On the Tjurina Number of Plane Curve Singularities

Masahiro Watari

Saitama University

Graduate School of Science and Engineering

Today's Contents

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§ 1 Introduction

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We call C ***a plane curve singularity***.

The ***Milnor number*** μ and ***Tjurina number*** τ of C are defined by

$$\mu := \dim_{\mathbb{C}} \left(\mathbb{C}[[x, y]]/ \left(f_x, f_y\right)\right),$$

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We easily see that $\mu \geq \tau$. Set $r := \mu - \tau$.

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If $r > 0$, then, up to analytic equivalence, C can be given by the parametrization

$$\begin{cases} x = t^n, \\ y = t^m + t^\lambda + \sum_{i \in G} a_i t^i, \end{cases}$$

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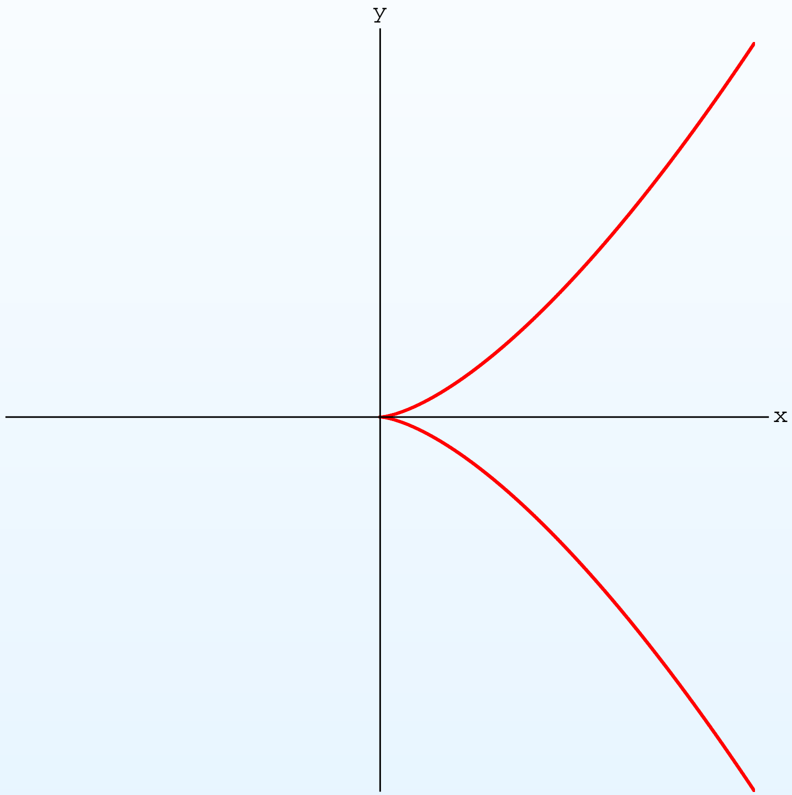
The integer λ is an analytic invariant. It is called the ***Zariski invariant*** of C.

Picture



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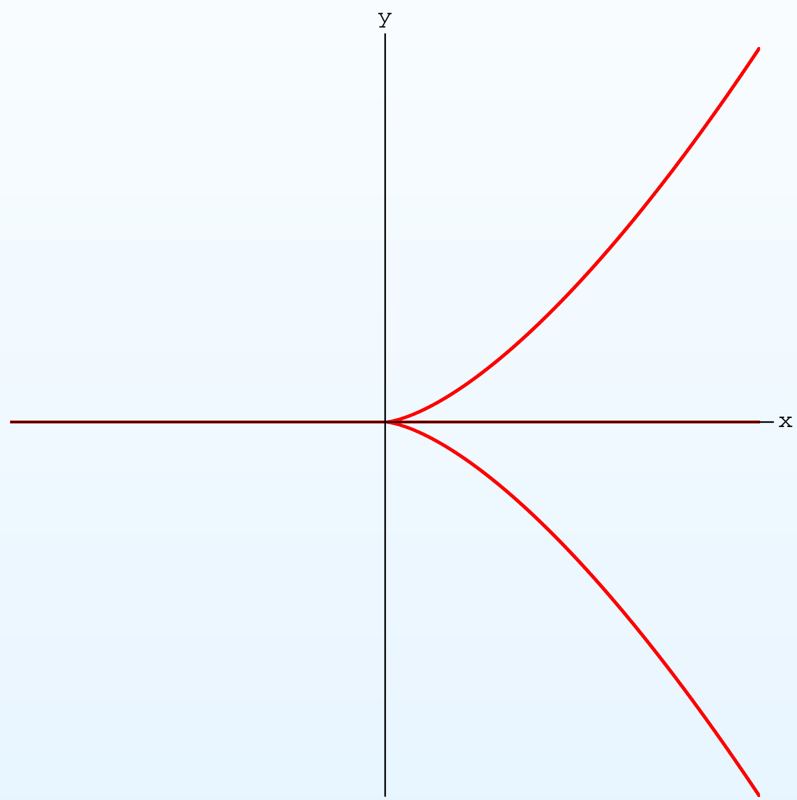

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Definition (Zariski)

We call the set $Ch(C) := \{\beta_0, \ldots, \beta_g\}$ and the positive integer g the ***characteristic*** of C and the ***genus*** of C.

Example 1

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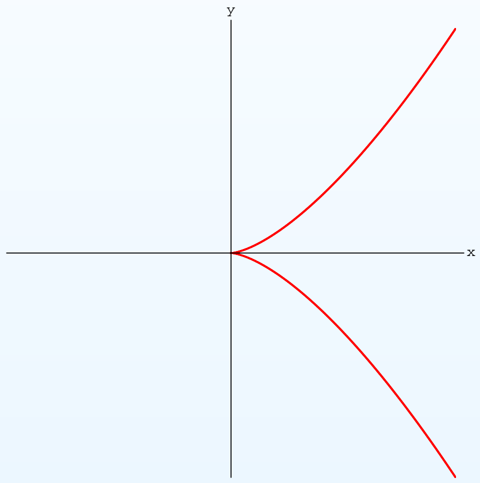
$$C : y^2 - x^3$$



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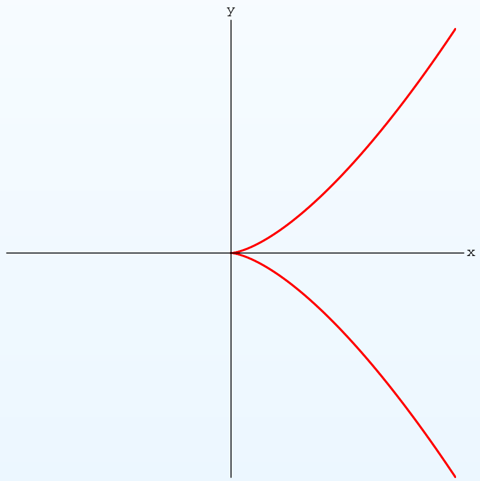

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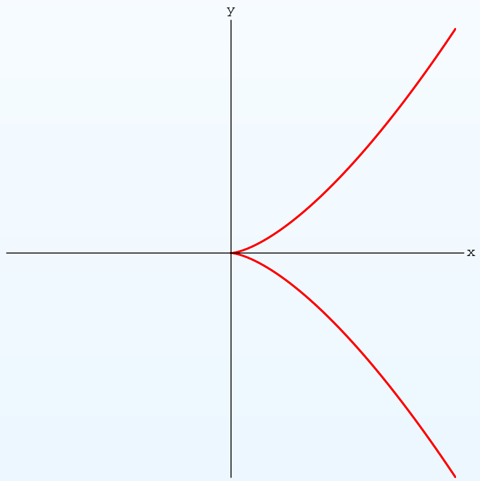

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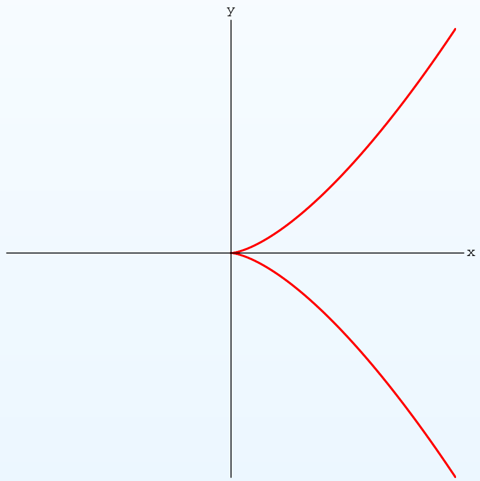

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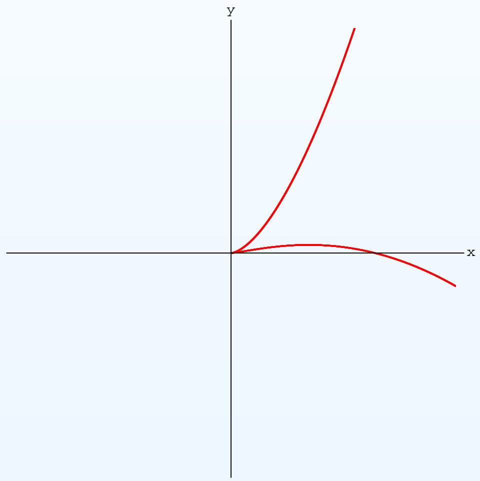

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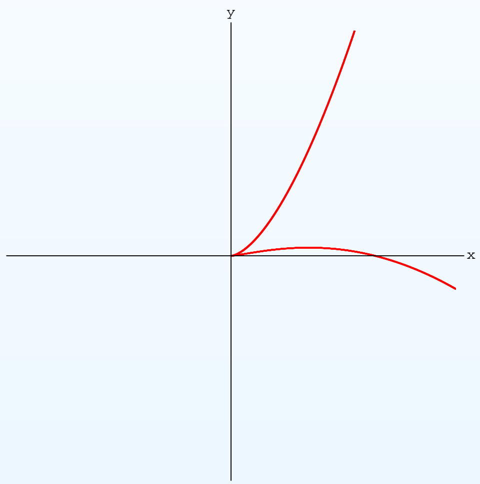

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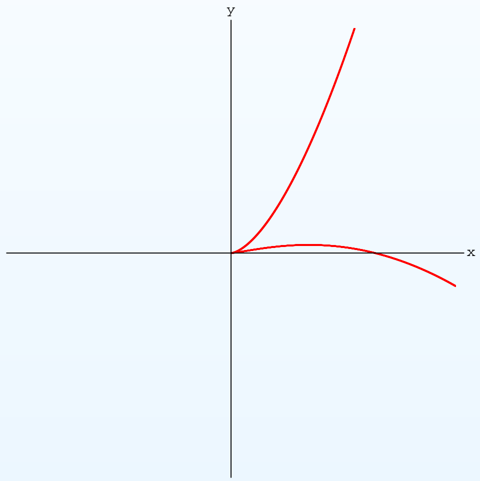

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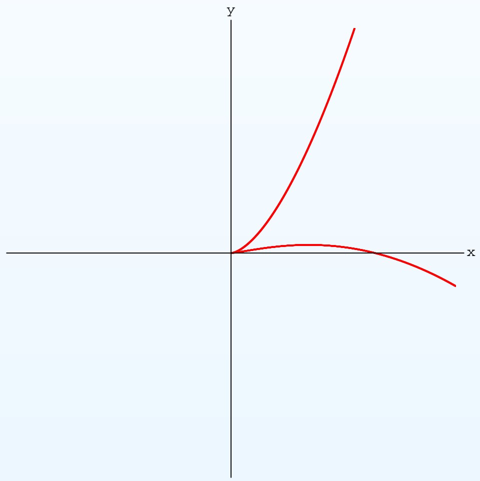

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$$\therefore g = 1.$$

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$\Longrightarrow$ We have $\tau = \beta + 3ab - a - 3b$.

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Case (1) $g = 2$

(a) $x = t^4, \ y = t^6 + t^\lambda$, where $\lambda \, (> 6)$ is odd.

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Case (2) $g = 1$.

(b) $x = t^n, \; y = t^m + t^\lambda,$

where $\lambda = (n-1)m - 3n$.

(c) $x = t^4 \; y = t^m + t^\lambda + \frac{3m-8}{2m} t^{3m-16} + a t^{3m-12},$

where $\lambda = (n-2)m - 2n$, $m > 8$.

(d) $x = t^n \; y = t^m + t^\lambda + a t^{(n-1)m-3n},$

where $\lambda = (n-2)m - 2n$, $n \geq 5$, $m > 2n/(n-3)$.

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♠ $r = 3 \Longrightarrow g = 1$ and the following three cases occur:

Type (i): $\lambda = (n-1)m - 4n$

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(A) $x = t^n, \; y = t^m + t^\lambda$, **where** $n \geq 3, p \geq 2$.

(B) $x = t^n, \; y = t^m + t^\lambda + at^{(n-2)m-2n}$,

where $n \geq 5$ **and** $p = 1$.

Type (ii): $\lambda = (n-2)m - 2n$

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(C) $x = t^n, \quad y = t^m + t^\lambda + at^{(n-1)m-4n} + bt^{(n-1)m-3n}$,

where $n \geq 5$, $p \geq 2$ and $a\ (\neq 0)$.

(D) $x = t^4, \quad y = t^m + t^\lambda + at^{3m-16} + bt^{3m-12}$,

where $p \geq 2$ and $a\ (\neq (3m-8)/2m)$.

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$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^{p}(a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p} b_i t^{n_i},$$

where

$$\begin{cases} n > 2q, n \geq 5, m > 2n/(n-4), \\ m_i = (n-2)m - (p+3-i)n, \\ n_i = (n-1)m - (2p+3-i)n. \end{cases}$$

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where

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Let C be a plane curve singularity with the multiplicity $n = 3$ or 4. Then τ of C is given by the following table:

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n	Ch(C)	μ	λ	τ	No.
3	$\{3, m\}$	$2(m-1)$	$-$	$2(m-1)$	(3.1)
			$2m - 3\lambda_0$	$2m - \lambda_0 - 1$	(3.2)
4	$\{4, m\}$	$3(m-1)$	$-$	$3(m-1)$	(4.1)
			$3m - 4\lambda_0$	$3m - \lambda_0 - 2$	(4.2)
			$2m - 4\lambda_0$	see Next	(4.3)
	$\{4, 2\alpha, \beta\}$	$4\alpha + \beta - 3$	β	$3\alpha + \beta - 2$	(4.4)

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Up to analytic equivalence, C is given by

$$x = t^4,\ y = t^m + t^\lambda + \sum_{i=1}^{p} b_i t^{3m-4(p+\lambda_0+1-i)},\quad \text{where } p = \left[\frac{m}{4}\right].$$

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Then τ of C is determined as in the following table:

τ	Conditions
$3m - 2\lambda_0 - 1$	$p = \lambda_0,\ b_1 = (3m - 4\lambda_0)/2m$
	$p > \lambda_0,\ b_i = 0\ (1 \le i \le p - \lambda_0),$ $b_{p-\lambda_0+1} = (3m - 4\lambda_0)/2m$
$3m - 2\lambda_0 - 2$	$p = \lambda_0,\ b_1 \neq (3m - 4\lambda_0)/2m$
	$p > \lambda_0,\ b_i = 0\ (1 \le i \le p - \lambda_0),$ $b_{p-\lambda_0+1} \neq (3m - 4\lambda_0)/2m$
$3m - \lambda_0 - (p - d + 3)$	$p > \lambda_0,\ \exists b_i \neq 0\ (1 \le i \le p - \lambda_0)$ $d := \min\{i \mid b_i \neq 0\}$

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ν : the order function of $\mathcal{O}_{\widetilde{C}}$,

Ω^1_C : the differential module of $\mathcal{O}_C$,

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$(x, y) = (t^n, \varphi(t))$: the parametrization of C.

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$\mathcal{O}_{\widetilde{C}} \cong \mathbb{C}[[t]]$: the local ring of $\widetilde{C}$,

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It is known that if the genus of C is g, then $\exists v_0, v_1, \cdots, v_g$ such that

$$S = \langle v_0, v_1, \cdots, v_g \rangle.$$

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$S = \{0, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\} = \langle 2, 3 \rangle$,

$G = \{1\}$.

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$\mathcal{O}_C = \mathbb{C}[[t^4, t^6 + t^7]]$,

$S = \{0, 4, 6, 8, 10, 12, 13, 14, 16, \ldots\} = \langle 4, 6, 13 \rangle$,

$G = \{1, 2, 3, 5, 7, 9, 11, 15\}$.

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We easily see that $v_1 = m$ and $v_0 < v_1 < \cdots < v_g$.

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So we define the order of $\xi \, (\in \Omega^1_C)$ by $\nu(\xi) := \nu(\pi^*(\xi))$.

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Definition

Define the set of orders V by

$$V := \nu\left(\Omega^1_C\right) \setminus \nu\left(d\mathcal{O}_C\right),$$

where $\nu\left(\Omega^1_C\right) = \{\nu(\xi) | \, \xi \in \Omega^1_C\}$, $\nu(d\mathcal{O}_C) = \{\nu(\xi) | \, \xi \in d\mathcal{O}_C\}$.

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Theorem (Zariski)

We have $r = \sharp\,(V)$.

§ 3 The determination of τ

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Fact

$$
\begin{aligned}
g = 1 \iff & \gcd(n, m) = 1 \\
\iff & \mathbf{Ch}(C) = \{n, m\} \\
\iff & S = \langle n, m \rangle \\
& \quad = \{am + bn \mid a, b \in \mathbb{Z}_{\geq 0}\}
\end{aligned}
$$

Angermüller's Lemma

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Lemma (Angermüller)

If $g = 1$, then any integer t is expressed uniquely as

$$t = t_1 m - t_0 n$$

where $t_0, t_1 \in \mathbb{Z}$ and $0 \leq t_1 \leq n - 1$.

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Fact $\mu = (n-1)m - n + 1$.

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If r is determined, then $\tau = \mu - r$.

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Fact

For the differential $\omega := mydx - nxdy$, we have

$$\nu(\omega) = \lambda + n - 1 = \min\{V\}.$$

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$$x = t^n, \;\; y = t^m + t^{(n-1)m-(R+1)n},$$

where $1 \leq R \leq p$, $(p = [m/n])$ and $\gcd(n, m) = 1$, then we have $r = R$.

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Remark

We have $\lambda = (n-1)m - (R+1)n.$

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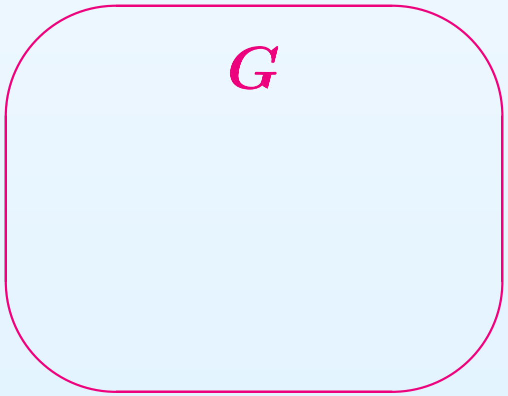

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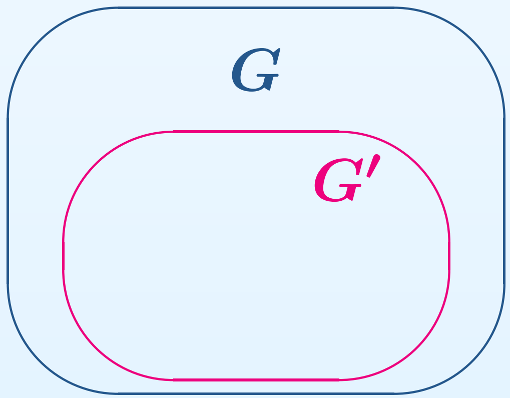

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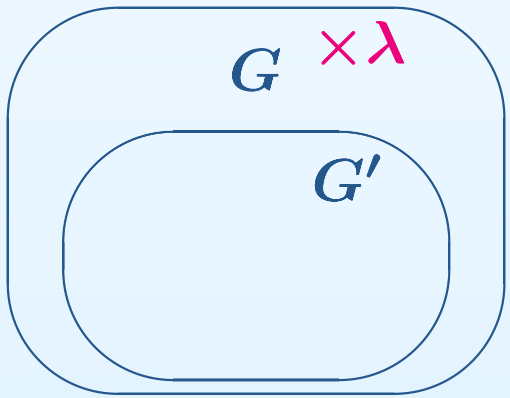

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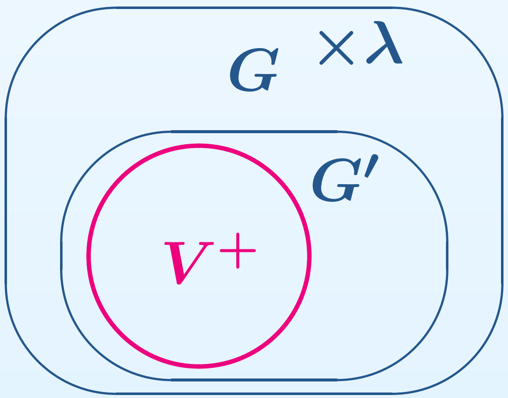

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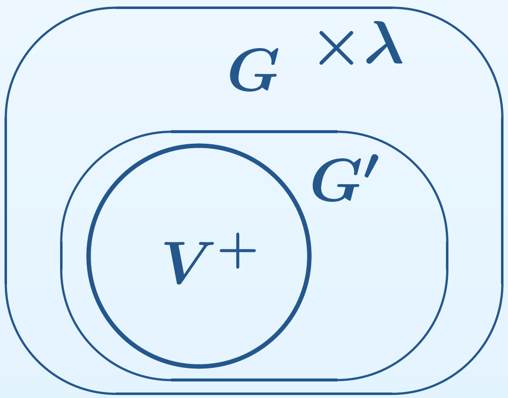

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&(n-1)m - Rn = \lambda + n, \\
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&\qquad\vdots \\
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By Zariski's Theorem, we have $r = \sharp(V^+) = R$. $\square$

§ 4 Problems

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What about τ for reducible plane curve singularities?

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F: a reduced homogeneus polynomial in $\mathbb{C}[x, y, z]$.

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For this C, consider *the polar map*

$\varphi_C : \mathbb{P}^2(\mathbb{C}) \longrightarrow \mathbb{P}^2(\mathbb{C})$ defined by $p \longmapsto \big(F_x(p), F_y(p), F_z(p)\big)$.

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We call the degree of φ_C ***the polar degree*** of C.

The polar degree of is denoted by ***Pdeg*** C.

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Problem: Classify C with Pdeg $C = 2, 3, 4$.

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What about the cases where $g \geq 3$?