SINGULARITIES OF PARALLEL SURFACES

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Abstract

We investigate singularities of all parallel surfaces to a given regular surface. In generic context, the types of singularities of parallel surfaces are cuspidal edge, swallowtail, cuspidal lips, cuspidal beaks, cuspidal butterfly and 3-dimensional $D_\pm^4$ singularities. We give criteria for these singularities types in terms of differential geometry (Theorem 3.4 and 3.5).

1 Introduction

Classically, a wave front is the locus of points having the same phase of vibration. A wave front is described by Huygens principle: The wave front of a propagating wave of light at any instant conforms to the envelope of spherical wavelets emanating from every point on the wave front at the prior instant (with the understanding that the wavelets have the same speed as the overall wave).

It is well known that a wave front may have singularities at some moment. Singularities of wave fronts are classified in generic context (see [1, p. 336]). The local classification of bifurcations in generic one parameter families of fronts in 3-dimensional spaces are also given in [1, p. 348]. To understand their singularities, it is important to know when the given front is generic and when the given one parameter family is generic.

In the differential geometric context, a wave front can be described as the parallel surface

$$g'(U \rightarrow \mathbb{R}^3, g'(u, v) := g(u, v) + m(u, v),$$

of a regular surface $g : U \rightarrow \mathbb{R}^3$ at time $t$. Here $U$ is an open set of $\mathbb{R}^2$ and $m$ denotes the unit normal vector given by $m = (g_u \times g_v)/||g_u \times g_v||$. It is well known that when $t$ is either of the principal radii of curvature at a point of the initial surface $g$, the parallel surface $g'$ has a singularity at the corresponding point (see, for example, [13]). So singularities of parallel surfaces should be investigated in terms of differential geometry of the regular map $g$.

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By Huygens principle, the wave front can be seen as the discriminant set of the distance squared unfolding

$$\Phi^t : U \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (u, v, x, y, z) \mapsto -\frac{1}{2} \left( \| (x, y, z) - g(u, v) \|^2 - t_0^2 \right),$$

where $t_0$ is a constant. Porteous [14, 15] investigated the (Thom-Boardman) singularities of the unfolding $(u, v, x, y, z) \mapsto \Phi^t + \frac{1}{2} \|(x, y, z)\|^2$ with $t_0 = 0$. He discovered that the notion of normal vectors, principal radii of curvature, and umbilics correspond to $A_1$-singularities, $A_2$-singularities, and $D_4$-singularities or worse, respectively. Moreover, he discovered the notion of ridge points corresponding to $A_3$-singularities or worse.

It is now natural to ask a description of the singularity types of $g^t$ in terms of differential geometry, which we answer in this paper. We fix a general regular map $g$ and investigate singularities of $g^t$ for all $t$. In other words, we investigate changes of singularities due to time evolution of fronts generated by $g$. To do this we need the notion of sub-parabolic points which is introduced by Bruce and Wilkinson [5] to study singularities of folding maps. The main theorem (Theorem 3.4) states criteria of the singularity types of $g^t$ for all $t$ in terms of differential geometry. For example, we show that, at a first order ridge point, $g^t$ has swallowtail singularity when it is not sub-parabolic where $t$ is the corresponding principal radius of curvature. This is enough to find a normal form when $\Phi^t$ is an unfolding of $A_1$, $A_2$, and $A_3$ singularities. This is proved by given a characterization for the unfolding $\Phi^t$ to be $\mathcal{K}$-versal in terms of differential geometry.

We now know that $\Phi^t$ is not a $\mathcal{K}$-versal unfolding at a sub-parabolic ridge point, a higher order ridge, and an umbilic. At these points, we are interested in the unfolding $\Phi$ defined by

$$\Phi : U \times \mathbb{R}^4 \rightarrow \mathbb{R}, \quad (u, v, x, y, z, t) \mapsto -\frac{1}{2} \left( \| (x, y, z) - g(u, v) \|^2 - t^2 \right).$$

Theorem 3.4 also gives a characterization for the unfolding $\Phi$ to be $\mathcal{K}$-versal in terms of differential geometry. For example, at a ridge point, we show that $\Phi$ is $\mathcal{K}$-versal without any other condition. The parallel surface is the section of discriminant set of this unfolding with the hyperplane defined by $t = \text{constant}$. For $A_4$-singularities, that is, at a second order ridge point, we also show (Theorem 3.5 (1)) that $g^t$ has cuspidal butterfly when it is not sub-parabolic where $t$ is the corresponding principal radius of curvature. At a sub-parabolic ridge point where $\Phi^t$ fails to be $\mathcal{K}$-versal, we show (Theorem 3.5 (2)) the singularities of $g^t$ are cuspidal beaks or cuspidal lips when the corresponding CPC (constant principal curvature) lines are Morse singularities. For $D_4$-singularities, we also show a similar result (Theorem 3.5 (3)). These results are satisfactory in the context of generic differential geometry.

2 Preliminary from differential geometry

We recall some differential geometric notions and their properties of regular surfaces in Euclidean space, which we need in this paper.
We present the definitions of ridge points, sub-parabolic points and umbilics, and their fundamental properties. We then discuss constant principal curvature (CPC) lines, which are the locus of singular points of the parallel surface. We state a characterization of these notions in terms of the coefficients of a Monge normal form of the surface.

2.1 Fundamental forms

Consider a surface $g$ defined by the Monge form:

$$g(u, v) = (u, v, f(u, v)),$$

$$f(u, v) = \frac{1}{2}(k_1u^2 + k_2v^2) + \sum_{i+j=3} \frac{1}{i!j!} a_{ij} u^i v^j.$$  

(2.1)

The coefficients of the first fundamental form are given by

$$E = \langle g_{uu}, g_{uu} \rangle = 1 + f_u^2,$$

$$F = \langle g_{uv}, g_{uv} \rangle = f_u f_v,$$

$$G = \langle g_{vv}, g_{vv} \rangle = 1 + f_v^2.$$  

Here subscripts denotes partial derivatives and $\langle , \rangle$ denotes the Euclidean inner product of $\mathbb{R}^3$. The unit normal vector is given by

$$n = \frac{1}{\sqrt{1 + f_u^2 + f_v^2} (-f_u, -f_v, 1)}.$$  

The coefficients of the second fundamental form are given by

$$L = \langle g_{uu}, n \rangle = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}},$$

$$M = \langle g_{uv}, n \rangle = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}},$$

$$N = \langle g_{vv}, n \rangle = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}.$$  

We consider the matrices of the first fundamental form and the second fundamental form:

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$  

2.2 Principal curvatures

We say that $\kappa$ is a principal curvature if there is a non-zero vector $(\xi, \zeta)$ such that

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \kappa \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}.$$  

(2.2)

or, equivalently,

$$\frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \kappa \begin{pmatrix} \xi \\ \zeta \end{pmatrix}.$$  

3
This is rewritten as
\[
\frac{1}{(1 + f_u^2 + f_v^2)^{3/2}} \begin{pmatrix}
1 + f_v^2 & -f_u f_v \\
-f_u f_v & 1 + f_u^2
\end{pmatrix}
\begin{pmatrix}
\frac{\xi}{\zeta}
\end{pmatrix}
= \kappa
\begin{pmatrix}
\frac{\xi}{\zeta}
\end{pmatrix}.
\]

The eigenvector \((\xi_i, \zeta_i) \ (i = 1, 2)\) of the equation (2.2) corresponding to the eigenvalue \(\kappa_i\) gives the principal vector \(v_i\). We can choose them so that the tangent vectors \(\xi_i g_u + \zeta_i g_v\) are of the unit length.

At a point on the surface where two principal curvatures are distinct, there are two principal vectors and these vectors are mutually orthogonal. These principal vectors are often colored (blue or red) to distinguish between the two vectors. We assume that \(v_1\) is the blue principal vector and \(v_2\) is the red principal vector.

If two principal curvatures are equal at a point on the surface, we call such a point an umbilic. At an umbilic every direction through the umbilic is principal and the umbilic is an isolated singularity of the direction field.

If only one principal curvature is zero, such a point is called a parabolic point. If both principal curvatures are zero, such a point is called a flat umbilic or a planer point.

We can consider the focal surface. Away form umbilics the focal surface consists of two sheets, the blue and red sheets given by \(g + n/\kappa_1\) and \(g + n/\kappa_2\), respectively. The two sheets come together at umbilics. We note that at parabolic points only one of the two sheets exits, and at flat umbilics the common focal point lies at infinity.

The focal surface might have a singular point where the same colored principal curvature has an extreme value along the same colored line of curvature. Such a point on \(g\) is called a ridge point and on focal surface a rib. Ridges were first studied in details by Porteous [14].

The locus of points where the principal curvature has extreme value along the other colored line of curvature is also of importance. This locus is called a sub-parabolic line. The sub-parabolic line were studied in details by Bruce and Wilkinson [5] in terms of folding maps. The sub-parabolic line is also the locus of points on the surface whose image is the parabolic line on the same colored sheet of the focal surface. In [12] the sub-parabolic line appear as the locus of points where the other colored line of curvature has the geodesic inflections.

### 2.3 Ridge points and sub-parabolic points

Let \(g(p)\) be not an umbilic of a regular surface \(g\), with principal vectors \(v_1\) (‘blue’), \(v_2\) (‘red’) corresponding principal curvature \(\kappa_1\), \(\kappa_2\).

We say that the point \(g(p)\) is a ridge point relative to \(v_i\) (‘blue ridge point’ for \(i = 1\), ‘red’ for \(i = 2\)) if \(v_i \kappa_i(p) = 0\), where \(v_i \kappa_i\) is the directional derivative of \(\kappa_i\) in \(v_i\). Moreover, \(g(p)\) is a \(k\)-th order ridge point relative to \(v_i\) if
\[
v_i^{(m)} \kappa_i(p) = 0 \quad (1 \leq m \leq k) \quad \text{and} \quad v_i^{(k+1)} \kappa_i(p) \neq 0,
\]
where \(v_i^{(k)} \kappa_i\) is the \(k\)-times directional derivative of \(\kappa_i\) in \(v_i\). The set of ridge points is called a ridge line or ridges.
Lemma 2.1. Suppose that a surface $g$ is given in Monge form as in (2.1), and that the origin is not an umbilic.

(1) The origin is a first order blue ridge point if and only if

$$a_{30} = 0 \quad \text{and} \quad 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) \neq 0.$$ 

(2) The origin is a second order blue ridge point if and only if

$$a_{30} = 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) = 0 \quad \text{and} \quad 15a_{21}^2a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2 \neq 0.$$ 

Proof. We remark that the principal curvatures at the origin are $k_1, k_2 \ (k_1 \neq k_2)$ with corresponding principal vectors $v_1 = (1, 0), v_2 = (0, 1)$.

The principal curvatures are the eigenvalues of $I^{-1} \Sigma$. So the principal curvature $\kappa_1$ is expressed as

$$\kappa_1(u, v) = k_1 + a_{30}u + a_{21}v + \frac{1}{2(k_1 - k_2)}[2a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2)]u^2 + 2[2a_{21}a_{12} + a_{31}(k_1 - k_2)]u + [2a_{12}^2 + (a_{22} - k_1k_2^2)(k_1 - k_2)]v^2 + O(u, v)^3,$$

and we have

$$\frac{\partial^3 \kappa_1}{\partial u^3}(0, 0) = \frac{6a_{21}^2(-a_{30} + a_{12}) + 6a_{21}a_{31}(k_1 - k_2) + (a_{50} - 18a_{30}k_1^2)(k_1 - k_2)^2}{6(k_1 - k_2)^2}.$$

Let $(\xi_1, \zeta_1)$ be the eigenvector of $I^{-1} \Sigma$ with the eigenvalue $\kappa_1$. It thus follows form (2.2) that there is a real number $\mu \neq 0$ such that

$$(\xi_1, \zeta_1) = \mu(N - \kappa_1 G, -M + \kappa_1 F).$$

Selection of the vector $(\xi_1, \zeta_1)$ in order for the tangent vector $\xi_1 g_u + \zeta_1 g_v$ to be of the unit length shows that the principal vector $v_1$ is expressed as

$$v_1(u, v) = \left(1 + O(u, v)^2\right) \frac{\partial}{\partial u} + \left(\frac{1}{k_1 - k_2} (a_{21}u + a_{12}v) + O(u, v)^2\right) \frac{\partial}{\partial v},$$

and that

$$\frac{\partial^2 \xi_1}{\partial u^2}(0, 0) = \frac{2a_{21}(a_{12} - a_{30}) + a_{31}(k_1 - k_2)}{2(k_1 - k_2)^2}.$$ 

Therefore, we have

$$v_1 \kappa_1(0, 0) = \frac{\partial \kappa_1}{\partial u}(0, 0) = a_{30},$$

$$v_1^2 \kappa_1(0, 0) = \frac{\partial^2 \kappa_1}{\partial u^2}(0, 0) + \frac{\partial \kappa_1}{\partial v}(0, 0) \frac{\partial \xi_1}{\partial u}(0, 0) = \frac{3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2)}{k_1 - k_2}.$$ 

5
Moreover, when \( v_1 \kappa_1(0, 0) = v_1^2 \kappa_1(0, 0) = 0 \), we obtain
\[
v_1^2 \kappa_1(0) = \frac{\partial^3 \kappa_1}{\partial u^2 v}(0) + \frac{3}{2} \frac{\partial^2 \kappa_1}{\partial u \partial v}(0) \frac{\partial \xi_1}{\partial u}(0) + \frac{\partial \kappa_1}{\partial v}(0) \left( \frac{\partial \xi_1}{\partial u}(0) \frac{\partial \xi_1}{\partial v}(0) + \frac{\partial^2 \xi_1}{\partial u^2}(0) \right)
\]
\[= \frac{15 a_{21}^2 a_{12} + 10 a_{21} a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2}{(k_1 - k_2)^2}.
\]

**Lemma 2.2.** Suppose that a surface \( g \) is given in Monge form as in (2.1), and that the origin is a blue ridge point. Then the blue ridge line through the origin fails to be smooth at the origin if and only if
\[3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) = 3a_{21}a_{12} + a_{31}(k_1 - k_2) = 0.
\]

**Proof.** It follows from form (2.3) and (2.4) that the equation of the blue ridge line through the origin is expressed as
\[
[3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2)]u + [3a_{21}a_{12} + a_{31}(k_1 - k_2)]v + \cdots = 0.
\]
This equation implies the assertion. \( \square \)

We turn to sub-parabolic points. A point \( g(p) \) which is not an umbilic is a *sub-parabolic point* relative to \( v_i \) (*blue* sub-parabolic point’ for \( i = 1 \), *red* for \( i = 2 \)) if \( v_i \kappa_j(p) = 0 \) \((i \neq j)\). The set of sub-parabolic points is called a *sub-parabolic line*.

**Lemma 2.3.** Suppose that a surface \( g \) is given in Monge form as in (2.1), and that the origin is not an umbilic. Then the origin is a red sub-parabolic point if and only if \( a_{21} = 0 \).

**Proof.** Since the principal vectors \( v_1 \) and \( v_2 \) are orthogonal, it follows from (2.4) that the principal vector \( v_2 \) is expressed the following form:
\[
v_2(u, v) = \left( \frac{1}{k_2 - k_1} (a_{21}u + a_{12}v) + O(u, v)^2 \right) \frac{\partial}{\partial u} + \left( 1 + O(u, v)^2 \right) \frac{\partial}{\partial v}.
\]
From (2.3) and (2.6), the directional derivative \( v_2 \kappa_1 \) is given by
\[
v_2 \kappa_1(u, v) = a_{21} + \frac{a_{21}(2a_{12} - a_{30}) + a_{31}(k_1 - k_2)}{k_1 - k_2} u
\]
\[+ \frac{a_{12}(2a_{12} - a_{30}) + (a_{22} - k_1k_2^2)(k_1 - k_2)}{k_1 - k_2} v + O(u, v)^2.
\]
This equation implies the assertion. \( \square \)

We can deduce from (2.7) that the equation of the red sub-parabolic line through the origin has the form
\[
a_{31}(k_1 - k_2)u + [a_{12}(2a_{12} - a_{30}) + (a_{22} - k_1k_2^2)(k_1 - k_2)]v + \cdots = 0.
\]
2.4 Umbilics

Umbilics of a regular surface are points where the two principal curvatures coincide. At these points the principal direction field is singular and the lines of curvature fail to cross at right angle. The classification of generic umbilics is due to Darboux [6]. He gave three configurations of the lines of curvature. The three configurations were given the names lemon, star, and monstar by Berry and Hannay [2]. Their classification was provided by Gutierrez and Sotomayor [7].

Suppose that the origin is an umbilic of a surface \( g \), and that \( g \) is given in Monge form

\[
g(u, v) = (u, v, f(u, v)), \quad f(u, v) = \frac{k}{2}(u^2 + v^2) + \sum_{i+j \geq 3} \frac{1}{i!j!} a_{ij} u^i v^j,\tag{2.9}
\]

where \( k \) is the common value for the principal curvatures at the origin.

At an umbilic the cubic part \( f_3 \) of \( f \) in (2.9) determines its type. An umbilic of the surface \( g \) is said to be elliptic or hyperbolic if \( f_3 \) has three real roots or one real root, respectively. Moreover, An umbilic is said to be right-angled if the root directions of the quadratic form which is the determinant of the Hessian matrix of \( f_3 \) are mutually orthogonal with respect to the standard scalar product on \( \mathbb{R}^2 \). Such an umbilic necessarily is a hyperbolic umbilic.

We shall present the conditions for types of umbilics in terms of the coefficients of the Monge form. We set

\[
\Gamma := \begin{vmatrix}
a_{30} & 2a_{21} & a_{12} & 0 \\
0 & a_{30} & 2a_{21} & a_{12} \\
a_{21} & 2a_{12} & a_{03} & 0 \\
0 & a_{21} & 2a_{12} & a_{03}
\end{vmatrix}, \quad \text{and} \quad \Gamma' := \begin{vmatrix}
1 & 0 & 1 \\
a_{30} & a_{21} & a_{12} \\
a_{21} & a_{12} & a_{03}
\end{vmatrix}.
\]

The discriminant of \( f_3 \) is given by \(-\Gamma\). Hence, the origin is an elliptic umbilic or hyperbolic umbilic if and only if \( \Gamma < 0 \) or \( \Gamma > 0 \), respectively. Moreover, the determinant of the Hessian matrix of \( f_3 \) is given by

\[
-36[(a_{21}^2 - a_{30}a_{12})u^2 + (a_{21}a_{12} - a_{30}a_{03})uw + (a_{12}^2 - a_{21}a_{03})v^2].
\]

It follows that the origin is a right-angled umbilic if and only if \( \Gamma' = 0 \).

It is shown in [15] that there is one ridge line passing through a hyperbolic umbilic and three ridge lines passing through an elliptic umbilic. It is also shown in [15] that ridge lines change their color as they pass through a generic umbilic.

It is known that when there is one direction for lines of curvature at an umbilic, there is one sub-parabolic line through the umbilic in the same direction, while, when there are three directions for lines of curvature at an umbilic, there are three sub-parabolic lines through the umbilic in the same three directions [5, 12].
2.5 Constant principal curvature lines

We set
\[ \Sigma_c := \{(u, v) \in U ; \kappa_i(u, v) = c \text{ for some } i \}. \]

We call \( \Sigma_c \) the constant principal curvature (CPC) line with the value of \( c \). There are two CPC lines \( \Sigma_{\kappa_1(p)} \) (colored by blue) and \( \Sigma_{\kappa_2(p)} \) (colored by red) locally through a non-umbilical point \( g(p) \). We recall that a point \( p \in U \) is a singular point of the parallel surface \( g' \) at distance \( t \) if and only if \( t = 1/\kappa_i(p) \) for some \( i \). This means that the set of singular points of \( g' \) is the CPC line \( \Sigma_{\kappa_i(p)} \).

Firstly, we investigate the CPC lines away from umbilics. Suppose that a surface \( g \) is given in Monge form as in (2.1). From (2.3), \( \kappa_1(u, v) = k_1 \) is expressed by the equation

\[ 0 = a_{30}u + a_{21}v + \frac{1}{2(k_1 - k_2)}[(2a_{21})^2 + (a_{40} - 3k_1^3)(k_1 - k_2)]u^2 + 2[2a_{21}a_{12} + a_{31}(k_1 - k_2)]uv + [2a_{12}^2 + (a_{22} - k_1k_2^2)(k_1 - k_2)]v^2 + \cdots. \]

(2.10)

Hence, the CPC line \( \Sigma_{k_1} \) is locally given by the equation (2.10) at the origin. The equation (2.10) shows that the CPC line \( \Sigma_{k_1} \) is singular at the origin if and only if \( a_{30} = a_{21} = 0 \), that is, the origin is a blue ridge point and a red sub-parabolic point (Lemma 2.1 and 2.3).

**Lemma 2.4.** Suppose that the origin is a blue ridge point which is not a red sub-parabolic point.

1. The CPC line \( \Sigma_{k_1} \) is transverse to the blue ridge line at the origin if and only if the order of the ridge is one.
2. the CPC line \( \Sigma_{k_1} \) is tangential to the blue ridge line at the origin if and only if the order of the ridge is more than one.

**Proof.** It follows from (2.5) and (2.10) that the CPC line \( \Sigma_{k_1} \) is transverse to the blue ridge line at the origin if and only if

\[ 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) \neq 0. \]

On the other hand, both lines are tangential at the origin if and only if

\[ 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) = 0. \]

Hence, the statement of the lemma follows from Lemma 2.1

**Lemma 2.5.** Suppose that the origin is a blue ridge point and red sub-parabolic point. Then the CPC line \( \Sigma_{k_1} \) is locally either an isolated point or two intersecting smooth curves at the origin, if the blue ridge line crosses the red sub-parabolic line at the origin.
Proof. First we remark that
\[
\frac{\partial \kappa_1}{\partial u}(0,0) = a_{30} = 0 \quad \text{and} \quad \frac{\partial \kappa_1}{\partial v}(0,0) = a_{21} = 0.
\]
The equations of the blue ridge line (2.5) and the red sub-parabolic line (2.8) reduce
\[
(a_{40} - 3k_1^3)(k_1 - k_2)u + a_{31}(k_1 - k_2)v + \cdots = 0
\]
and
\[
a_{31}(k_1 - k_2)u + [2a_{12}^2 + (a_{22} - k_1k_2^2)(k_1 - k_2)]v + \cdots = 0,
\]
respectively. From these equations, the blue ridge line crosses the red sub-parabolic line at the origin if and only if
\[
(a_{40} - 3k_1^3)(k_1 - k_2)[2a_{12}^2 + (a_{22} - k_1k_2^2)(k_1 - k_2)] - a_{31}^2(k_1 - k_2)^2 \neq 0.
\]
In addition, from (2.3), the determinant of the Hessian matrix of \( \kappa_1 \) at \((0,0)\) is given by
\[
(a_{40} - 3k_1^3)(k_1 - k_2)[2a_{12}^2 + (a_{22} - k_1k_2^2)(k_1 - k_2)] - a_{31}^2(k_1 - k_2)^2.
\]
By the Morse lemma (see, for example, [3]), we complete the proof. \(\square\)

Secondly, we investigate the CPC line near an umbilic.

Theorem 2.6. (1) The CPC line \( \Sigma_k \) is locally an isolated point at the elliptic umbilic, where \( k \) is the common value for the principal curvatures at the umbilic.

(2) The CPC line \( \Sigma_k \) is locally two intersecting smooth curves at a hyperbolic umbilic. The locally two curves change their color as they pass through the hyperbolic umbilic.

Proof. We suppose that the origin is an umbilic of a surface \( g \), and that the surface \( g \) is given in Monge form as in (2.9). The principal curvatures are the roots of the quadric equation
\[
(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (LN - M^2) = 0.
\]
Replacing \( \kappa \) by \( k \) which is the common value for the principal curvatures at the origin, we can express the equation in the form
\[
(a_{30}a_{12} - a_{21}^2)u^2 + (a_{30}a_{03} - a_{21}a_{12})uw + (a_{21}a_{03} - a_{12}^2)v^2 + \cdots = 0.
\]
The locus of this equation is the CPC line \( \Sigma_k \). We denote the quadric part of (2.11) by \( \alpha u^2 + 2\beta uv + \gamma v^2 \). Then we have \( \beta^2 - \alpha \gamma = \Gamma/4 \), where \( \Gamma \) is as in Section 2.4. Hence, the CPC line \( \Sigma_k \) at an umbilic is locally either an isolated point if \( \Gamma < 0 \) (i.e., the origin is an elliptic umbilic) or two smooth intersecting curves if \( \Gamma > 0 \) (i.e., the origin is a hyperbolic umbilic).
We investigate the case of hyperbolic umbilics in detail. For a hyperbolic umbilic, we may assume that \( g \) is locally given in the form

\[
(2.12) \quad g(u, v) = (u, v, f(u, v)), \quad f(u, v) = \frac{k}{2}(u^2 + v^2) + \frac{P}{6}u(u^2 + 2Qu + Rv^2) + \cdots
\]

for some \( P, Q, \) and \( R \) with \( P \neq 0 \) and \( Q^2 - R < 0 \). Then the principal curvatures \( \kappa_1 \) (maximum curvature), \( \kappa_2 \) (minimum curvature) are expressed as

\[
(2.13) \quad \kappa_1(u, v) = k + \frac{1}{6}\left(P[(R + 3)u + 2Qu]\right)
\]

\[
+ |P| \sqrt{[16Q^2 + (R - 3)^2]u^2 + 12Q(R + 1)uv + 4(Q^2 + R^2)v^2} + \cdots,
\]

\[
\kappa_2(u, v) = k + \frac{1}{6}\left(P[(R + 3)u + 2Qu]\right)
\]

\[
- |P| \sqrt{[16Q^2 + (R - 3)^2]u^2 + 12Q(R + 1)uv + 4(Q^2 + R^2)v^2} + \cdots.
\]

Therefore, the locally two smooth curves change their color as they through the hyperbolic umbilic.

Remark 2.7. (1) A simple calculation gives \( \Gamma' = \alpha + \gamma \), where \( \Gamma' \) is as in Section 2.4. It follows that the tangents to the locally two smooth curves of the CPC line through the right-angled umbilic are mutually orthogonal. We note that the right-angled umbilic necessarily is a hyperbolic umbilic.

(2) Equation (2.11) shows that the CPC line \( \Sigma_k \) is approximated by a conic near the origin when the origin is not a parabolic umbilic.

Finally, we investigate bifurcations of the CPC lines at an umbilic. We start with the case of an elliptic umbilic. There are three ridge lines through the elliptic umbilic. The bifurcation of the CPC lines at the elliptic umbilic is shown in Figure 1 (i), (ii) (cf. [4], Figure 2). We now turn to the case of a hyperbolic umbilic. We may assume that the surface given in the from (2.12). There is one ridge line through the hyperbolic umbilic. Calculations show that the ridge line is tangent to \( 2Qu + Rv = 0 \) at the origin (cf. [15], corollary (iii) of Theorem 11.10) and that the locally two smooth curves of the CPC line \( \Sigma_k \) are tangent to \( QR \pm \sqrt{R^2 - Q^2 + R^2}u + R^2v = 0 \). Thus it follows that the bifurcation of the CPC lines at the hyperbolic umbilic is given in Figure 1 (iii)–(v) (cf. [4], Figure 2), in the generic context.

As shown in Figure 1, there are three intersection points of the CPC line and the same colored ridge line near an elliptic umbilic, and there is one such intersection point near a hyperbolic umbilic, in the generic context.

3 Singularities of parallel surfaces In this section we present our main theorem.
Figure 1: Bifurcations of the CPC lines near an elliptic umbilic (i) and (ii), and a hyperbolic umbilic (iii)–(v), where $\varepsilon$ is a small positive number.
3.1 Augmented distance squared functions Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be a smooth function germ. We say that a smooth function germ \( F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0) \) is an unfolding of \( f \) if \( F(u, 0) = f(u) \). We define the discriminant set of \( F \) by

\[
\mathcal{D}(F) = \left\{ x \in \mathbb{R}^r : F(u, x) = \frac{\partial F}{\partial u_1}(u, x) = \cdots = \frac{\partial F}{\partial u_n}(u, x) = 0 \text{ for some } u \in U \right\},
\]

where \((u, x) = (u_1, \ldots, u_n, x_1, \ldots, x_r) \in (\mathbb{R}^n \times \mathbb{R}^r, 0)\). We say that the \( F \) is a \( \mathcal{K} \)-versal unfolding if any unfolding \( G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0) \) of \( f \) is representable in the form

\[
G(u, y) = h(u, y) \cdot F(\Psi(u, y), \psi(y)),
\]

where \( \Psi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n, 0) \) is a smooth map germ with \( \Psi(u, 0) = u \), \( \psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^r, 0) \) is a smooth map germ with \( \psi(0) = 0 \) and \( h : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to \mathbb{R} \) is a smooth function germ with \( h(0, 0) \neq 0 \) (cf. [1, §8]). This condition is equivalent to that

\[
\mathcal{E}_n = \left\{ \frac{\partial f}{\partial u_1}, \cdots, \frac{\partial f}{\partial u_n}, f \right\}_{\mathcal{E}_n} + \left\{ \frac{\partial F}{\partial x_1}|_{\mathbb{R}^n \times \{0\}}, \cdots, \frac{\partial F}{\partial x_r}|_{\mathbb{R}^n \times \{0\}} \right\} + \mathcal{M}_n^{k+1}
\]

when \( f(u) \) is \( k \)-determined (see [17, §3] and [11, p.75]). Here, \( \mathcal{E}_n \) is the set of smooth function germs \((\mathbb{R}^n, 0) \to \mathbb{R}, \) which is the local ring with the unique maximal ideal \( \mathcal{M}_n = \{ f \in \mathcal{E}_n : f(0) = 0 \}. \) We say that two function germs \( f \) and \( g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) are \( \mathcal{K} \)-equivalent if there exist a diffeomorphism germ \( \psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and a smooth function germ \( h : (\mathbb{R}^n, 0) \to \mathbb{R} \) with \( h(0) \neq 0 \) such that \( g(u) = h(u) \cdot f \circ \psi(u) \). If \( F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0) \) are \( \mathcal{K} \)-versal unfoldings of \( \mathcal{K} \)-equivalent function germs \( f, g \), respectively. Then, there exist a diffeomorphism germ \( \Psi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n \times \mathbb{R}^r, 0), \) \((u, x) \mapsto (\Psi(u, x), \psi(x)) \) and a smooth function germ \( h : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to \mathbb{R} \) with \( h(0, 0) \neq 0 \) such that

\[
G(u, x) = h(u, x) \cdot F(\Psi(u, x), \psi(x)).
\]

(cf. [1, §8]). Moreover, calculation shows that \( \mathcal{D}(F) = \psi(\mathcal{D}(G)) \).

In order to investigate singularities of parallel surfaces, we consider the functions

\[
\Phi' : U \times \mathbb{R}^3 \to \mathbb{R}, \quad \text{defined by} \quad (u, v, x, y, z) \mapsto -\frac{1}{2} \left( \|(x, y, z) - g(u, v)\|^2 - t_0^2 \right),
\]

where \( t_0 \in \mathbb{R} \setminus \{0\} \), and

\[
\Phi : U \times \mathbb{R}^4 \to \mathbb{R}, \quad \text{defined by} \quad (u, v, x, y, z, t) \mapsto -\frac{1}{2} \left( \|(x, y, z) - g(u, v)\|^2 - t^2 \right).
\]

We call them augmented distance squared functions.

Calculating the discriminant set of \( \Phi' \), we have

\[
\mathcal{D}(\Phi') = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) = g(u, v) + t_0 u, v \text{ for some } (u, v) \in \mathbb{R}^2 \},
\]

12
which is the parallel surface of \( g \) at a distance \( t_0 \). Besides, the discriminant set of \( \Phi \) is given by

\[
\mathcal{D}(\Phi) = \{(x, y, z, t) \in \mathbb{R}^3 : (x, y, z) = g(u, v) + \mathbf{m}(u, v) \text{ for some } (u, v) \in \mathbb{R}^2\}.
\]

Its intersection with the hyperplane \( t = t_0 \) is the parallel surface of \( g \) at distance \( t_0 \).

We take a point \( q_0 = (x_0, y_0, z_0) \) or \( q_0 = (x_0, y_0, z_0, t_0) \) where

\[
(x_0, y_0, z_0) = g(u_0, v_0) + t_0 \mathbf{n}(u_0, v_0), \quad t_0 = \frac{1}{\kappa(u_0, v_0)}
\]
possibly with \( \kappa_1(u_0, v_0) = \kappa_2(u_0, v_0) \), and set

\[
\varphi(u, v) = \Phi'(u, v, q_0) \quad \text{or} \quad \varphi(u, v) = \Phi(u, v, q_0).
\]

Then the augmented distance functions \( \Phi \) and \( \Phi' \) are the unfoldings of \( \varphi \).

If \( \varphi \) is \( \mathcal{K} \)-equivalent to \( A_2 \) (resp. \( A_3 \)) and \( \Phi' \) is a \( \mathcal{K} \)-versal unfolding of \( \varphi \), then the discriminant set of \( \Phi' \) is locally diffeomorphic to the discriminant set of the versal unfolding \( G : (U \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0) \),

\[
G(u, v, x, y, z) = u^4 \pm v^2 + x + yu + zu^2
\]

( resp. \( G(u, v, x, y, z) = u^4 \pm v^2 + x + yu + z^2u \) ) of \( g(u, v) = u^4 \pm v^2 \) ( resp. \( g(u, v) = u^4 \pm v^2 \) ). The singularity of the discriminant set of \( G \) is the cuspidal edge ( resp. swallowtail ).

Here, the **cuspidal edge** is a set locally diffeomorphic to the image of a map germ \( CE : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \), \((u, v) \mapsto (u, v^2, v^3)\) and the **swallowtail** is a a set locally diffeomorphic to the image of a map germ \( SW : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \), \((u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)\). The pictures of the cuspidal edge and the swallowtail are shown in Figure 2.

![Figure 2: From left to right: Cuspidal edge, Swallowtail.](image)

If \( \varphi \) is \( \mathcal{K} \)-equivalent to \( A_4 \) (resp. \( D_4^+ \)) and \( \Phi : (U \times \mathbb{R}^4, (u_0, v_0, q_0)) \to (\mathbb{R}, 0) \) is a \( \mathcal{K} \)-versal unfolding of \( \varphi \), then the discriminant set of \( \Phi \) is locally diffeomorphic to the discriminant set of the versal unfolding \( G : (U \times \mathbb{R}^4, 0) \to (\mathbb{R}, 0) \),

\[
G(u, v, x, y, z, t) = u^4 \pm v^2 + x + yu + zu^2 + tu^3
\]

( resp. \( G(u, v, x, y, z, t) = u^2v \pm v^3 + x + yu + zv + tu^2 \) ) of \( g(u, v) = u^4 \pm v^2 \) ( resp. \( g(u, v) = u^2v \pm v^3 \) ). The singularity of the discriminant set of \( G \) is a butterfly ( resp. \( D_4^+ \) singularities ).
Here, the **butterfly** is a set locally diffeomorphic to the image of a map germ $BF : (R^3, 0) \to (R^4, 0)$, $(u, v, w) \mapsto (u, 5v^4 + 2uw + 3v^2w, 4v^5 + uw^2 + 2v^3w, w)$ and the 4-dimensional $D^+_4$ singularity is a set locally diffeomorphic to the image of a map germ $FD^+ : (R^3, 0) \to (R^4, 0)$, $(u, v, w) \mapsto (uw, u^2 + 2vw + 3v^2, 2uv^2 + v^2w \pm 2v^3, w)$.

### 3.2 Criteria for singularities of fronts in $R^3$

It is well known that the parallel surface $g'$ is a front. Fronts were first studied in details by Arnol’d and Zakalyukin. They showed that the generic singularities of fronts in $R^3$ are cuspidal edges and swallowtails. Moreover, they showed that the singularities of the bifurcations in generic one parameter families of fronts in $R^3$ are cuspidal lips, cuspidal beaks, cuspidal butterflies and 3-dimensional $D^+_4$ singularities (cf. [1]).

Here, the **cuspidal lips** is a set locally diffeomorphic to the image of a map germ $CLP : (R^2, 0) \to (R^3, 0)$, $(u, v) \mapsto (3u^4 + 2u^2v^2, u^3 + uw^2, v)$, the **cuspidal beaks** is a set locally diffeomorphic to the image of a map germ $CBK : (R^2, 0) \to (R^3, 0)$, $(u, v) \mapsto (3u^4 - 2u^2v^2, u^3 - uw^2, v)$, the **cuspidal butterfly** is a set of the image of a map germ $CBF : (R^2, 0) \to (R^3, 0)$, $(u, v) \mapsto (4u^5 + u^2v, 5u^4 + 2uv, v)$ and the 3-dimensional $D^+_4$ singularity (resp. $D^-_4$ singularity) is a set of the image of a map germ $TD^+ : (R^2, 0) \to (R^3, 0)$, $(u, v) \mapsto (uw, u^2 + 3v^2, u^2v + v^3)$ (resp. $TD^- : (u, v) \mapsto (uw, u^2 - 3v^2, u^2v - v^3)$). Their pictures are shown in Figure 3.

![Figure 3: From top left to bottom right: Cuspidal lips, Cuspidal beaks, Cuspidal butterfly, 3-dimensional $D^+_4$ singularity, 3-dimensional $D^-_4$ singularity.](image-url)
criteria, we prepare basic notions of fronts in $\mathbb{R}^3$. A smooth map $f : U \to \mathbb{R}^3$ is called a \textit{front} if there exists a unit vector field $\nu$ of $\mathbb{R}^3$ along $f$ such that $L_f = (f, \nu) : U \to T_1\mathbb{R}^3$ is a Legendrian immersion, where $T_1\mathbb{R}^3$ is the unit tangent bundle of $\mathbb{R}^3$ (cf. [1], see also [10]). For a front $f$, we define a function $\lambda : U \to \mathbb{R}$ by $\lambda(u, v) = \det(f_u, f_v, \nu)$. The function $\lambda$ is called a \textit{discriminant function} of $f$. The set of singular points $S(f)$ of $f$ is the zero set of $\lambda$. A singular point $p \in U$ of $f$ is said to be non-degenerate if $d\lambda(p) \neq 0$. Let $p$ be a non-degenerate singular point of a front $f$. Then $S(f)$ is parameterized by a regular curve $\gamma(t) : (-\epsilon, \epsilon) \to U$ near $p$. Moreover, there exists a a unique direction $\eta(t) \in T_{\gamma(t)}U$ up to scalar multiplications such that $df(\eta(t)) = 0$. We call $\eta(t)$ the \textit{null direction}. Under these notations, we present the criterion for the cuspidal butterfly.

**Theorem 3.1 ([8]).** Let $f : U \to \mathbb{R}^3$ be a front and $p$ a non-degenerate singular point of $f$. Then the germ of the front $f$ at $p$ is $\mathcal{A}$-equivalent to the map germ $\mathcal{CBF}$ if and only if $\eta\lambda(p) = \eta^2\lambda(p) = 0$ and $\eta^3\lambda(p) \neq 0$.

Here, two map germs $f_1, f_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\psi_1 : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\psi_2 : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\psi_2 \circ f_1 = f_2 \circ \psi_1$, and $\eta\lambda$ denotes the directional derivative of $\lambda$ in the direction of $\eta$.

We now turn to degenerate singularities. Let $p$ be a degenerate singular point of the front $f$. If $\text{rank}(df_p) = 1$, then there exists the non-zero vector field $\eta$ near $p$ such that if $q \in S(f)$ then $df_q(\eta(q)) = 0$. Criteria for degenerate singularities are as follows:

**Theorem 3.2 ([9]).** Let $f : U \to \mathbb{R}^3$ be a front and $p$ a degenerate singular point of $f$.

1. The germ of the front $f$ at $p$ is $\mathcal{A}$-equivalent to the map germ $\mathcal{CLP}$ if and only if $\text{rank}(df_p) = 1$ and the $\det(\text{Hess} \lambda(p)) > 0$, where $\det(\text{Hess} \lambda(p))$ denotes the determinant of the Hessian matrix of $\lambda$ at $p$.

2. The germ of the front $f$ at $p$ is $\mathcal{A}$-equivalent to the map germ $\mathcal{CBK}$ if and only if $\text{rank}(df_p) = 1$, $\det(\text{Hess} \lambda(p)) < 0$ and $\eta^2\lambda(p) \neq 0$.

**Theorem 3.3 ([16]).** Let $f : U \to \mathbb{R}^3$ be a front and $p$ a degenerate singular point of $f$. Then the germ of the front $f$ at $p$ is $\mathcal{A}$-equivalent to the map germ $\mathcal{T}D^+$ (resp. $\mathcal{T}D^-$) if and only if $\text{rank}(df_p) = 0$ and $\det(\text{Hess} \lambda(p)) < 0$ (resp. $\det(\text{Hess} \lambda(p)) > 0$).

### 3.3 Singularities of parallels surfaces

Now we are ready to state our main theorem.

**Theorem 3.4.** Let $g : U \to \mathbb{R}^3$ be a regular surface and $g'$ the parallel surface of $g$ at distance $t$, where $U$ is an open subset of $\mathbb{R}^2$. Assume that $\Phi$, $\Phi'$, and $\varphi$ is defined as in Section 3.1.

1. If $g(u_0, v_0)$ is neither a ridge point relative to the principal vector $v_i$ nor an umbilic, and $\kappa_i(u_0, v_0) \neq 0$, then $\varphi$ has an $A_2$ singularity at $(u_0, v_0)$. In this case, $\Phi'$ is a $\mathcal{K}$-versal unfolding of $\varphi$. Moreover, the parallel surface $g'$ at distance $t = 1/\kappa_i(u_0, v_0)$ is locally diffeomorphic to the cuspidal edge at $g'(u_0, v_0)$. 

15
Moreover, the Jacobian matrix $g$ of the surface $g$ at distance $t$ is given by

$$J_g = J_g - t J_d \Gamma^{-1} I_l = J_g (I_l - t I^{-1}),$$

where $J_g$ is the Jacobian matrix of $g$ and $I_l$ is the $2 \times 2$ identity matrix. Applying criteria for singularities of fronts (Theorem 3.1–3.3) to the parallel surface $g'$, we obtain Theorem 3.5 as corollaries of these criteria.

**Theorem 3.5.** Let $g : U \to \mathbb{R}^3$ be a regular surface and $g'$ the parallel surface of $g$ at distance $t$, where $U$ is an open subset of $\mathbb{R}^2$.

(1) Suppose that $g(p)$ is a second order ridge point relative to the principal vector $v_i$, which is not a sub-parabolic point relative to the other principal direction $v_j$, and that $\kappa_i(p) \neq 0$. Then the parallel surface $g'$ at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to the cuspidal butterfly at $g(p)$. 

(2) If $g(u_0, v_0)$ is a first order ridge point relative to the principal vector $v_i$, and $\kappa_i(u_0, v_0) \neq 0$, then $\varphi$ has an $A_2$ singularity at $(u_0, v_0)$. In this case, $\Phi^i$ is a $K$-versal unfolding of $\varphi$ if and only if $g(u_0, v_0)$ is not a sub-parabolic point relative to the other principal vector $v_j$. Moreover, the parallel surface $g'$ at distance $t = 1/\kappa_i(u_0, v_0)$ is locally diffeomorphic to the swallowtail at $g'(u_0, v_0)$.

(3) If $g(u_0, v_0)$ is a second order ridge point relative to the principal vector $v_i$, and $\kappa_i(u_0, v_0) \neq 0$, then $\varphi$ has an $A_2$ singularity at $(u_0, v_0)$. In this case, $\Phi^i$ is a $K$-versal unfolding of $\varphi$ if and only if $(u_0, v_0)$ is a regular point of the ridge line relative to the same principal vector $v_i$. Moreover, the parallel surface $g'$ at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $D(\Phi)$, which is locally diffeomorphic to the butterfly, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.

(4) If $g(u_0, v_0)$ is a hyperbolic umbilic and not a flat umbilic, then $\varphi$ has a $D_4^+$ singularity at $(u_0, v_0)$. In this case, $\Phi$ is a $K$-versal unfolding of $\varphi$ if and only if $g(u_0, v_0)$ is not a right-angled umbilic. Moreover, the parallel surface $g'$ at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $D(\Phi)$, which is locally diffeomorphic to the $4$-dimensional $D_4^+$ singularity, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.

(5) If $g(u_0, v_0)$ is an elliptic umbilic and not a flat umbilic, then $\varphi$ has a $D_4^-$ singularity at $(u_0, v_0)$. In this case, $\Phi$ is a $K$-versal unfolding of $\varphi$. Moreover, the parallel surface $g'$ at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $D(\Phi)$, which is locally diffeomorphic to the $4$-dimensional $D_4^-$ singularity, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.

A proof of this theorem is given in Section 5.

Again, we remark that the parallel surfaces $g'$ of a regular surface $g$ are the front. Since the unit normal vector of the parallel surface $g'$ coincides with the unit normal vector $n$ of the initial surface $g$, the discriminant function of $g'$ is given by

$$\lambda(u, v) = \det(g'_i(u, v), g'_i(u, v), n(u, v)).$$

Moreover, the Jacobian matrix $J_g'$ of $g'$ is given by

$$J_g' = J_g - t J_d \Gamma^{-1} I_l = J_g (I_l - t I^{-1}),$$

where $J_g$ is the Jacobian matrix of $g$ and $I_l$ is the $2 \times 2$ identity matrix.
(2) Suppose that \( g(p) \) is a ridge point relative to the principal direction \( \mathbf{v}_i \) and a sub-parabolic point relative to the other principal direction \( \mathbf{v}_j \), and that \( \kappa_i(p) \neq 0 \). Then the parallel surface \( g' \) at distance \( t = 1/\kappa_1(p) \) is locally diffeomorphic to the cuspidal lips (resp. cuspidal beaks) at \( g'(p) \) if \( \det(\text{Hess}(\mathbf{v}_1, \mathbf{v}_2)\kappa_i(p)) > 0 \) (resp. \( \det(\text{Hess}(\mathbf{v}_1, \mathbf{v}_2)\kappa_i(p)) < 0 \) and the order of ridge is one), where \( \text{Hess}(\mathbf{v}_1, \mathbf{v}_2)\kappa_i \) is the Hessian matrix of \( \kappa_i \) with respect to \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

(3) Suppose that \( g(p) \) is an umbilic which is not a flat umbilic. Then the parallel surface \( g' \) at distance \( t = 1/\kappa_1(0, 0) = 1/\kappa_2(0, 0) \) is locally diffeomorphic to a 3-dimensional \( D^+ \) singularity (resp. \( D^- \) singularity) at \( g(p) \) if \( g(p) \) is a hyperbolic umbilic (resp. elliptic umbilic).

**Proof.** (1) We may assume that \( p = (0, 0) \) and that the initial regular surface \( g \) given in Monge form as in (2.1). We remark that \( \kappa_1 \neq \kappa_2 \). Now we prove in the case \( t = 1/\kappa_1(0, 0) = 1/\kappa_1 \).

From Lemma 2.1 and 2.3, we have

\[
\begin{align*}
a_{30} &= 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) = 0, \\
15a_{21}^2a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2 &= 0, \quad \text{and} \quad a_{21} \neq 0.
\end{align*}
\]

(3.2)

Suppose that \( t = 1/\kappa_1 \). Then we have \( \lambda(0, 0) = 0 \). Moreover, from (3.2), we have

\[
\lambda_u(0, 0) = a_{30}(k_2 - k_1)k_1^2 = 0 \quad \text{and} \quad \lambda_v(0, 0) = a_{21}(k_2 - k_1)k_1^2 \neq 0.
\]

(3.3)

It turns out that \( (0, 0) \) is a non-degenerate singular point of \( g' \). Therefore, the set of singular points of \( g' \) is a locally smooth curve near \( (0, 0) \), which is the CPC line \( \Sigma_{k_1} \), and there exists a null direction \( \eta \) with \( dg'(|\eta|) = 0 \) along this smooth curve. It follows form (3.1) that the null direction \( \eta \) has the same direction as the principal vector \( \mathbf{v}_1 \). From (3.2), we have

\[
\begin{align*}
\mathbf{v}_1 \lambda(0, 0) &= -a_{30}(k_1 - k_2)k_1^2 = 0, \\
\mathbf{v}_1^2 \lambda(0, 0) &= -(k_1 - k_2)^2[a_{30}(3a_{30} - 3a_{12}) + 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2)]k_1^2 = 0, \quad \text{and} \\
\mathbf{v}_1^3 \lambda(0, 0) &= -(k_1 - k_2)^2[15a_{21}^2a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2]k_1^2 \neq 0.
\end{align*}
\]

Therefore, we obtain that \( \eta \lambda(0, 0) = \eta^2 \lambda(0, 0) = 0, \eta^3 \lambda(0, 0) \neq 0 \).

If the two map germs are \( \mathcal{A} \)-equivalent, their images are locally diffeomorphic. Hence, by Theorem 3.1 the parallel surface \( g' \) at distance \( t = 1/k_1 \) is locally diffeomorphic to the cuspidal butterfly.

(2) We may assume that \( p = (0, 0) \) and that the initial regular surface \( g \) given in Monge form as in (2.1). We remark that \( \kappa_1 \neq \kappa_2 \). Now we prove in the case \( t = 1/\kappa_1(0, 0) = 1/\kappa_1 \).

From Lemma 2.1 and 2.3, we have

\[
a_{30} = a_{21} = 0.
\]

(3.4)
Suppose that \( t = 1/k_1 \). Then we have \( \lambda(0, 0) = 0 \) and

\[
J_{\gamma'}(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & (k_1 - k_2)/k_1 \end{pmatrix}.
\]

Moreover, from (3.4), we have \( \lambda_u(0, 0) = \lambda_v(0, 0) = 0 \). It follows that (0, 0) is a degenerate singular point of \( g' \) with \( \text{rank}(dg'_p) = 1 \). Using (3.4), we obtain that

\[
\det(\text{Hess}(0, 0)) = \frac{(k_1 - k_2)^2}{k_1^4} \begin{vmatrix} a_{40} - 3k_1^3 & a_{31} \\ a_{31} & 2a_{12}^2 + (a_{22} - k_1k_2^2) \\ \end{vmatrix} k_1 - k_2.
\]

and

\[
\det(\text{Hess}(v_1, v_2)k_1(0, 0)) = \begin{vmatrix} a_{40} - 3k_1^3 & a_{31} \\ a_{31} & 2a_{12}^2 + (a_{22} - k_1k_2^2) \\ \end{vmatrix} k_1 - k_2.
\]

Therefore, the sign of \( \det(\text{Hess}(0, 0)) \) is the same as the sign of \( \det(\text{Hess}(v_1, v_2)k_1(0, 0)) \). Besides, since \( \text{rank}(dg'_p) = 1 \), there exists a non-zero vector \( \eta \) with \( dg'_p(\eta) = 0 \). From (3.1), the non-zero vector \( \eta \) has the same direction as the principal vector \( v_1 \). Hence, \( \eta^2 \lambda(0, 0) \neq 0 \) if and only if \( v_1^2 \lambda(0, 0) \neq 0 \). From (3.4), we have

\[
v_1^2 \lambda(0, 0) = -\frac{(a_{40} - 3k_1^3)(k_1 - k_2)^3}{k_1^2}.
\]

Therefore, this shows that (0, 0) is a first order blue ridge point if and only if \( \eta^2 \lambda(0, 0) \neq 0 \) (cf. Lemma 2.1). Applying Theorem 3.2 to the argument indicated above, we obtain (2).

(3) We may assume that \( p = (0, 0) \) and that the initial regular surface \( g \) given in Monge form as in (2.9). We remark that \( \kappa_1(0, 0) = \kappa_2(0, 0) = k \). Suppose that \( t = 1/k \). Then we have \( \lambda(0, 0) = 0 \),

\[
\lambda_u(0, 0) = t(kt - 1)(a_{30} + a_{21}) = 0, \quad \lambda_v(0, 0) = t(kt - 1)(a_{12} + a_{03}) = 0,
\]

and

\[
J_{\gamma'}(0, 0) = \begin{pmatrix} 1 - kt & 0 \\ 0 & 1 - kt \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Hence, (0, 0) is a degenerate singular point of \( g' \) with \( \text{rank}(dg'_p) = 0 \). Moreover, we have

\[
\det(\text{Hess}(0, 0)) = -\frac{1}{k^4} (a_{30}^2a_{03}^2 - 6a_{30}a_{21}a_{12}a_{03} + 4a_{30}a_{12}^3 + 4a_{21}^3a_{03} - 3a_{21}^2a_{12}^2) = -\frac{1}{k^4} \Gamma,
\]

where \( \Gamma \) is as in Section 2.4. It follows that \( \det(\text{Hess}(0, 0)) < 0 \) (resp. \( \det(\text{Hess}(0, 0)) > 0 \)) if and only if \( g(0, 0) \) is a hyperbolic umbilic (resp. elliptic umbilic). Therefore, using Theorem 3.3, we obtain (3). \( \square \)
Remark 3.6. Suppose that \( g(p) \) is a ridge point relative to the principal direction \( v_i \) and a sub-parabolic point relative to the other principal direction \( v_j \). It follow from (2.5), (2.8) and (3.5) that \( \det(\text{Hess}(v_1, v_2)\kappa(p)) = 0 \) if and only if the ridge line relative to \( v_i \) and the sub-parabolic line relative to \( v_j \) are tangent at \( p \).

These theorems imply that the configuration of CPC lines, ridge lines, and sub-parabolic lines determines types of singularities of parallel surfaces. For example, it follows from Theorem 3.4 (1) and Lemma 2.4 (1) that if the CPC line \( \Sigma_{\kappa(u_0, v_0)}(\kappa(u_0, v_0) \neq 0) \) does not meet the ridge line relative \( v_i \) at \( (u_0, v_0) \) then the parallel surface \( g' \) at distance \( t = 1/\kappa(u_0, v_0) \) is the cuspidal edge at \( g'(u_0, v_0) \). Moreover, it follows from Theorem 3.4 (2) and Lemma 2.4 (1) that if CPC line \( \Sigma_{\kappa(u_0, v_0)}(\kappa(u_0, v_0) \neq 0) \) crosses the ridge line relative to the principal vector \( v_i \) and does not cross the sub-parabolic line relative to the other principal vector \( v_j \) at \( (u_0, v_0) \) then the parallel surface \( g' \) at distance \( t = 1/\kappa(u_0, v_0) \) is the swallowtail at \( g'(u_0, v_0) \). Therefore, Figure 1 (i) and (ii) show that there are three swallowtails near \( g'(u_0, v_0) \pm \epsilon \) if \( g(u_0, v_0) \) is an elliptic umbilic which is not flat. Similarly, Figure 1 (iii)–(v) show that there is one swallowtail near \( g'(u_0, v_0) \pm \epsilon \) if \( g(u_0, v_0) \) is a hyperbolic umbilic which is either flat nor right-angled. These bifurcations of parallel surfaces near umbilics are depicted in Figure 4. These are also shown in [1, p. 384].

Figure 4: From top to bottom: Elliptic umbilic, Hyperbolic umbilic.

4 Criteria for \( A_1, A_2, A_3, A_4 \) and \( D_4^+ \) singularities Before we present proof of Theorem 3.4, we shall provide a convenient criteria for \( A_{\leq 4} \) and \( D_4 \) singularities in this section.
We consider the function \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) whose Taylor expansion at \((0, 0)\) is

\[
f(u, v) = \sum_{i,j} \frac{1}{i! j!} c_{ij} u^i v^j.
\]

4.1 Criteria for \(A_k\)-singularities \((k \leq 4)\) We assume that \( f \) is singular at \((0, 0)\) (i.e., \(c_{10} = c_{01} = 0\)). It is well known that the function \( f \) has an \(A_1\)-singularity at \((0, 0)\) if and only if

\[
\begin{pmatrix}
c_{20} & c_{11} \\
c_{11} & c_{02}
\end{pmatrix}
\]

is of full rank. Now we set

\[
c_n(u, v) := \sum_{i+j=n} \frac{1}{i! j!} c_{ij} u^i v^j.
\]

It is easy to see that the following conditions are equivalent.

1. The matrix \(\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix}\) is of rank 1.

2. There exists a non-zero vector \((\lambda, \mu)\) such that

\[
\begin{pmatrix}
c_{20} & c_{11} \\
c_{11} & c_{02}
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

3. There exist a non-zero vector \((\lambda, \mu)\) and non-zero real number \(s\) such that

\[
\begin{pmatrix}
c_{20} & c_{11} \\
c_{11} & c_{02}
\end{pmatrix}
= s \begin{pmatrix}
\mu^2 & -\lambda \mu \\
-\lambda \mu & \lambda^2
\end{pmatrix}.
\]

The rank of the Hesse’s matrix of \( f \) is 1 if and only if one of these conditions holds. Under this assumption, we have the followings.

**Theorem 4.1.**

1. The function \( f \) is \(A_2\)-singularity at \((0, 0)\) if and only if \(c_3(\lambda, \mu) \neq 0\).

2. The function \( f \) is \(A_3\)-singularity at \((0, 0)\) if and only if \(c_3(\lambda, \mu) = 0\),

\[
\hat{c}_4(\lambda, \mu) := c_4(\lambda, \mu) + \frac{1}{8s^3} \begin{vmatrix}
\mu^2 & -\lambda \mu & \lambda^2 \\
c_{30} & c_{21} & c_{12} \\
c_{21} & c_{12} & c_{03}
\end{vmatrix} \neq 0.
\]

3. The function \( f \) is \(A_4\)-singularity at \((0, 0)\) if and only if \(c_3(\lambda, \mu) = \hat{c}_4(\lambda, \mu) = 0\) and one of the following conditions holds.

   (a) \(\lambda \neq 0\), \(c_5(\lambda, \mu) - \frac{1}{s^2 \lambda^2} c_{4e}(\lambda, \mu) c_{3e}(\lambda, \mu) + \frac{1}{2s^2 \lambda^4} c_{3e}(\lambda, \mu)^2 c_{3e}(\lambda, \mu) \neq 0\).

   (b) \(\mu \neq 0\), \(c_5(\lambda, \mu) - \frac{1}{s^2 \mu^2} c_{4e}(\lambda, \mu) c_{3e}(\lambda, \mu) + \frac{1}{2s^2 \mu^4} c_{3e}(\lambda, \mu)^2 c_{3e}(\lambda, \mu) \neq 0\).

Here, \((\lambda, \mu)\) is a non-zero vector and \(s\) is a non-zero real number that satisfy (4.1).
Proof. (1) If $\lambda \neq 0$, the coefficient of $u^2$, $v^2$, and $u^3$ in $f(u, v + (\mu/\lambda)u)$ are $0$, $s\lambda^2/2$, and $c_3(\lambda, \mu)/\lambda^3$ respectively. Hence, we obtain the result. The case that $\mu \neq 0$ is similar.

(2) We assume that $c_3(\lambda, \mu) = 0$. Suppose that $\lambda \neq 0$. Setting $c = c_3(\lambda, \mu)/(s\lambda^4)$, we obtain that the coefficients of $v^2$, $u^2v$, and $u^4$ in $f(u, v + (\mu/\lambda)u - cu^2)$ are $s\lambda^2/2$, 0, and

$$
\frac{1}{s^4} \left( c_3(\lambda, \mu) - \frac{1}{2s^2\lambda^2} c_3^2(\lambda, \mu)^2 \right),
$$

respectively. Since

$$
\begin{vmatrix}
\lambda^2 & -\lambda \mu & \mu^2 \\
c_{30} & c_{21} & c_{12} \\
c_{21} & c_{12} & c_{03}
\end{vmatrix}
$$

$c_3(\lambda, \mu) \neq 0$ implies that (4.2) is not zero. The case that $\mu \neq 0$ is similar.

(3) We keep the notation above and assume $c_3(\lambda, \mu) = \hat{c}_3(\lambda, \mu) = 0$. We shall consider case (a). (Case (b) is similar and we omit the detail.) If $\lambda \neq 0$, the coefficients of $v^2$, $u^2v$, $u^4$, and $u^5$ in $f(u, v + (\mu/\lambda)u - cu^2)$ are $s\lambda^2/2$, 0, 0, and

$$
\frac{1}{s^5} \left( c_3(\lambda, \mu) - \frac{1}{s^2\lambda^2} c_4(\lambda, \mu)c_3(\lambda, \mu) + \frac{1}{2s^2\lambda^4} c_3^2(\lambda, \mu)^2 c_3(\lambda, \mu) \right),
$$

respectively. The case that $\mu \neq 0$ is similar.

\[\Box\]

### 4.2 Criterion for $D_4^+$-singularity

We assume that $c_{10} = c_{01} = c_{20} = c_{11} = c_{02} = 0$. Then $f$ is at least $D_4^+$-singularity at $(0, 0)$. We have the following.

**Theorem 4.2.** The function $f$ is $D_4^+$-singularity (resp. $D_4^-$-singularity) at $(0, 0)$ if and only if

$$
\begin{vmatrix}
c_{30} & 2c_{21} & c_{12} & 0 \\
0 & c_{30} & 2c_{21} & c_{12} \\
c_{21} & 2c_{12} & c_{03} & 0 \\
0 & c_{21} & 2c_{12} & c_{03}
\end{vmatrix}
$$

takes positive values (resp. negative values).

**Proof.** The function $f$ is $D_4^+$-singularity or $D_4^-$-singularity at $(0, 0)$ if the cubic part $c_3$ of $f$ has one real root or three real roots, respectively. The discriminant $\Delta$ of $c_3$ is given by

$$
\Delta = -\frac{1}{48} (a_{30}^2 a_{03}^2 - 6a_{03}a_{21}a_{12}a_{30} + 4a_{30}a_{12}^3 + 4a_{21}^2 a_{03} - 3a_{21}^2 a_{12}^2).
$$

Expanding (4.3), we have

$$
\begin{vmatrix}
c_{30} & 2c_{21} & c_{12} & 0 \\
0 & c_{30} & 2c_{21} & c_{12} \\
c_{21} & 2c_{12} & c_{03} & 0 \\
0 & c_{21} & 2c_{12} & c_{03}
\end{vmatrix} = -48\Delta,
$$

and we complete the proof.  \[\Box\]
5 Singularities of ϕ and K-versality

Let g be given in Monge form as (2.1). If we write down Φ as

\[ \Phi = c_{00} + xu + yv + \frac{1}{2}(\hat{k}_1 u^2 + \hat{k}_2 v^2) + \sum_{i+j\geq 3} \frac{1}{i!j!} c_{ij} u^i v^j, \]

then we obtain that

\[ c_{00} = \frac{t^2 - x^2 - y^2 - z^2}{2}, \quad \hat{k}_i = k_i z - 1 \quad (i = 1, 2), \quad c_{ij} = a_{ij} z \quad (i + j = 3), \]

\[ c_{40} = a_{40} z - 3k_1^2, \quad c_{31} = a_{31} z, \quad c_{22} = a_{22} z - k_1 k_2, \quad c_{13} = a_{13} z, \]

\[ c_{04} = a_{04} z - 3k_2^2, \quad c_{50} = a_{50} z - 10k_1 a_{30}, \quad c_{05} = a_{05} z - 10k_2 a_{03}. \]

We recall that we take a point \( q_0 = (x_0, y_0, z_0) \) or \( q_0 = (x_0, y_0, z_0, t_0) \), where

\[ (x_0, y_0, z_0) = g(u_0, v_0) + t_0 n(u_0, v_0) \quad \text{and} \quad t_0 = \frac{1}{k_i(u_0, v_0)}, \]

and that we set \( \varphi(u, v) = \Phi(u, v, q_0) \) or \( \varphi(u, v) = \Phi'(u, v, q_0) \). Now we assume that \( (u_0, v_0) = (0, 0) \). So we have

\[ (x_0, y_0, z_0) = (0, 0, \frac{1}{k_i}), \quad t_0 = \frac{1}{k_i}. \]

We note that Φ (resp. Φ') is a K-versal unfolding of ϕ if and only if

\[ \mathcal{E}_2 = \langle \varphi, \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{R^2 \times \mathbb{R}^0}, \Phi_y|_{R^2 \times \mathbb{R}^0}, \Phi_z|_{R^2 \times \mathbb{R}^0}, \Phi_i|_{R^2 \times \mathbb{R}^0} \rangle_R + \langle u, v \rangle^{k+1} \]

(resp. \( \mathcal{E}_2 = \langle \varphi, \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi'_x|_{R^2 \times \mathbb{R}^0}, \Phi'_y|_{R^2 \times \mathbb{R}^0}, \Phi'_z|_{R^2 \times \mathbb{R}^0} \rangle_R + \langle u, v \rangle^{k+1} \))

when ϕ is k-determined.

5.1 A₂-singularity

Proof of Theorem 3.4 (1). Using Theorem 4.1 (1), we have ϕ is K-equivalent to A₂ at (0, 0) if and only if one of the following conditions holds:

(a) \( \hat{k}_1 = 0, \hat{k}_2 \neq 0, c_{30} \neq 0 \);

(b) \( \hat{k}_1 \neq 0, \hat{k}_2 = 0, c_{03} \neq 0 \).

We consider Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

\[ z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} \neq 0 \]

in the original coefficients of the Monge form. From Lemma 2.1, this implies the first assertion. We next remark that A₂-singularity is 3-determined. To show K-versality of Φ', it is enough to verify that

\[ \mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi'_x|_{R^2 \times \mathbb{R}^0}, \Phi'_y|_{R^2 \times \mathbb{R}^0}, \Phi'_z|_{R^2 \times \mathbb{R}^0} \rangle_R + \langle u, v \rangle^4. \]

Then the coefficients of \( u^i v^j \) of functions appearing in (5.1) are given by the following table:
We consider Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to the first assertion. We next remark that in the original coefficients of the Monge form, Form Lemma 2.1, these conditions imply the first assertion. We next remark that $A_3$-singularity is 4-determined. To show $\mathcal{K}$-versality of $\Phi'$, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{E_2} + \langle \Phi'_x |_{\mathcal{R}^2} \times \mathbb{Q}_0, \Phi'_y |_{\mathcal{R}^2} \times \mathbb{Q}_0, \Phi'_z |_{\mathcal{R}^2} \times \mathbb{Q}_0 \rangle_{\mathcal{R}} + \langle 1 \rangle_{\mathcal{R}} + \langle u, v \rangle^4.$$  

when $\varphi$ is 3-determined. By using the table appearing in the proof of Theorem 3.4 (1), it follows that $\Phi'$ is also an $\mathcal{R}^*\text{-versal unfolding of } \varphi \text{ when } \varphi \text{ is } A_2\text{-singularity.}$

### 5.2 $A_3$-singularity

**Proof of Theorem 3.4 (2).** Using Theorem 4.1 (2), we have $\varphi$ is $\mathcal{K}$-equivalent to $A_3$ at $(0, 0)$ if and only if one of the following conditions holds:

(a) $\hat{k}_1 = 0, \hat{k}_2 \neq 0, c_{30} = 0, \hat{k}_2 c_{40} - 3c_{21}^2 \neq 0$;

(b) $\hat{k}_1 \neq 0, \hat{k}_2 = 0, c_{03} = 0, \hat{k}_1 c_{04} - 3c_{12}^2 \neq 0$.

We consider Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

$$z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} = 0, \quad 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) \neq 0$$

in the original coefficients of the Monge form. Form Lemma 2.1, these conditions imply the first assertion. We next remark that $A_3$-singularity is 4-determined. To show $\mathcal{K}$-versality of $\Phi'$, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{E_2} + \langle \Phi'_x |_{\mathcal{R}^2} \times \mathbb{Q}_0, \Phi'_y |_{\mathcal{R}^2} \times \mathbb{Q}_0, \Phi'_z |_{\mathcal{R}^2} \times \mathbb{Q}_0 \rangle_{\mathcal{R}} + \langle u, v \rangle^5.$$
Setting $c = c_{21}/(2\hat{k}_2)$ and replacing $v$ by $v - cu^2$, we see that the coefficients of $u^iv^j$ of functions appearing in (5.2) are given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$u$</th>
<th>$v$</th>
<th>$u^2$</th>
<th>$uv$</th>
<th>$v^2$</th>
<th>$u^3$</th>
<th>$u^2v$</th>
<th>$uv^2$</th>
<th>$v^3$</th>
<th>$u^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_i$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_y$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-c$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_z$</td>
<td>$-\hat{z}_0$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}k_1$</td>
<td>$\frac{1}{2}k_2$</td>
<td>0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>$\varphi_u$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}c_{12}$</td>
<td>$\frac{1}{6}\hat{c}_{40}$</td>
<td>$\frac{1}{2}\hat{c}_{31}$</td>
<td>$\frac{1}{2}\hat{c}_{22}$</td>
<td>$\frac{1}{6}c_{13}$</td>
<td>0</td>
<td>$\frac{1}{24}\hat{c}_{30}$</td>
</tr>
<tr>
<td>$\varphi_v$</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>$c_{12}$</td>
<td>$\frac{1}{2}c_{30}$</td>
<td>$\frac{1}{2}c_{31}$</td>
<td>$\frac{1}{2}c_{22}$</td>
<td>$\frac{1}{2}c_{13}$</td>
<td>0</td>
<td>$\frac{1}{24}\hat{c}_{41}$</td>
</tr>
<tr>
<td>$u\varphi_u$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}c_{12}$</td>
<td>$\frac{1}{2}c_{12}$</td>
<td>$\frac{1}{6}c_{13}$</td>
<td>0</td>
<td>$\frac{1}{6}\hat{c}_{40}$</td>
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<tr>
<td>$u\varphi_v$</td>
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<td>0</td>
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</tr>
<tr>
<td>$u^2\varphi_v$</td>
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<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
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<tr>
<td>$u^2\varphi_v$</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The condition $c = 0$ is equivalent to $a_{21} = 0$, that is, the origin is a sub-parabolic point relative to $v_2$ (see Lemma 2.3). Therefore, by using Gauss’s elimination method using boxed elements as pivots, it follows that the matrix presented by this table is full rank, that is, (5.2) holds if and only if the origin is not a sub-parabolic point relative to $v_2$.  

\[ \hat{c}_{40} = (\hat{k}_2c_{40} - 3c_{21})/\hat{k}_2, \quad \hat{c}_{31} = (\hat{k}_2c_{31} - 3c_{21}c_{12})/\hat{k}_2, \quad \hat{c}_{22} = (\hat{k}_2c_{22} - c_{21}c_{13})/\hat{k}_2, \]

\[ \hat{c}_{50} = (\hat{k}_2^2c_{50} - 10\hat{k}_2c_{21}c_{31} + 15c_{21}^2c_{12})/\hat{k}_2^2, \quad \hat{c}_{41} = (\hat{k}_2^2c_{41} - 6\hat{k}_2c_{21}c_{22} + 3c_{21}^2c_{13})/\hat{k}_2^2, \]

and so on. The coefficients mentioned by “*” are not important. The condition $c = 0$ is equivalent to $a_{21} = 0$, that is, the origin is a sub-parabolic point relative to $v_2$.

Remark 5.2. The function $\Phi'$ is an $R^+$-versal unfolding of $\varphi$ if and only if

\[ \mathcal{E}_2 = \langle \varphi_{u}, \varphi_{v} \rangle_{E_2} + \langle \Phi'_{x}, \Phi'_{y}, \Phi'_{z} \rangle_{R^{x,y,z}} + \langle 1 \rangle_{R} + \langle u, v \rangle_{R} \]

when $\varphi$ is 4-determined. By using the table appearing in the proof of Theorem 3.4 (2), it follows that $\Phi'$ is an $R^+$-versal unfolding of $\varphi$ when $\varphi$ is $A_3$-singularity.
5.3 $A_4$-singularity

Proof of Theorem 3.4 (3). From Theorem 4.1 (3), $\varphi$ is $\mathcal{K}$-equivalent to $A_4$ at $(0, 0)$ if and only if one of the following conditions holds:

(a) $\hat{k}_1 = 0, \hat{k}_2 \neq 0, c_{30} = 0, \hat{k}_2 c_{40} - 3 c_{21}^2 = 0, \hat{k}_2^2 c_{50} - 10 \hat{k}_2 c_{21} c_{31} + 15 c_{21}^2 c_{12} \neq 0$;

(b) $\hat{k}_1 \neq 0, \hat{k}_2 = 0, c_{03} = 0, \hat{k}_1 c_{04} - 3 c_{12}^2 = 0, \hat{k}_1^2 c_{05} - 10 \hat{k}_1 c_{12} c_{13} + 15 c_{21} c_{12}^2 \neq 0$.

We work on Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

\[
z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} = 0, \quad 3a_{21}^2 + (a_{40} - 3k_1^3)(k_1 - k_2) = 0, \quad 15a_{21}^2 a_{12} + 10a_{21} a_{31} (k_1 - k_2)^2 + a_{50}(k_1 - k_2)^2 \neq 0,
\]

in the original coefficients of the Monge form. By Lemma 2.1, we obtain the first assertion. We next remark that $A_4$-singularity is 5-determined. To show $\mathcal{K}$-versality of $\Phi$, it is enough to verify that

\begin{equation}
\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle c_2 + \langle \Phi_x |_{\mathcal{R}^{2} \times \mathcal{Q}_0}, \Phi_y |_{\mathcal{R}^{2} \times \mathcal{Q}_0}, \Phi_z |_{\mathcal{R}^{2} \times \mathcal{Q}_0}, \Phi_t |_{\mathcal{R}^{2} \times \mathcal{Q}_0} \rangle_R + \langle u, v \rangle^6
\end{equation}

Setting $c = c_{21}/(2 \hat{k}_2)$ and replacing $v$ by $v - cu^2$, we see that the coefficients of $u^i v^j$ of functions appearing in (5.3) are given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$u$</th>
<th>$v$</th>
<th>$u^2$</th>
<th>$uv$</th>
<th>$v^2$</th>
<th>$u^3$</th>
<th>$u^2v$</th>
<th>$uv^2$</th>
<th>$v^3$</th>
<th>$u^4$</th>
<th>$u^5$</th>
</tr>
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<tbody>
<tr>
<td>$\Phi_x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_y$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_z$</td>
<td>$-z_0$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} \hat{k}_1$</td>
<td>$\frac{1}{2} k_2$</td>
<td>0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\Phi_t$</td>
<td>$l_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\varphi_u$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} c_{12}$</td>
<td>$\frac{1}{2} \hat{c}_{31}$</td>
<td>$\frac{1}{2} \hat{c}_{22}$</td>
<td>$\frac{1}{2} \hat{c}_{13}$</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{50}$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\varphi_v$</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>$c_{12}$</td>
<td>$\frac{1}{2} c_{03}$</td>
<td>$\frac{1}{2} \hat{c}_{31}$</td>
<td>$\frac{1}{2} \hat{c}_{22}$</td>
<td>$\frac{1}{2} \hat{c}_{13}$</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{41}$</td>
<td>*</td>
</tr>
<tr>
<td>$u \varphi_u$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} \hat{k}_2$</td>
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<td>0</td>
<td>$\frac{1}{2} \hat{c}_{12}$</td>
<td>$\frac{1}{2} \hat{c}_{03}$</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{50}$</td>
<td>*</td>
</tr>
<tr>
<td>$v \varphi_u$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{12}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{50}$</td>
</tr>
<tr>
<td>$u \varphi_v$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>$c_{12}$</td>
<td>$\frac{1}{2} c_{03}$</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{31}$</td>
</tr>
<tr>
<td>$v \varphi_v$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>$c_{12}$</td>
<td>$\frac{1}{2} c_{03}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$u^2 \varphi_v$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2} \hat{c}_{31}$</td>
</tr>
<tr>
<td>$u^2 \varphi_v$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v^2 \varphi_v$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\hat{k}_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
We claim that $\Phi$ is $\mathcal{K}$-versal when $\hat{c}_{31} \neq 0$. The condition $\hat{c}_{31} \neq 0$ is equivalent to
\[
3a_{12}a_{21} + a_{31}(k_1 - k_2) \neq 0
\]
in the original coefficients of the Monge form. From Lemma 2.2, $\Phi$ is $\mathcal{K}$-versal unfolding of $\varphi$ if and only if $(0, 0)$ is a regular point of the ridge line relative to $v_1$. 

5.4 $D_4^+$-singularity

Proof of Theorem 3.4 (4). From Theorem 4.2, $\varphi$ is $\mathcal{K}$-equivalent to $D_4^+$ at $(0, 0)$ if
\[
\hat{k}_1 = \hat{k}_2 = 0, \quad \text{and} \quad \begin{vmatrix} c_{30} & 2c_{21} & c_{12} & 0 \\ 0 & c_{30} & 2c_{21} & c_{12} \\ c_{21} & 2c_{12} & c_{03} & 0 \\ 0 & c_{21} & 2c_{12} & c_{03} \end{vmatrix} > 0
\]
These conditions are equivalent to
\[
k_1 = k_2 = \frac{1}{z_0}, \quad \text{and} \quad \begin{vmatrix} a_{30} & 2a_{21} & a_{12} & 0 \\ 0 & a_{30} & 2a_{21} & a_{12} \\ a_{21} & 2a_{12} & a_{03} & 0 \\ 0 & a_{21} & 2a_{12} & a_{03} \end{vmatrix} > 0
\]
in the original coefficients of the Monge form. Therefore, $\varphi$ is $\mathcal{K}$-equivalent to $D_4^+$ at $(0, 0)$ if the origin is an elliptic (see Section 2.4). Since $D_4^+$-singularity is 3-determined, $\Phi$ is $\mathcal{K}$-versal unfolding of $\varphi$ if and only if
\[
E_2 = \langle \varphi, \varphi, \varphi \rangle_{E_2} + \langle \Phi_{r_1} | R^{2 \times \mathbb{R}_0}, \Phi_{r_2} | R^{2 \times \mathbb{R}_0}, \Phi_{r_3} | R^{2 \times \mathbb{R}_0}, \Phi_{r_4} | R^{2 \times \mathbb{R}_0} \rangle_R + \langle u, v \rangle^4
\]
The coefficients of $u'^j v^j$ of functions appearing in (5.4) are given by the following tables:
Thus we obtain that $\Phi$ is $\mathcal{K}$-versal if and only if
\[
\begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{vmatrix}
\neq 0.
\]
This condition is equivalent to
\[
\begin{vmatrix}
1 & 0 & 1 \\
a_{30} & a_{21} & a_{12} \\
a_{21} & a_{12} & a_{03}
\end{vmatrix}
\neq 0
\]
in the original coefficients of the Monge form. Hence, we complete the proof.

Since the proof of Theorem 3.4 (5) is completely parallel to that of Theorem 3.4 (4), we omit the detail. We remark that an elliptic umbilic is not right-angled.

References


28

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