	not exists	not exist	exist

$\mu = (\lambda_1^2 - 1)(\lambda_2^2 - 1).$

Breather surfaces

For $\lambda \in \mathbb{C}$, with $\operatorname{Re}\lambda \neq 0$, $\operatorname{Im}\lambda \neq 0$, we have

$$\phi_{\lambda,\bar{\lambda}} = -4\tan^{-1}\left(\cot\arg\lambda \cdot \frac{\sin \operatorname{Im}\xi}{\cosh \operatorname{Re}\xi}\right)$$

where $\xi = \lambda u + v/\lambda$.
The ∂_u-flecnodal, ∂_v-flecnodal, ∂_x-ridge and ∂_y-ridge are defined by

$$\frac{(\tanh \operatorname{Re}\xi)(\tan \operatorname{Im}\xi)}{\tan\arg\lambda} = 1,\ \ -1,\ \ \frac{|\lambda| - |\lambda|^{-1}}{|\lambda| + |\lambda|^{-1}},\ \ \frac{|\lambda| + |\lambda|^{-1}}{|\lambda| - |\lambda|^{-1}},$$

respectively.