The gonality of singular plane curves
Masahito Ohkouchi and Fumio Sakai

1 Introduction

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d$ over the complex number field $\mathbb{C}$. We denote by $\mathbb{C}(C)$ the field of rational functions on $C$. Let $\tilde{C}$ be the non-singular model of $C$. Since $\mathbb{C}(\tilde{C}) \cong \mathbb{C}(C)$, a non-constant rational function $\varphi$ on $C$ induces a non-constant morphism $\varphi : \tilde{C} \to \mathbb{P}^1$. Let $\deg \varphi$ denote the degree of this morphism $\varphi$. We remark that $\deg \varphi = [\mathbb{C}(\tilde{C}) : \mathbb{C}(\varphi)] = \deg (\varphi)_{0} = \deg (\varphi)_{\infty}$. The gonality of $C$, denoted by $\text{Gon}(C)$, is defined to be $\min \{ \deg \varphi \mid \varphi \in \mathbb{C}(C) \setminus \mathbb{C} \}$. So by definition, the gonality of $C$ is nothing but the gonality of $\tilde{C}$. Let $\nu$ denote the maximal multiplicity of $C$. We easily see that $\text{Gon}(C) \cdot d \leq \nu$. We know that the genus of $C$ is equal to $(d - 1)(d - 2)/2 - \delta$ with $\delta \geq 0$.

**Theorem 1.** Let $C$ be an irreducible plane curve of degree $d$ with $\delta \geq \nu$. Letting $d \equiv i \pmod{\nu}$, define

$$R(\nu, \delta, i) = \frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)} + \sqrt{\frac{\delta - \nu}{\nu - 1} + \frac{(\nu - 2 + i)^2}{2(\nu - 1)}}.$$

If $d/\nu > R(\nu, \delta, i)$, then $\text{Gon}(C) = d - \nu$.

**Remark 1.** Theorem 1 is a generalization of Theorem 2.1 in Coppens and Kato [1] where they considered the case in which $C$ has only nodes and ordinary cusps. Note that $R(2, \delta, 0) = 1 + \sqrt{\delta - 2}$, $R(2, \delta, 1) = 1 + \sqrt{\delta - 7/4}$. In general, we have the estimation: $R(\nu, \delta, i) < 1 + \sqrt{\delta/(\nu - 1)}$.

We have $\delta < \nu$ if either (i) $C$ is a smooth curve ($\delta = 0, \nu = 1$ and $\text{Gon}(C) = d - 1$ for all $d \geq 2$), or (ii) $C$ has one node or one ordinary cusp ($\delta = 1$ and $\nu = 2$ and $\text{Gon}(C) = d - 2$ for all $d \geq 3$). Cf. [1], [3], [5].

**Definition.** Let $m_1, \ldots, m_n$ denote the multiplicities of all singular points (we include infinitely near singular points) of $C$. Set $\eta = \sum (m_i/\nu)^2$. Clearly, we have $\eta \geq 1$. 

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THEOREM 2. Let $C$ be an irreducible plane curve of degree $d$ with $\nu \geq 3$. We have $\text{Gon}(C) = d - \nu$ if

$$d/\nu \begin{cases} > (\eta + 1)/2, & \text{for } \eta < a(\nu), \eta \geq 5 \\ > 2\sqrt{\eta} - (1 + 1/\nu), & \text{for } a(\nu) \leq \eta < 4, \\ \geq 3, & \text{for } 4 \leq \eta < 5, \end{cases}$$

where $a(\nu) = (2 - \sqrt{1 - 2/\nu})^2$.

REMARK 2. Note that $a(3) = 2.023\ldots$ and $1 < a(\nu) \leq 1.671\ldots$ for $\nu \geq 4$.

We shall show that if $\eta \geq 2\nu + 5$, then the criterion in Theorem 1 is more effective than that in Theorem 2. We also prove some subtle criterions.

THEOREM 3. Let $C$ be an irreducible plane curve of degree $d$ with $n$ singular points (infinitely near singular points are also counted). We renumber the multiplicities $m_i$'s as $\nu = m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_n$. We have $\text{Gon}(C) = d - \nu$ if either

(i) $n \leq 2$, or

(ii) $n = 3$ and $d/\nu > 2$, or

(iii) $n \geq 4$, $d \geq m_2 + m_3 + m_4$ and

$$d/\nu > \begin{cases} (\eta + 1)/2 & \text{if } \nu = 3, 4, \\ (1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\} & \text{if } \nu \geq 5 \text{ and } \eta < b(\nu), \eta \geq c(\nu), \\ (1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\} & \text{if } \nu \geq 5 \text{ and } b(\nu) \leq \eta < c(\nu), \end{cases}$$

where $b(\nu) = (3/2 - \sqrt{1/4 - 1/\nu})^2$ and $c(\nu) = (3/2 + \sqrt{1/4 - 1/\nu})^2$.

REMARK 3. In view of Theorem 2, the condition (iii) is meaningful only if $a(\nu) \leq \eta < 5$. We remark that $a(\nu) < b(\nu) < c(\nu)$ and $1 < b(\nu) \leq 1.629\ldots$ and $2.970\ldots \leq c(\nu) < 4$ for $\nu \geq 5$.

2 Rational functions on $C$ and on $\mathbb{P}^2$

Let $\varphi$ be a rational function on $C$. Set $r = \deg \varphi$. We know that a rational function $\varphi$ of a plane curve $C$ is a restriction of a rational function $\Phi = g(x, y, z)/h(x, y, z)$ on $\mathbb{P}^2$, where $g$ and $h$ are relatively prime homogeneous
polynomials of the same degree, say \( k \). We call \( k \) the degree of the rational function \( \Phi \). A rational function \( \Phi \) is called a \textit{linear function} if \( k = 1 \).

Classically, one says that \( \varphi \) is cut out by the pencil \( \Lambda : t_0g - t_1h = 0 \) on \( \mathbb{P}^2 \).

Let us consider the rational map

\[
\Phi : \mathbb{P}^2 \ni P \mapsto (h(P), g(P)) \in \mathbb{P}^1.
\]

By a sequence of blowing-ups \( \pi : X \rightarrow \mathbb{P}^2 \), one can resolve the base points of \( \Phi \) and the singularities of \( C \), so that \( \Phi \circ \pi : X \rightarrow \mathbb{P}^1 \) becomes a morphism and the strict transform \( \tilde{C} \) of \( C \) is non-singular. Write \( \pi = \pi_1 \circ \cdots \circ \pi_s \), where \( \pi_i : X_i \rightarrow X_{i-1} \) is the blowing-up at a point \( P_i \in X_{i-1} \) and \( X_0 = \mathbb{P}^2 \), \( X_s = X \). Let \( E_i \) be the total transform on \( X \) of the exceptional curve of the blowing-up \( \pi_i \). We have a relation of divisors: \( \tilde{C} = \pi^*C - \sum m_iE_i \), where \( m_i \) is the multiplicity of the strict transform of \( C \) on \( X_{i-1} \) at \( P_i \).

Set \( H = \pi^*L \), where \( L \) is a line on \( \mathbb{P}^2 \). Then, we have the linear equivalence: \( \tilde{C} \sim dH - \sum a_iE_i \). It follows from this and the adjunction formula that \( \delta = \sum m_i(m_i-1)/2 \). Any fibre \( D \) of the morphism \( \Phi \circ \pi \) is linearly equivalent to a divisor \( kH - \sum a_iE_i \) with some integers \( a_i \). Since \( DE_i \geq 0 \) and \( D^2 = 0 \), we must have the relation:

\[
k^2 = \sum a_i^2
\]

and also we must have \( a_i \geq 0 \) for all \( i \). We then obtain the formula:

\[
r = dk - \sum a_im_i.
\]

If \( k = 1 \), then we must have \( r = d - m_i \) for some \( i \). In particular, there is a rational function \( \varphi \) with \( r = d - \nu \). Note that a rational function \( g^*/h^* \in C(\mathbb{P}^2) \) also induces \( \varphi \) if and only if \( gh^* - hg^* \) is divisible by the defining polynomial of \( C \). So many different rational functions on \( \mathbb{P}^2 \) can induce the same rational function \( \varphi \) on \( C \).

**LEMMA 1.** We have the inequality: \( r + \delta \geq dk - k^2 \).

**PROOF.** It suffices to show that \( k^2 + \delta \geq \sum a_im_i \). We see that

\[
k^2 + \delta - \sum a_im_i = \sum_{m_i \neq 1} (2a_i - m_i)^2/4 + \sum_{m_i \neq 1} m_i(m_i - 2)/4 + \sum_{m_i = 1} a_i(a_i - 1).
\]

If \( m_i \geq 2 \) or \( m_i = 0 \), then \( m_i(m_i - 2) \geq 0 \). Since \( a_i \) is an integer, we have \( a_i(a_i - 1) \geq 0 \). Thus we get the desired inequality. \( \square \)

Let \( b \) denote the number of \( a_i \) with \( a_i \neq 0 \).
LEMMA 2. If \( r < d - \nu \), then \( k \geq 2 \) and \( d/\nu < (k\sqrt{b} - 1)/(k - 1) \).

PROOF. If \( k = 1 \), then we have \( r \geq d - \nu \). So assume \( k \geq 2 \). By Schwarz’ inequality, we have \( \sum a_i \leq \sqrt{b}k \). We obtain

\[
r \geq dk - (\sum a_i)\nu \geq k(d - \nu \sqrt{b}) = d - \nu + (k - 1)\nu \{d/\nu - (k\sqrt{b} - 1)/(k - 1)\},
\]

which implies the assertion. \( \square \)

LEMMA 3. If \( r < d - \nu \), then \( k \geq 2 \) and \( k > d/\nu - 1 \). Furthermore, if \( r = d - \nu + s \) with \( s \geq 0 \) and \( k \geq 2 \), then \( k \geq d/\nu - s - 1 \).

PROOF. In view of the inequality in Lemma 2, it suffices to note that \( b \leq k^2 \). Suppose \( r = d - \nu + s \) with \( s \geq 0 \). If \( k \geq 2 \), then we obtain

\[
k + s \geq k + s/(k - 1) \geq d/\nu - 1.
\]

We renumber \( a_i \)'s so that \( a_1 \geq a_2 \geq \ldots \geq a_b \geq 1 \), \( a_i = 0 \) for \( i > b \).

LEMMA 4. We have \( r \geq d - \nu \) either if \( b \leq 2 \), or if \( b = 3 \) and \( d/\nu \geq 2 \).

PROOF. (i) \( b = 1 \). We have \( r = k(d - m_1) \geq k(d - \nu) \geq d - \nu \). (ii) \( b = 2 \). By Bezout’s theorem applied to the curve \( C \) and the line passing through \( P_1 \) and \( P_2 \), we have the inequality: \( d \geq m_1 + m_2 \). On the other hand, since \( k^2 = a_1^2 + a_2^2 \), we must have \( a_i < k \) for \( i = 1, 2 \). Thus, we obtain

\[
r = d - \nu + (\nu - m_1) + (k - 1)d - (a_1 - 1)m_1 - a_2m_2 \geq d - \nu + (k - a_1)m_1 + (k - a_2 - 1)m_2 \geq d - \nu.
\]

(iii) \( b = 3 \). In case \( k \geq 4 \), by Lemma 2, we have \( r > d - \nu \), since \((4\sqrt{3} - 1)/3 = 1.976 \ldots < 2 \). In case \( k \leq 3 \), the equation: \( k^2 = a_1^2 + a_2^2 + a_3^2 \), \((3 \leq a_1 + a_2 + a_3 \leq 5) \) has only one integer solution: \( k = 3, a_1 = a_2 = 2, a_3 = 1 \). Note that \( d \geq \nu + m_2 \). Under the assumption \( d \geq 2\nu \), we get \( r = d - \nu + (\nu - m_1) + 2d - (m_1 + 2m_2 + m_3) \geq d - \nu \). \( \square \)

LEMMA 5. If \( r < d - \nu \) and \( k = 2 \), then \( b = 4 \) and \( d < m_1 + m_2 + m_3 + m_4 - \nu \).
PROOF. We have $b = 1$ or $b = 4$. In case $b = 4$, we must have $a_1 = a_2 = a_3 = a_4 = 1$. So we obtain $d - \nu > r = 2d - m_1 - m_2 - m_3 - m_4$, which gives the assertion.

LEMMA 6. We have the inequality: $r \geq k(d - \sqrt{\sum m_i^2})$.

PROOF. By Schwarz’ inequality, we have

$$\sum a_i m_i \leq \sqrt{\sum a_i^2} \sqrt{\sum m_i^2} = k \sqrt{\sum m_i^2},$$

which gives the assertion.

3 Proof of Theorem 1

Let $C$ be an irreducible plane curve of degree $d$.

LEMMA 7 (Cf. Coppens and Kato[1, 2]). Let $\varphi$ be a rational function on $C$ with $r = \deg \varphi$. Let $l$ be a positive integer with $l < d$. Suppose $r + \delta < (l + 1)(d - l - 1)$. Then there exists a rational function on $\mathbb{P}^2$ of degree $k \leq l$ which induces $\varphi$ on $C$.

PROOF. Assume to the contrary that there are no rational functions of degree $\leq l$ on $\mathbb{P}^2$ which induces $\varphi$ on $C$. Following the arguments in [1, 2], one can prove that there exists a rational function of degree $k$ on $\mathbb{P}^2$ which induces $\varphi$ on $C$ with $l < k \leq d - 3 - l$. Using Lemma 1, we have $dk - k^2 \leq r + \delta < (l + 1)(d - l - 1)$, from which we infer that $(l + 1 - k)(d - k - l - 1) > 0$. This is absurd, because $l + 1 - k \leq 0$ and $d - k - l - 1 \geq 2$.

PROPOSITION 1. Assume there is a positive integer $l$ such that $l \leq (d/\nu) - 1$ and $\delta - \nu < l(d - l - 2)$. Then we have $\text{Gon}(C) = d - \nu$.

PROOF. Suppose there exists a rational function $\varphi$ on $C$ with $r = \deg \varphi < d - \nu$. In this case, we have the inequality:

$$r + \delta \leq d - \nu - 1 + \delta < l(d - l - 2) + d - 1 = (l + 1)(d - l - 1).$$

So by Lemma 7, there exists a rational function $\Phi$ of degree $k \leq l$ on $\mathbb{P}^2$ which induces $\varphi$ on $C$. But, since $k \leq l \leq (d/\nu) - 1$, by Lemma 3, there cannot exist such a rational function $\Phi$. 

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PROPOSITION 2. If \([d/\nu] \geq 2\) and \(((d/\nu) - 1)(d - [d/\nu] - 1) > \delta - \nu\), then we have \(\text{Gon}(C) = d - \nu\).

REMARK 4. In case \(\nu = 2\), this criterion is best possible. See [1], Examples 4,1 and 4,2. We see that the assertion of Proposition 1 is equivalent to that of Proposition 2. Take a positive integer \(l\) which satisfies the two assumptions in Proposition 1. We find that \(1 \leq l \leq [d/\nu] - 1 \leq (d/\nu) - 1\). The quadratic function \(Q(x) = x(d - x - 2)\) is a monotone increasing function for the interval \(0 \leq x \leq (d/2) - 1\). Hence we infer that \(Q(l) \leq Q([d/\nu] - 1)\). Thus the integer \([d/\nu] - 1\) also satisfies the two assumptions in Proposition 1.

Using the latter assertion in Lemma 3, we obtain the following

PROPOSITION 3. Let \(s\) be a non-negative integer. Set \(l = d/\nu - s - 2\) (if \(d \equiv 0 \pmod{\nu}\)), \([d/\nu] - s - 1\) (otherwise). If \(l \geq 1\) and \(\delta - \nu + s + 1 < l(d - l - 2)\), then \(\text{Gon}(C) = d - \nu\) and any rational function \(\phi\) with \(d - \nu \leq \deg \phi \leq d - \nu + s\) is induced by a linear function on \(\mathbf{P}^2\).

Proof of Theorem 1. We reformulate Proposition 2. Letting \(d = [d/\nu]\nu + i\) with \(0 \leq i < \nu\), the inequality \(\delta - \nu < ([d/\nu] - 1)(d - [d/\nu] - 1)\) can be written as:

\[
\frac{\delta - \nu}{\nu - 1} + \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2 < \left\{d - \frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)}\right\}^2.
\]

If \(\delta - \nu \geq 0\), then the above inequality is equivalent to the inequality \(d/\nu > R(\delta, \nu, i)\). Furthermore, we easily see that \(R(\delta, \nu, i) \geq 1 + i/\nu\). So it follows from the inequality \(d/\nu > R(\delta, \nu, i)\) that \(d > \nu + i\), which gives \(d \geq 2\nu + i\) if \(d \equiv i \pmod{\nu}\) and hence \(d/\nu \geq 2\).

REMARK 5. If \(\delta - \nu < 0\), then the left hand side of the above inequality is negative. It follows that the above inequality always holds. In case \(\delta = 1, \nu = 2\), we have \(\text{Gon}(C) = d - 2\) for \(d \geq 4\). It is well known that \(\text{Gon}(C) = 1\) if \(d = 3\). In case \(\delta = 0, \nu = 1\), we have \(\text{Gon}(C) = d - 1\) for \(d \geq 2\).

LEMMA 8. We have the estimation:

\[
R(\nu, \delta, i) < 1 + \sqrt{\delta/(\nu - 1)}.
\]
PROOF. Since $i \leq \nu - 1$, we have
\[
\nu^2 + (\nu - 2)i \leq \nu^2 + (\nu - 2)(\nu - 1) = 2\nu(\nu - 1) - (\nu - 2)
\]
and $\nu - 2 + i \leq 2(\nu - 1)$. Thus, we obtain
\[
\frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)} \leq 1 \quad \text{and} \quad \frac{\nu}{\nu - 1} - \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2 > 0,
\]
which gives the desired inequality. \(\square\)

4 Proof of Theorems 2 and 3

Let $C$ be an irreducible plane curve of degree $d$. Now let $\pi : X \to \mathbb{P}^2$ be the minimal resolution of the singularities of $C$. We do not require that the inverse image $\pi^{-1}(C)$ has normal crossings. In this case, $m_i \geq 2$ for all $i$.

LEMMA 9. Assume $d/\nu > (\eta + 1)/2$. Let $\varphi$ be a rational function on $C$ with $r = \deg \varphi < d - \nu$. Then we can find a rational function $\Phi$ on $\mathbb{P}^2$ which induces $\varphi$ on $C$ such that $\Phi \circ \pi : X \to \mathbb{P}^1$ becomes a morphism. Furthermore, the degree $k$ of $\Phi$ satisfies the inequality:

\[
k \leq 1 + \frac{\sqrt{\eta} - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}}.
\]

PROOF. According to Theorem 3.1 in Serrano[5](See also [4]), such a rational function exists if $C^2 > (r + 1)^2$. On $X$, we have
\[
\hat{C}^2 - (r + 1)^2 \geq d^2 - \sum m_i^2 - (d - \nu)^2 = 2d\nu - \sum m_i^2 - \nu^2 = 2\nu^2\{d/\nu - (\eta + 1)/2\} > 0.
\]
By Lemma 6, we have $d - \nu - 1 \geq r \geq k\nu(d/\nu - \sqrt{\eta})$. Thus we obtain
\[
k \leq \frac{d/\nu - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}} = 1 + \frac{\sqrt{\eta} - (1 + 1/\nu)}{d/\nu - \sqrt{\eta}}.
\]
\(\square\)

REMARK 6. Under the hypothesis $d/\nu > (\eta + 1)/2$, we see that $d/\nu - \sqrt{\eta} = d/\nu - (\eta + 1)/2 + (\sqrt{\eta} - 1)^2/2 > 0$. Since $k \geq 1$, we must have $\sqrt{\eta} - (1 + 1/\nu) \geq 0$.

In a similar manner to that in the proof of Lemma 9, we can show the following
**LEMMA 10.** Let $s$ be a non-negative integer with $s < \nu - 1$. Let $\varphi$ be a rational function on $C$ with $r = \deg \varphi = d - \nu + s$. If
\[
d/\nu > (\eta + 1)/2 + \frac{s + 1}{2(\nu - s - 1)} \left\{ \eta - 1 + \frac{s + 1}{\nu} \right\},
\]
then we can find a rational function $\Phi$ on $P^2$ which induces $\varphi$ on $C$ such that $\Phi \circ \pi : X \to P^1$ becomes a morphism. Furthermore, the degree $k$ of $\Phi$ satisfies the inequality:
\[
k \leq 1 + \frac{\sqrt{\eta} - 1 + s/\nu}{d/\nu - \sqrt{\eta}}.
\]

**PROPOSITION 4.** Suppose $d/\nu > (\eta + 1)/2$. We get $Gon(C) = d - \nu$ if either

(i) $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$, or

(ii) $\eta \geq 5$, or

(iii) $d/\nu \geq 3$ and $\eta < 5$, or

(iv) $d/\nu > (1/2) \{3\sqrt{\eta} - (1 + 1/\nu)\}$ and $d \geq m_2 + m_3 + m_4$ (if $n \geq 4$), where the multiplicities $m_i$’s are renumbered as $m_1 \geq m_2 \geq m_3 \geq \cdots$.

**PROOF.** Assume there is a rational function $\varphi$ on $C$ with $r = \deg \varphi < d - \nu$. By Lemma 9, we can find a rational function $\Phi$ on $P^2$ which induces $\varphi$ on $C$ such that $\pi$ has already resolved the base points of $\Phi$. The degree $k$ of $\Phi$ must satisfy the inequality in Lemma 9.

(i) If $d/\nu > 2\sqrt{\eta} - (1 + 1/\nu)$, then we infer that $k < 2$. So we get $k = 1$, which is impossible by Lemma 3.

(ii) If $\eta \geq 5$, then we have $d/\nu > 3$. We obtain
\[
k < 1 + \frac{\sqrt{\eta} - 1}{(\eta + 1)/2 - \sqrt{\eta}} = 1 + \frac{2}{\sqrt{\eta} - 1} \leq 1 + \frac{2}{\sqrt{5} - 1} = (3 + \sqrt{5})/2 < 3.
\]
So $k \leq 2$, which contradicts Lemma 3.

(iii) We have
\[
k < 1 + \frac{\sqrt{5} - 1}{3 - \sqrt{5}} = (3 + \sqrt{5})/2 < 3.
\]
So \( k \leq 2 \), which again contradicts Lemma 3.

(iv) In a similar manner to that in the proof of (i), under the assumption on \( d/\nu \), we obtain \( k < 3 \). In case \( k = 2 \), by Lemma 5, we get a contradiction. □

Proof of Theorem 2. By Proposition 4, (i), we get \( \text{Gon}(C) = d - \nu \) if \( d/\nu > \max\{2\sqrt{\eta} - (1 + 1/\nu), (\eta + 1)/2\} \). We easily see that \( 2\sqrt{\eta} - (1 + 1/\nu) \geq (\eta + 1)/2 \) if and only if \( 2 - \sqrt{1 - 2/\nu} \leq \sqrt{\eta} \leq 2 + \sqrt{1 - 2/\nu} \). In case \( \nu \geq 3 \), we have the relation: \( a(\nu) = (2 - \sqrt{1 - 2/\nu})^2 < 5 < (2 + \sqrt{1 - 2/\nu})^2 \). Using also Proposition 4, (ii), we get \( \text{Gon}(C) = d - \nu \) if

\[
d/\nu > \begin{cases} (\eta + 1)/2, & \text{for } \eta < a(\nu), \eta \geq 5 \\ 2\sqrt{\eta} - (1 + 1/\nu), & \text{for } a(\nu) \leq \eta < 5 \end{cases}.
\]

On the other hand, by Proposition 4, (iii), for \( \eta < 5 \), we also get \( \text{Gon}(C) = d - \nu \) if \( d/\nu \geq 3 \). Obviously, \( 2\sqrt{\eta} - (1 + 1/\nu) > 3 \) if and only if \( \sqrt{\eta} > 2 + 1/(2\nu) \). Thus, for \( (2 + 1/(2\nu))^2 < \eta < 5 \), the condition \( d/\nu \geq 3 \) is sharper than the condition \( d/\nu > 2\sqrt{\eta} - (1 + 1/\nu) \). Finally, for the interval \( 4 \leq \eta \leq (2 + 1/(2\nu))^2 \), we find that \( 3 \geq 2\sqrt{\eta} - (1 + 1/\nu) \geq 3 - 1/\nu \). The inequality \( d/\nu > 3 - 1/\nu \) implies \( d > 3\nu - 1 \), hence \( d \geq 3\nu \). As a consequence, the conditions \( d/\nu \geq 3 \) and \( d/\nu > 2\sqrt{\eta} - (1 + 1/\nu) \) have the same effect.

REMARK 7. In case \( \nu = 2 \), we infer from Proposition 4, (i) that if \( d/2 > (\eta + 1)/2 \), then \( \text{Gon}(C) = d - 2 \). In this case, \( \delta = \eta \). But the criterion in Theorem 1 is sharper than this one.

PROPOSITION 5. Suppose \( \nu \geq 3 \). If \( \eta \geq 2\nu + 5 \), then the criterion in Theorem 1 is sharper than that in Theorem 2.

PROOF. It suffices to prove the inequality: \( (\eta + 1)/2 > R(\nu, \delta, i) \). By definition, we have \( \delta < \sum m_i^2/2 = \nu^2\eta/2 \). Using Lemma 8, we obtain

\[
R(\nu, \delta, i) < R(\nu, \nu^2\eta/2, i) < 1 + \nu\sqrt{\eta}/2(\nu - 1).
\]

By an easy manipulation, the inequality: \( (\eta + 1)/2 \geq 1 + \nu\sqrt{\eta}/2(\nu - 1) \) can be reduced to the inequality: \( \eta \geq t(\nu) \), where

\[
t(\nu) = \nu + 2 + \frac{1}{\nu - 1} + \sqrt{\left(\nu + 2 + \frac{1}{\nu - 1}\right)^2 - 1}.
\]

Clearly, we have \( t(\nu) \leq 2\nu + 5 \). Thus, if \( \eta \geq 2\nu + 5 \), then \( (\eta + 1)/2 > R(\nu, \delta, i) \). □
PROPOSITION 6. Assume
\[
d/\nu > (\eta + 1)/2 + \frac{1}{2(\nu - 1)}\left\{\eta - 1 + \frac{1}{\nu}\right\}.
\]
If either
(i) \(d/\nu > 2\sqrt{\eta} - 1\), or
(ii) \(\eta > 5\), or
(iii) \(d/\nu > 3\), \(\eta \leq 5\),
then we have \(\text{Gon}(C) = d - \nu\) and any rational function \(\varphi\) with \(\deg \varphi = d - \nu\) is induced by a linear function on \(\mathbb{P}^2\).

**Proof of Theorem 3.** Suppose \(d/\nu > (\eta + 1)/2\). Assume there is a rational function \(\varphi\) on \(C\) with \(r = \deg \varphi < d - \nu\). We infer from Lemma 9 that there is a rational function \(\Phi\) on \(\mathbb{P}^2\) which induces \(\varphi\) on \(C\) such that \(\pi\) resolves the base points of \(\Phi\). It follows that \(b \leq n\).

(i), (ii) We first show that \(d/\nu > (\eta + 1)/2\). (i) If \(n = 1\), then we have \(\eta = 1\) and so \(d/\nu > 1 = (\eta + 1)/2\). If \(n = 2\), then, as we have noticed, we have \(d \geq m_1 + m_2\). It follows that \(d/\nu \geq 1 + (m_2/\nu) > 1 + (1/2)(m_2/\nu)^2 = (\eta + 1)/2\). (ii) Since \(\eta \leq n = 3\), we have \(d/\nu > 2 \geq (\eta + 1)/2\). Thus, we obtain \(b \leq n\). By Lemma 4, we derive a contradiction.

(iii) We easily see that \((1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\} \geq (\eta + 1)/2\) if \(\nu \leq 4\), or if \(\nu \geq 5\) and \(b(\nu) \leq \eta \leq c(\nu)\). Thus, under the assumptions in (iii), we have
\[
d/\nu > \max\left\{(1/2)\{3\sqrt{\eta} - (1 + 1/\nu)\}, (\eta + 1)/2\right\}.
\]
By Proposition 4, (iv), we arrive at a contradiction.

5 **Examples**

**EXAMPLE 1.** Let \(C\) be an irreducible plane curve of degree \(d = km + 1\) defined by the equation:
\[
y \prod_{i=1}^{k} (x - a_i)^m - c \prod_{j=1}^{k} (y - b_j)^m = 0,
\]
where the \(a_i\)’s and the \(b_j\)’s are mutually distinct, respectively, \(b_j \neq 0\) for all \(j\) and \(c\) is a general constant. We have \(\text{Gon}(C) = k\).
PROOF. By Eisenstein’s criterion applied to the homegenization of the above polynomial, we easily see that the curve $C$ is irreducible. If $m = 1$, then $C$ is a smooth curve with $Gon(C) = d - 1 = k$. In what follows, we assume that $m \geq 2$. Under the assumption that the constant $c$ is general, the curve $C$ has $k^2$ ordinary $m$-fold singular points $P_{ij} = (a_i, b_j)$ for $1 \leq i, j \leq k$. Thus $\nu = m$ and $\eta = k^2$. In this case, $Gon(C) < d - \nu$. Indeed, the rational function $\Phi = \prod (y - b_j)/\prod (x - a_i)$ of degree $k$ on $\mathbb{P}^2$ induces a rational function $\varphi$ on $C$. The function $\Phi$ has $k^2$ base points $P_{ij}$ on $C$. This proves that $\deg \varphi = (km + 1)k - k^2m = k$. Note that $k > d/\nu - 1$.

We now prove that $Gon(C) = k$. We first see that $C(C) \cong C(\varphi, x)$. For simplicity’s sake, we also denote by $x, y$ the rational functions on $C$ induced by $x, y$. Clearly, we have $C(\varphi, x) \subset C(C)$. Since $\varphi^m = y/c$, we obtain $y \in C(\varphi, x)$, which implies $C(\varphi, x) = C(C)$. Now $C(\varphi, x)$ is the rational function field of the curve $C' : \varphi \prod (x - a_i) - c \prod (c\varphi^m - b_j) = 0$. The curve $C'$ is of degree $d' = mk$ and has one singular point with multiplicity sequence $((m - 1)k, k_{m-2}, k - 1)$ on the line at infinity, where by $k_{m-2}$ we mean $k$’s repeated $m - 2$ times. For $C'$, we use the notation $d', \nu'$ and $\eta'$. We have

$$\eta' = 1 + (m - 2)/(m - 1)^2 + (k - 1)/k(m - 1)^2 < m/(m - 1).$$

We obtain $2\sqrt{\eta'} - (1 + 1/\nu') < 2\sqrt{m/(m - 1)} - 1 - 1/(m - 1)k$. Hence, we have $d'/\nu' - 2\sqrt{\eta'} - (1 + 1/\nu') > (\sqrt{m/(m - 1)} - 1)^2 + 1/(m - 1)k > 0$. We can show that $\eta' > a(\nu')$. We therefore conclude from Theorem 2 that $Gon(C') = d' - \nu = k$ if $\nu \geq 3$. In case $\nu \leq 2$, by Theorem 3, we can easily check that $Gon(C') = k$. Since $C$ and $C'$ are birational, we get $Gon(C) = k$. $$\Box$$

EXAMPLE 2. Let $C$ be an irreducible plane curve of degree $d$. Suppose $C$ has 9 ordinary triple points. By Theorem 1, we get $Gon(C) = d - 3$ if $d \geq 14$. Let $C$ be the curve of degree 11 defined by the equation:

$$y \prod_{i=1}^3 (x - a_i)^3(x - a_4) - c \prod_{j=1}^3 (y - b_j)^3(y - b_4) = 0,$$

where the $a_i$’s and the $b_j$’s are mutually distinct, respectively, $b_j \neq 0$ for all $j$ and $c$ is a general constant. This curve $C$ has 9 ordinary triple points. But we see that $Gon(C) \leq 6 < 11 - 3$.

PROOF. We consider the rational function $\Phi = \prod_{j=1}^3 (y - b_j)/\prod_{i=1}^3 (x - a_i)$ on $\mathbb{P}^2$. Let $\varphi$ be the rational function on $C$ induced by $\Phi$. It turns out that $\deg \varphi = 6$. $$\Box$$
EXAMPLE 3. Let $C$ be an irreducible plane curve of degree $d = em$ defined by the equation: $y^m = \prod_{i=1}^{em}(x - a_i)$, where the $a_i$’s are mutually distinct. We have $\text{Gon}(C) = m$ if $e \geq 2$ or $= m - 1$ if $e = 1$.

PROOF. If $e = 1$ or if $e = 2$ and $m = 1$, then $C$ is smooth. Otherwise, the curve $C$ has one singular point with multiplicity sequence $((e - 1)m, m_{e-1})$ on the line at infinity. We have $\nu = (e - 1)m$, $\eta = e/(e - 1)$ and so $d/\nu = e/(e - 1) = \eta$. In case $\nu \geq 3$, we can apply Theorem 2 and we conclude that $\text{Gon}(C) = d - \nu = m$. In case $\nu = 2$, we see that the genus of $C$ is equal to 1 (if $m = e = 2$) or 0 (if $m = 1$ and $e = 3$). Thus we also get $\text{Gon}(C) = m$. \hfill \Box

EXAMPLE 4. Let $C$ be the transform of an irreducible plane curve $\Gamma$ of degree $m$ by a general quadratic transformation. Then $C$ is of degree $2m$ and has three ordinary $m$-fold singular points other than the singular points of $\Gamma$. Since a general line is transformed into a conic, we have a rational function $\Phi$ on $\mathbb{P}^2$ of degree two which induces a rational function $\phi$ on $C$ with $\deg \phi \leq m - 1$. In this case, we have $d/\nu = 2$, but $\text{Gon}(C) = \text{Gon}(\Gamma) < d - \nu$. Cf.Lemma 5. As a consequence, we conclude that the condition in Theorem 3, (ii) is sharp.

EXAMPLE 5. Let $C$ be the plane curve of degree $2m + 1$ with $m \geq 2$ defined by the equation: $y^{m+1} - (x^m + x^{2m+1}) = 0$. We have $\text{Gon}(C) = m + 1$.

PROOF. The point $(0, 0)$ is a singular point with multiplicity sequence $(m)$ and $C$ also has a singular point with multiplicity sequence $(m, m)$ on the line at infinity. We have $d = 2m + 1$, $\nu = m$, $n = 3$ and $\eta = 3$. Thus $d/\nu = 2 + 1/m > 2$. By Theorem 3, (ii), we infer that $\text{Gon}(C) = d - \nu = m + 1$. \hfill \Box

EXAMPLE 6. Let $C$ be the Fermat curve: $x^m + y^m - 1 = 0$. Take a rational function $\Phi = y/(x - 1)$ on $\mathbb{P}^2$. Let $\varphi$ be the rational function on $C$ induced by $\Phi$. We know that $\text{Gon}(C) = m - 1 = \deg \varphi$. By the way, we have $C(C) = C(x, \varphi) = C(C')$, where the curve $C'$ is defined by the equation:

$$\varphi^m(x - 1)^m - (x^m - 1)/(x - 1) = 0.$$ 

In this case, the curve $C'$ has two singular points with multiplicity sequences $(m)$ and $(m - 1, m - 1)$.
EXAMPLE 7. Let $C$ be the curve of degree 9 defined by the equation:

$$y(x - a_1)^5(x - a_2)^3 - c(y - b_1)^5(y - b_2)^3 = 0,$$

where the $a_i$'s and the $b_i$'s are mutually distinct, respectively and the constant $c$ is generally chosen. Then we have $\text{Gon}(C) = 4$.

PROOF. The curve $C$ has two ordinary singular points of multiplicities 5 and 3, two singular points with multiplicity sequence $(3, 2)$. We have $\nu = 5$ and $\eta = 12/5$. By Theorem 3, (iii), we conclude that $\text{Gon}(C) = 9 - 5 = 4$. In this example, we cannot apply Theorem 2. \qed

References


Department of Mathematics
Faculty of Science
Saitama University
Shimo-Okubo 255
Sakura-ku, Saitama 338-8570, Japan
E-mail: mohkouch@rimath.saitama-u.ac.jp
fsakai@rimath.saitama-u.ac.jp