BOUNDING THE VOLUMES OF SINGULAR WEAK LOG DEL PEZZO SURFACES

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Abstract. We give an optimal upper bound for the anti-canonical volume of an \( \epsilon \)-lc weak log del Pezzo surface. Moreover, we consider the relation between the bound of the volume and the Picard number of the minimal resolution of the surface. And we consider blowing up some points on a Hirzebruch surface in general position and give some examples of smooth weak log del Pezzo surfaces.

This is a report for my talk on Mini Conference in Saitama University explaining the content of [3].
Throughout this article, we work over an algebraically closed field of arbitrary characteristic. We start by some basic definitions.

Definition 1. Let \( X \) be a normal projective surface and \( \Delta \) be an \( \mathbb{R} \)-divisor on \( X \) with coefficients in \([0,1]\) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. We say that \((X, \Delta)\) is a weak log del Pezzo surface if \(- (K_X + \Delta)\) is nef and big.

Definition 2. Let \( X \) be a normal projective variety and let \( \Delta \) be an \( \mathbb{R} \)-divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. Let \( f : Y \to X \) be a log resolution of \((X, \Delta)\), we call the coefficients \( a_i \) in the following formula the discrepancies.

\[
K_Y = f^*(K_X + \Delta) + \sum a_i F_i,
\]
where \( F_i \) is a prime divisor. For some \( \epsilon \in (0,1] \), the pair \((X, \Delta)\) is called
(a). \( \epsilon \)-kawamata log terminal (\( \epsilon \)-klt, for short) if \( a_i > -1 + \epsilon \) for all \( i \), or
(b). \( \epsilon \)-log canonical (\( \epsilon \)-lc, for short) if \( a_i \geq -1 + \epsilon \) for all \( i \).

In this article we first consider the following question.

Question 3. Let \((X, \Delta)\) be an \( \epsilon \)-lc weak log del Pezzo surface. Then what is the upper bound of the anti-canonical volume \( \text{Vol}(- (K_X + \Delta)) = (K_X + \Delta)^2 \)?

The motivation of this kind of problem is the following B-A-B Conjecture due to A. Borisov, L. Borisov and V. Alexeev.

Definition 4. Let \( X \) be a normal projective variety and \( \Delta \) be a \( \mathbb{Q} \)-divisor on \( X \) with coefficients in \([0,1]\) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. We say that \((X, \Delta)\) is a \log \( \mathbb{Q} \)-Fano variety if \(- (K_X + \Delta)\) is ample.

Definition 5. A collection of varieties \( \{X_\lambda\}_{\lambda \in \Lambda} \) is said to be bounded if there exists \( h : \mathcal{X} \to S \) a morphism of finite type of Neotherian schemes such that for each \( X_\lambda \), \( X_\lambda \simeq \mathcal{X}_s \) for some \( s \in S \).

Conjecture 6 (Borisov-Alexeev-Borisov). Fix \( 0 < \epsilon < 1 \), an integer \( n > 0 \), and consider the set of all \( n \)-dimensional \( \epsilon \)-klt log \( \mathbb{Q} \)-Fano varieties \((X, \Delta)\). The set of underlying varieties \( \{X\} \) is bounded.
The B-A-B Conjecture is still open in dimension three and higher. We are mainly interested in the following weak conjecture for anti-canonical volumes which is a consequence of B-A-B Conjecture.

**Conjecture 7** (Boundedness of anti-canonical volumes). Fix \(0 < \epsilon < 1\), an integer \(n > 0\), and consider the set of all \(n\)-dimensional \(\epsilon\)-klt log \(\mathbb{Q}\)-Fano varieties \((X, \Delta)\). The volume \(\text{Vol}(-(K_X + \Delta)) = (- (K_X + \Delta))^n\) is bounded from above by a fixed number \(M(n, \epsilon)\) depending only on \(n\) and \(\epsilon\).

In dimension two, Conjecture 7 is well-researched in history. Alexeev establishes the two-dimensional B-A-B Conjecture in [1] which implies the boundedness for the anti-canonical volumes, but no clear bound is written down.

Alexeev and Mori in [2] give a simplified argument for two-dimensional B-A-B Conjecture and give an upper bound for the pair \((X, 0)\) which is

\[K_X^2 \leq \left(\left\lceil \frac{2}{\epsilon} \right\rceil + 2\right)^2.\]

Recently Lai in [4] gives an upper bound using McKernan’s covering-families-of-tigers method, which is

\[(K_X + \Delta)^2 \leq \max \left\{ 64, \frac{8}{\epsilon} + 4 \right\}.\]

Here we remark that one could use Lai’s method to get a refinement of his bound by carefully computation, which is

\[(K_X + \Delta)^2 \leq \max \left\{ 36, \left\lceil \frac{2}{\epsilon} \right\rceil + 4 + \left\lfloor \frac{4}{\left\lceil \frac{2}{\epsilon} \right\rceil} \right\rfloor \right\}.\]

This is very close to be optimal. But we should also remark that Lai’s method only works for complex number field and rational boundary and the method used in this article is totally different from Lai’s.

For dimension three, recently in [4], Lai proves the following theorem:

**Theorem 8** ([4], Theorem B). Let \((X, \Delta)\) be an \(\epsilon\)-klt \(\mathbb{Q}\)-factorial log \(\mathbb{Q}\)-Fano threefold of \(\rho(X) = 1\). The degree \(-K_X^3\) satisfies

\[-K_X^3 \leq \left( \frac{24M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 12 \right)^3,\]

where \(R(2, \epsilon)\) is an upper bound of the Cartier index of \(K_S\) for \(S\) any \(\epsilon/2\)-klt log del Pezzo surface of \(\rho(S) = 1\) and \(M(2, \epsilon)\) is an upper bound of the volume \(\text{Vol}(-K_S) = K_S^2\) for \(S\) any \(\epsilon/2\)-klt log del Pezzo surface of \(\rho(S) = 1\).

This is another motivation to find an optimal bound in dimension two. Recently the author hears from Lai that the assumption on \(\rho(X)\) in the above theorem is removed.

In this article, we solve Conjecture 7 in dimension two (i.e. Question 3) by giving an optimal upper bound:

**Theorem 9.** Let \((X, \Delta)\) be an \(\epsilon\)-lc weak log del Pezzo surface. Then the anti-canonical volume \(\text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2\) satisfies

\[(K_X + \Delta)^2 \leq \max \left\{ 9, \left\lfloor \frac{2}{\epsilon} \right\rfloor + 4 + \frac{4}{\left\lfloor \frac{2}{\epsilon} \right\rfloor} \right\},\]

where \([\ ]\) means round down.

Moreover, the equality holds if and only if at least one of the following holds:
Theorem 12. Let \( (X, \Delta) \) be an \( \epsilon \)-lc weak log del Pezzo surface with \( \rho(X_{\text{min}}) \geq 3 \). Then the anti-canonical volume \( \text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2 \) satisfies
\[
(K_X + \Delta)^2 \leq \left\lfloor \frac{2}{\epsilon} \right\rfloor + 3 + \frac{4}{\left\lfloor \frac{2}{\epsilon} \right\rfloor + 1},
\]
which is optimal.

Theorem 11. Let \( (X, \Delta) \) be an \( \epsilon \)-lc weak log del Pezzo surface with \( \rho(X_{\text{min}}) \geq 4 \). Then the anti-canonical volume \( \text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2 \) satisfies
\[
(K_X + \Delta)^2 \leq \max \left\{ \left\lfloor \frac{2}{\epsilon} \right\rfloor + 2 + \frac{4}{\left\lfloor \frac{2}{\epsilon} \right\rfloor + 1}, \left\lfloor \frac{3 + \epsilon}{2\epsilon} \right\rfloor + \frac{5}{2} + \frac{9}{4\left\lfloor \frac{3 + \epsilon}{2\epsilon} \right\rfloor - 2} \right\},
\]
which is optimal.

After this, since almost all smooth weak log del Pezzo surfaces come from blowing up the Hirzebruch surfaces, we consider the case when blowing up the Hirzebruch surfaces at points in general position by the same idea.

Theorem 10. Let \( (X, \Delta) \) be an \( \epsilon \)-lc weak log del Pezzo surface with \( \rho(X_{\text{min}}) \geq 3 \). Then the anti-canonical volume \( \text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2 \) satisfies
\[
(K_X + \Delta)^2 \leq \left\lfloor \frac{2}{\epsilon} \right\rfloor + 3 + \frac{4}{\left\lfloor \frac{2}{\epsilon} \right\rfloor + 1},
\]
which is optimal.

Theorem 12. Let \( X \) be a smooth surface which is given by blowing up \( k \) points on \( \mathbb{P}^n \). Assume that these points are not on \( S_n \), and no two of them are on the same fiber. Assume that \( -(K_X + \Delta) \) is nef and big. Then the anti-canonical volume \( \text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2 \) satisfies
\[
(K_X + \Delta)^2 \leq \left\lfloor \frac{n + 4 + \frac{2}{n} - k}{8 - k} \right\rfloor \quad \text{if} \quad n \geq 2;
\]
\[
8 - k \quad \text{if} \quad n = 0, 1.
\]

Finally we give some examples of smooth weak log del Pezzo surfaces which make the above bounds optimal.

Theorem 13. Let \( X \) be a smooth surface which is given by blowing up \( k \) points on \( \mathbb{P}^n \), \( n \geq 2 \). Assume that these points are not on \( S_n \), and no two of them are on the same fiber. \( S'_n \) is the strict transform of \( S_n \). Then

1. if \( k \leq n + 2 \), then \( (X, (1 - \frac{2}{n})S'_n) \) is a weak log del Pezzo surface.
2. if \( n + 2 < k < \frac{(n+2)^2}{n} \) and the points are in general position, then \( (X, (1 - \frac{2}{n})S'_n) \) is a weak log del Pezzo surface.
3. if \( k \geq \frac{(n+2)^2}{n} \), then \( (X, \Delta) \) can not be a weak log del Pezzo surface for any boundary \( \Delta \).
We also have similar examples for $n = 0, 1$. And when $n = 1$, the examples are just weak del Pezzo surfaces in the common sense, which is well-known, and the meaning for "general position" can be stated more precisely. So the examples we give here can be viewed as some kind of generalization of the traditional examples.

In the above theorem, taking $n = \lfloor 2/\epsilon \rfloor$, it gives examples of $\epsilon$-lc weak log del Pezzo surfaces with $\rho(X_{\text{min}}) = O(\frac{1}{\epsilon})$. So it is interesting to ask for the existence of $\epsilon$-lc weak log del Pezzo surfaces with large $\rho(X_{\text{min}})$.

**Question 14.** Is there any $\epsilon$-lc weak log del Pezzo surface $(X, \Delta)$ with $\rho(X_{\text{min}}) = O(\frac{1}{\epsilon^c})$ for $c \geq 2$?

**References**


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