# Versality of central projections of regular surfaces

Shuhei Honda

**Abstract**

We discuss singularities of central projections of a regular surface in $\mathbb{R}^3$. We describe criteria of singularity types of central projections of a given surface in terms of its Monge normal form and discuss their geometric meaning, which is often not clearly understood. We consider all possible central projections of a fixed surface as a central projection unfolding and discuss their $\mathcal{A}$-versality. We obtain geometric criteria of versality for central projection unfoldings. We also observe that geometric meaning of criteria of singularity types of central projections become clear assuming the versality of central projection unfoldings.

## Contents

# 1 Introduction

The **central projection** of a point $z$ from the center $y$ ($\neq z$) onto a plane $H$ in 3-dimensional Euclidean space $\mathbb{R}^3$, that does not contains $y$, is the intersection of the line $\mathcal{L}$ containing $y$ and $z$ with the plane $H$. The center $y = (y_1, y_2, y_3)$ of this projection in often called a **viewpoint** and the line $\mathcal{L}$ is called **viewline**. We are going to investigate central projections regarding centers $y$ as parameters.

Historically speaking, central projections (perspective projections) have been used since the ancient Greece. For star charts Thales of Miletus used the gnomonic projection, which is the central projection of the sphere from the center onto a plane tangent to the sphere. In the Renaissance period, there was interest in central projections as the drawing in perspective. G. Desargues (see, for instance, [6, Theorem 2.32]) gave a mathematical comprehension to the central projections.

Nowadays, computer vision (for instance, [3]) motivates to study singularities of central projection. One good example is to analyze view of pinhole camera model, which is also a central projection. To understand the shape of a surface in $\mathbb{R}^3$, it is important to analyze its contours by central projections. The key step is understanding the distribution of singularities of central projections, which often have geometric meaning. Our main results contribute to give criteria for distribution of singularities in generic context. The theory of singularities enables us to analyze more complex image in computer vision.

In this paper, we investigate singularities of central projections of regular surfaces in $\mathbb{R}^3$ changing center as parameter. Let us prepare several notation. Let $S$ be a surface parameterized by $f(x) : U \to \mathbb{R}^3$. Here $U$ is an open set of $\mathbb{R}^2$ containig the origin. We are interestrd in local behavior of $S$ at $f(0)$, and we express $f$ as in the following form:

$$f = (f_1(x), f_2(x), f_3(x)) : U \to \mathbb{R}^3 : (x_1, x_2) \mapsto f(0) + x_1\boldsymbol{u} + x_2\boldsymbol{v} + Q(x)\boldsymbol{w} \tag{1.1}$$

where $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal frame of $\mathbb{R}^3$. Here, $Q(x)$ denotes a $C^\infty$-function whose $k$-th Taylor polynomial is

$$\sum_{l\geq 2}^{k} H_l(x_1, x_2),\ H_l(x_1, x_2) := \sum_{i+j=l} \frac{a_{ij}}{i!\,j!} {x_1}^i {x_2}^j, \tag{1.2}$$

for any $k$ at the origin. We call the expression (1.1) Monge normal form for $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

We can fix $H$ to be the $z_1z_2$-plane, since we can send $H$ to the $z_1z_2$-plane by certain rotation and translation. In the rest of this paper, we fix $H$ to be the $z_1z_2$-plane. We denote Euclidean inner product by $\langle\ ,\ \rangle$. Then, a restriction of a central projection to $S$ is written as follows:

$$\pi : U \times \mathbb{R}^3 \longrightarrow \mathbb{R}^2 : (x_1, x_2, y) \mapsto \pi_y(x_1, x_2) \tag{1.3}$$

where

$$\pi_y(x_1, x_2) := \begin{pmatrix} \langle t(x,y)f(x) + (1 - t(x,y))\, y,\ \boldsymbol{e}_1\rangle \\ \langle t(x,y)f(x) + (1 - t(x,y))\, y,\ \boldsymbol{e}_2\rangle \end{pmatrix},$$

$$t(x,y) := \frac{y_3}{y_3 - \langle f(x),\ \boldsymbol{e}_3\rangle} \quad \text{and} \quad \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\ :\ \text{the standard basis of } \mathbb{R}^3.$$

$\pi_y$ is written as

$$\frac{1}{y_3 - f_3(x)} \left( y_3 \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} - f_3(x) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right). \tag{1.4}$$

We regard $\pi$ as an unfolding of $\pi_y$ with parameters $y$. We call $\pi$ a **central projection unfolding**. Our main result for versality of $\pi$ is the following Theorem:

**Theorem 1.1.** *Let $S$ be a regular surface parameterized by $f$ as in* (1.1)*. Then, the criteria of the family $\pi$ to be a versal unfolding of the singularities of $\mathcal{A}_e$-codim. $\leq 3$ of $\pi_y$ are given as in Table* 1*.*

| type | criteria for $\mathcal{A}_e$-versality | geometric interpretation |
|---|---|---|
| fold | always | |
| cusp | always | |
| swallowtail | always | |
| butterfly | $2a_{31}\,k_1 - 3a_{21}^2 \neq 0$ | the flecnodal curve is not singular |
| elder butterfly | $2a_{31}\,k_1 - 3a_{21}^2 \neq 0$ and $(a_{60}\,k_1 - 3a_{21}\,a_{50})\,p_1 - 18a_{50}\,k_1 \neq 0$ | the flecnodal curve is not singular and $y$ is not in a special position |
| unimodal | not versal | |
| lips | always | |
| beaks | always | |
| goose<br>ugly goose | $k_2 \neq 0$ | $f$ is not flat umbilic |
| gulls<br>ugly gulls | $a_{40}\,k_2 - 3a_{21}^2 \neq 0$ | $f$ is the first order blue ridge |
| type 12 | not versal | |
| type 16 | not versal | |

Table 1: Criteria of versality of $\pi$ at each singularity

We quickly review the history of mathematical research on central projection from singularity theory viewpoint (cf. [4]). C. T. C. Wall [25] started to consider central projections from a perspective of singularity theory and stated a general transversality theorem due to his student J. M. S. David. He considered "generic" projections including central projections in [7].

J. H. Rieger and M. A. S. Ruas [21, 22] classified all corank one map germs with $\mathcal{A}_e$-codimension $\leq 3$. Criteria of singularities of $\pi_y$ have been given by O. A. Platnova [19] and V. I. Arnold [1, 2]. O. A. Platnova recognized that asymptotic straight lines appear as a set of viewpoints $y$ so that $\pi_y$ is not fold. She states the following paragragh ([19], p. 2798):

> The only exclusions concern some points on isolated asymptotic lines in a hyperbolic domain with fourth order contact (no more than two on a line) and on asymptotic lines passing through parabolic points of the surface (not more than one on a line).

The asymptotic line here is that we call asymptotic straight line (in §2, Definition 2.5). She called the excluded points "$h$-focal" ("$h$" for "hyperbolic") and "$p$-focal" ("$p$" for "parabolic") respectively. Since she mentioned "$p'$-focal" in [19, Table 1], the author believes that she was aware of the following treatment: Once we fix $f$, $\pi_y$ has the same type $\Sigma$ of singularity for a viewpoint $y$ on an asymptotic straight line except several points $y$. We call such a point $\Sigma$-focal point.

Y. Kabata [15] has written criteria of singularities of $\mathcal{A}_e$-codimension $\leq 4$ from plane-to-plane map-germs and applied them to central projections of regular surfaces in the projective space $\mathbb{P}^3$. He also gave the conditions of $\Sigma$-focal point in terms of the coefficients of the Monge normal form $f$. We recall these results in our terminology for criteria of singularity types of a central projection $\pi_y$ in §3.

As an applications of singularities of $\pi_y$, H. Sano, Y. Kabata and J. L. Deolindo Silva and T. Ohmoto [24] classified regular surfaces on $\mathbb{P}^3$ by using the classification of singularities of central projections of them. And related to bifurcations, they have determined local topological types of binary differential equations of asymptotic curves at parabolic point in $\mathbb{P}^3$ ([8]). From these, we are motivated to investigate certain criteria of versality of central projection unfoldings.

Versality for several geometric unfoldings are already investigated in [10] and [14]. T. Fukui and M. Hasegawa show ($\mathcal{K}$)-versality of distance squared unfoldings ([10]). In [14], criteria of $\mathcal{A}$-versality of orthogonal projection unfoldings are given. Both of them are concerned with geometric interpretation of conditions of versality. In this article, we show criteria of $\mathcal{A}$-versality of the central projection in §4. The key step is to compute the $\mathcal{A}_e$-tangent space. The computation is often complicated and we completed them using the aid of computer. The source code of Maxima scripts are available at `https://github.com/Shuhei-singularity123/Versality_of_central_projection_of_regular_surface`.

In §5 we show an application of our criteria of versality of $\pi$ to geometric interpretations of singularities of $\pi_y$. Versal gulls series singularity of central projections is related to contact type with a cone. J. Montaldi [18] defines the notion of contact between two submanifolds and established the relation to $\mathcal{K}$-equivalence which is introduced by J. Mather ([16, §2]). For criteria of contact types of a surface, for instance, T. Fukui, M. Hasegawa, and K. Nakagawa [11] investigated contact type of a regular surface with right circular cylinders in $\mathbb{R}^3$.

# 2 Preliminary

## 2.1 Definitions from differential geometry

We consider a regular surface $S$ parameterized by $f$ as in (1.1). Let

$$E := \langle f_{x_1}, f_{x_1}\rangle,\ F := \langle f_{x_1}, f_{x_2}\rangle,\ G := \langle f_{x_2}, f_{x_2}\rangle$$

and

$$L := \langle f_{x_1x_1}, \mathbf{n}\rangle,\ M := \langle f_{x_1x_2}, \mathbf{n}\rangle,\ N := \langle f_{x_2x_2}, \mathbf{n}\rangle$$

where $\mathbf{n}$ is the unit normal vector $\frac{f_{x_1}\times f_{x_2}}{|f_{x_1}\times f_{x_2}|}$. The function $E$, $F$ and $G$ (resp. $L$, $M$ and $N$) are called the first (resp. second) fundamental of $S$. The **Gauss curvature** is given by

$$K := \frac{LN - M^2}{EG - F^2}.$$

Then,

- If $K > 0$ at $x$, the point $f(x)$ is **elliptic**,
- If $K = 0$ at $x$, the point $f(x)$ is **parabolic**,
- If $K < 0$ at $x$, the point $f(x)$ is **hyperbolic**.

If there is a non-zero vector $\mathbf{v}$ such that

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}\mathbf{v} = \kappa \begin{pmatrix} E & F \\ F & G \end{pmatrix}\mathbf{v} \ \text{ for some } \ \kappa,$$

we call $\kappa$ a **principal curvature** and a unit eigenvector generated by $\mathbf{v}$ for $\kappa$ a **principal vector** on $\mathbb{R}^3$. We set $\kappa_1$ and $\kappa_2$ the principal curvatures of $f$ at $x$. If $\kappa_1 = \kappa_2$ at $x$, we call a point $f(x)$ **umbilic point**. We call a point $f(x)$ **flat umbilic point** if $\kappa_1 = \kappa_2 = 0$ at $x$.

**Definition 2.1.** We assume that $f(0)$ is not an umbilic of a regular surface $S$ parametrized by $f$, with principal vectors $\mathbf{v}_1$ ('blue') and $\mathbf{v}_2$ ('red') corresponding to principal curvature $\kappa_1$, $\kappa_2$. We say that the point $f(0)$ is a $\mathbf{v}_i$**-ridge point** ('blue ridge point' for $i = 1$, 'red ridge point' for $i = 2$) if $\mathbf{v}_i\kappa_i(0) = 0$, where $\mathbf{v}_i\kappa_i$ is the directional derivative of $\kappa_i$ in $\mathbf{v}_i$.
Moreover, $f(0)$ is a $k$**-th order ridge point** relative to $\mathbf{v}_i$ if

$$\mathbf{v}_i^{(m)}\kappa_i(0) = 0 \quad (1 \leq m \leq k) \qquad \text{and} \qquad \mathbf{v}_i^{(k+1)}\kappa_i(0) \neq 0,$$

where $\mathbf{v}_i^{(m)}\kappa_i(0)$ is the $m$-times directional derivative of $\kappa_i$ in $\mathbf{v}_i$. We call the set of ridge points a **ridge line** or **ridges**.

The notion of ridge was introduced by Porteous [20] for the first time.

**Lemma 2.2.** *Let $S$ be the regular surface parameterized by $f$ as in 1.1 and $f(0)$ is a parabolic point. Then, the origin is a first order blue ridge point if and only if*

$$a_{30} = 0 \quad \text{and} \quad 3a_{21}^2 - a_{40}k_2 \neq 0.$$

*Proof.* See [10, Lemma 2.1], for example. □

Bruce and Wilkinson [5] studied subparabolic points in terms of folding maps in details.

**Definition 2.3.** We assume that $f(0)$ is not an umbilic of a regular surface $S$, with principal vectors $\mathbf{v}_1$ ('blue') and $\mathbf{v}_2$ ('red') corresponding to principal curvature $\kappa_1$, $\kappa_2$. We say that the point $f(0)$ is a $\mathbf{v}_i$**-sub-parabolic point** ('blue sub-parabolic point' for $i = 1$, 'red sub-parabolic point' for $i = 2$) if $\mathbf{v}_i\kappa_j(0) = 0 \quad (i \neq j)$. We call the set of sub-parabolic points a **sub-parabolic line**.

**Lemma 2.4.** *Let $S$ be the regular surface parameterized by $f$ as in 1.1 and $f(0)$ is a parabolic point. Then, the origin is not red sub-parabolic point if and only if*

$$a_{21} \neq 0.$$

*Proof.* See [10, Lemma 2.3], for example. □

**Definition 2.5.** We say $(dx_1, dx_2)$ represents an **asymptotic direction** of $S$ at $f(0)$ if the second fundamental form

$$II := Ldx_1^2 + 2Mdx_1dx_2 + Ndx_2^2$$

vanishes at $x = 0$. The tangent space of $S$ at $x = 0$ contains a line $\mathcal{L}$ which is generated by the corresponding direction. We call $\mathcal{L}$ an **asymptotic straight line** of $S$ at $f(0)$.

**Remark 2.6.** Asymptotic lines usually means the integral curves of asymptotic directions on the surface. Thus, we do not call asymptotic straight line as asymptotic line in order to avoid confusion.

**Definition 2.7.** Let $\alpha(t) := (x_1(t), x_2(t))$ be a regular plane curve and let $\beta$ another plane curve given as the zero set of a smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$. We say that the curve $\alpha$ has $(k+1)$**-point contact** ($k$**-th order contact**) at $t_0$ with the curve $\beta$ if $t_0$ is a zero of order $k$ of the function $g(t) = \Phi(\alpha(t)) = \Phi(x_1(t), x_2(t))$, that is,

$$g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0 \;\text{ and }\; g^{(k+1)}(t_0) \neq 0$$

where $g^{(i)}$ denotes the $i^{th}$-derivative of the function $g$.

**Definition 2.8.** A point $p$ on $S$ is a **flecnodal point** if there is an asymptotic straight line through $p$ which has at least 4-point contact with $S$ at $p$. Equivalently, $p$ is a flecnodal point if it is in the closure of the set of points where the projection along an asymptotic direction has a swallowtail singularity. The **flecnodal curve** of $S$ is the set of flecnodal points.

**Theorem 2.9** ([14, Theorem 6.6 (ii)]). *We assume that the origin of regular surface $S$ is hyperbolic and $\pi_y$ has the butterfly singularity at this point. Then, the flecnodal curve of $S$ is not singular if and only if $2k_1a_{31} - 3a_{21}^2 \neq 0$.*

## 2.2 Definitions from singularity theory

In this paper, "smooth" means $C^\infty$. We set $\mathcal{E}_m$ to be the $\mathbb{R}$-algebra of smooth map-germs $\mathbb{R}^m, 0 \to \mathbb{R}$ with a unique maximal ideal $\mathrm{m}_m \coloneqq \langle x_1, \cdots, x_m \rangle_{\mathcal{E}_m}$. We define

$$\mathcal{E}_m^n \coloneqq \{f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, f(0)) : f \text{ is a smooth map germ at } 0\}$$

which is an $\mathcal{E}_m$-module. In particular,

$$\mathrm{m}_m\mathcal{E}_m^n \coloneqq \{f \in \mathcal{E}_m^n : f(0) = 0\}.$$

In this section, suppose that $f$ and $f_i$ $(i = 1, 2)$ be smooth map germs in $\mathcal{E}_m^n$. We say $f_1$ and $f_2$ are **$\mathcal{A}$-equivalent** ($f_1 \sim_{\mathcal{A}} f_2$) if there exist diffeomorphism germs $\varphi$ and $\psi$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^m, 0 & \xrightarrow{f_1} & \mathbb{R}^n, f_1(0) \\ \varphi \downarrow & & \downarrow \psi \\ \mathbb{R}^m, 0 & \xrightarrow{f_2} & \mathbb{R}^n, f_2(0) \end{array} .$$

**Definition 2.10** ($\mathcal{A}$-stability).
1. Let $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \longrightarrow (\mathbb{R}^n, F(0,0))$ be a smooth map germ. If $F(x, 0) = f(x)$, $F$ is called an unfolding of $f$.
2. An unfolding $F$ is trivial if there exist germs of diffeomorphisms $h : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n \times \mathbb{R}^k, 0 \times 0)$ such that
   (*i*) $h(x, 0) = (x, 0)$ and $H(X, 0) = (X, 0)$.
   (*ii*) The following diagram is commutative ;
   $$\begin{array}{ccccc} \mathbb{R}^m \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{(F,\Pi)} & \mathbb{R}^n \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{\Pi'} & \mathbb{R}^k, (0) \\ h \downarrow & & \downarrow H & & \downarrow id \\ \mathbb{R}^m \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{(f,\Pi)} & \mathbb{R}^n \times \mathbb{R}^k, 0 \times 0 & \xrightarrow{\Pi'} & \mathbb{R}^k, (0) \end{array}$$
   where $\Pi : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to \mathbb{R}^k, 0$ is the canonical projection.
3. We call $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ is **$\mathcal{A}$-stable** if every unfolding of $f$ is trivial.

**Definition 2.11** ($\mathcal{A}_e$-versal unfolding).
1. Let $F_i : (\mathbb{R}^m \times \mathbb{R}^{k_i}, 0 \times 0) \to (\mathbb{R}^n, F_i(0,0))(i = 1, 2)$ be unfoldings of $f$. A triplet $(s, t, \varphi)$ is an $\mathcal{A}_e$-morphism from $F_1$ to $F_2$ if $\varphi : (\mathbb{R}^{k_1}, 0) \to (\mathbb{R}^{k_2}, 0)$ is a smooth map germ, $s : (\mathbb{R}^m \times \mathbb{R}^{k_1}, 0 \times 0) \to (\mathbb{R}^m, 0)$ and $t : (\mathbb{R}^n \times \mathbb{R}^{k_1}, F_2(0,0) \times 0) \to (\mathbb{R}^n, F_1(0,0))$ are unfoldings of $id_m$ and $id_n$ respectively such that
$$F_1(x, y) = t(F_2(s(x, y), \varphi(y)), y).$$

2. Let $F : (\mathbb{R}^m \times \mathbb{R}^k, 0) \to (\mathbb{R}^n, F(0,0))$ be unfoldings of $f$ with parameter $y$ in $\mathbb{R}^k$. $F$ is called an $\mathcal{A}_e$**-versal unfolding** if for any unfolding $(\mathbb{R}^m \times \mathbb{R}^l, 0) \to (\mathbb{R}^n, G(0,0))$ of $f$, there exists an $\mathcal{A}_e$-morphism from $G$ to $F$.

Let $\xi : \mathbb{R}^m, 0 \to T\mathbb{R}^n$ be a smooth map germ such that $\Pi \circ \xi = f$ where $\Pi$ is a projection of tangent vector bundle. We call $\xi$ the vector field along $f$ or infinitesimal deformation of $f$. We write $\theta(f)$ for the set of all the vector field along $f$. $\theta(f)$ is a $\mathcal{E}_m$-module. For the identity maps $id_m : \mathbb{R}^m, 0 \to \mathbb{R}^m, 0$ and $id_n : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, we write $\theta_m = \theta(id_m)$ and $\theta_n = \theta(id_n)$ which are the module of vector field germs. We define

$$tf : \theta_m \to \theta(f) : \xi \mapsto df \circ \xi, \quad \omega f : \theta_n \to \theta(f) : \eta \mapsto \eta \circ f$$

and $\mathcal{A}_e$-tangent space of $f$

$$T\mathcal{A}_e(f) \coloneqq tf(\theta_m) + \omega f(\theta_n) \subset \theta(f).$$

Then, the $\mathcal{A}_e$-codimension of $f$ is defined by

$$cod(\mathcal{A}_e, f) \coloneqq \dim_{\mathbb{R}} \frac{\theta(f)}{T\mathcal{A}_e(f)}.$$

**Definition 2.12** ($\mathcal{A}_e$-infinitesimal versal unfolding)**.** Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ, and $(\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of $f$ with parameter $y$ in $\mathbb{R}^k$. Then, $F$ is called an **infinitesimal** $\mathcal{A}_e$**-versal unfolding** if

$$T\mathcal{A}_e(f) + \sum_{i=1}^{k} \mathbb{R} \frac{\partial F}{\partial y_i}(x, 0) = \theta(f)$$

**Theorem 2.13.** *Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ and $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of $f$. Then, $F$ is $\mathcal{A}_e$-versal if and only if $F$ is infinitesimal $\mathcal{A}_e$-versal.*

*Proof.* See [26, Theorem 3. 3 and Theorem 3. 4 (*i*)]. □

**Theorem 2.14.** *Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, f_i(0))$ be a smooth map germ and $F : (\mathbb{R}^m \times \mathbb{R}^k, 0 \times 0) \to (\mathbb{R}^n, F(0,0))$ be an unfolding of $f$. If $f$ is $\mathcal{A}$-stable, any $F$ is $\mathcal{A}_e$-versal.*

*Proof.* From [17, Theorem 1], we know $T\mathcal{A}_e(f) = \theta(f)$ if $f$ is $\mathcal{A}$-stable. □

**Definition 2.15** (finite $\mathcal{A}$-determinacy)**.** A germ $f$ is said to be $k$-$\mathcal{A}$-determined if any $g$ with $j^k g = j^k f$ is $\mathcal{A}$-equivalent to $f$. The least integer $k$ with this property is called the degree of determinacy of $f$. A finitely $\mathcal{A}$-determined germ is a $k$-$\mathcal{A}$-determined germ for integer $k$.

The following Theorem for $k$-$\mathcal{A}$-determinacy is important to prove versality of unfoldings.

**Theorem 2.16** ([26, Thorem 1.2 (*i*)])**.** *For a smooth map germ $f$ in $\mathcal{E}_m^n$,*

$$\mathrm{m}_n^{k+1}\theta(f) \subset T\mathcal{A}_e(f)$$

*if $f$ is $k$-$\mathcal{A}$-determined.*

## 2.3 Criteria of singularity types of central projections

First of all, we recall several results for criteria of singularity type of smooth map germ $g : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}^2, 0$ with corank one at the origin. Let $(x_1, x_2)$ be coordinates of source. We define $\lambda(x_1, x_2) := \det\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right)$ and take an arbitrary vector field $\eta$ near the origin of the source such that $\eta$ spans $\ker dg$ on $\lambda = 0$. We denote $\eta^k\lambda := \eta(\eta^{k-1}\lambda)$.

**Theorem 2.17** (Whitney [27, §4], Saji [23, Theorem 3])**.** *For a plane-to-plane map-germ $g$, $\mathcal{A}$-types of fold, cusp, swallowtail, lips and beaks are characterized by the following table:*

| *Type* | *Normal form* | *Criteria* |
|---|---|---|
| *fold* | $(x_1, x_2^2)$ | $d\lambda(0) \neq 0$, $\eta\lambda(0) \neq 0$ |
| *cusp* | $(x_1, x_1x_2 + x_2^3)$ | $d\lambda(0) \neq 0$, $\eta\lambda(0) = 0$, $\eta^2\lambda(0) \neq 0$ |
| *swallowtail* | $(x_1, x_1x_2 + x_2^4)$ | $d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$ |
| *lips(+), beaks(-)* | $(x_1, x_2^3 \pm x_1^2x_2)$ | $d\lambda(0) = 0$, $\det H_\lambda(0) \neq 0$, $\eta^2\lambda(0) \neq 0$ |

*where* $\det H_\lambda(0)$ *is the Hessian of* $\lambda$ *at the origin.*

We introduce a well-known fact as the following Lemma 2.18.

**Lemma 2.18.** *The projection $\pi_y$ has a singular point at $x = 0$ if and only if the viewline $\mathcal{L}$ is contained in the tangent space of $S$ at $f(0)$.*

Now, we consider criteria of singularity types of $\pi_y$. Thus, from Theorem 2.18, we suppose that $\mathcal{L}$ is a tangent of $f$ at the origin and $p_1, p_2$ are coefficients which satisfy

$$y - f(0) = p_1 f_{x_1}(0) + p_2 f_{x_2}(0). \tag{2.1}$$

Using the results of Kabata [15], we obtain criteria of $\mathcal{A}$-types of $\mathcal{A}_e$-codimension 2 to 4 for corank 1 plane-to-plane map-germ. We summarize preliminal results of criteria of $\mathcal{A}_e$-codimension $\leq 3$ singularity types of $\pi_y$ as the following theorem.

**Theorem 2.19** (Kabata [15])**.** *Suppose the regular surface $S$ is parameterized by $f$ as in* (1.1) *and a viewpoint $y$ is in $\boldsymbol{u}$-axis, that is, $y - f(0) = p_1\,\boldsymbol{u}$. Then, criteria of $\mathcal{A}_e$-codimension $\leq 3$ singularities of $\pi_y$ are written as in table 2.*

**Remark 2.20.** If $f(0)$ is elliptic, $\pi_y$ has only the fold singualrity at 0 for any $y$.

**Remark 2.21.** From criteria of butterfly and elder butterfly singualrites, it turns out that the only exclusions concern some points on isolated asymptotic straight lines in a hyperbolic domain with 4-th order contact (no more than two on a line). We call the excluded points ***h*-focal** ("*h*" for "hyperbolic"). This is introduced by Platnova [19] and is characterized by the coefficients of Monge form $f$ from Kabata [15]. We often call this point **butterfly-focal** point.

In the same way, we define ***u*-focal** point ("u" for "unimodal") as exceptional points characterized as the formula in the table 2.

**Remark 2.22.** 1. As seen in Remark 2.21, we also have an exceptional point on asymptotic straight lines passing through parabolic points of the surface (not more than one on a line). If $a_{30} \neq 0$ for parabolic surface at the origin, the lips or beaks singularities appears from viewpoints on the line except for the point. The exceptional point are called ***p*-focal** point ("*p*" for parabolic) by Platonova [19] and characterized by the condition in the table 2 from Kabata [15]. We often call *p*-focal point **goose-focal** point.

| type | $\mathcal{A}$-normal form | criteria for $\mathcal{A}$-type |
|---|---|---|
| fold | $(x_1, x_2^2)$ | $a_{20} \neq 0$ |
| cusp | $(x_1, x_1x_2 + x_2^3)$ | $a_{20} = 0,\ a_{11} \neq 0,\ a_{30} \neq 0$ |
| swallowtail | $(x_1, x_1x_2 + x_2^4)$ | $a_{20} = 0,\ a_{11} \neq 0,\ a_{30} = 0,\ a_{40} \neq 0$ |
| butterfly | $(x_1,\ x_1x_2 + x_2^5 \pm x_2^7)$ | $a_{20} = 0,\ a_{11} \neq 0,\ a_{30} = a_{40} = 0,$<br>$\begin{pmatrix} (48a_{50}\,a_{70} - 35a_{60}^2)\,a_{11}^2 + 42(a_{21}\,a_{60} - 40a_{31}\,a_{50})\,a_{50}\,a_{11} \\ +\,2205a_{21}^2\,a_{50}^2 \end{pmatrix} p_1^2$<br>$+(-84a_{50}\,a_{60}\,a_{11}^2 + 252a_{21}\,a_{50}^2\,a_{11})\,p_1 + 756a_{50}^2\,a_{11}^2 \neq 0$ |
| elder butterfly | $(x_1,\ x_1x_2 + x_2^5)$ | $a_{20} = 0,\ a_{11} \neq 0,\ a_{30} = a_{40} = 0,$<br>$\begin{pmatrix} (48a_{50}\,a_{70} - 35a_{60}^2)\,a_{11}^2 + 42(a_{21}\,a_{60} - 40a_{31}\,a_{50})\,a_{50}\,a_{11} \\ +\,2205a_{21}^2\,a_{50}^2 \end{pmatrix} p_1^2$<br>$+(-84a_{50}\,a_{60}\,a_{11}^2 + 252a_{21}\,a_{50}^2\,a_{11})\,p_1 + 756a_{50}^2\,a_{11}^2 = 0$ |
| unimodal | $(x_1, x_1x_2 + x_2^6 \pm x_2^8 + \alpha x_2^9)$ | $a_{20} = 0,\ a_{11} \neq 0,\ a_{30} = a_{40} = a_{50} = 0$<br>$\begin{pmatrix} (35a_{60}\,a_{80} - 24a_{70}^2)\,a_{11}^2 + 28(a_{21}\,a_{60}\,a_{70} - 70a_{31}\,a_{60}^2)\,a_{11} \\ +\,2646a_{21}^2\,a_{60}^2 \end{pmatrix} p_1^2$<br>$-28a_{60}\,a_{11}\,(2a_{70}\,a_{11} - 7a_{21}\,a_{60})\,p_1 + 784a_{60}^2\,a_{11}^2 \neq 0$ |
| lips<br>(resp. beaks) | $(x_1, x_2^3 + x_1^2x_2)$<br>(resp. $(x_1, x_2^3 + x_1^2x_2)$) | $a_{20} = a_{11} = 0,\ a_{30} \neq 0,$<br>$H_2(-a_{21}, a_{30}) < H_{3x_1}(-a_{21}, a_{30})\,p_1$ (resp. $>$) |
| goose | $(x_1, x_2^3 + x_1^3x_2)$ | $a_{20} = a_{11} = 0,\ a_{30} \neq 0,$<br>$H_2(-a_{21}, a_{30}) = H_{3x_1}(-a_{21}, a_{30})\,p_1$ and<br>$H_3(-a_{21}, a_{30}) \neq \frac{1}{2}H_{4x_1}(-a_{21}, a_{30})\,p_1$ |
| ugly goose | $(x_1, x_2^3 \pm x_1^4x_2)$ | $a_{20} = a_{11} = 0,\ a_{30} \neq 0,$<br>$H_2(-a_{21}, a_{30}) = H_{3x_1}(-a_{21}, a_{30})\,p_1,$<br>$H_3(-a_{21}, a_{30}) = \frac{1}{2}H_{4x_1}(-a_{21}, a_{30})\,p_1$ and<br>$a_{30}(H_{5x_1}(-a_{21}, a_{30})p_1 - 3H_4(-a_{21}, a_{30}))p_1$<br>$\neq \frac{1}{2}(H_{4x_1x_1}(-a_{21}, a_{30})p_1 - 2H_{3x_1}(-a_{21}, a_{30}))^2$ |
| gulls | $(x_1, x_1x_2^2 + x_2^4 + x_2^5)$ | $a_{20} = a_{11} = 0,\ a_{30} = 0,\ a_{40} \neq 0,\ a_{21} \neq 0,$<br>$(3a_{21}^2\,a_{50} + 5a_{12}\,a_{40}^2 - 10a_{21}\,a_{31}\,a_{40})\,p_1 \neq 5a_{40}\,(a_{40}\,a_{02} - 3a_{21}^2)$ |
| ugly gulls | $(x_1,\ x_1x_2^2 + x_2^4 + x_2^7)$ | $a_{20} = a_{11} = 0,\ a_{30} = 0,\ a_{40} \neq 0,\ a_{21} \neq 0,$<br>$(3a_{21}^2\,a_{50} + 5a_{12}\,a_{40}^2 - 10a_{21}\,a_{31}\,a_{40})\,p_1 = 5a_{40}\,(a_{40}\,a_{02} - 3a_{21}^2),$<br>$\begin{pmatrix} 225a_{21}^3\,a_{40}^2\,a_{70} - 315a_{21}^2\,a_{40}\,(3a_{21}\,a_{50} - 5a_{31}\,a_{40})\,a_{60} \\ -\,1575a_{21}^2\,a_{40}^3\,a_{51} \\ +\begin{pmatrix} 756a_{21}^3\,a_{50}^2 - 3150a_{21}^2\,a_{40}\,(a_{31}\,a_{50} - a_{40}\,a_{41}) \\ -\,1575a_{21}\,a_{22}\,a_{40}^3 + 4200a_{21}\,a_{31}^2\,a_{40}^2 \end{pmatrix} a_{50} \\ -\,5250a_{21}\,a_{31}\,a_{40}^3\,a_{41} \\ -875a_{13}\,a_{40}^5 + 2625(a_{21}\,a_{32} + a_{22}\,a_{31})\,a_{40}^4 - 1750a_{31}^3\,a_{40}^3 \end{pmatrix} p_1^2$<br>$-70a_{40} \begin{pmatrix} 5a_{40} \begin{pmatrix} 9a_{21}^2\,(a_{21}\,a_{60} - 5a_{40}\,a_{41}) \\ -\,5a_{40}^2\,(a_{03}\,a_{40} - 9a_{21}\,a_{22}) \end{pmatrix} \\ -3(3a_{21}\,a_{50} - 5a_{31}\,a_{40}) \begin{pmatrix} 9a_{21}^2\,a_{50} + 5a_{12}\,a_{40}^2 \\ -\,20a_{21}\,a_{31}\,a_{40} \end{pmatrix} \end{pmatrix} p_1$<br>$+3150a_{21}\,a_{40}^2\,(3a_{21}^2\,a_{50} + 5a_{12}\,a_{40}^2 - 10a_{21}\,a_{31}\,a_{40}) \neq 0$ |
| type 12 | $(x_1,\ x_1x_2^2 + x_2^5 + x_2^6)$ | $a_{20} = a_{11} = 0,\ a_{30} = a_{40} = 0,\ a_{50} \neq 0,\ a_{21} \neq 0,$<br>$(a_{21}\,a_{60} - 5a_{31}\,a_{50})p_1 + 6a_{21}\,a_{50} \neq 0$ |
| type 16 | $(x_1,\ x_1^2x_2 + x_2^4 \pm x_2^5)$ | $a_{20} = a_{11} = 0,\ a_{30} = 0,\ a_{40} \neq 0,\ a_{21} = 0,$<br>$(a_{12}\,a_{50} - 10a_{22}\,a_{40} + 10a_{31}^2)p_1^2$<br>$-(a_{50}\,a_{02} - 25a_{12}\,a_{40})p_1 - 5a_{40}\,a_{02} \neq 0$ |

Table 2: Criteria of $\mathcal{A}$-type of $\pi_y$

2. As seen in the above, there is an exceptional point on asymptotic straight lines passing through parabolic points of the surface (not more than one on a line) if $a_{30} = 0$ and $a_{40} \neq 0$ for parabolic surface at the origin. It is called $p'$**-focal** by Platonova [19]. $pi_y$ has the ugly gulls singularity where a viewpoint $y$ is $p'$-focal. Kabata [15] has characterized by the condition in the table 2. We often call $p'$-focal point **gulls-focal** point.

3. In the same way, we define 12**-focal** point ("12" for "type 12 singularity") and 16**-focal** point ("16" for "type 16 singularity") as exceptional points characterized as the formula in the table 2.

# 3 Versality of central projection unfoldings

We can set an orthonormal frame

$$\boldsymbol{u} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

with $\theta$ in $(0, \frac{\pi}{2}]$ by certain rotation and translation. Let $S$ be given by Monge form as in (1.1) where the degree 2 polynomial of $Q(x)$ is written as

$$H_2(x) = k_1\, x_1\, x_2 + \frac{k_2}{2} x_2^2. \tag{3.1}$$

In the rest of this paper, we define the central projection $\pi_y$ and its unfolding $\pi$ in 1 as germs at $x = 0$. Our main claims summarized the table 1 are the following Theorem 3.1 to 3.4.

**Theorem 3.1.** *Suppose the origin is a singularity of $\pi_y$ with $\mathcal{A}_e$-*codimension $\leq 1$. *Then, $\pi$ is an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$.*

**Theorem 3.2.** *1. If the origin is a butterfly singularity of $\pi_y$, then the following two conditions are equivalent.*

(*i*) *$\pi$ is an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$;*

(*ii*) *the flecnodal curve is not singular at* 0, *that is,* $2a_{31}\, k_1 - 3a_{21}^2 \neq 0$.

*2. Suppose that $\pi_y$ has the elder butterfly singularity at the origin. Then, $\pi$ is an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$ if and only if*

$$2a_{31}\, k_1 - 3a_{21}^2 \neq 0 \quad \textit{and} \quad (a_{60}\, k_1 - 3a_{21}\, a_{50})\, p_1 - 18a_{50}\, k_1 \neq 0$$

*The later condition means that there is a special degenerate position of a viewpoint $y$ for $\mathcal{A}_e$-versality.*

*3. Suppose that $\pi_y$ has an unimodal singularity at the origin. Then, $\pi$ is not an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$.*

**Theorem 3.3.** *1. Suppose that $\pi_y$ has a gulls or ugly gulls singularity at the origin. Then, the following two conditions are equivalent.*

(*i*) *$\pi$ is an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$;*

(*ii*) *the origin is the first order blue ridge point, that is,* $a_{40}\, k_2 - 3\, a_{21}^2 \neq 0$.

2. *If $\pi_y$ has a type 12 singularity at the origin, then $\pi$ is not an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$.*

**Theorem 3.4.** *1. If the origin is a goose or ugly goose singularity of $\pi_y$, then the following two conditions are equivalent.*

(*i*) *$\pi$ is an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$;*

(*ii*) *the origin is not flat umbilic point, that is, $k_2 \neq 0$.*

2. *Suppose that the origin is a type 16 singularity of $\pi_y$. Then, $\pi$ is not an $\mathcal{A}_e$-*versal unfolding *of the singularity of $\pi_y$.*

**Remark 3.5.** The conditions (Theorem 3.2 (*ii*), Theorem 3.3 (*ii*) and Theorem 3.4 (*ii*)) above have already appeared as criteria of versality of orthogonal projection (cf.[14, Theorem 6.8]).

From Theorem 2.13 and 2.16, to prove the versality of $\pi$ of a singularity of $\pi_y$ which is $k$-$\mathcal{A}$-determined, we only need to show that the following equality

$$T\mathcal{A}_e\pi_y + \left\langle \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial y_3} \right\rangle_{\mathbb{R}} = \theta(\pi_y). \tag{3.2}$$

holds modulo $\mathrm{m}^k\varepsilon_2$.

We write the $k$-th order Taylor polynomial of the central projection $\pi_y$ at the origin as follows:

$$\pi_y(x) = \sum_{i+j\geq 1}^{k} \begin{pmatrix} c_{ij} \\ d_{ij} \end{pmatrix} \frac{x_1^i\, x_2^j}{i!\, j!}. \tag{3.3}$$

In the proof of the theorems above, we assume that $H_2(x) = k_1\, x_1 x_2 + \frac{k_2}{2} x_2^2$ in (1.2) and suppose that $\mathcal{L}$ is the asymptotic straight line written as $f(0) + t\,\boldsymbol{u}$. Then, $p_1 \neq 0$ and $p_2 = 0$ in (2.1). Thus, the coefficients of the 3-jet of $\pi_y$ are as follows:

$c_{10} = d_{10} = c_{01} = 0,\ d_{01} = c := \frac{y_3}{p_1 \sin\theta} \neq 0,$

$c_{20} = d_{20} = 0,\ c_{11} = -\frac{k_1 c}{\sin\theta},\ d_{11} = \frac{c}{p_1} \neq 0,\ c_{02} = -\frac{k_2 c}{\sin\theta},\ d_{02} = 0,$

$c_{30} = -\frac{a_{30} c}{y_3 \sin\theta},\ c_{21} = -(a_{21}\, p_1 + 2k_1)\frac{c}{p_1 \sin\theta},\ c_{12} = -(a_{12}\, p_1 + k_2)\frac{c}{p_1 \sin\theta},\ c_{03} = -\frac{a_{03} c}{\sin\theta},$
$d_{30} = 0,\ d_{21} = \frac{2c}{p_1^2} \neq 0,\ d_{12} = \frac{2 k_1 c \cos\theta}{p_1 \sin\theta},\ d_{03} = \frac{3 k_2 c \cos\theta}{p_1 \sin\theta}.$

We also assume that $\pi_y(0,0) = (0,0)$.

## 3.1 Proof of Theorem 3.1 for the case of $\mathcal{A}_e$-cod.$\pi_y \leq 1$

### 3.1.1 Fold and cusp

These singularities are stable. It is clear that the central projection unfolding $\pi$ is $\mathcal{A}_e$-versal unfolding in this case by Theorem 2.14.

### 3.1.2 Swallowtail

*Proof of Theorem 3.1 in the hyperbolic case.* The swallowtail singularity is 4-$\mathcal{A}$-deternined. Thus, it is enough to show (3.2) that

$$T\mathcal{A}_e\pi_y + \left\langle \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial y_3} \right\rangle_{\mathbb{R}} \tag{3.4}$$

spans $\theta(\pi_y)$ over $\mathbb{R}$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. From criteria of the swallowtail singularity, $k_1 \neq 0$, $a_{40} \neq 0$ and $a_{30} = 0$. Thus, we have $c_{30} = 0$ and several coefficients of degree 4 monomials of $\pi_y$ at 0 as follows:

$$c_{40} = -\frac{a_{40}\, c}{\sin\theta} \quad \text{and} \quad d_{40} = 0.$$

Since $\begin{pmatrix} 0 \\ O_4 \end{pmatrix} = \frac{1}{d_{01}} O_4 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^5\mathcal{E}_2^2$, we know all degree 4 monomials of the second component $\begin{pmatrix} 0 \\ O_4 \end{pmatrix}$ are contained in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^5\mathcal{E}_2^2$. Working modulo these monomials, $\begin{pmatrix} x_2\, O_3 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_3 \frac{\partial \pi_y}{\partial x_1}$ is contained in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^5 \\ \mathrm{m}_2^4 \end{pmatrix}$. This means that all degree 4 monomials except $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$.

Using $\begin{pmatrix} 0 \\ x_2 O_2 \end{pmatrix} = \frac{1}{d_{01}} x_2\, O_2 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^3 + \mathrm{m}_2^5 \\ \mathrm{m}_2^4 \end{pmatrix}$, degree 3 monomials of the second component except $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ is in $T\mathcal{A}_e\pi_y$. From $\begin{pmatrix} x_2 O_2 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_2 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^3 + \mathrm{m}_2^5 \\ x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \end{pmatrix}$, we know that the degree 3 monomial $\begin{pmatrix} x_2 O_2 \\ 0 \end{pmatrix}$ is contained in $T\mathcal{A}_e\pi_y$.
We also know that degree 2 monomials $\begin{pmatrix} 0 \\ x_2 O_1 \end{pmatrix}$ and $\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}^2 + \mathrm{m}^5 \\ x_2\mathrm{m}^2 + \mathrm{m}^4 \end{pmatrix}$ from

$$\begin{pmatrix} 0 \\ x_2 O_1 \end{pmatrix} = \frac{1}{d_{01}} x_2 O_1 \frac{\partial \pi_y}{\partial x_2} \quad \text{and} \quad \begin{pmatrix} x_2^2 \\ \frac{d_{11}}{c_{11}} x_2^2 \end{pmatrix} = \frac{1}{c_{11}} x_2 \frac{\partial \pi_y}{\partial x_1}.$$

From this, $\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}^2 + \mathrm{m}^5 \\ x_2\mathrm{m} + \mathrm{m}^4 \end{pmatrix}$ from the following two vectors

$$\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 \\ x_1 \frac{\partial \pi_y}{\partial x_1} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{40}/24 \\ c_{11} & c_{40}/6 \end{pmatrix} \begin{pmatrix} x_1 x_2 \boldsymbol{e}_1 \\ x_1^4 \boldsymbol{e}_1 \end{pmatrix}.$$

The determinant of the above matrix is $\frac{a_{40}\, k_1\, p_1^2\, c^4}{8 y_3^2}$. This does not vanish from criteria of swallowtail singularity. Working modulo these monomials, the following elements are written as

$$\begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \frac{1}{d_{01}} \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \frac{1}{d_{01}} \begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2 \rangle \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} = \frac{1}{d_{01}} x_1^3 \frac{\partial \pi_y}{\partial x_2}$$

and are in $T\mathcal{A}_e\pi_y$ modulo $(x_2\mathrm{m} + \mathrm{m}^4)\varepsilon_2$. Therefore, the three monomials $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ are contained in $T\mathcal{A}_e\pi_y/\mathrm{m}^5\varepsilon_2$. From this, we know $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} = \frac{1}{c_{40}} \frac{\partial \pi_y}{\partial x_1}$ is in $T\mathcal{A}_e\pi_y$ modulo $(x_2 + x_2\mathrm{m})\varepsilon_2 + \begin{pmatrix} \mathrm{m}^4 \\ \mathrm{m}^3 \end{pmatrix}$.

Finally, we consider the following four vectors

$$\begin{pmatrix} \frac{\partial \pi_y}{\partial y_2} \\ \frac{\partial \pi_y}{\partial x_2} \\ x_1 \frac{\partial \pi_y}{\partial x_2} \\ x_1^2 \frac{\partial \pi_y}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & -c/p_1 & 0 & -c/p_1^2 \\ c_{11} & d_{11} & c_{21}/2 & d_{21}/2 \\ 0 & d_{01} & c_{11} & d_{11} \\ 0 & 0 & 0 & d_{01} \end{pmatrix} \begin{pmatrix} x_1 \boldsymbol{e}_1 \\ x_1 \boldsymbol{e}_2 \\ x_1^2 \boldsymbol{e}_1 \\ x_1^2 \boldsymbol{e}_2 \end{pmatrix}$$

in (3.4) modulo $(x_2 + x_2\mathrm{m} + \mathrm{m}^3)\varepsilon_2$. The determinant of the above matrix is $\frac{k_1^2\, p_1\, c^6}{y_3^2}$ and does not vanish. It follows that $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$ are contained in (3.4).
Therefore equality (3.2) is satisfied. □

### 3.1.3 Lips/Beaks

*Proof of Theorem 3.1 in the parabolic case.* From criteria of the lips/beaks singularity, $k_1 = 0$, $k_2 \neq 0$ and $a_{30} \neq 0$. Then, we know $c_{11} = 0$ and $c_{02} \neq 0$.
The lips/beaks singularities are 3-$\mathcal{A}$-deternined. Thus, we need to prove equality (3.2) where $k = 3$.

Since $\begin{pmatrix} 0 \\ O_3 \end{pmatrix} = \frac{1}{d_{01}} O_3 \frac{\partial \pi_y}{\partial x_2}$ and $\begin{pmatrix} x_2^3 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^3} \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^3 \\ 0 \end{pmatrix}$, degree 3 monomials $\begin{pmatrix} 0 \\ O_3 \end{pmatrix}$ and $\begin{pmatrix} x_2^3 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e \pi_y / \mathrm{m}_2^4 \mathcal{E}_2^2$. In the same way, we know $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$ is in $T\mathcal{A}_e \pi_y$ modulo $\begin{pmatrix} x_2^3 + \mathrm{m}_2^4 \\ \mathrm{m}_2^3 \end{pmatrix}$ from $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} = \frac{1}{d_{01}} x_2^2 \frac{\partial \pi_y}{\partial x_2}$.

The determinant of the following $6 \times 6$ matrix $D$ defined by the following:

$$\begin{aligned} &{}^t\left( \begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2 \rangle \end{pmatrix}, \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^2 \\ 0 \end{pmatrix}, x_1 \frac{\partial \pi_y}{\partial x_1}, x_2 \frac{\partial \pi_y}{\partial x_1}, x_2 \frac{\partial \pi_y}{\partial x_2}, x_1 x_2 \frac{\partial \pi_y}{\partial x_2} \right) \\ &= D\, {}^t\left( \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 x_2^2 \\ 0 \end{pmatrix} \right) \end{aligned}$$

where

$$D := \begin{pmatrix} d_{01} & d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{01}^2 & 0 & 0 & 2d_{01}\, d_{11} \\ 0 & 0 & 0 & 0 & c_{30}/2 & c_{21} \\ 0 & d_{11} & 0 & c_{30}/2 & c_{21} & c_{12}/2 \\ d_{01} & d_{11} & c_{02} & 0 & c_{21}/2 & c_{12} \\ 0 & d_{01} & 0 & 0 & 0 & c_{02} \end{pmatrix}$$

on $T\mathcal{A}_e \pi_y$ modulo $\begin{pmatrix} x_2^3 + \mathrm{m}_2^4 \\ x_2^2 + \mathrm{m}_2^3 \end{pmatrix}$ is $a_{30} \frac{(a_{12}\, a_{30} - a_{21}^2) p_1 - k_2\, a_{30}}{4 {p_1}^8 \sin^{10}\theta}\, y_3^7$. From criteria of lips and beaks, this does not vanish. Thus, we get monomials $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}$, $\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1 x_2^2 \\ 0 \end{pmatrix}$ in $T\mathcal{A}_e \pi_y / \mathrm{m}_2^4 \mathcal{E}_2^2$.

Next, we consider the generation of degree 1 monomials and remaining degree 2 monomials. A degree 2 monomial $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} = \frac{1}{d_{01}} x_1^2 \frac{\partial \pi_y}{\partial x_2}$ is contained in $T\mathcal{A}_e \pi_y$ modulo $\begin{pmatrix} x_2^2 \\ x_2 + x_2 \mathrm{m}_2 \end{pmatrix} + \mathrm{m}_2^3 \mathcal{E}_2^2$. Furthemore, we consider linear independence of the following elements in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + \mathrm{m}_2^3 \\ x_2 + \mathrm{m}_2^2 \end{pmatrix}$:

$$\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle \boldsymbol{e}_1 \\ \frac{\partial \pi_y}{\partial x_1} \\ \frac{\partial \pi_y}{\partial x_2} \\ x_1 \frac{\partial \pi_y}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & d_{01} & 0 & d_{11} \\ 0 & 0 & c_{30}/2 & c_{21} \\ d_{11} & c_{02} & c_{21}/2 & c_{12} \\ d_{01} & 0 & 0 & c_{02} \end{pmatrix} \begin{pmatrix} x_1 \boldsymbol{e}_2 \\ x_2 \boldsymbol{e}_1 \\ x_1^2 \boldsymbol{e}_1 \\ x_1 x_2 \boldsymbol{e}_1 \end{pmatrix}.$$

The determinant of the above matrix is $-\frac{(a_{12}\,a_{30} - a_{21}^2)p_1 - k_2\,a_{30}}{2{p_1}^5 \sin^6\theta} y_3^4$. From criteria of lips and beaks, This does not vanish. Therefore, $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^4\mathcal{E}_2^2$.

Finally, we get the remaining monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ in (3.4) modulo $\begin{pmatrix} x_2 + \mathrm{m}_2^2 \\ \mathrm{m}_2 \end{pmatrix}$ since $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = -\frac{p_1}{c}\frac{\partial \pi_y}{\partial y_1}$. □

## 3.2 Hyperbolic surfaces with $\mathcal{A}_e$-cod.$\pi_y = 2, 3$

Using criteria of the butterfly singularity, we know two coefficients both of two coefficients $k_1$ and $a_{50}$ does not vanish and $a_{30} = a_{40} = 0$. Thus, coefficients of the 3-jet of $\pi_y$ is the same as in the case of the swallowtail singularity. The coefficients of the 7-jet of $\pi_y$ at 0 are as follows:

$$c_{40} = d_{40} = 0,\ c_{31} = -(a_{31}\,p_1^2 + 3a_{21}\,p_1 + 6k_1)\frac{c^2}{p_1\,y_3},\ d_{31} = \frac{6\,c}{p_1^3},$$

$$c_{50} = -\frac{a_{50}\,p_1\,c^2}{y_3} \neq 0,\ d_{50} = 0,\ c_{41} = -(a_{41}\,p_1^3 + 4a_{31}\,p_1^2 + 12a_{21}\,p_1 + 24k_1)\frac{c^2}{p_1^2\,y_3},\ d_{41} = \frac{24\,c}{p_1^4},$$

$$c_{60} = -(a_{60}\,p_1 + 6a_{50})\frac{c^2}{y_3},\ d_{60} = 0,\ c_{70} = -(a_{70}\,p_1^2 + 7a_{60}\,p_1 + 42a_{50})\frac{c^2}{p_1\,y_3},\ d_{70} = 0.$$

### 3.2.1 Butterfly

*Proof of 1 in Theorem 3.2.* Since the butterfly singularity is 7-$\mathcal{A}$-deternined, it is enough to show that (3.4) spans $\theta(\pi_y)$ over $\mathbb{R}$ modulo $\mathrm{m}_2^8\mathcal{E}_2^2$.

Since $\begin{pmatrix} 0 \\ O_7 \end{pmatrix} = \frac{1}{d_{01}} O_7 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$, we get all monomials $\begin{pmatrix} 0 \\ O_7 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}^8\varepsilon_2$. $\begin{pmatrix} x_2 O_6 \\ 0 \end{pmatrix}$ is generated by $\begin{pmatrix} x_2 O_6 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} O_6 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$ over $\mathbb{R}$.

In the same way, all monomials $\begin{pmatrix} 0 \\ x_2 O_k \end{pmatrix}$ and $\begin{pmatrix} x_2 O_l \\ 0 \end{pmatrix}$ for $k = 3$ to 5, $l = 4$ to 6 are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ \mathrm{m}_2^7 \end{pmatrix}$ since

$$\frac{1}{d_{01}} x_2 O_k \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2 O_k \end{pmatrix} \quad \text{and} \quad \frac{1}{c_{11}} O_l \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2 O_l \\ \frac{d_{11}}{c_{11}} x_2 O_l \end{pmatrix}.$$

From $\begin{pmatrix} x_2^2 O_2 \\ 0 \end{pmatrix} = \frac{1}{c_{11}} {x_2}^2 O_2 \frac{\partial \pi_y}{\partial x_1}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$, $\begin{pmatrix} x_2^2 O_2 \\ 0 \end{pmatrix}$ is in $T\mathcal{A}_e\pi_y$. Thus, degree 3 monomials $\begin{pmatrix} x_2^2 O_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2 O_1 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2^2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$ since

$$\frac{1}{c_{11}} x_2 O_1 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2^2 O_1 \\ \frac{d_{11}}{c_{11}} x_2^2 O_1 \end{pmatrix} \quad \text{and} \quad \frac{1}{d_{01}} x_2^2 O_1 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2^2 O_1 \end{pmatrix}.$$

Using the following linearly independent elements in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2^2\mathrm{m}_2 + x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$

$$\frac{1}{c_{11}} x_2 \frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} x_2^2 \\ \frac{d_{11}}{c_{11}} x_2^2 \end{pmatrix} \quad \text{and} \quad \frac{1}{d_{01}} x_2^2 \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix},$$

we know degree 2 monomials $\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$.

We consider the following fifteen elements

$$\begin{aligned} {}^t &\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1,\, \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_2,\, \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1,\, \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2, \\ \frac{\partial \pi_y}{\partial x_1},\, x_1\frac{\partial \pi_y}{\partial x_1},\, x_2\frac{\partial \pi_y}{\partial x_2},\, x_1^2\frac{\partial \pi_y}{\partial x_1},\, x_1x_2\frac{\partial \pi_y}{\partial x_2},\, x_1^3\frac{\partial \pi_y}{\partial x_1},\, x_1^3\frac{\partial \pi_y}{\partial x_2},\, x_1^2x_2\frac{\partial \pi_y}{\partial x_2},\, x_1^4\frac{\partial \pi_y}{\partial x_2},\, x_1^5\frac{\partial \pi_y}{\partial x_2},\, x_1^6\frac{\partial \pi_y}{\partial x_2} \end{pmatrix} \\ &= D_1\, {}^t\begin{pmatrix} x_2\boldsymbol{e}_1,\, x_2\boldsymbol{e}_2,\, x_1x_2\boldsymbol{e}_1,\, x_1x_2\boldsymbol{e}_2,\, x_1^3\boldsymbol{e}_2,\, x_1^2x_2\boldsymbol{e}_1,\, x_1^2x_2\boldsymbol{e}_2, \\ x_1^4\boldsymbol{e}_1,\, x_1^4\boldsymbol{e}_2,\, x_1^3x_2\boldsymbol{e}_1,\, x_1^5\boldsymbol{e}_1,\, x_1^5\boldsymbol{e}_2,\, x_1^6\boldsymbol{e}_1,\, x_1^6\boldsymbol{e}_2,\, x_1^7\boldsymbol{e}_1 \end{pmatrix}, \end{aligned}$$

$$D_1 := \begin{pmatrix}
0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & c_{31}/6 & c_{50}/120 & 0 & c_{60}/720 & 0 & c_{70}/5040 \\
0 & 0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & c_{50}/120 & 0 & c_{60}/720 & 0 \\
d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & d_{31}/6 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{11} & d_{11} & c_{21} & d_{21} & 0 & c_{31}/2 & d_{31}/2 & c_{50}/24 & 0 & c_{41}/6 & c_{60}/120 & 0 & c_{70}/720 & 0 & * \\
0 & 0 & c_{11} & d_{11} & 0 & c_{21} & d_{21} & 0 & 0 & c_{31}/2 & c_{50}/24 & 0 & c_{60}/120 & 0 & c_{70}/720 \\
0 & d_{01} & c_{11} & d_{11} & 0 & c_{21}/2 & d_{21}/2 & 0 & 0 & c_{31}/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{11} & d_{11} & 0 & 0 & c_{21} & 0 & 0 & c_{50}/24 & 0 & c_{60}/120 \\
0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & c_{50}/24 \\
0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & d_{11} & 0 & c_{21}/2 & d_{21}/2 & c_{31}/6 & d_{31}/6 & c_{41}/24 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & c_{21}/2 & d_{21} & c_{31}/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11}
\end{pmatrix}.$$

in $T\mathcal{A}_e\pi_y$ modulo $x_2^2\mathcal{E}_2^2 + \begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$. The determinant of $D_1$ is

$$\left\{ \begin{pmatrix} (48a_{50}\, a_{70} - 35a_{60}^2)\, k_1^2 \\ +42(a_{21}\, a_{60} - 40a_{31}\, a_{50})\, a_{50}\, k_1 \\ + 2205a_{21}^2\, a_{50}^2 \end{pmatrix} p_1^2 - 84a_{50}\, k_1\, (a_{60}\, k_1 - 3a_{21}\, a_{50})\, p_1 + 756a_{50}^2\, k_1^2 \right\} \frac{a_{50}^2\, k_1^2\, p_1^6\, c^{23}}{10450944000y_3^8}.$$

Thus, the fifteen monomials above are in $T\mathcal{A}_e\pi_y$ since $y$ is not butterfly-focal point.

Finally, we consider the following five elements

$${}^t\left(\frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_2}, \frac{\partial \pi_y}{\partial x_2}, x_1\frac{\partial \pi_y}{\partial x_2}, x_1^2\frac{\partial \pi_y}{\partial x_2}\right) = D_2\, {}^t\left( \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix} \right)$$

where

$$D_2 := \begin{pmatrix} -c/p_1 & 0 & -c/p_1^2 & 0 & c/p_1^3 \\ 0 & -c/p_1 & 0 & -c/p_1^2 & 0 \\ c_{11} & d_{11} & c_{21}/2 & d_{21}/2 & c_{31}/6 \\ 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\ 0 & 0 & 0 & d_{01} & c_{11} \end{pmatrix},$$

$$\frac{\partial^2 \pi_y}{\partial y_1 \partial x_1} = \begin{pmatrix} -\frac{c}{p_1} \\ 0 \end{pmatrix}, \ \frac{1}{2}\frac{\partial^3 \pi_y}{\partial y_1 \partial x_1^2} = \begin{pmatrix} -\frac{c}{p_1^2} \\ 0 \end{pmatrix} \ \text{ and } \ \frac{1}{6}\frac{\partial^3 \pi_y}{\partial y_1 \partial x_1^3} = \begin{pmatrix} -\frac{c}{p_1^3} \\ 0 \end{pmatrix}.$$

The determinant of $D_2$ is $(2a_{31}\,k_1 - 3a_{21}^2)\frac{c^7}{12y_3^2}$. Therefore, the five monomials above are in (3.4) if and only if $2a_{31}k_1 - 3a_{21}^2$ does not vanish. □

**Remark 3.6.** Our source code for computation the determinant of $D_1$ and $D_2$ is available at [12].

### 3.2.2 Elder butterfly

*Proof of 2 in Theorem 3.2.* The elder butterfly singularity is 7-$\mathcal{A}$-determined which is equal to the determinacy of the butterfly singularity. Thus, we should prove equality (3.2) holds for $k = 7$. We know the fifteen elements expressed by $D_1$ in the subsection 3.2.1 are not linearly independent since $y$ is butterfly-focal. The other elements used in the subsection 3.2.1 of (3.4) are linearly independent if $2a_{31}k_1 - 3a_{21}^2$ does not vanish. Thus, we retake the following fifteen elements in (3.4) modulo $x_2^2\mathcal{E}_2^2 + \begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^3 + \mathrm{m}_2^7 \end{pmatrix}$:

$$
{}^t\begin{pmatrix}
\frac{\partial \pi_y}{\partial y_3} + \frac{\partial \pi_y}{\partial y_1}\tan\theta, \\
\langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2, \\
\frac{\partial \pi_y}{\partial x_1},\ x_1\frac{\partial \pi_y}{\partial x_1},\ x_1^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^3\frac{\partial \pi_y}{\partial x_1},\ x_1^3\frac{\partial \pi_y}{\partial x_2},\ x_1^2x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^4\frac{\partial \pi_y}{\partial x_2},\ x_1^5\frac{\partial \pi_y}{\partial x_2},\ x_1^6\frac{\partial \pi_y}{\partial x_2}
\end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{d}_{11} \\ D_{12} \end{pmatrix} {}^t\begin{pmatrix} x_2\boldsymbol{e}_1,\ x_2\boldsymbol{e}_2,\ x_1x_2\boldsymbol{e}_1,\ x_1x_2\boldsymbol{e}_2,\ x_1^3\boldsymbol{e}_2,\ x_1^2x_2\boldsymbol{e}_1,\ x_1^2x_2\boldsymbol{e}_2, \\ x_1^4\boldsymbol{e}_1,\ x_1^4\boldsymbol{e}_2,\ x_1^3x_2\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_2,\ x_1^6\boldsymbol{e}_1,\ x_1^6\boldsymbol{e}_2,\ x_1^7\boldsymbol{e}_1 \end{pmatrix},
$$

where the $(14, 15)$-matrix $D_{12}$ is

$$
\begin{pmatrix}
0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & c_{31}/6 & c_{50}/120 & 0 & c_{60}/720 & 0 & c_{70}/5040 \\
0 & 0 & 0 & c_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & c_{50}/120 & 0 & c_{60}/720 & 0 \\
d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & d_{31}/6 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{01} & 0 & d_{11} & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{11} & d_{11} & c_{21} & d_{21} & 0 & c_{31}/2 & d_{31}/2 & c_{50}/24 & 0 & c_{41}/6 & c_{60}/120 & 0 & c_{70}/720 & 0 & * \\
0 & 0 & c_{11} & d_{11} & 0 & c_{21} & d_{21} & 0 & 0 & c_{31}/2 & c_{50}/24 & 0 & c_{60}/120 & 0 & c_{70}/720 \\
0 & 0 & 0 & 0 & 0 & c_{11} & d_{11} & 0 & 0 & c_{21} & 0 & 0 & c_{50}/24 & 0 & c_{60}/120 \\
0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & 0 & 0 & c_{21}/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & c_{50}/24 \\
0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & d_{11} & c_{02} & c_{21}/2 & d_{21}/2 & c_{31}/6 & d_{31}/6 & c_{41}/24 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & c_{11} & d_{11} & c_{21}/2 & d_{21}/2 & c_{31}/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21}/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & c_{11}
\end{pmatrix}
$$

and

$$
\boldsymbol{d}_{11} := \frac{c^3}{{y_3}^3}\begin{pmatrix}
0,\ -\frac{f_3(0)\,y_3}{c},\ f_3(0)\,k_1\,p_1,\ -\frac{(y_3+f_3(0))\,y_3}{p_1\,c},\ 0, \\
\frac{(2k_1\,(y_3+f_3(0))+a_{21}\,f_3(0)\,p_1)}{2},\ -\frac{(2y_3+f_3(0))\,y_3}{p_1^2\,c},\ 0,\ 0,\ \frac{3a_{21}\,p_1\,(y_3+f_3(0))+6k_1\,(2y_3+f_3(0))+a_{31}\,f_3(0)\,p_1^2}{6\,p_1}, \\
\frac{a_{50}\,f_3(0)\,p_1}{120},\ 0,\ \frac{6a_{50}\,(y_3+f_3(0))+a_{60}\,f_3(0)\,p_1}{720},\ 0,\ \frac{7a_{60}\,p_1\,(y_3+f_3(0))+42a_{50}\,(2a_{50}\,y_3+f_3(0))+a_{70}\,f_3(0)\,p_1^2}{5040\,p_1}
\end{pmatrix}.
$$

The determinant of $D_{12}$ is $a_{50}^3\,{k_1}^3\,((a_{60}\,k_1 - 3a_{21}\,a_{50})\,p_1 - 18a_{50}\,k_1)\frac{p_1^6\,c^{24}}{62208000\,y_3^9}$. Therefore, we get the claim. Our source code for the computation of the determinant of $D_{12}$ is available at [12]. □

### 3.2.3 Unimodal

*Proof of 3 in Theorem 3.2.* From the assumption and criteria, $a_{50} = 0$ and $a_{60}$ does not vanish. The unimodal singularity is 8-$\mathcal{A}$-deternined. If we know versality of this type, we check equality (3.2) where $k = 8$. We consider whether seven monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$, $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ and $\begin{pmatrix} x_1^4 \\ 0 \end{pmatrix}$ are in (3.4) modulo $\mathrm{m}_2^9\mathcal{E}_2^2$. However, we can only choose the following elements in (3.4) modulo $\mathrm{m}_2^9\mathcal{E}_2^2$ to generate the monomials above:

$$
{}^t\left(\frac{\partial \pi_y}{\partial y_1},\ \frac{\partial \pi_y}{\partial y_2},\ \frac{\partial \pi_y}{\partial y_3},\ \frac{\partial \pi_y}{\partial x_2},\ x_1\frac{\partial \pi_y}{\partial x_2},\ x_1^2\frac{\partial \pi_y}{\partial x_2},\ x_1^3\frac{\partial \pi_y}{\partial x_2}\right)
$$

$$
= \begin{pmatrix}
-\frac{y_3}{p_1^2 \sin\theta} & 0 & -\frac{y_3}{p_1^3 \sin\theta} & 0 & -\frac{y_3}{p_1^4 \sin\theta} & 0 & -\frac{y_3}{p_1^5 \sin\theta} \\
0 & -\frac{y_3}{p_1^2 \sin\theta} & 0 & -\frac{y_3}{p_1^3 \sin\theta} & 0 & -\frac{y_3}{p_1^4 \sin\theta} & 0 \\
\frac{y_3\ \cos\theta}{p_1^2 \sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{p_1^3 \sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{p_1^4 \sin^2\theta} & 0 & \frac{y_3\ \cos\theta}{p_1^5 \sin^2\theta} \\
c_{11} & d_{11} & c_{21} & d_{21} & c_{31} & d_{31} & c_{41} \\
0 & d_{01} & c_{11} & d_{11} & c_{21} & d_{21} & c_{31} \\
0 & 0 & 0 & d_{01} & c_{11} & d_{11} & c_{21} \\
0 & 0 & 0 & 0 & 0 & d_{01} & c_{11}
\end{pmatrix}
\begin{pmatrix} x_1\boldsymbol{e}_1 \\ x_1\boldsymbol{e}_2 \\ x_1^2\boldsymbol{e}_1 \\ x_1^2\boldsymbol{e}_2 \\ x_1^3\boldsymbol{e}_1 \\ x_1^3\boldsymbol{e}_2 \\ x_1^4\boldsymbol{e}_1 \end{pmatrix} + \cdots .
$$

From $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ are not linearly independent in this part, these monomials cannot generate the seven elements. Therefore, we know that an unfolding $\pi$ is not versal at the unimodal singularity. □

## 3.3 Parabolic surfaces so that $\pi_y$ has gulls series singularities with $\mathcal{A}_e$-cod.$\pi_y \leq 3$

The Taylor series of central projection $\pi_y$ is (3.3) where $c_{21} \neq 0$ from criteria of the gulls series singularities which are $a_{30} = 0$ and $a_{21} \neq 0$. Several coefficients of the 7-jet of $\pi_y$ are expressed as follows:

$c_{40} = -\frac{a_{40}\, p_1\, c^2}{y_3},\ d_{40} = 0,\ c_{31} = -(a_{31}\, p_1 + 3a_{21})\frac{c^2}{2y_3},\ d_{31} = \frac{6\, c}{p_1^3},$
$c_{22} = -(a_{22}\, p_1^2 + 2a_{12}\, p_1 + 2k_2)\frac{c^2}{p_1\, y_3},\ c_{13} = -(a_{13}\, p_1 + a_{03})\frac{c^2}{y_3},$

$c_{50} = -(a_{50}\, p_1 + 5a_{40})\frac{c^2}{y_3},\ c_{41} = -(a_{41}\, p_1^2 + 4a_{31}\, p_1 + 12a_{21})\frac{c^2}{p_1\, y_3},$
$c_{32} = -(a_{32}\, p_1^3 + 3a_{22}\, p_1^2 + 6a_{12}\, p_1 + 6k_2)\frac{c^2}{p_1^2\, y_3},$

$c_{60} = -(a_{60}\, p_1^2 + 6a_{50}\, p_1 + 30a_{40})\frac{c^2}{p_1\, y_3},\ c_{51} = -(a_{51}\, p_1^3 + 5a_{41}\, p_1^2 + 20a_{31}\, p_1 + 60a_{21})\frac{c^2}{p_1^2\, y_3},$

$c_{70} = -(a_{70}\, p_1^3 + 7a_{60}\, p_1^2 + 42a_{50}\, p_1 + 210a_{40})\frac{c^2}{p_1^2\, y_3}.$

### 3.3.1 Gulls

*Proof of 1 in Theorem 3.3 at gulls singularity.* Since gulls type is 5-$\mathcal{A}$-determined, we should show that equality (3.2) holds for $k = 5$. From criteria of gulls singularity, $a_{40} \neq 0$ and $c_{40} \neq 0$. From the element $\frac{1}{d_{01}}O_5\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ O_5 \end{pmatrix}$, all degree 5 monomial of second component are in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^6\mathcal{E}_2^2$. Thus, we know a monomial $\begin{pmatrix} 0 \\ x_2^4 \end{pmatrix} = \frac{1}{d_{01}^4}\begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2\rangle^4 \end{pmatrix}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^6 \\ \mathrm{m}_2^5 \end{pmatrix}$. As same, a monomial $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix}$ is in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^6 \\ x_2^4 + \mathrm{m}_2^5 \end{pmatrix}$ since $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^5}\langle \pi_y, \boldsymbol{e}_2\rangle^5\boldsymbol{e}_1$. Using the following linearly independent elements of $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^5 + \mathrm{m}_2^6 \\ x_2^4 + \mathrm{m}_2^5 \end{pmatrix}$:

$$x_1 O_3\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02}x_1x_2O_3 \\ d_{01}x_1O_3 \end{pmatrix},\ x_1 O_2\frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} c_{21}x_1^2x_2O_2 + \frac{c_{12}}{2}x_1x_2^2O_2 \\ d_{11}x_1x_2O_2 \end{pmatrix} \text{ and } x_2^3\frac{\partial \pi_y}{\partial x_1} = \begin{pmatrix} c_{21}x_1x_2^4 \\ 0 \end{pmatrix},$$

we get monomials $\begin{pmatrix} 0 \\ x_1O_3 \end{pmatrix}$ and $\begin{pmatrix} x_1x_2O_3 \\ 0 \end{pmatrix}$. Thus, $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^4}\langle \pi_y, \boldsymbol{e}_2\rangle^4\boldsymbol{e}_1$ is in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \\ \mathrm{m}_2^4 \end{pmatrix}$. An degree 3 monomial $\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix} = \frac{1}{d_{01}}\begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2\rangle^3 \end{pmatrix}$ is in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^4 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \\ \mathrm{m}_2^4 \end{pmatrix}$.

We consider the following fourteen elements

$$\begin{aligned} &{}^t\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1\rangle\boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1\rangle\boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle\boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle^2\boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle^2\boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle^3\boldsymbol{e}_1, \\ x_1\frac{\partial \pi_y}{\partial x_1},\ x_2\frac{\partial \pi_y}{\partial x_1},\ x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_1},\ x_2^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_2},\ x_2^2\frac{\partial \pi_y}{\partial x_2} \end{pmatrix} \\ &= D\,{}^t\begin{pmatrix} x_2\boldsymbol{e}_2,\ x_1x_2\boldsymbol{e}_2,\ x_2^2\boldsymbol{e}_1,\ x_2^2\boldsymbol{e}_2,\ x_1^2x_2\boldsymbol{e}_1,\ x_1^2x_2\boldsymbol{e}_2,\ x_1x_2^2\boldsymbol{e}_1,\ x_1x_2^2\boldsymbol{e}_2,\ x_2^3\boldsymbol{e}_1, \\ x_1^4\boldsymbol{e}_1,\ x_1^3x_2\boldsymbol{e}_1,\ x_1^2x_2^2\boldsymbol{e}_1,\ x_1x_2^3\boldsymbol{e}_1,\ x_1^5\boldsymbol{e}_1 \end{pmatrix} \end{aligned}$$

where

$$D := \begin{pmatrix}
0 & 0 & c_{02}/2 & 0 & c_{21}/2 & 0 & c_{12}/2 & 0 & c_{03}/6 & c_{40}/24 & c_{31}/6 & c_{22}/4 & c_{13}/6 & c_{50}/120 \\
0 & 0 & 0 & c_{02}/2 & 0 & c_{21}/2 & 0 & c_{12}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{01} & d_{11} & 0 & 0 & 0 & d_{21}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{01}^2 & 0 & 0 & 0 & 2d_{01}\,d_{11} & 0 & 0 & 0 & 0 & d_{01}\,d_{21}+d_{11}^2 & 0 & 0 \\
0 & 0 & 0 & d_{01}^2 & 0 & 0 & 0 & 2d_{01}\,d_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01}^3 & 0 & 0 & 0 & 3d_{01}^2\,d_{11} & 0 \\
0 & d_{11} & 0 & 0 & c_{21} & d_{21} & c_{12}/2 & 0 & 0 & c_{40}/6 & c_{31}/2 & c_{22}/2 & c_{13}/6 & c_{50}/24 \\
0 & 0 & 0 & d_{11} & 0 & 0 & c_{21} & d_{21} & c_{12}/2 & 0 & c_{40}/6 & c_{31}/2 & c_{22}/2 & 0 \\
d_{01} & d_{11} & c_{02} & 0 & c_{21}/2 & d_{21}/2 & c_{12} & 0 & c_{03}/2 & 0 & c_{31}/6 & c_{22}/2 & c_{13}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & c_{21} & c_{12}/2 & 0 & c_{40}/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & c_{21} & c_{12}/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 \\
0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & c_{21}/2 & c_{12} & c_{03}/2 & 0 \\
0 & 0 & 0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & c_{21}/2 & c_{12} & 0
\end{pmatrix}.$$

The determinant of the matirx $D$ is

$$-\frac{a_{21}^4\,a_{40}(3a_{21}^2\,a_{50}\,p_1 + 5a_{12}\,a_{40}^2\,p_1 - 10a_{21}\,a_{31}\,a_{40}\,p_1 - 5a_{40}^2\,k_2 + 15a_{21}^2\,a_{40})\,p_1^6\,c^{26}}{23040y_3^8}$$

Our source code for computation of the determinant of $D$ is available at [12]. Thus, we can get the monomials above at gulls type.

A monomial $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ is in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}_2^2 \\ x_2\varepsilon_1 \end{pmatrix} + \mathrm{m}_2^4\mathcal{E}_2^2$ from $\frac{1}{d_{01}}x_1^3\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$. In the same way, we know $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} = \frac{1}{d_{01}}x_1^2\frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \\ x_2\varepsilon_1 + \mathrm{m}_2^3 \end{pmatrix}$. A degree 1 monomial $\begin{pmatrix} 0 \\ x_1 \end{pmatrix} = -\frac{p_1}{c}\frac{\partial \pi_y}{\partial y_2}$ is in (3.4) modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \\ x_2 + \mathrm{m}_2^2 \end{pmatrix}$.

We have no other way to generate two monomials $\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$ which is to use pair of elements

$$\left(\frac{\partial \pi_y}{\partial x_1},\ x_1\frac{\partial \pi_y}{\partial x_2}\right) = \left(\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix},\ \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}\right)\begin{pmatrix} c_{21} & c_{02} \\ \frac{c_{40}}{6} & \frac{c_{21}}{2} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2 + x_2\mathrm{m}_2^2 + \mathrm{m}_2^4 \\ \mathrm{m}_2 \end{pmatrix}$. This two elements are linearly independent to each other if and only if $f$ is the 1-st order blue ridge point at the origin (that is, $a_{40}\,k_2 - 3\,a_{21}^2 \neq 0$). Finally, we get remaining monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ by linearly independent elements

$$\frac{\partial \pi_y}{\partial y_1} = \begin{pmatrix} -\frac{c}{p_1}x_1 - \frac{c}{p_1^2}x_1^2 \\ 0 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}x_2 \\ 0 \end{pmatrix} \ \text{ and } \ \frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02}x_2 + \frac{c_{21}}{2}x_1^2 \\ 0 \end{pmatrix}$$

in (3.4) modulo $\begin{pmatrix} x_2\mathrm{m}_2 + \mathrm{m}_2^3 \\ \mathrm{m}_2 \end{pmatrix}$ respectively.

□

### 3.3.2 Ugly gulls

*Proof of 1 in Theorem 3.3 at ugly gulls singularity.* The ugly gulls singularity is 7-$\mathcal{A}$-deternined. Thus, if we know versality of this type, we check the equality (3.2) in the case of $k = 7$. The 4-jet of each derivative of central projection $\pi_y$ is the same as the case of gulls singularity.

If $a_{40}\,k_2 - 3\,a_{21}^2 = 0$, $\pi_y$ is not versal at the origin from the same reason in the gulls case. We assume that $f$ is the 1-st order blue ridge at the origin.

Since $d_{01} \neq 0$ and $\begin{pmatrix} 0 \\ O_7 \end{pmatrix} = \frac{1}{d_{01}} O_7 \frac{\partial \pi_y}{\partial x_2}$ in $T\mathcal{A}_e\pi_y/\mathrm{m}_2^8\mathcal{E}_2^2$, degree 7 monomials of the second component are in $T\mathcal{A}_e\pi_y$. Degree 7 monomials of the first component except $\begin{pmatrix} x_1^7 \\ 0 \end{pmatrix}$ and all degree 6 monomials $\begin{pmatrix} 0 \\ O_6 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} \mathrm{m}_2^8 \\ \mathrm{m}_2^7 \end{pmatrix}$ from the following linearly independent vectors

$$O_5\frac{\partial \pi_y}{\partial x_1} = O_5\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^7 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^7 x_2^7 \\ 0 \end{pmatrix} \ \text{ and } \ O_6\frac{\partial \pi_y}{\partial x_2} = O_6\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}.$$

In the same way, monomials $\begin{pmatrix} x_2^2O_4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2O_4 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ \mathrm{m}_2^6 \end{pmatrix}$ from the following linearly independent vectors

$$x_2O_3\frac{\partial \pi_y}{\partial x_1} = x_2O_3\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^6 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^6 x_2^6 \\ 0 \end{pmatrix} \ \text{ and } \ x_2O_4\frac{\partial \pi_y}{\partial x_2} = x_2O_4\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}.$$

Furthermore, we get $\begin{pmatrix} x_2^3O_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^2O_2 \end{pmatrix}$ from the following linearly independent vectors

$$x_2^2O_1\frac{\partial \pi_y}{\partial x_1} = x_2^2O_1\begin{pmatrix} c_{21}x_1x_2 + c_{12}x_2^2/2 \\ d_{11}x_2 \end{pmatrix}, \ \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^5 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^5 x_2^5 \\ 0 \end{pmatrix} \ \text{ and } \ x_2^2O_2\frac{\partial \pi_y}{\partial x_2} = x_2^2O_2\begin{pmatrix} c_{02}x_2 \\ d_{01} \end{pmatrix}$$

in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^2\mathrm{m}_2^4 + x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \end{pmatrix}$ respectively. We know two elements $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^3\mathrm{m}_2^2 + x_2^2\mathrm{m}_2^4 + x_2\mathrm{m}_2^6 + \mathrm{m}_2^8 \\ x_2^2\mathrm{m}_2^2 + x_2\mathrm{m}_2^4 + \mathrm{m}_2^6 \end{pmatrix}$ from

$$\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^4 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{01}^4 x_2^4 \\ 0 \end{pmatrix} \ \text{ and } \ x_2^3\frac{\partial \pi_y}{\partial x_2} = \begin{pmatrix} c_{02}x_2^4 \\ d_{01}x_2^3 \end{pmatrix}.$$

To show that $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and remaining monomials whose degree is degree 2 or more except $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$, $\begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix}$ are in (3.4), we consider the elements of (3.4) given by the following elemnts:

$$
{}^t\begin{pmatrix}
\frac{\partial \pi_y}{\partial y_3} + \frac{1}{\tan\theta}\frac{\partial \pi_y}{\partial y_1} + \frac{f_3(0)}{y_3\, p_1\, \sin\theta}\begin{pmatrix} \langle \pi_y, \boldsymbol{e}_1\rangle \\ \langle \pi_y, \boldsymbol{e}_2\rangle \end{pmatrix}, \\
\langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_1\rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2,\ \langle \pi_y, \boldsymbol{e}_1\rangle\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle^2 \boldsymbol{e}_1,\ \langle \pi_y, \boldsymbol{e}_2\rangle^3 \boldsymbol{e}_1, \\
x_1\frac{\partial \pi_y}{\partial x_1},\ x_2\frac{\partial \pi_y}{\partial x_1},\ x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_1},\ x_2^2\frac{\partial \pi_y}{\partial x_1},\ x_1x_2\frac{\partial \pi_y}{\partial x_2},\ x_2^2\frac{\partial \pi_y}{\partial x_2}, \\
x_1^2x_2\frac{\partial \pi_y}{\partial x_1},\ x_1^2x_2\frac{\partial \pi_y}{\partial x_2},\ x_1x_2^2\frac{\partial \pi_y}{\partial x_2},\ x_1^4\frac{\partial \pi_y}{\partial x_1},\ x_1^4\frac{\partial \pi_y}{\partial x_2},\ x_1^3x_2\frac{\partial \pi_y}{\partial x_2},\ x_1^5\frac{\partial \pi_y}{\partial x_2}
\end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix} {}^t\begin{pmatrix}
x_2\,\boldsymbol{e}_2,\ x_1x_2\,\boldsymbol{e}_2,\ x_2^2\,\boldsymbol{e}_1,\ x_2^2\,\boldsymbol{e}_2,\ x_1^2x_2\,\boldsymbol{e}_1, \\
x_1^2x_2\,\boldsymbol{e}_2,\ x_1x_2^2\,\boldsymbol{e}_1,\ x_1x_2^2\,\boldsymbol{e}_2,\ x_2^3\,\boldsymbol{e}_1, \\
x_1^4\,\boldsymbol{e}_1,\ x_1^4\,\boldsymbol{e}_2,\ x_1^3x_2\,\boldsymbol{e}_1,\ x_1^3x_2\,\boldsymbol{e}_2,\ x_1^2x_2^2\,\boldsymbol{e}_1,\ x_1x_2^3\,\boldsymbol{e}_1, \\
x_1^5\,\boldsymbol{e}_1,\ x_1^5\,\boldsymbol{e}_2,\ x_1^4x_2\,\boldsymbol{e}_1,\ x_1^3x_2^2\,\boldsymbol{e}_1,\ x_1^6\,\boldsymbol{e}_1,\ x_1^5x_2\,\boldsymbol{e}_1,\ x_1^7\,\boldsymbol{e}_1
\end{pmatrix}
$$

where $D_1$ is the $(21, 22)$-matrix expressed as follows:

$$
\left(\begin{array}{cccccccccccccccccccccc}
0 & 0 & \frac{c_{02}}{2} & 0 & \frac{c_{21}}{2} & 0 & \frac{c_{12}}{2} & 0 & \frac{c_{03}}{6} & \frac{c_{40}}{24} & 0 & \frac{c_{31}}{6} & 0 & \frac{c_{22}}{4} & \frac{c_{13}}{6} & \frac{c_{50}}{120} & 0 & \frac{c_{41}}{24} & \frac{c_{32}}{12} & \frac{c_{60}}{720} & \frac{c_{51}}{120} & \frac{c_{70}}{5040} \\
0 & 0 & 0 & \frac{c_{02}}{2} & 0 & \frac{c_{21}}{2} & 0 & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{24} & 0 & \frac{c_{31}}{6} & 0 & 0 & 0 & \frac{c_{50}}{120} & 0 & 0 & 0 & 0 & 0 \\
d_{01} & d_{11} & 0 & 0 & 0 & \frac{d_{21}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{d_{31}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c_{02}\,d_{01}}{2} & 0 & 0 & 0 & 0 & \frac{c_{21}\,d_{01}}{2} & \frac{\alpha_{13}}{2} & 0 & 0 & \frac{c_{40}\,d_{01}}{24} & \frac{\alpha_{32}}{6} & 0 & \frac{\alpha_{51}}{120} & 0 \\
0 & 0 & d_{01}^2 & 0 & 0 & 0 & 2d_{01}\,d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{22} & 0 & 0 & 0 & 0 & \frac{\beta_{32}}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01}^3 & 0 & 0 & 0 & 0 & 0 & 3d_{01}^2\,d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{11} & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{d_{31}}{2} & \frac{c_{22}}{2} & \frac{c_{13}}{6} & \frac{c_{50}}{24} & 0 & \frac{c_{41}}{6} & \frac{c_{32}}{4} & \frac{c_{60}}{120} & \frac{c_{51}}{24} & \frac{c_{70}}{720} \\
0 & 0 & 0 & d_{11} & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{c_{22}}{2} & 0 & 0 & \frac{c_{50}}{24} & \frac{c_{41}}{6} & 0 & \frac{c_{60}}{120} & 0 \\
d_{01} & d_{11} & c_{02} & 0 & \frac{c_{21}}{2} & \frac{d_{21}}{2} & c_{12} & 0 & \frac{c_{03}}{2} & 0 & 0 & \frac{c_{31}}{6} & \frac{d_{31}}{6} & \frac{c_{22}}{2} & \frac{c_{13}}{2} & 0 & 0 & \frac{c_{41}}{24} & \frac{c_{32}}{6} & 0 & \frac{c_{51}}{120} & 0 \\
0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & c_{21} & d_{21} & \frac{c_{12}}{2} & 0 & \frac{c_{40}}{6} & 0 & \frac{c_{31}}{2} & \frac{c_{22}}{2} & \frac{c_{50}}{24} & \frac{c_{41}}{6} & \frac{c_{60}}{120} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & c_{21} & \frac{c_{12}}{2} & 0 & 0 & \frac{c_{40}}{6} & \frac{c_{31}}{2} & 0 & \frac{c_{50}}{24} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 & 0 & 0 & \frac{c_{40}}{6} & 0 & 0 & 0 \\
0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & 0 & \frac{c_{21}}{2} & \frac{d_{21}}{2} & c_{12} & \frac{c_{03}}{2} & 0 & 0 & \frac{c_{31}}{6} & \frac{c_{22}}{2} & 0 & \frac{c_{41}}{24} & 0 \\
0 & 0 & 0 & d_{01} & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & 0 & \frac{c_{21}}{2} & c_{12} & 0 & 0 & 0 & \frac{c_{31}}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & 0 & \frac{c_{40}}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & c_{02} & 0 & 0 & 0 & \frac{c_{21}}{2} & c_{12} & 0 & \frac{c_{31}}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & 0 & c_{02} & 0 & 0 & 0 & \frac{c_{21}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & \frac{c_{40}}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & d_{11} & c_{02} & 0 & \frac{c_{21}}{2} & c_{12} & \frac{c_{31}}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & 0 & 0 & c_{02} & 0 & \frac{c_{21}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01} & 0 & 0 & 0 & c_{02} & \frac{c_{21}}{2}
\end{array}\right),
$$

$\alpha_{13} := c_{02}\,d_{11} + c_{12}\,d_{01}$, $\alpha_{32} := d_{01}\,c_{31} + 3c_{21}\,d_{11}$, $\alpha_{51} := 5c_{40}\,d_{11} + c_{50}\,d_{01}$,
$\beta_{22} := d_{01}\,d_{21} + d_{11}^2$, $\beta_{32} := d_{01}\,d_{31} + 3d_{11}\,d_{21}$
and

$$
\boldsymbol{d} := \frac{c^2}{p_1\,y_3}\begin{pmatrix} 0, -1, 0, 0, 0, -\frac{2}{p_1}, \frac{k_2}{2\sin\theta}, 0, 0, 0, 0, \frac{a_{21}}{2\sin\theta}, -\frac{3}{p_1^2}, \frac{a_{12}\,p_1+2k_2}{2p_1\sin\theta}, \frac{a_{03}}{6\sin\theta}, \frac{a_{40}}{24\sin\theta}, 0, \\ \frac{a_{31}\,p_1+6a_{21}}{6p_1\sin\theta}, \frac{a_{22}\,p_1^2+4a_{12}\,p_1+6k_2}{4p_1^2\sin\theta}, \frac{a_{50}\,p_1+10a_{40}}{120p_1\sin\theta}, \frac{a_{41}\,p_1^2+8a_{31}\,p_1+36a_{21}}{24p_1^2\sin\theta}, \frac{a_{60}\,p_1^2+12a_{50}\,p_1+90a_{40}}{720p_1^2\sin\theta} \end{pmatrix}.
$$

The determinant of $\begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix}$ does not vanish from the non-degenerate condition of ugly gulls singularity. Our source code for Gauss elimination method of the determinant of $\begin{pmatrix} \boldsymbol{d} \\ D_1 \end{pmatrix}$ is available at [12].

The elements which generate remaining degree 1 to 3 monomials are nothing else the following eight elements:

$${}^t\left(\frac{\partial\pi_y}{\partial y_1},\ \frac{\partial\pi_y}{\partial y_2},\ \begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle\\ 0\end{pmatrix},\ \frac{\partial\pi_y}{\partial x_1},\ \frac{\partial\pi_y}{\partial x_2},\ x_1\frac{\partial\pi_y}{\partial x_2},\ x_1^2\frac{\partial\pi_y}{\partial x_2},\ x_1^3\frac{\partial\pi_y}{\partial x_2}\right)$$
$$= D_2{}^t\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}0\\x_1\end{pmatrix},\begin{pmatrix}x_2\\0\end{pmatrix},\begin{pmatrix}x_1^2\\0\end{pmatrix},\begin{pmatrix}0\\x_1^2\end{pmatrix},\begin{pmatrix}x_1x_2\\0\end{pmatrix},\begin{pmatrix}x_1^3\\0\end{pmatrix},\begin{pmatrix}0\\x_1^3\end{pmatrix}\right)$$

where

$$D_2 := \begin{pmatrix}
-\frac{c}{p_1} & 0 & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3} & 0\\
0 & -\frac{c}{p_1} & 0 & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3}\\
0 & 0 & d_{01} & 0 & 0 & d_{11} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & c_{21} & c_{40}/6 & 0\\
0 & d_{11} & c_{02} & c_{21}/2 & d_{21}/2 & c_{12} & c_{31}/6 & d_{31}/6\\
0 & d_{01} & 0 & 0 & d_{11} & c_{02} & c_{21}/2 & d_{21}/2\\
0 & 0 & 0 & 0 & d_{01} & 0 & 0 & d_{11}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{01}
\end{pmatrix}$$

These are in (3.4) if and only if $a_{40}\,k_2 - 3a_{21}^2 \neq 0$. □

### 3.3.3 Type 12

*Proof of 2 in Theorem 3.3.* From the assumption and criteria, $a_{40} = 0$ and $a_{50}$ does not vanish. The type 12 singularity is 6-$\mathcal{A}$-deternined. Thus, we need to prove equality (3.2) where $k = 6$. We consider whether several seven elements in (3.4) modulo $\mathrm{m}_2^6\mathcal{E}_2^2$ generate seven elements $\begin{pmatrix}x_1\\0\end{pmatrix}$, $\begin{pmatrix}x_2\\0\end{pmatrix}$, $\begin{pmatrix}x_1^2\\0\end{pmatrix}$, $\begin{pmatrix}0\\x_1^2\end{pmatrix}$, $\begin{pmatrix}x_1x_2\\0\end{pmatrix}$, $\begin{pmatrix}x_1^3\\0\end{pmatrix}$ and $\begin{pmatrix}x_1^4\\0\end{pmatrix}$. However, we can only choose the following elements in (3.4) modulo $\mathrm{m}_2^6\mathcal{E}_2^2$:

$${}^t\left(\frac{\partial\pi_y}{\partial y_1},\ \frac{\partial\pi_y}{\partial y_3},\ \begin{pmatrix}\langle\pi_y,\boldsymbol{e}_2\rangle\\ 0\end{pmatrix},\ \frac{\partial\pi_y}{\partial x_1},\ \frac{\partial\pi_y}{\partial x_2},\ x_1\frac{\partial\pi_y}{\partial x_2},\ x_1^2\frac{\partial\pi_y}{\partial x_2}\right)$$
$$= \begin{pmatrix}
-\frac{c}{p_1} & 0 & -\frac{c}{p_1^2} & 0 & 0 & -\frac{c}{p_1^3} & -\frac{c}{p_1^4}\\
\frac{c\,\cos\theta}{y_3} & 0 & \frac{c\,\cos\theta}{p_1\,y_3} & 0 & 0 & \frac{c\,\cos\theta}{p_1^2\,y_3} & \frac{c\,\cos\theta}{p_1^3\,y_3}\\
0 & d_{01} & 0 & 0 & d_{11} & 0 & 0\\
0 & 0 & 0 & 0 & c_{21} & 0 & c_{50}/24\\
0 & c_{02} & c_{21}/2 & d_{21}/2 & c_{12} & c_{31}/6 & c_{41}/24\\
0 & 0 & 0 & d_{11} & c_{02} & c_{21}/2 & c_{31}/6\\
0 & 0 & 0 & d_{01} & 0 & 0 & c_{21}/2
\end{pmatrix}
\begin{pmatrix}x_1\boldsymbol{e}_1\\ x_2\boldsymbol{e}_1\\ x_1^2\boldsymbol{e}_1\\ x_1^2\boldsymbol{e}_2\\ x_1x_2\boldsymbol{e}_1\\ x_1^3\boldsymbol{e}_1\\ x_1^4\boldsymbol{e}_1\end{pmatrix} + \cdots$$

to generate the monomials above. Since $\frac{\partial\pi_y}{\partial y_1}$ and $\frac{\partial\pi_y}{\partial y_3}$ are not linearly independent in this part, these elements cannot generate the seven elements and we know an unfolding $\pi$ is not versal at the type 12 singularity. □

## 3.4 Parabolic surfaces so that $\pi_y$ has goose series singularities with $\mathcal{A}_e$-cod.$\pi_y \leq 3$

The Taylor series of central projection $\pi_y$ is (3.3) where $c_{30} \neq 0$. Thus, the cofficients of terms whose degree is upto 3 are expressed as the same in the case of lips/beaks. Several coefficients of the 5-jet of $\pi_y$ are written as follows:

$c_{40} = -(a_{40}\,p_1 + 4a_{30})\frac{c^2}{y_3},\ d_{40} = 0,$
$c_{31} = -(a_{31}\,p_1 + 3a_{21})\frac{c^2}{y_3},\ d_{31} = (6\sin\theta + a_{30}\,p_1^2\cos\theta)\frac{c^2}{p_1^2\,y_3},$

$c_{22} = -(a_{22}\,p_1^2 + 2a_{12}\,p_1 + 2k_2)\frac{c^2}{p_1\,y_3},\ d_{22} = (2a_{21}\cos\theta)\frac{c^2}{y_3},$
$c_{13} = -(a_{13}\,p_1 + a_{03})\frac{c^2}{y_3},\ d_{13} = (3(a_{12}\,p_1 + 2k_2)\cos\theta)\frac{c^2}{p_1\,y_3},$
$c_{04} = -p_1\,(a_{04} + 6k_2^2\,\frac{c}{y_3}\cos\theta)\frac{c^2}{y_3},\ d_{04} = 4a_{03}\cos\theta\frac{c^2}{y_3},$

$c_{50} = -(a_{50}\,p_1^2 + 5a_{40}\,p_1 + 20a_{30})\frac{c^2}{p_1\,y_3},\ d_{50} = 0,$
$c_{41} = -(a_{41}\,p_1^2 + 4a_{31}\,p_1 + 12a_{21})\frac{c^2}{p_1\,y_3},\ d_{41} = \{24\sin\theta + (a_{40}\,p_1 + 8a_{30})\,p_1^2\cos\theta\}\frac{c^2}{p_1^3\,y_3},$
$c_{32} = -\{(a_{32}\,p_1^3 + 3a_{22}\,p_1^2 + 6a_{12}\,p_1 + 6k_2)\sin\theta + 2a_{30}\,k_2\,p_1^2\cos\theta\}\frac{c^3}{p_1\,y_3^2},$
$d_{32} = 2(a_{31}\,p_1 + 6a_{21})\cos\theta\frac{c^2}{p_1\,y_3},$
$c_{23} = -\{(a_{23}\,p_1^2 + 2a_{13}\,p_1 + 2a_{03})\sin\theta + 6a_{21}\,k_2\,p_1\cos\theta\}\frac{c^3}{y_3^2},$
$d_{23} = 3\,(a_{22}\,p_1^2 + 4a_{12}\,p_1 + 6k_2)\,\cos\theta\frac{c^2}{p_1^2\,y_3},$
$c_{14} = -12k_2\,(a_{12}\,p_1 + k_2)\cos\theta\frac{c^3}{y_3^2} - (a_{14}\,p_1 + a_{04})\frac{c^2}{y_3},\ d_{14} = 4(a_{13}\,p_1 + 2a_{03})\,\cos\theta\frac{c^2}{p_1\,y_3},$
$c_{05} = -20a_{03}\,k_2\cos\theta\frac{p_1\,c^3}{y_3^2} - a_{05}\frac{p_1\,c^2}{y_3}.$

### 3.4.1 Goose

*Proof of 1 in Theorem 3.4 at goose singularity.* Since the goose singularity is 4-$\mathcal{A}$-determined, we should show equality (3.2) holds for $k = 4$. We consider whether all monomial bases of $\mathrm{m}_2\mathcal{E}_2^2/\mathrm{m}_2^5\mathcal{E}_2^2$ are in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. First, we assume that the surface $f$ is not flat umbilic at the origin. Since $d_{01} \neq 0$, $\begin{pmatrix} 0 \\ O_4 \end{pmatrix} = \frac{1}{d_{01}} O_4 \frac{\partial \pi_y}{\partial x_2}$ and $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^4} \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2 \rangle^4 \\ 0 \end{pmatrix}$, degree 4 monomials $\begin{pmatrix} 0 \\ O_4 \end{pmatrix}$ and $\begin{pmatrix} x_2^4 \\ 0 \end{pmatrix}$ are in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^5\mathcal{E}_2^2$. Furthermore, $\begin{pmatrix} 0 \\ x_2^3 \end{pmatrix} = \frac{1}{d_{01}} x_2^3 \frac{\partial \pi_y}{\partial x_2}$ is contained in $T\mathcal{A}_e\pi_y$ modulo $\begin{pmatrix} x_2^4 + \mathrm{m}_2^5 \\ \mathrm{m}_2^4 \end{pmatrix}$.

To show that other monomials except a monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ are in (3.4) modulo $\begin{pmatrix} x_2^4 + \mathrm{m}_2^5 \\ x_2^3 + \mathrm{m}_2^4 \end{pmatrix}$, we consider the elements expressed as the following table:

| | $x_1\,\boldsymbol{e}_2$ | $x_2\,\boldsymbol{e}_1$ | $x_2\,\boldsymbol{e}_2$ | $x_1^2\,\boldsymbol{e}_1$ | $x_1^2\,\boldsymbol{e}_2$ | $x_1x_2\,\boldsymbol{e}_1$ | $x_1x_2\,\boldsymbol{e}_2$ | $x_2^2\,\boldsymbol{e}_1$ | $x_2^2\,\boldsymbol{e}_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\partial\pi_y}{\partial y_2}$ | $-\frac{c}{p_1}$ | 0 | 0 | 0 | $-\frac{c}{p_1^2}$ | 0 | 0 | 0 | $-\frac{k_2\,c^2\,\cos\theta}{2y_3}$ |
| $\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_1$ | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_2$ | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle^2\boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}^2$ | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle^3\boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | $d_{11}$ | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 |
| $\frac{\partial\pi_y}{\partial x_2}$ | $d_{11}$ | $c_{02}$ | 0 | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | $d_{03}/2$ |
| $x_1\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | 0 | 0 |
| $x_2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{11}$ |
| $x_1\,\frac{\partial\pi_y}{\partial x_2}$ | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 |
| $x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 |
| $x_1^2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 |
| $x_2^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}$ |
| $x_1^3\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $x_1^3\,\boldsymbol{e}_1$ | $x_1^3\,\boldsymbol{e}_2$ | $x_1^2x_2\,\boldsymbol{e}_1$ | $x_1^2x_2\,\boldsymbol{e}_2$ | $x_1x_2^2\,\boldsymbol{e}_1$ | $x_1x_2^2\,\boldsymbol{e}_2$ | $x_2^3\,\boldsymbol{e}_1$ | $x_1^4\,\boldsymbol{e}_1$ | $x_1^3x_2\,\boldsymbol{e}_1$ | $x_1^2x_2^2\,\boldsymbol{e}_1$ | $x_1x_2^3\,\boldsymbol{e}_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\partial\pi_y}{\partial y_2}$ | 0 | $-d_{31}/6$ | 0 | $-d_{22}/4$ | 0 | $-d_{13}/6$ | 0 | 0 | 0 | 0 | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_1$ | 0 | 0 | $d_{21}/2$ | 0 | 0 | 0 | $d_{03}/6$ | 0 | $d_{31}/6$ | $d_{22}/4$ | $d_{13}/6$ |
| $\langle\pi_y,\boldsymbol{e}_2\rangle\boldsymbol{e}_2$ | 0 | 0 | 0 | $d_{21}/2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle^2\boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | $2d_{01}\,d_{11}$ | 0 | 0 | 0 | 0 | $d_{01}\,d_{21}+d_{11}^2$ | 0 |
| $\langle\pi_y,\boldsymbol{e}_2\rangle^3\boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}^3$ | 0 | 0 | 0 | $3d_{01}^2\,d_{11}$ |
| $\frac{\partial\pi_y}{\partial x_1}$ | $c_{40}/6$ | 0 | $c_{31}/2$ | $d_{31}/2$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/6$ | $c_{50}/24$ | $c_{41}/6$ | $c_{32}/4$ | $c_{23}/6$ |
| $\frac{\partial\pi_y}{\partial x_2}$ | $c_{31}/6$ | $d_{31}/6$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/2$ | $d_{13}/2$ | $c_{04}/6$ | $c_{41}/24$ | $c_{32}/6$ | $c_{23}/4$ | $c_{14}/6$ |
| $x_1\,\frac{\partial\pi_y}{\partial x_1}$ | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 | 0 | $c_{40}/6$ | $c_{31}/2$ | $c_{22}/2$ | $c_{13}/6$ |
| $x_2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 | $c_{40}/6$ | $c_{31}/2$ | $c_{22}/2$ |
| $x_1\,\frac{\partial\pi_y}{\partial x_2}$ | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | $d_{03}/2$ | 0 | $c_{31}/6$ | $c_{22}/2$ | $c_{13}/2$ | $c_{04}/6$ |
| $x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | 0 | $c_{31}/6$ | $c_{22}/2$ | $c_{13}/2$ |
| $x_1^2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 | 0 | $c_{30}/2$ | $c_{21}$ | $c_{12}/2$ | 0 |
| $x_1x_2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | 0 | 0 | $c_{30}/2$ | $c_{21}$ | $c_{12}/2$ |
| $x_2^2\,\frac{\partial\pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $c_{30}/2$ | $c_{21}$ |
| $x_1^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 | 0 | $c_{21}/2$ | $c_{12}$ | $c_{03}/2$ | 0 |
| $x_1x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 | $c_{21}/2$ | $c_{12}$ | $c_{03}/2$ |
| $x_2^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | $c_{21}/2$ | $c_{12}$ |
| $x_1^3\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | $d_{01}$ | 0 | 0 | 0 | 0 | 0 | 0 | $c_{02}$ | 0 | 0 |
| $x_1^2x_2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | 0 | 0 | $c_{02}$ | 0 |
| $x_1x_2^2\,\frac{\partial\pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | 0 | $c_{02}$ |

where $\alpha_{30} := -\frac{\left(6\sin\theta+a_{30}\,p_1^2\cos\theta\right)c^2}{p_1^2\,y_3}$, $\alpha_{21} := -\frac{a_{21}\,c^2\,\cos\theta}{y_3}$ and $\alpha_{12} := -\frac{(a_{12}\,p_1+2k_2)\,c^2\,\cos\theta}{p_1\,y_3}$.
From Gauss elimination method of which our source code is available on Github [12], the condition of fullrank of this matrix expressed as the table above is the same as criteria of goose singularity. Thus, they are in (3.4).

The remaining monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ is in (3.4) modulo $\begin{pmatrix} x_2+\mathrm{m}_2^2 \\ \mathrm{m}_2 \end{pmatrix}$ since $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = -\frac{p_1}{c}\frac{\partial\pi_y}{\partial y_1}$. Thus, $\pi$ is a versal unfolding of the singularity of $\pi_y$ if $f(0)$ is not flat umbilic.

Next, we consider in the case of flat umbilic, that is, $k_2 = 0$. In this case, we have only way to generate monomials $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}$. That is using the following five elements:

$$
{}^t\left(\begin{pmatrix}\langle \pi_y, \boldsymbol{e}_2\rangle \\ 0\end{pmatrix}, \frac{\partial \pi_y}{\partial y_1}, \frac{\partial \pi_y}{\partial y_3}, \frac{\partial \pi_y}{\partial x_1}, \frac{\partial \pi_y}{\partial x_2}\right) = D\,{}^t\left(\begin{pmatrix}x_1 \\ 0\end{pmatrix}, \begin{pmatrix}x_2 \\ 0\end{pmatrix}, \begin{pmatrix}x_1^2 \\ 0\end{pmatrix}, \begin{pmatrix}x_1x_2 \\ 0\end{pmatrix}\right),
$$

$$
D := \begin{pmatrix} 0 & d_{01} & 0 & d_{11} \\ -\frac{c}{p_1} & 0 & -\frac{c}{p_1^2} & 0 \\ \frac{c}{p_1\tan\theta} & 0 & \frac{c}{p_1^2\tan\theta} & 0 \\ 0 & 0 & c_{30}/2 & c_{21} \\ 0 & 0 & c_{21}/2 & c_{12} \end{pmatrix}.
$$

From criteria of goose singularity and non linearly independency of $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ in this part, the rank of the above matrix is less than 4.
Therefore, we get criteria of versality of $\pi$ at goose singularity. □

#### 3.4.2 Ugly goose

*Proof of 1 in Theorem 3.4 at ugly goose singularity.* From assumption and criteria, $a_{30} \neq 0$. The ugly goose singularity is 5-$\mathcal{A}$-deternined. We should show equality (3.2) holds for $k = 5$. The 3-jet of each derivative of central projection $\pi_y$ is the same in the case of the goose singularity. In the same way of proof at goose singularity, we know $\pi$ is not an $\mathcal{A}_e$-versal if $f(x)$ is flat umbilic. We enough to consider in the case of not flat umbilic.

Since

$$
\begin{pmatrix} 0 \\ O_5 \end{pmatrix} = \frac{1}{d_{01}} O_5 \frac{\partial \pi_y}{\partial x_2}, \quad \begin{pmatrix} 0 \\ x_2^4 + \frac{2d_{01}^3 d_{11}}{d_{01}^4} x_1 x_2^4 \end{pmatrix} = \frac{1}{d_{01}^4} \begin{pmatrix} 0 \\ \langle \pi_y, \boldsymbol{e}_2\rangle^4 \end{pmatrix}, \begin{pmatrix} x_2^5 \\ 0 \end{pmatrix} = \frac{1}{d_{01}^5} \begin{pmatrix} \langle \pi_y, \boldsymbol{e}_2\rangle^5 \\ 0 \end{pmatrix}
$$

in $T\mathcal{A}_e\pi_y$ modulo $\mathrm{m}_2^6\mathcal{E}_2^2$ and $d_{01} \neq 0$, we get $\begin{pmatrix} 0 \\ O_5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_2^4 \end{pmatrix}$ and $\begin{pmatrix} x_2^5 \\ 0 \end{pmatrix}$.

To show that the remaining monomials except $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ are in (3.4) modulo $\begin{pmatrix} x_2^5 + \mathrm{m}_2^6 \\ x_2^4 + \mathrm{m}_2^5 \end{pmatrix}$, we consider the elements in (3.4) expressed as the following table:

| | $x_1\,\boldsymbol{e}_2$ | $x_2\,\boldsymbol{e}_1$ | $x_2\,\boldsymbol{e}_2$ | $x_1^2\,\boldsymbol{e}_1$ | $x_1^2\,\boldsymbol{e}_2$ | $x_1x_2\,\boldsymbol{e}_1$ | $x_1x_2\,\boldsymbol{e}_2$ | $x_2^2\,\boldsymbol{e}_1$ | $x_2^2\,\boldsymbol{e}_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\partial \pi_y}{\partial y_2}$ | $-d_{11}$ | 0 | 0 | 0 | $-d_{21}/2$ | 0 | 0 | 0 | $-d_{03}/6$ |
| $\boldsymbol{d}$ | 0 | 0 | $-\frac{c^2\,f_3(0)}{y_3^2}$ | 0 | 0 | 0 | $-\frac{c^2\,(y_3+f_3(0))}{p_1 y_3^2}$ | $\langle \boldsymbol{d}_{02}, \boldsymbol{e}_1\rangle$ | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1$ | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2$ | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^2 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}^2$ | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^3 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^4 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | $d_{11}$ | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 |
| $\frac{\partial \pi_y}{\partial x_2}$ | $d_{11}$ | $c_{02}$ | 0 | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | $d_{03}/2$ |
| $x_1\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | 0 | 0 |
| $x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{11}$ |
| $x_1\,\frac{\partial \pi_y}{\partial x_2}$ | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 |
| $x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 |
| $x_1^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 |
| $x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}$ |
| $x_1^3\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^3\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^4\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | $x_1^3\,\boldsymbol{e}_1$ | $x_1^3\,\boldsymbol{e}_2$ | $x_1^2x_2\,\boldsymbol{e}_1$ | $x_1^2x_2\,\boldsymbol{e}_2$ | $x_1x_2^2\,\boldsymbol{e}_1$ | $x_1x_2^2\,\boldsymbol{e}_2$ | $x_2^3\,\boldsymbol{e}_1$ | $x_2^3\,\boldsymbol{e}_2$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{\partial \pi_y}{\partial y_2}$ | 0 | $-d_{31}/6$ | 0 | $-d_{22}/4$ | 0 | $-d_{13}/6$ | 0 | $-d_{04}/24$ |
| $\boldsymbol{d}$ | $\langle \boldsymbol{d}_{30}, \boldsymbol{e}_1\rangle$ | 0 | $\langle \boldsymbol{d}_{21}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{21}, \boldsymbol{e}_2\rangle$ | $\langle \boldsymbol{d}_{12}, \boldsymbol{e}_1\rangle$ | 0 | $\langle \boldsymbol{d}_{03}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{03}, \boldsymbol{e}_2\rangle$ |
| $\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_1$ | 0 | 0 | $d_{21}/2$ | 0 | 0 | 0 | $d_{03}/6$ | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle \boldsymbol{e}_2$ | 0 | 0 | 0 | $d_{21}/2$ | 0 | 0 | 0 | $d_{03}/6$ |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^2 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | $2d_{01}\,d_{11}$ | 0 | 0 | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^3 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}^3$ | 0 |
| $\langle \pi_y, \boldsymbol{e}_2\rangle^4 \boldsymbol{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{\partial \pi_y}{\partial x_1}$ | $c_{40}/6$ | 0 | $c_{31}/2$ | $d_{31}/2$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/6$ | $d_{13}/6$ |
| $\frac{\partial \pi_y}{\partial x_2}$ | $c_{31}/6$ | $d_{31}/6$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/2$ | $d_{13}/2$ | $c_{04}/6$ | $d_{04}/6$ |
| $x_1\,\frac{\partial \pi_y}{\partial x_1}$ | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 | 0 | 0 |
| $x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | $c_{30}/2$ | 0 | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | 0 |
| $x_1\,\frac{\partial \pi_y}{\partial x_2}$ | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | $d_{03}/2$ | 0 | 0 |
| $x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | 0 | $c_{03}/2$ | $d_{03}/2$ |
| $x_1^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | $d_{11}$ | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | 0 | 0 |
| $x_2^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{11}$ |
| $x_1^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 | 0 | 0 |
| $x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | $d_{11}$ | $c_{02}$ | 0 |
| $x_1^3\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^2\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^3\,\frac{\partial \pi_y}{\partial x_1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | $d_{01}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | $d_{01}$ | 0 | 0 | 0 | 0 |
| $x_1x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | $d_{01}$ | 0 | 0 |
| $x_2^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d_{01}$ |
| $x_1^4\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3x_2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2^2\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^3\,\frac{\partial \pi_y}{\partial x_2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $x_1^4\,\boldsymbol{e}_1$ | $x_1^4\,\boldsymbol{e}_2$ | $x_1^3x_2\,\boldsymbol{e}_1$ | $x_1^3x_2\,\boldsymbol{e}_2$ | $x_1^2x_2^2\,\boldsymbol{e}_1$ | $x_1^2x_2^2\,\boldsymbol{e}_2$ | $x_1x_2^3\,\boldsymbol{e}_1$ | $x_1x_2^3\,\boldsymbol{e}_2$ | $x_2^4\,\boldsymbol{e}_1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $-d_{41}/24$ | $0$ | $-d_{32}/12$ | $0$ | $-d_{23}/12$ | $0$ | $-d_{14}/24$ | $0$ |
| $\langle \boldsymbol{d}_{40}, \boldsymbol{e}_1\rangle$ | $0$ | $\langle \boldsymbol{d}_{31}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{31}, \boldsymbol{e}_2\rangle$ | $\langle \boldsymbol{d}_{22}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{22}, \boldsymbol{e}_2\rangle$ | $\langle \boldsymbol{d}_{13}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{13}, \boldsymbol{e}_2\rangle$ | $\langle \boldsymbol{d}_{04}, \boldsymbol{e}_1\rangle$ |
| $0$ | $0$ | $d_{31}/6$ | $0$ | $d_{22}/4$ | $0$ | $d_{13}/6$ | $0$ | $d_{04}/24$ |
| $0$ | $0$ | $0$ | $d_{31}/6$ | $0$ | $d_{22}/4$ | $0$ | $d_{13}/6$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $d_{01}\,d_{21} + d_{11}^2$ | $0$ | $0$ | $0$ | $d_{01}\,d_{03}/3$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $3d_{01}^2\,d_{11}$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $d_{01}^4$ |
| $c_{50}/24$ | $0$ | $c_{41}/6$ | $d_{41}/6$ | $c_{32}/4$ | $d_{32}/4$ | $c_{23}/6$ | $d_{23}/6$ | $c_{14}/24$ |
| $c_{41}/24$ | $d_{41}/24$ | $c_{32}/6$ | $d_{32}/6$ | $c_{23}/4$ | $d_{23}/4$ | $c_{14}/6$ | $d_{14}/6$ | $c_{05}/24$ |
| $c_{40}/6$ | $0$ | $c_{31}/2$ | $d_{31}/2$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/6$ | $d_{13}/6$ | $0$ |
| $0$ | $0$ | $c_{40}/6$ | $0$ | $c_{31}/2$ | $d_{31}/2$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/6$ |
| $c_{31}/6$ | $d_{31}/6$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/2$ | $d_{13}/2$ | $c_{04}/6$ | $d_{04}/6$ | $0$ |
| $0$ | $0$ | $c_{31}/6$ | $d_{31}/6$ | $c_{22}/2$ | $d_{22}/2$ | $c_{13}/2$ | $d_{13}/2$ | $c_{04}/6$ |
| $c_{30}/2$ | $0$ | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $c_{30}/2$ | $0$ | $c_{21}$ | $d_{21}$ | $c_{12}/2$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $c_{30}/2$ | $0$ | $c_{21}$ | $d_{21}$ | $c_{12}/2$ |
| $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | $0$ | $c_{03}/2$ | $d_{03}/2$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | $0$ | $c_{03}/2$ | $d_{03}/2$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $c_{21}/2$ | $d_{21}/2$ | $c_{12}$ | $0$ | $c_{03}/2$ |
| $0$ | $0$ | $0$ | $d_{11}$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $d_{11}$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $d_{11}$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $d_{11}$ | $c_{02}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $d_{11}$ | $c_{02}$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $d_{11}$ | $c_{02}$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $d_{11}$ | $c_{02}$ |
| $0$ | $d_{01}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $d_{01}$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $d_{01}$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $d_{01}$ | $0$ |

| $x_1^5\,\boldsymbol{e}_1$ | $x_1^4x_2\,\boldsymbol{e}_1$ | $x_1^3x_2^2\,\boldsymbol{e}_1$ | $x_1^2x_2^3\,\boldsymbol{e}_1$ | $x_1x_2^4\,\boldsymbol{e}_1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\langle \boldsymbol{d}_{50}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{41}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{32}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{23}, \boldsymbol{e}_1\rangle$ | $\langle \boldsymbol{d}_{14}, \boldsymbol{e}_1\rangle$ |
| $0$ | $d_{41}/24$ | $d_{32}/12$ | $d_{23}/12$ | $d_{14}/24$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $(d_{01}\,d_{31} + 3d_{11}\,d_{21})/3$ | $d_{01}\,d_{22}/2$ | $(d_{01}\,d_{13} + d_{03}\,d_{11})/3$ |
| $0$ | $0$ | $0$ | $3d_{01}\,(d_{01}\,d_{21} + 2d_{11}^2)/2$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $4d_{01}^3\,d_{11}$ |
| $*$ | $*$ | $*$ | $*$ | $*$ |
| $*$ | $*$ | $*$ | $*$ | $*$ |
| $c_{50}/24$ | $c_{41}/6$ | $c_{32}/4$ | $c_{23}/6$ | $c_{14}/24$ |
| $0$ | $c_{50}/24$ | $c_{41}/6$ | $c_{32}/4$ | $c_{23}/6$ |
| $c_{41}/24$ | $c_{32}/6$ | $c_{23}/4$ | $c_{14}/6$ | $c_{05}/24$ |
| $0$ | $c_{41}/24$ | $c_{32}/6$ | $c_{23}/4$ | $c_{14}/6$ |
| $c_{40}/6$ | $c_{31}/2$ | $c_{22}/2$ | $c_{13}/6$ | $0$ |
| $0$ | $c_{40}/6$ | $c_{31}/2$ | $c_{22}/2$ | $c_{13}/6$ |
| $0$ | $0$ | $c_{40}/6$ | $c_{31}/2$ | $c_{22}/2$ |
| $c_{31}/6$ | $c_{22}/2$ | $c_{13}/2$ | $c_{04}/6$ | $0$ |
| $0$ | $c_{31}/6$ | $c_{22}/2$ | $c_{13}/2$ | $c_{04}/6$ |
| $0$ | $0$ | $c_{31}/6$ | $c_{22}/2$ | $c_{13}/2$ |
| $c_{30}/2$ | $c_{21}$ | $c_{12}/2$ | $0$ | $0$ |
| $0$ | $c_{30}/2$ | $c_{21}$ | $c_{12}/2$ | $0$ |
| $0$ | $0$ | $c_{30}/2$ | $c_{21}$ | $c_{12}/2$ |
| $0$ | $0$ | $0$ | $c_{30}/2$ | $c_{21}$ |
| $c_{21}/2$ | $c_{12}$ | $c_{03}/2$ | $0$ | $0$ |
| $0$ | $c_{21}/2$ | $c_{12}$ | $c_{03}/2$ | $0$ |
| $0$ | $0$ | $c_{21}/2$ | $c_{12}$ | $c_{03}/2$ |
| $0$ | $0$ | $0$ | $c_{21}/2$ | $c_{12}$ |
| $0$ | $c_{02}$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $c_{02}$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $c_{02}$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $c_{02}$ |

where $\boldsymbol{d} \coloneqq \frac{\partial \pi_y}{\partial y_3} + \frac{1}{\tan\theta}\,\frac{\partial \pi_y}{\partial y_1}$ and $\boldsymbol{d}_{ij} \coloneqq \frac{1}{i!\,j!}\,\frac{\partial^{(1+i+j)}\boldsymbol{d}}{\partial x_1^i\,\partial x_2^j}(0)$.
From Gauss elimination method, we know that the matrix expressed as the table above is of fullrank from criteria of ugly goose singularity and the assumption $k_2 \neq 0$. Our source code is available on GitHub [12].

The degree 1 monomial $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = -\frac{p_1}{c}\,\frac{\partial \pi_y}{\partial y_1}$ is in (3.4). Therefore, if $f(0)$ is not flat umbilic, $\pi$ is versal unfolding of the singularity of $\pi_y$. □

### 3.4.3 Type 16

*Proof of 2 in Theorem 3.4.* From assumption and criteria, $a_{30} = a_{21} = 0$ and $a_{40} \neq 0$. The type 16 singularity is 5-$\mathcal{A}$-deternined. Thus, we need to check equality (3.2) holds for $k = 5$.

We consider whether two elements $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$ are generated by several elements in (3.4). However, we can only choose the following elements in (3.4) to generate elements above in $j^5\theta(\pi_y)$:

$$^t\left(\frac{\partial \pi_y}{\partial y_1},\ \frac{\partial \pi_y}{\partial y_3}\right) = \begin{pmatrix} -\frac{c}{p_1} & -\frac{c}{p_1^2} \\ \frac{c}{p_1\ \tan\theta} & \frac{c}{p_1^2 \tan\theta} \end{pmatrix} {}^t\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix},\ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}\right) + \cdots.$$

Since $\frac{\partial \pi_y}{\partial y_1}$ and $\frac{\partial \pi_y}{\partial y_3}$ are not linearly independent, these elements cannot generates the seven elements and we know that an unfolding $\pi$ is not versal at the type 16 singularity. $\square$

## 4 Geometric conditions of singularities for versality

We consider the contact of the surface $S$ with cones. Since a cone in $\mathbb{R}^3$ is determined by its vertex, direction of central axis and angle, the moduli space of cones is of dimension six.

Consider a cone which has a vertex $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, a direction vector of central axis $\boldsymbol{d} = (d_1, d_2, d_3)$ in $S^2$ and an angle $\theta$ in $(0, \pi/2)$ where $\langle \boldsymbol{d}, y\rangle \neq 0$ and $\boldsymbol{d}$ is not parallel to the position vector of $y$. Then, its implicit function is given by

$$C_{y,\boldsymbol{d},\theta}(z_1, z_2, z_3) \coloneqq \langle \boldsymbol{d}, z - y\rangle^2 - |z - y|^2 \cos^2\theta = 0. \tag{4.1}$$

The contact between the cone (4.1) and the regular surface $S$ parameterized by $f(x_1, x_2) = (x_1, x_2, Q(x))$ is measured by the $\mathcal{K}$-singularities of the function

$$\begin{aligned} C(x_1, x_2) &= C_{y,\boldsymbol{d},\theta}(x_1, x_2, Q(x)) \\ &= \sum_{k\geq 2}^{m} C_k(x_1, x_2) + o(x_1, x_2)^{m+1} \quad \text{where} \quad C_k(x_1, x_2) \coloneqq \sum_{i+j=k} \frac{c_{ij}}{i\, j} x_1^i x_2^j. \end{aligned} \tag{4.2}$$

We call $C(x_1, x_2)$ the **contact function with cones**.

According to [18], we define the notion of contact type. We recall that two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ are $\mathcal{K}$-equivalent if there are a diffeomorphism $\varphi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and a smooth map $A : (\mathbb{R}^m, 0) \to GL(\mathbb{R}^n)$ such that $g(\varphi(x)) = A(x)\, f(x)$.

In this section, we consider the $A_k$ (or $A_k^{\pm}$)-contact type which is a $\mathcal{K}$-modal $x_1^2 \pm x_2^{k+1}$. We introduce some results of $A_{\leq 6}$-contact of cones with regular surfaces at a parabolic point. Before stating the results, we need the following Lemma 4.1. In this section, we assume that the vertex of cones is not the origin in $\mathbb{R}^3$.

**Lemma 4.1.** *One of generatrix is passing through the origin in $\mathbb{R}^3$ if and only if the angle of cones $\theta$ is equal to the angle between the position vector of the vertex $y$ and the unit direction vector of the central axis $\boldsymbol{d}$ of cones.*

This lemma is shown by checking the condition of $C(0) = 0$.

**Lemma 4.2.** *We assume that one of generatrix is passing through the origin in $\mathbb{R}^3$. Then, the contact function $C(x_1, x_2)$ with cones has critical point at 0 if and only if*

$$y_3 = d_1\, y_2 - d_2\, y_1 = 0. \tag{4.3}$$

*This condition means that the vertex $y$ is in the tangent plane of the regular surface $S$ and the orthogonal projection of the direction of central axis $\boldsymbol{d}$ belong to $v = (0, 0, 1)$ is parallel to the position vector of $y$.*

The lemma above is proved by checking the rank of Jacobian matrix.

**Lemma 4.3.** *We consider a cone $C_{y,\boldsymbol{d},\theta}$ whose vertex is satisfied (4.3) and is not origin in $\mathbb{R}^3$. We measure contact between this cone with the regular surface $S$ as follows:*

1. *The cone $C_{y,\boldsymbol{d},\theta}$ has $A_1$-contact with $S$ if and only if none of the following conditions hold.*

   ($A_2a$) *the origin is flat umbilic.*

   ($A_2b$) *the origin is parabolic but not flat umbilic and the vertex $y$ is contained in an asymptotic straight line of $S$ at 0, that is, $y_2 = 0$ as same $d_2 = 0$.*

2. *Suppose that $S$ is parabolic but not flat umbilic at the origin. The cone $C_{y,\boldsymbol{d},\theta}$ has $A_2$-contact with $S$ if and only if the condition ($A_2b$) holds and none of the following conditions hold.*

   ($D_4a$) *the vertex $y$ is contained in the asymptotic straight line of $S$ at $x = 0$ and*

$$d_3 = -\frac{d_1}{k_2\, y_1}.$$

   *This is the condition in which the rank of the Hesse matrix is 0.*

   ($A_3b$) *the vertex $y$ is contained in the asymptotic straight line of $S$ at $x = 0$, $d_3 \neq -\frac{d_1}{k_2\, y_1}$ and the asymptotic straight line is 3-rd or higher order contact with $S$.*

The first item *1.* in Lemma 4.3 is proved by checking the rank of Hesse matrix of $C(x_1, x_2)$. To prove the second item *2.* in Lemma 4.3, we use criteria of $A_3$ and $D_4$-singularity type (for example, see Theorem 1.1. in [9]).

We consider more degenerate $A_k$-contact in the case of ($A_2b$) in Lemma 4.3. It is relevant to gulls series singularity of $\pi_y$ if $\pi$ is versal at gulls series singularity. Using criteria of $A_{\leq 6}$-singularity (for example, see Theorem 1.2 to 1.4 in [9]), we have the following Theorem 4.4. The proof is similar to that of the item *2.* in Lemma 4.3. See [13] also, which is available in [12].

**Theorem 4.4.** *Assume that the origin of the regular surface $S$ has parabolic but not flat umbilic and the vertex of the cone $y$ is contained in the asymptotic straight line of $S$.*

1. *The cone $C_{y,\boldsymbol{d},\theta}$ has $A_3$-contact with $S$ if and only if $S$ has the 3-rd or higher order contact with the asymptotic straight line at the origin and*

$$(k_2\, a_{40} - 3a_{21}^2)\, d_3\, y_1 + a_{40}\, d_1 \neq 0. \tag{4.4}$$

   *After this, we assume that both of $a_{40}$ and $k_2\, a_{40} - 3a_{21}^2$ do not vanish.*

2. *The cone $C_{y,\boldsymbol{d},\theta}$ has $A_4$-contact with $S$ if and only if $S$ has the 3-rd order contact with the asymptotic straight line at the origin, (4.4) vanishes and*

$$(3a_{21}^2\, a_{50} + 5a_{12}\, a_{40}^2 - 10a_{21}\, a_{31}\, a_{40})\, y_1 - 5a_{40}\, (k_2\, a_{40} - 3a_{21}^2) \neq 0. \tag{4.5}$$

3. *The cone $C_{y,\boldsymbol{d},\theta}$ has $A_5$-contact with $S$ if and only if $S$ has the 3-rd order contact with the asymptotic straight line at the origin, both of (4.4) and (4.5) vanish and*

   $AC_5 := (45a_{21}^3\, a_{40}\, a_{60} - 54a_{21}^3\, a_{50}^2 + 180a_{21}^2\, a_{31}\, a_{40}\, a_{50} - 225a_{21}^2\, a_{40}^2\, a_{41} - 25a_{03}\, a_{40}^4 + 225a_{21}\, a_{22}\, a_{40}^3 - 150a_{21}\, a_{31}^2\, a_{40}^2)y_1 - 270a_{21}^3\, a_{40}\, a_{50} - 450a_{12}\, a_{21}\, a_{40}^3 + 900a_{21}^2\, a_{31}\, a_{40}^2$

   *does not vanish.*

4. *The cone $C_{y,\boldsymbol{d},\theta}$ has $A_6$-contact with $S$ if and only if $S$ has the 3-rd order contact with the asymptotic straight line at the origin, (4.4), (4.5) and $AC_5$ vanish and*

   $$\begin{aligned} AC_6 := &\begin{pmatrix} 225a_{21}^3\, a_{40}^2\, a_{70} - 945a_{21}^3\, a_{40}\, a_{50}\, a_{60} + 1575a_{21}^2\, a_{31}\, a_{40}^2\, a_{60} - 1575a_{21}^2\, a_{40}^3\, a_{51} \\ + 756a_{21}^3\, a_{50}^3 - 3150a_{21}^2\, a_{31}\, a_{40}\, a_{50}^2 + 3150a_{21}^2\, a_{40}^2\, a_{41}\, a_{50} - 1575a_{21}\, a_{22}\, a_{40}^3\, a_{50} \\ + 4200a_{21}\, a_{31}^2\, a_{40}^2\, a_{50} - 5250a_{21}\, a_{31}\, a_{40}^3\, a_{41} - 875a_{13}\, a_{40}^5 + 2625a_{21}\, a_{32}\, a_{40}^4 \\ + 2625a_{22}\, a_{31}\, a_{40}^4 - 1750a_{31}^3\, a_{40}^3 \end{pmatrix} y_1^2 \\ &+ 210a_{40}\,(3a_{21}\, a_{50} - 5a_{31}\, a_{40})\,(3a_{21}^2\, a_{50} + 5a_{12}\, a_{40}^2 - 10a_{21}\, a_{31}\, a_{40})y_1 \\ &- 3150a_{21}\, a_{40}^2\,(3a_{21}^2\, a_{50} + 5a_{12}\, a_{40}^2 - 10a_{21}\, a_{31}\, a_{40}) \end{aligned}$$

   *does not vanish.*

**Remark 4.5.** Suppose that $f(0)$ is not red subparabolic, that is, $a_{21} \neq 0$.

1. The non degenerate condition of $A_4$-contact in Theorem 4.4 means that criteria of gulls singularity type of the central projection $\pi_y$.
2. It follows from *3 and 4 in Theorem 4.4 that the sum of the non degenerate conditions of $A_5$-contact and $A_6$-contact*

   $$AC_6 - 70\, a_{40}\, AC_5 \tag{4.6}$$

   *is equal to the non degenerate condition of ugly gulls singularity of $\pi_y$. We call (4.6) the ug-**focal** condition.*

Finally, we summarize geometric criteria of singularities of $\pi_y$ for versality.

**Lemma 4.6.** *Suppose the regular surface $S$ is parameterized by $f$ as in (1.1) and a viewpoint $y$ is in $\boldsymbol{u}$-axis, that is, $y - f(0) = p_1\, \boldsymbol{u}$. Then, geometric criteria of $\mathcal{A}_e$-codimension $\leq 3$ singularities of $\pi_y$ are written as in table 3 if $\pi$ is versal at $x = 0$.*

| type | $\mathcal{A}$-cod. | $c$ | position of $y$ | other condition |
|---|---|---|---|---|
| fold | 0 | 1 | | ($\mathcal{L}$ is not asymptotic straight line.) |
| cusp | 0 | 2 | | |
| swallowtail | 1 | 3 | | |
| butterfly | 2 | 4 | not $h$-focal | |
| elder butterfly | 3 | 4 | $h$-focal | |
| unimodal | 3 | 5 | not $u$-focal | |
| lips (resp. beaks) | 1 | 2 | not $p$-focal | $y$ is farther (resp. nearer ) $f(0)$ than $p$-focal |
| goose | 2 | 2 | $p$-focal | viewlines passing through parabolic points of $S$ form cusipidal edge (cf. Platnova [19]) |
| ugly goose | 3 | 2 | $p$-focal | viewlines passing through parabolic points of $S$ form swallowtail (cf. Platnova [19]) |
| type 16 | 3 | 3 | not 16-focal | 1-st or higher order red subparabolic |
| gulls | 2 | 3 | not $p'$-focal | not red subparabolic |
| ugly gulls | 3 | 3 | $p'$-focal | not red subparabolic and not $ug$-focal condition |
| type 12 | 3 | 4 | not 12-focal | not red-subparabolic |

Table 3: Geometric criteria of $\mathcal{A}$-singularity of $\pi_y$ where $c$ is contact order of $S$ with $\mathcal{L}$ at $x = 0$.


### Acknowledgement

The author is very grateful to his advisor Toshizumi Fukui for a lot of instructions and encouragements. He also appreciates very much lots of advice from Yutaro Kabata and the referee's valuable comments.


## References


[1] V. I. Arnold. Indices of singular points of 1-forms on a manifold with boundary, convolution of invariants of reflection groups, and singular projections of smooth surfaces. *Russian Math. Surveys*, Vol. 34, No. 2, pp. 1–42, 1979.

[2] V. I. Arnold. *Singularities of Caustics and Wave Fronts.* Mathematics and its Applications. Springer Netherlands, 2013.

[3] G Bradski and A Kaehler. *Learning OpenCV - computer vision with the OpenCV library: software that sees.* O'Reilly, 2008.

[4] J.W. Bruce. Seeing—the mathematical viewpoint. *Math. Intelligencer*, Vol. 6, No. 4, pp. 18–25, 1984.

[5] J.W. Bruce and T.C. Wilkinson. *Folding maps and focal sets*, Vol. 1462, pp. 63–72. Springer Verlag, Germany, 1991.

[6] H. S. M. Coxeter. *Projective Geometry.* Springer-Verlag New York, 2-nd edition, 1987.

[7] J. M. S. David. Projection-generic curves. *J. London Math. Soc. (2)*, Vol. 27, No. 3, pp. 552–562, 1983.

[8] J. L. Deolindo-Silva, Y. Kabata, and T. Ohmoto. Binary differential equations at parabolic and umbilical points for 2-parameter families of surfaces. *Topology and its Applications*, Vol. 234, pp. 457 – 473, 2018.

[9] T. Fukui. Criteria of singularities and an application to differential geometry. In *Geometry on Real Closed Field and its Application to Singularity Theory*, Vol. 1764 of *RIMS Kôkyûroku*, pp. 16 – 29. Research Institute for Mathematical Sciences Kyoto University, 9. 2011. (in Japanese).

[10] T. Fukui and M. Hasegawa. Singularities of parallel surfaces. *Tohoku Math. J. (2)*, Vol. 64, No. 3, pp. 387–408, 2012.

[11] T. Fukui, M. Hasegawa, and K. Nakagawa. Contact of a regular surface in Euclidean 3-space with cylinders and cubic binary differential equations. *J. Math. Soc. Japan*, Vol. 69, No. 2, pp. 819 – 847, 2017.

[12] S. Honda. Computation of versality of central projections at singularities, 2021. https://github.com/Shuhei-singularity123/Computation-of-versality-of-central-projections-at-singularities.

[13] S. Honda. Singularity types of central projections and thier versality, March 2021. master thesis in English, Saitama university.

[14] S. Izumiya, M. D. C. R. Fuster, M. A. S. Ruas, and F. Tari. *Differential geometry from a singularity theory viewpoint.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[15] Y. Kabata. Recognition of plane-to-plane map-germs. *Topology and its Applications*, Vol. 202, pp. 216 – 238, 2016.

[16] J. N. Mather. Stability of $C^\infty$ mappings. III. Finitely determined mapgerms. *Inst. Hautes Études Sci. Publ. Math.*, No. 35, pp. 279–308, 1968.

[17] J. N. Mather. Stability of $C^\infty$ mappings. II. Infinitesimal stability implies stability. *Annals of Math.*, Vol. 89, No. 2, pp. 254–291, 1969.

[18] J. Montaldi. On contact between submanifolds. *Michigan Math. J.*, Vol. 33, No. 2, pp. 195 – 199, 1986.

[19] O. A. Platonova. Projections of smooth surfaces. *J. Soviet Math.*, Vol. 35, pp. 2796–2808, 1986.

[20] I. R. Porteous. The normal singularities of surfaces in $\mathbb{R}^3$. *Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math.* 40, *Amer. Math. Soc., Providence, RI*, pp. 379–393, 1983.

[21] J. H. Rieger. Families of maps from the plane to the plane. *J. London Math. Soc. (2)*, Vol. 36, No. 2, pp. 351–369, 1987.

[22] J. H. Rieger and M. A. S. Ruas. Classification of $\mathcal{A}$-simple germs from $k^n$ to $k^2$. *Compositio Math.*, Vol. 79, No. 1, pp. 99–108, 1991.

[23] K. Saji. Criteria for singularities of smooth maps from the plane into the plane and their applications. *Hiroshima Math. J.*, Vol. 40, No. 2, pp. 229–239, 2010.

[24] H. Sano, Y. Kabata, J. L. Deolindo Silva, and T. Ohmoto. Classification of jets of surfaces in projective 3-space via central projection. *Bull. Brazilian Math. Soc.*, Vol. 48, No. 4, pp. 623–639, 2017.

[25] C. T. C. Wall. Geometric properties of generic differentiable manifolds. In *Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)*, pp. 707–774. Lecture Notes in Math., Vol. 597, 1977.

[26] C. T. C. Wall. Finite Determinacy of Smooth Map-Germs. *Bull. London Math. Soc.*, Vol. 13, No. 6, pp. 481–539, 1981.

[27] H. Whitney. On singularities of mappings of Euclidean spaces. I. mappings of the plane into the plane. *Annals of Math.*, Vol. 62, No. 3, pp. 374–410, 1955.