The gonality of singular plane curves revisited II

Fumio Sakai
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1 Introduction

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d$ over $\mathbb{C}$. Let $\nu$ be the maximum multiplicity of the singular points of $C$. We say that $C$ is of type $(d, \nu)$. To a non-constant rational function $\varphi$ on $C$, we can associate a holomorphic map $\varphi : \tilde{C} \to \mathbb{P}^1$, where the $\tilde{C}$ is the non-singular model of $C$. The gonality, denoted by $G$ (or by $\text{Gon}(C)$), of $C$ is defined to be the minimum of the degrees of such holomorphic maps. We always have $G \leq d - \nu$. Indeed, if we consider the projection from the point with multiplicity $\nu$ to a line, then we obtain a rational function of degree $d - \nu$.

In 2004 (Ohkouchi-Sakai[4]), we proved several criteria for the equality $G = d - \nu$. One of them is a generalization of Coppens-Kato Criterion (1990). In the last conference, I reported an improvement of this result. This time, I reported an improvement of another type of criterion proved in 2004, which follows from the following Serrano Extension Theorem.

Theorem (Serrano [6]). Let $C$ be a smooth curve on a smooth surface $X$. Let $\varphi$ be a rational function on $C$ with $r = \text{deg}(\varphi)$. If $C^2 > (r + 1)^2$, then the holomorphic map $\varphi : C \to \mathbb{P}^1$ can be extended to a holomorphic map from $X$ to $\mathbb{P}^1$.

2 Main Theorem

Let $C$ be an irreducible plane curve of type $(d, \nu)$. Let $\pi : X \to \mathbb{P}^2$ be the minimal resolution of the singularities of $C$. We do not require that the
inverse image $\pi^{-1}(C)$ has normal crossings. We only require that the strict transform $\tilde{C}$ is non-singular. The minimal resolution $\pi$ consists of blowing ups of all singular points $P_1, \ldots, P_n$ (including infinitely near singular points) of $C$. We denote by $m_i$ the multiplicity of $P_i$. We write as $\text{Data}(C) = [m_1, \ldots, m_n]$. Let $E_i$ denote the total transform of the exceptional curve over $P_i$. Since $\tilde{C} \sim \pi^*O(d) - \sum m_i E_i$, we have

$$\tilde{C}^2 = d^2 - \sum m_i^2$$

Let $q$ be a non-negative integer such that $d - \nu - q > 1$. In order to prove the inequality:

$$G \geq d - \nu - q,$$

we have to show the non-existence of rational functions $\varphi$ with

$$\deg(\varphi) < d - \nu - q.$$

**Definition 1 (Serrano Condition $(q)$).** The condition

$$d^2 - \sum_{i=1}^{n} m_i^2 > (d - \nu - q)^2$$

is called the Serrano condition $(q)$. This condition implies the inequality:

$$d > \sqrt{\sum m_i^2}.$$

**Definition 2.** Define the invariant

$$k^*(q) = \frac{d - \nu - q - 1}{d - \sqrt{\sum m_i^2}}.$$

**Remark 1.** Given a singular point $P$ on $C$. Let $P_1, \ldots, P_r$ be the infinitely near singular points above $P$. Letting $P_0 = P$, we denote by $m_i$ the multiplicity of $C$ at $P_i$. Letting $M(P) = \sum_{i=0}^{r} m_i^2$, the amount $\sum m_i^2$ in the above definition, can be written as

$$\sum_{P \in \text{Sing}(C)} M(P).$$

For the simple singularity $y^a - x^b = 0$ $(a \leq b)$, the value $M(P)$ can be computed by the Euclidean algorithm. In particular, we have the inequality $M(P) \leq ab$ ([5]).
For a positive integer $k$, we discuss combinations of positive integers $(a_1, \ldots, a_b)$ with
\[ \sum_{i=1}^{b} a_i^2 = k^2. \]
Renumbering $a_i$’s, we rearrange them as: $a_1 \geq a_2 \geq \ldots \geq a_b$. We abbreviate $(2, 1, 1, 1, 1)$ as $(2, 1_5)$.

**Definition 3.** Let $A(k)$ denote the set of all such combinations $a = (a_1, \ldots, a_b)$ with the property: $\sum a_i^2 = k^2$. We set $l(a) = b$. For instance, we have

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A(k)$</th>
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<tbody>
<tr>
<td>1</td>
<td>(1)</td>
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<tr>
<td>2</td>
<td>(2), (14)</td>
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<tr>
<td>3</td>
<td>(3), (2, 1), (2, 15), (19)</td>
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<tr>
<td>4</td>
<td>(4), (3, 2, 13), (3, 17), (24), (2, 14), (2, 18), (2, 112), (116)</td>
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**Definition 4.** Let $C$ be a plane curve with $\text{Data}(C) = [m_1, \ldots, m_n]$. Rearrange as $m_1 \geq m_2 \geq \ldots \geq m_n$. For $a \in A(k)$ with $l(a) \leq n$, set
\[ S_C(a) = \sum_{i=1}^{b} a_i m_i, \quad S_C(k) = \max \left\{ S_C(a) \mid a \in A(k) \mid l(a) \leq n \right\}. \]

We define
\[ r_C(k) = dk - S_C(k). \]

**Proposition 1.** Assume that the Serrano Condition $(q)$ is satisfied. Let $\varphi$ be a rational function on $C$ with $r = \deg(\varphi) < d - \nu - q$. Then

(i) There is a rational function $\Phi$ on $\mathbb{P}^2$ which induces $\varphi$ such that $\tilde{\Phi} = \Phi \circ \pi$ gives a holomorphic map from $X$ to $\mathbb{P}^1$.

(ii) Letting $k$ be the degree of $\Phi$, we must have the upper bound:
\[ k \leq k^*(q). \]

(iii) We have the lower bound:
\[ r \geq r_C(k). \]
Proof. Under the Serrano condition \((q)\), since \(r \leq d - \nu - q - 1\), We have

\[
\tilde{C}^2 = d^2 - \sum m_i^2 > (d - \nu - q)^2 \geq (r + 1)^2.
\]

Therefore, by Serrano Extension Theorem, the rational function \(\Phi = G/H\) in (i) exists, where the \(G, H\) are relatively prime homogeneous polynomials having the same degree \(k\). Let \(f\) be a fibre of \(\Phi : X \to \mathbb{P}^1\). We can write as

\[
f \sim \pi^*O(k) - \sum a_iE_i \quad (a_i's \text{ are non-negative integers}),
\]

Using the relations: \(r = \tilde{C}f\) and \(f^2 = 0\), we have

\[
r = dk - \sum a_im_i, \quad \sum a_i^2 = k^2.
\]

By Cauchy-Schwartz inequality, we have

\[
\sum a_im_i \leq k\sqrt{\sum m_i^2}.
\]

Thus, we obtain the inequality in (ii):

\[
k \leq \frac{r}{d - \sqrt{\sum m_i^2}} \leq \frac{d - \nu - q - 1}{d - \sqrt{\sum m_i^2}} = k^*(q).
\]

By the definition of \(r_C(k)\), the inequality (iii) is clear. \(\square\)

**Theorem 1.** Let \(C\) be an irreducible plane curve of type \((d, \nu)\). Assume that the Serrano Condition \((q)\) holds. If \(r_C(k) \geq d - \nu - q\), for all positive integers \(k\) with \(k \leq k^*(q)\), then we have the inequality:

\[
\text{Gon}(C) \geq d - \nu - q.
\]

**Proof.** The assertion follows from Proposition 1. \(\square\)

**Corollary.** We have \(\text{Gon}(C) \geq d - \nu - q\), either if

(i) \(k^*(q) < 3\) if \(d \geq m_2 + m_3 + m_4 - q\), or if

(ii) \(k^*(q) < 2\) otherwise.
Proof. Note that $r_C(1) = d - \nu \geq d - \nu - q$. We have

$$
\begin{align*}
r_C(2) &= \min \{2(d - \nu), 2d - \nu - m_2 - m_3 - m_4\} \\
&= (d - \nu - q) + \min \{d - \nu + q, d - (m_2 + m_3 + m_4 - q)\}
\end{align*}
$$

if $n \geq 4$ and $r_C(2) = 2(d - \nu)$ if $n < 4$. Thus, if $d \geq m_2 + m_3 + m_4 - q$, then we obtain the inequality: $r_C(2) \geq d - \nu - q$. \hfill \Box$

I stated the content of this Corollary in a somewhat different way:

**Corollary** ([5]). Let $C$ be an irreducible singular plane curve of type $(d, \nu)$. Let $q$ be as above. Then, we have the inequality:

$$
\text{Gon}(C) \geq d - \nu - q,
$$

if the following two conditions are satisfied:

(a) \hspace{1em} d/\nu > h(\eta, \nu, q),

(b) \hspace{1em} d/\nu > f_{k_0}(\eta, \nu, q),

where $k_0 = 3$ if $d \geq m_2 + m_3 + m_4 - q$ and $k_0 = 2$ otherwise. We introduced the following invariants:

$$
\begin{cases}
\eta &= \sum_{i=1}^{n} \left( \frac{m_i}{\nu} \right)^2, \\
h(\eta, \nu, q) &= \frac{\eta}{2(1 + q/\nu)} + \frac{1 + q/\nu}{2}, \\
f_k(\eta, \nu, q) &= k\sqrt{\eta} - (1 + 1/\nu + q/\nu) \\
\end{cases}
$$

Proof. The condition $d/\nu > h(\eta, \nu, q)$ is nothing but the Serrano Condition $(q)$. Also the condition $k^*(q) < 3$ (resp. $k^*(q) < 2$) is equivalent to the condition $d/\nu > f_3(\eta, \nu, q)$ (resp. $d/\nu > f_2(\eta, \nu, q)$). \hfill \Box
Example 1. We examine the following curve considered by N.Wangyu.

\[ C : y^{31} = x^5(x - 1) \]

We have \( \text{Data}(C) = [25, 6, 4, 5] \). We prove that \( G = 6 \). Since

\[
31^2 - (25^2 + 6^2 \times 4 + 5^2 \times 6) = 42 > 6^2,
\]
the Serrano Condition (0) is satisfied. We have

\[
k^*(0) = \frac{5}{31 - \sqrt{(25^2 + 6^2 \times 4 + 5^2 \times 6)}} = 7.299 \ldots
\]

So \( k \leq 7 \). Since \( 31 > 6 \times 3 \), we have \( k \geq 3 \). We have to check whether \( r_C(k) \geq 6 \) for \( k = 3, 4, 5, 6, 7 \).

(i) \( r_C(3) \). For simplicity, we write \( S(a) \) for \( S_C(a) \). We have

\[
\begin{align*}
S((3)) &= 25 \times 3 = 75 \\
S((2, 1)) &= 2 \cdot 25 + 2 \cdot 6 + 1 \cdot 6 = 68 \\
S((2, 1)) &= 2 \cdot 25 + 6 \times 4 + 5 = 79 \\
S((1)) &= 25 + 6 \times 4 + 5 \times 4 = 69
\end{align*}
\]

So we obtain \( S_C(3) = 79 \), hence \( r_C(3) = 14 \).

(ii) \( r_C(4) \). It suffices to check \( a \in A(4) \) with \( l(a) \leq 11 \). We have

\[
\begin{align*}
S((4)) &= 25 \times 4 = 100 \\
S((3, 2, 1, 3)) &= 3 \cdot 25 + 2 \cdot 6 + 1 \cdot 6 \times 3 = 105 \\
S((3, 1, 7)) &= 3 \cdot 25 + 6 \times 4 + 5 \times 3 = 114 \\
S((2, 4)) &= 2 \cdot 25 + 2 \cdot 6 \times 3 = 86 \\
S((2, 1)) &= 2 \cdot 25 + 2 \cdot 6 \times 2 + 6 \times 2 + 5 \times 2 = 96 \\
S((2, 1)) &= 2 \cdot 25 + 2 \cdot 6 + 6 \times 3 + 5 \times 5 = 105
\end{align*}
\]

We have \( S_C(4) = 114 \), hence \( r_C(4) = 10 \).

(iii) \( r_C(5), r_C(6), r_C(7) \). Similarly, we obtain \( r_C(5) = 6, r_C(6) = 6, r_C(7) = 7 \).

We omit the details.

Remark 2. In case \( k^*(q) \) is large, the list of elements in \( A(k) \) is very large. I made a Maple program to compute the values \( r_C(k) \) up to \( k = 7 \).
3 Cyclic coverings of $\mathbb{P}^1$

We examine some cyclic coverings of $\mathbb{P}^1$. Using Corollary, we obtained the following result.

**Theorem 2 ([5]).** Let us consider the curve

$$C_{n,m} : y^n = \prod_{i=1}^{m} (x - a_i),$$

where the $a_i$'s are mutually distinct. We have

$$\text{Gon}(C_{n,m}) = \begin{cases} 
  m & \text{if } n \geq m + 1, \\
  n - 1 & \text{if } n = m, \\
  n & \text{if } n < m.
\end{cases}$$

**Definition 5.** Now we consider the following curves for $n \geq m + e$ with $m > e \geq 2$:

$$C_{n,m,e} : y^n = \prod_{i=1}^{m-e} (x - a_i) \cdot (x - b)^e,$$

where the $a_i$'s and the $b$ are mutually distinct. There are two singular points $P = (b, 0, 1)$ and $Q = (1, 0, 0)$. They are locally defined by the equations: $(x - b)^e - y^n = 0$ and $z^{n-m} - y^n = 0$. Note that $C_{n,m,e}$ has type$(n, n - m)$.

**Lemma 1.** *The Serrano Condition* (0) *is satisfied if*

$$n - m - e > \frac{e^2}{m - e}.$$ 

*Proof.* Note that $M(P) \leq en$ and $M(Q) \leq (n - m)n$ (See Remark 1). We have

$$n^2 - M(P) - M(Q) \geq n^2 - en - (n - m)n = n(m - e).$$

It follows that

$$n(m - e) - m^2 = (m - e)\{n - m - e - \frac{e^2}{m - e}\},$$

which gives the assertion. \qed
Lemma 2. We have
\[ k^*(0) < 2 + \frac{2(e - 1)}{m - e}. \]

Proof. By definition, we have
\[
k^*(0) = \frac{m - 1}{n - \sqrt{n(n - m + e)}} = \frac{(m - 1)\{n + \sqrt{n(n - m + e)}\}}{(m - e)n}
\]
\[
< \frac{2(m - 1)}{m - e} = 2 + \frac{2(e - 1)}{m - e}.
\]

\[ \square \]

Lemma 3. For \( e = 2, 3 \), we have the inequality: \( d \geq m_2 + m_3 + m_4 \) except for the following curves: \( C_{5,3,2}, C_{6,3,2}, C_{7,4,3}, C_{9,4,3}, C_{8,5,3}, C_{9,5,3}, C_{10,5,3} \).

Proof. Set \( R = m_2 + m_3 + m_4 \). If \( n \geq 3(n - m) \) or equivalently, if \( n \leq 3m/2 \), then the assertion is clear. (i) \( n \geq 3m \). we have \( m_2 = m_3 = m \geq m_4 \). So \( R \leq 3m \leq n \) (ii) \( 2m \leq n < 3m \). In case \( n - 2m \geq e \), we have \( m_2 = m, m_3 = n - 2m \geq m_4 \), hence \( R \leq 2n - 3m < n \). In case \( n - 2m < e \), if \( n \geq m + 2e \), then we have \( n \geq R \). It turns out that the required inequality does not hold for \( C_{6,3,2}, C_{9,4,3}, C_{10,5,3} \). (iii) \( 3m/2 < n < 2m \). In case \( 2m - n \geq e \), we have \( m_2 = n - m, m_3 = 2m - n \geq m_4 \), hence \( R \leq 3m - n < n \). In case \( 2m - n < e \), if \( n \geq n - m + 2e \), then we have \( n \geq R \). Checking the remaining cases, we see that the required inequality does not hold for \( C_{5,3,2}, C_{7,4,3}, C_{8,5,3}, C_{9,5,3} \) \[ \square \]

Theorem 3. For \( n \geq m + 2 \) with \( m > 2 \), we have
\[ \text{Gon}(C_{n,m,2}) = m \]
except for the following cases:
\[ \text{Gon}(C_{5,3,2}) = 2, \quad \text{Gon}(C_{6,3,2}) = 2, \quad \text{Gon}(C_{6,4,2}) = 3. \]

Proof. We write \( C = C_{n,m,2} \). First we consider the case in which \( n = m + 2 \). In this case, \( \text{Data}(C) = [2] \), where \( l = 2[n/2] \). By Theorem 1 in [5], we obtain \( G = n - 2 = m \) for \( n \geq 7 \) and \( G \geq 3 \) for \( n = 6 \). For \( n = 6 \), since \( G \leq 3 \) (Brill-Noether bound), we have \( G = 3 \). For \( n = 5 \), \( C \) has genus 2, hence \( G = 2 \).
In what follows we assume that \( n > m + 2 \). By Lemma 1 with some computation, we see that the Serrano Condition \((0)\) is satisfied except for the following cases: (1) \( m = 6, n = 9, 10 \), (2) \( m = 5, n = 8 \), (3) \( m = 4, n = 8 \), (4) \( m = 3, n = 6 \).

**Case 1.** We consider the case where the Serrano Condition \((0)\) is satisfied. By Lemma 2, we have \( k^*(0) \leq 2 \) (for \( m \geq 4 \)), \( \leq 3 \) (for \( m = 3 \)). In case \( m \geq 4 \), in view of Lemma 3, by Corollary, we get \( G = m \). In case \( m = 3 \), we have to compute the term \( r_C(3) \). First of all, noting that \( n \geq 7 \), the Data consists of \([n - 3, 3t, 2s]\) for some \( t, s \). We have

\[
S((3)) = 3(n - 3) = 3n - 6
\]
\[
S((2, 1)) \leq 2(n - 3) + 2 \times 3 + 1 \times 3 = 2n + 3
\]
\[
S((2, 15)) \leq 2(n - 3) + 1 \times 3 \times 5 = 2n + 9
\]
\[
S((1)) \leq (n - 3) + 8 \times 3 = n + 21
\]

If \( S_C(3) \leq 3(n - 1) \), then we obtain \( r_C(3) \geq 3 \). For \( n \geq 12 \), this is the case. So we consider the remaining cases where \( 7 \leq n \leq 11 \). We obtain \( r_C(3) \geq 3 \) for all these cases. Thus, by Theorem 1, we obtain \( G = 3 \).

**Case 2.** Now we consider the exceptional cases in which the Serrano Condition \((0)\) does not hold.

(1) \( m = 6 \). We have \( G = 6 \) for \( n = 9 \) by Theorems 2 in [5]. We also have \( G = 6 \) for \( n = 10 \) by Theorem 1 in [5]. Indeed, we have \( \delta = 19 \) and \( V = 4 \). So the Conditions (A), (B) there are satisfied.

(2) \( m = 5 \). For \( n = 8 \), we have \( G = 5 \) (Theorems 2 in [5]).

(3) \( m = 4 \). It is known that the curve \( C_{8,4,2} \) is birational to the octic curve \( C : y^8 = x^2(x - 1)(x - a) \) with some \( a \). Letting \( X = y^4/x(x - 1), Y = y \), the curve \( C \) is birationally transformed to the octic:

\[
C' : (X^2 - 1)^2Y^4 + (a - 1)X(X^2 - a) = 0,
\]

which has four singular points with \([3], [3], [2], [2]\). Let \( C'' \) be the quadratic transformation of \( C' \). We have

\[
C'' : (X^2 - 1)^2 - (a - 1)X(aX^2 - 1)Y^4 = 0.
\]

Substituting \((x, y, z) = (X + Z, Y, Z)\), \( C'' \) is transformed to the curve:

\[
C^{(3)} : x^2(x + 2)^2 - (a - 1)(x + 1)(ax^2 + 2ax + a - 1)y^4.
\]
By quadratic transformation, $C^{(3)}$ is transformed to the following curve:

$$C^{(4)} : (2x + 1)^2y^4 - (a - 1)x((a - 1)x^2 + 2ax + a) = 0.$$  

It turns out that $C^{(4)}$ is a sextic with two double points with $[2_3]$ and $[2_2]$. We conclude that $G = 4$ (Theorem 1 in [5]).

(4) $m = 3$. For $n = 6$, we have $g = 1$. So $G = 2$. \hfill $\Box$

**Theorem 4.** For $n \geq m + 3$ with $m > 3$, we have

$$\text{Gon}(C_{n,m}) = m$$

except for the following cases (including undetermined cases):

1. $m = 10$. $\text{Gon}(C_{14,10}) = 9$ or 10,
2. $m = 7$. $\text{Gon}(C_{11,7}) = 6$,
3. $m = 6$. $\text{Gon}(C_{n,6}) = 5$ or 6 for $n = 9, 10, 12$,
4. $m = 5$. $\text{Gon}(C_{8,5}) = \text{Gon}(C_{9,5}) = \text{Gon}(C_{10,5}) = 4$,
5. $m = 4$. $\text{Gon}(C_{7,4}) = \text{Gon}(C_{8,4}) = 2$ and $\text{Gon}(C_{n,4}) = 3$ for $n = 9, 10, 11, 12$.

**Proof.** We write $C = C_{n,m}$. First we consider the case in which $n = m + 3$. If $n \geq 10$, then applying Theorem 2 in [5], we obtain $G = n - 3 = m$. We separately deal with the cases $n = 7, 8, 9$.

(a) The curve $C_{7,4}$ is birational to the curve $y^7 = x(x - 1)$, which is a hyperelliptic curve, hence $G = 2$.

(b) The curve $C_{8,5}$ is birational to the curve $C_8 : y^8 = x^3(x - 1)(x - a)$ for some $a$. Letting $X = y^3/x$, $C_8$ is birationally transformed to the septic $C' : X^5 = (Y^3 - X)(Y^3 - aX)Y$, with Data($C'$) = $[3, 2_5]$. By applying Theorem 2 in [5], we have $G = 4$.

(c) For $C_{9,6,3}$, the Serrano Condition (1) is satisfied and $k^*(1) < 3$. Since $n \geq m_2 + m_3 + m_4 - 1$, by Corollary, we obtain $G \geq 5$. 

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In what follows we assume that \( n > m + 3 \). By Lemma 1 with some direct computation, we see that the Serrano Condition (0) is satisfied except for the following cases:

(1) \( m = 10, n = 14 \), (2) \( m = 7, n = 11 \), (3) \( m = 6, n = 10, 12 \),
(4) \( m = 5, n = 9, 10 \), (5) \( m = 4, 8 \leq n \leq 12, 14 \).

**Case 1.** We consider the case where the Serrano Condition (0) is satisfied. We know that \( k^*(0) < 3 \), if \( m \geq 7 \) (Lemma 2). With the help of Lemma 3, by Corollary, we obtain \( G = m \) for \( m \geq 7 \). We consider the remaining cases: 

(1) \( m = 6 \). In this case, for \( n \geq 13 \), we have \( \text{Data}(C) = [n - 6, 6_i, \ldots] \). By computation, we get \( r_C(3) \geq 7 \) if \( n \geq 24 \). In the remaining cases: \( n = 11 \) and \( 13 \leq n \leq 23 \), we have \( r_C(3) \geq 7 \). Thus, by Theorem 1, we have \( G = 6 \).

(2) \( m = 5 \). In this case, for \( n \geq 11 \), we have \( \text{Data}(C) = [n - 5, 5_i, \ldots] \). By computation, we get \( r_C(3) \geq 5 \) if \( n \geq 20 \). In the remaining cases: \( 11 \leq n \leq 19 \), we have \( r_C(3) \geq 5 \). By Theorem 1, we have \( G = 5 \).

(3) \( m = 4 \). In this case, for \( n \geq 9 \), we have \( \text{Data}(C) = [n - 4, 4_i, \ldots] \). Computing the terms \( r_C(3), r_C(4), r_C(5) \), we obtain the following results.

\[ r_C(3) \geq 4 \text{ (if } n \geq 16), \quad r_C(4) \geq 4 \text{ (if } n \geq 22), \quad r_C(5) \geq 4 \text{ (if } n \geq 28). \]

In the remaining cases: \( n = 13 \) and \( 15 \leq n \leq 27 \), we obtain \( r_C(i) \geq 4 \) for \( i = 3, 4, 5 \). By Theorem 1, we conclude that \( G = 4 \).

**Case 2.** We now consider the case where the Serrano Condition (0) is not satisfied.

(1) \( m = 10, n = 14 \). We have \( \text{Data}(C) = [4_3, 3_4, 2_3] \). The Serrano Condition (1) is satisfied and \( k^*(1) < 2 \). By Corollary, we obtain \( G \geq 9 \).

(2) \( m = 7, n = 11 \). We have \( \text{Data}(C) = [4_2, 3_4, 2] \). The Serrano Condition (1) is satisfied and \( k^*(1) < 2 \). By Corollary, we obtain \( G \geq 6 \).

(3) \( m = 6, n = 10, 12 \). In both cases, the Serrano Condition (1) is satisfied and \( k^*(1) < 3 \). With the help of Lemma 3, by Corollary, we obtain \( G \geq 5 \).
(4) $m = 5, n = 9, 10$. Note that the curve $C_{n,4,3}$ is birational to the curve $C_n : y^n = x^3(x - 1)(x - a)$ for some $a$. Letting $X = y^3/x, Y = y$, the curve $C_9$ (resp. $C_{10}$) is birationally transformed to the curve $C' : X^5 = (Y^3 - X)(Y^3 - aX)$ (resp. $C'' : X^5Y = (Y^3 - X)(Y^3 - aX)$). The curves $C'$ and $C''$ are of type $(5, 2)$ with genus 7. We have $G = 4$ (Theorem 1 in [5]).

(5) $m = 4, n = 8, 9, 10, 11, 12, 14$. Note that the curve $C_{n,4,3}$ is birational to the curve $C_n : y^n = x^3(x - 1)$.

(a) Since genus($C_8$) = 2, we have $G = 2$.

(b) The curves $C_9$ and $C_{12}$ are birational to smooth quartics, hence we have $G = 3$.

(c) Letting $X = y^3/x, Y = y$ (resp. $X = y^4/x, Y = y$, the curve $C_{10}$ (resp. $C_{11}$) is birationally transformed to the curve $C' : X^4Y = Y^3 - X$ (resp. $C'' : X^4 = (Y^4 - X)Y$). The curves $C'$ and $C''$ are of type $(5, 2)$ with genus 4 and 5, respectively. It follows that $G = 3$.

(d) Letting $X = y^5/x, Y = y$, the curve $C_{14}$ is birationally transformed to the curve $\Gamma : X^4 = (Y^5 - X)Y$, with Data($\Gamma$) = $[24]$. By Theorem 1 in [5], we have $G = 4$.

\[ \square \]

**Remark 3.** Recently, N.Wangyu [7, 8, 9] classified hyperelliptic curves and trigonal curves among cyclic coverings of $\mathbb{P}^1$. 

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References


[9] N.Wangyu, Cyclic coverings of the projective line with prime gonality, Preprint

Fumio SAKAI
Department of Mathematics
Graduate School of Science and Engineering
Saitama University
Shimo–Okubo 255
Sakura–ku, Saitama 338–8570, Japan
E–mail: fsakai@rimath.saitama-u.ac.jp