

Survey on the theory of G -zips

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1 Introduction

The theory of F-zips was first introduced by Moonen-Wedhorn in [2]. Roughly speaking, this theory aims at classifying geometric objects in positive characteristic. For example, let E be an elliptic curve over an algebraically closed field k of characteristic p and consider the p -torsion part $E(k)[p]$ of the group $E(k)$. There are two cases:

- If $E(k)[p] \simeq \mathbb{Z}/p\mathbb{Z}$, we say that E is ordinary.
- If $E(k)[p] = \{0\}$, we say that E is supersingular.

Hence, the group $E(k)[p]$ is a discrete invariant for elliptic curves over k , in the sense that the number of possible cases is finite. The theory of F-zips is a similar attempt to attach invariants to geometric objects.

If we consider a family of elliptic curves $\mathcal{E} \rightarrow S$ over a base scheme S of characteristic p , then the fibers $\mathcal{E}_s$ for $s \in S$ are usual elliptic curves over fields. Hence S is naturally the (set-theoretic) disjoint union

$$S = S^{\text{ord}} \sqcup S^{\text{ss}}$$

where S^{ord} (resp. S^{ss}) is the set of $s \in S$ such that $\mathcal{E}_s$ is ordinary (resp. supersingular). It turns out that the ordinary locus S^{ord} is always open. This is related to the fact that an ordinary elliptic curve has a good deformation theory.

A concrete way of seeing that S^{ord} is open, is to express it as the non-vanishing locus of a section of a line bundle over S . Denote by ω the line bundle over S obtained by pulling back the sheaf $\Omega_{\mathcal{E}/S}^1$ along the unit section $S \rightarrow \mathcal{E}$ of the S -group scheme $\mathcal{E}$. There exists a section $\text{Ha} \in H^0(S, \omega^{p-1})$ called the Hasse invariant such that

$$S^{\text{ord}} = \{s \in S \mid \text{Ha}(s) \neq 0\}.$$

The theory of F-zips provides a geometric object (an algebraic stack), which carries naturally the line bundle ω and the Hasse invariant Ha . Concretely, let B denote the group of upper-triangular matrices in GL_2 , let B_- be the lower-triangular ones, and let T be the diagonal torus. Consider the set of pairs $(x, y) \in B_- \times B$ such that the diagonal coefficients of y are the p -th powers of the diagonal coefficients of x . In other words, x, y have the form

$$x = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad y = \begin{pmatrix} a^p & b \\ 0 & d^p \end{pmatrix}$$

for some $a, d \in k^\times$ and $b, c \in k$. The set of such pairs forms a group $E \subset B_- \times B$. Let E act on GL_2 by the rule $(x, y) \cdot g = xgy^{-1}$. Then one sees that there are exactly two orbits for this action, an open orbit and a closed one. The open orbit is

$$GL_2^{\text{ord}} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2 \mid a \neq 0 \right\}.$$

The closed one is GL_2^{ss} , defined by the condition $a = 0$.

The stack of F-zips in this context is the quotient stack $\mathcal{X} = [E \backslash GL_2]$, its underlying topological space consists of two points. We will see in this survey that the datum of an elliptic curve $\mathcal{E}$ over S induces a map $S \rightarrow \mathcal{X}$. The ordinary and supersingular loci are the fibers of this map. This geometrization is very useful. The stack $\mathcal{X}$ has a rich structure, and S inherits it by way of pulling back. For example, the section Ha previously mentioned is actually pulled back from a section $\text{Ha} \in H^0(\mathcal{X}, \omega^{p-1})$ for a certain line bundle ω on $\mathcal{X}$.

In the papers [3] and [4], Pink-Wedhorn-Ziegler define the notion of G -zips. The formalism of G -zips makes it possible to work with arbitrary reductive groups G , in place of GL_2 in the previous example. We will see that the Hasse invariant Ha possesses a vast generalization as well.

2 The category of F-zips

We start by recalling the definition of F-zips, as introduced by Moonen-Wedhorn in [2]. Basically, an F-zip over a scheme S of characteristic p is a locally free module endowed with two filtrations and Frobenius-linear isomorphisms between the graded pieces. Specifically, let S be a scheme and let $\mathcal{M}$ be a locally free $\mathcal{O}_S$ -module. By a descending filtration on $\mathcal{M}$, we mean a sequence of locally free $\mathcal{O}_S$ -submodules $(\mathcal{C}^i)_{i \in \mathbb{Z}}$ such that

- (i) For all $i \in \mathbb{Z}$, $\mathcal{C}^{i+1} \subset \mathcal{C}^i$ is Zariski locally a direct factor of $\mathcal{C}^i$.
- (ii) One has $\mathcal{C}^i = 0$ for $i \gg 0$ and $\mathcal{C}^i = \mathcal{M}$ for $i \ll 0$.

We define $\text{gr}^i(\mathcal{C}^\bullet) := \mathcal{C}^i / \mathcal{C}^{i+1}$, by assumption (i) it is a locally free $\mathcal{O}_S$ -module. We say that $(\mathcal{D}_i)_{i \in \mathbb{Z}}$ is an ascending filtration if $(\mathcal{D}_{-i})_{i \in \mathbb{Z}}$ is a descending filtration. In this case, we write $\text{gr}_i(\mathcal{D}_\bullet) := \mathcal{D}_i / \mathcal{D}_{i-1}$. For an $\mathcal{O}_S$ -module $\mathcal{F}$, we denote by $\mathcal{F}^{(p)}$ the pullback of $\mathcal{F}$ under the absolute Frobenius map $\text{Fr}_S : S \rightarrow S$.

Definition 2.1. An F-zip over S is a tuple $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$, where

- (1) $\mathcal{M}$ is a locally free $\mathcal{O}_S$ -module of finite rank.
- (2) $\mathcal{C}^\bullet = (\mathcal{C}^i)_{i \in \mathbb{Z}}$ is a descending filtration on $\mathcal{M}$.
- (3) $\mathcal{D}_\bullet = (\mathcal{D}_i)_{i \in \mathbb{Z}}$ is an ascending filtration on $\mathcal{M}$.
- (4) For each $i \in \mathbb{Z}$, ι_i is an isomorphism $\text{gr}^i(\mathcal{C}^\bullet)^{(p)} \rightarrow \text{gr}_i(\mathcal{D}_\bullet)$ of $\mathcal{O}_S$ -modules.

Next we describe homomorphisms of F-zips. If $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are F-zips, a homomorphism $f : \underline{\mathcal{M}} \rightarrow \underline{\mathcal{N}}$ is a morphism of $\mathcal{O}_S$ -modules $f : \mathcal{M} \rightarrow \mathcal{N}$ such that $f(\mathcal{C}^i \mathcal{M}) \subset \mathcal{C}^i \mathcal{N}$ and $f(\mathcal{D}^i \mathcal{M}) \subset \mathcal{D}^i \mathcal{N}$ for all $i \in \mathbb{Z}$, and such that the following diagram commutes

$$\begin{array}{ccc} \text{gr}^i(\mathcal{C}^\bullet \mathcal{M})^{(p)} & \xrightarrow{\iota_i} & \text{gr}_i(\mathcal{D}_\bullet \mathcal{M}) \\ \downarrow \text{gr}^i(\mathcal{C}^\bullet f)^{(p)} & & \downarrow \text{gr}_i(\mathcal{D}_\bullet f) \\ \text{gr}^i(\mathcal{C}^\bullet \mathcal{N})^{(p)} & \xrightarrow{\iota_i} & \text{gr}_i(\mathcal{D}_\bullet \mathcal{N}). \end{array}$$

The category of F-zips is denoted by $\mathbf{F}\text{-Zip}(S)$, it is $\mathbb{F}_p$ -linear (but not $\mathcal{O}_S$ -linear, due to the presence of the Frobenius isogeny). We will see now that it is a tensor category. First, if $\mathcal{M}, \mathcal{N}$ are locally free $\mathcal{O}_S$ -modules endowed with descending filtrations $\mathcal{C}^\bullet \mathcal{M}$ and $\mathcal{C}^\bullet \mathcal{N}$ respectively, then $\mathcal{M} \otimes \mathcal{N}$ has a descending filtration defined by:

$$\mathcal{C}^i(\mathcal{M} \otimes \mathcal{N}) := \sum_{j \in \mathbb{Z}} \mathcal{C}^j \mathcal{M} \otimes \mathcal{C}^{i-j} \mathcal{N}.$$

There is of course a similar statement with ascending filtrations. Hence if $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are F-zips over S , the locally free $\mathcal{O}_S$ -module $\mathcal{M} \otimes \mathcal{N}$ is endowed with a descending filtration $\mathcal{C}^\bullet(\mathcal{M} \otimes \mathcal{N})$ and an ascending filtration $\mathcal{D}_\bullet(\mathcal{M} \otimes \mathcal{N})$. Furthermore, it is easy to see that the isomorphisms $\iota_\bullet$ for $\mathcal{M}$ and $\mathcal{N}$ induces similar isomorphisms between the graded pieces of these filtrations. We obtain an F-zip structure on $\mathcal{M} \otimes \mathcal{N}$, which we call the tensor product of $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ and denote it by $\underline{\mathcal{M}} \otimes \underline{\mathcal{N}}$. This shows that $\mathbf{F}\text{-Zip}(S)$ is an $\mathbb{F}_p$ -linear tensor category.

There is a notion of type for an F-zip $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ over a base scheme S . Let $\eta : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ be a function with finite support (i.e $\eta(i) \neq 0$ for finitely many $i \in \mathbb{Z}$), then we say that $\underline{\mathcal{M}}$ has type η if the locally free sheaf $\text{gr}^i(\mathcal{C}^\bullet)$ has rank $\eta(i)$ for all $i \in \mathbb{Z}$. Denote by $\mathbf{F}\text{-Zip}^\eta(S)$ the full subcategory of F-zips of type η over S .

For an F-zip $\underline{\mathcal{M}}$, it is possible to shift the indexation of the filtrations by an index $r \in \mathbb{Z}$. The F-zip $\mathcal{M}[r]$ is defined by $\mathcal{C}^\bullet(\mathcal{M}[r]) := \mathcal{C}^{\bullet+r}(\underline{\mathcal{M}})$ and $\mathcal{D}_\bullet(\mathcal{M}[r]) := \mathcal{D}_{\bullet+r}(\underline{\mathcal{M}})$. Denote by $\mathcal{O}_S[r]$ the unique F-zip whose underlying sheaf is $\mathcal{O}_S$ and whose type has support $\{r\}$. Then $\mathcal{M}[r]$ is simply $\underline{\mathcal{M}} \otimes \mathcal{O}_S[r]$. Hence if we denote by $\eta[r]$ the function $i \mapsto \eta(r+i)$, then the categories $\mathbf{F}\text{-Zip}^\eta(S)$ and $\mathbf{F}\text{-Zip}^{\eta[r]}(S)$ are equivalent.

3 F-zips and Dieudonne spaces

We say that a commutative group scheme G over k is n -torsion (for an integer $n \in \mathbb{Z}_{\geq 1}$) if multiplication by n is the zero map of $G(R)$ for any k -algebra R . Recall the classification of finite, commutative, p -torsion group schemes over k by Dieudonne theory.

Theorem 3.1 ([14, page 69]). *There is a contravariant functor $\mathbb{D} : G \mapsto \mathbb{D}(G)$ between the category of finite, commutative, p -torsion group schemes over k and the category of triples (M, F, V) where M is a finite-dimensional k -vector space, $F : M \rightarrow M$ is a σ -linear map, $V : M \rightarrow M$ is a σ^{-1} -linear map, satisfying the conditions $FV = 0$ and $VF = 0$. Furthermore, this functor is an equivalence of categories.*

Furthermore, if a triple (M, F, V) satisfies the extra conditions $\text{im}(F) = \text{Ker}(V)$ and $\text{im}(V) = \text{Ker}(F)$, then we call it a Dieudonne space over k . For example, if A is an abelian variety over k , the p -torsion $A[p]$ is a finite, commutative, p -torsion group scheme over k and its associated object $\mathbb{D}(A[p])$ is a Dieudonne space over k ([15, §3.3.8]). A Dieudonne space (M, F, V) over k gives rise to an F-zip over k as follows.

- (i) The filtration $\mathcal{C}^\bullet$ of M is defined by $\mathcal{C}^0 = M$, $\mathcal{C}^1 = \text{Ker}(F)$, $\mathcal{C}^2 = 0$.
- (ii) The filtration $\mathcal{D}_\bullet$ of M is defined by $\mathcal{D}_{-1} = 0$, $\mathcal{D}_0 = \text{Ker}(V)$, $\mathcal{D}_1 = M$.
- (iii) The isomorphism $\iota_0 : (\mathcal{C}^1)^{(p)} \rightarrow M/\mathcal{D}_0$ is the inverse of the map induced by V . The isomorphism $\iota_1 : (M/\mathcal{C}^1)^{(p)} \rightarrow \mathcal{D}_0$ is the one induced by F .

Proposition 3.2. *This construction gives an equivalence of categories between the category of Dieudonne spaces over k to the full subcategory of $\mathbf{F}\text{-Zip}(k)$ of F-zips whose type $\eta : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ has support in $\{0, 1\}$.*

In particular, if A is an abelian scheme over k , we can attach an F-zip to A by applying Proposition 3.2 to the Dieudonne space $\mathbb{D}(A[p])$.

4 F-zips arising in geometry

It is not obvious at first glance why Definition 2.1 is relevant. We will see that F-zips arise naturally in geometry via de Rham cohomology. To define it, first recall the construction of hypercohomology. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and suppose that $\mathcal{A}$ has enough injective objects (i.e., any object has a monomorphism to an injective object). Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor. Let $K^\bullet$ be a bounded below complex of objects in $\mathcal{A}$. Choose a quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ to a complex of injective objects (this is always possible). Then one defines the hypercohomology of $K^\bullet$ as

$$\mathbb{R}^i T(K^\bullet) := H^i(T(I^\bullet)).$$

This gives a well-defined object in $\mathcal{B}$, that we call the hypercohomology of $K^\bullet$. It is possible to construct several spectral sequences that converge to the hypercohomology, by considering different filtrations on a complex $K^\bullet$. In this survey, the two spectral sequences of importance are the following two:

$$\begin{aligned} {}^I E_1^{a,b} &= R^b T(K^a) \implies \mathbb{R}^{a+b} T(K^\bullet). \\ {}^{II} E_2^{a,b} &= R^a T(H^b(K^\bullet)) \implies \mathbb{R}^{a+b} T(K^\bullet). \end{aligned}$$

For example, let X be a scheme of finite-type over an arbitrary field k . The sheaves $\Omega_{X/k}^i$ are coherent $\mathcal{O}_X$ -modules. However, since differentials are not $\mathcal{O}_X$ -linear, the de Rham complex $\Omega_{X/k}^\bullet$ is a complex in the category of sheaves of k -vector spaces on X . Denote this abelian category by $\mathcal{A}$. Let $\mathcal{B}$ be the category of k -vector spaces, and let $T : \mathcal{A} \rightarrow \mathcal{B}$ be the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. The formalism of hypercohomology yields a k -vector space $H_{\text{dR}}^i(X/k) := \mathbb{R}^i T(\Omega_{X/k}^\bullet)$ and spectral sequences converging to $H_{\text{dR}}^*(X/k)$:

$$\begin{aligned} {}^H E_1^{a,b} &:= H^b(X, \Omega_{X/k}^a) \implies H_{\text{dR}}^{a+b}(X/k). \\ {}^{\text{conj}} E_2^{a,b} &:= H^a(X, \mathcal{H}^b(\Omega_{X/k}^\bullet)) \implies H_{\text{dR}}^{a+b}(X/k). \end{aligned}$$

We call them respectively the Hodge and the conjugate spectral sequences. When k has characteristic zero, things are particularly simple as both sequences degenerate immediately. Furthermore, if $k = \mathbb{C}$, a standard fact in Hodge theory states that the conjugate filtration is obtained from the Hodge filtration by applying complex conjugation.

When k has positive characteristic p , the degeneracy is no longer true, not even when X is proper smooth over k . Hence, we make the following assumption:

Assumption 4.1. *The Hodge spectral sequence of X degenerates at E_1 .*

Let $F_X : X \rightarrow X^{(p)}$ be the relative p -power Frobenius map and consider the complex $F_{X,*}(\Omega_{X/k}^\bullet)$. This is a complex of $\mathcal{O}_{X^{(p)}}$ -modules whose maps are $\mathcal{O}_{X^{(p)}}$ -linear (easy computation). Hence the cohomology sheaves $\mathcal{H}^a(F_{X,*}(\Omega_{X/k}^\bullet))$ are $\mathcal{O}_{X^{(p)}}$ -modules. The Cartier isomorphisms are natural isomorphisms

$$\Omega_{X^{(p)}}^a \xrightarrow{\sim} \mathcal{H}^a(F_{X,*}(\Omega_{X/k}^\bullet)).$$

for all $a \geq 0$ (see [16, Theorem 7.2]). Taking the b -th cohomology group over $X^{(p)}$ on each side, we obtain σ -linear isomorphisms for all $a, b \geq 0$:

$${}^H E_1^{a,b} \simeq {}^{\text{conj}} E_2^{b,a}$$

It also follows from this that the conjugate spectral sequence automatically degenerates at the second page. We now give the construction of the F-zip $\underline{M} := (M, C^\bullet, D_\bullet, \iota_\bullet)$ over k attached to X .

- (i) Take $M = H_{\text{dR}}^n(X/k)$.
- (ii) Denote by $C^\bullet = (C^i)_{i \in \mathbb{Z}}$ the filtration obtained by the Hodge spectral sequence, indexed such that $C^\bullet$ is descending and $\text{gr}^i(C^\bullet) = H^{n-i}(X, \Omega_{X/k}^i)$.
- (iii) Denote by $D_\bullet = (D_i)_{i \in \mathbb{Z}}$ the filtration obtained by the conjugate spectral sequence, indexed such that $D_\bullet$ is ascending and $\text{gr}_i(D_\bullet) = H^{n-i}(X, \mathcal{H}^i(\Omega_{X/k}^\bullet))$.
- (iv) Let $\iota_i : \text{gr}_i(C^\bullet)^{(p)} \rightarrow \text{gr}_i(D_\bullet)$ be the linearized Cartier isomorphism.

We have just seen that if X is a proper smooth scheme over k satisfying Assumption 4.1, then $H_{\text{dR}}^n(X/k)$ is naturally endowed with an F-zip structure. There are many such schemes, for example abelian varieties, K3 surfaces, complete intersections in projective bundles... Also a theorem of Deligne-Illusie states that a proper smooth scheme X over k which lifts to $W_2(k)$ and of dimension $\dim(X) < p$ satisfies Assumption 4.1.

We now give the generalization of this construction to an arbitrary base scheme. Let $f : X \rightarrow S$ be a proper, smooth morphism of $\mathbb{F}_p$ -schemes. Following the terminology of [2], we say that $f : X \rightarrow S$ satisfies condition (D) if the following properties hold.

- (1) The $\mathcal{O}_S$ -modules $R^b f_*(\Omega_{X/S}^a)$ are locally free of finite rank for all $a, b \geq 0$.
- (2) The Hodge spectral sequence degenerates at E_1 .

Assume that f satisfies condition (D), then just as in the previous case, for any integer n such that $0 \leq n \leq 2 \dim(X/S)$ the locally free $\mathcal{O}_S$ -module $H_{\text{dR}}^n(X/S)$ is naturally endowed with an F-zip structure over S . We write $\underline{H}_{\text{dR}}^n(X/S)$ for this F-zip. This construction can be promoted to a contravariant functor

$$\underline{H}_{\text{dR}}^n : \left(\begin{array}{c} \text{Proper smooth } X \rightarrow S \\ \text{satisfying condition (D)} \end{array} \right) \longrightarrow \text{F-Zip}(S).$$

An important example is the case of abelian schemes $A \rightarrow S$. If g denotes the relative dimension of A/S , the F-zip $\underline{H}_{\text{dR}}^1(A/S)$ has type $\eta : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ where η is defined by

$$\eta(i) = \begin{cases} g & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

In the case $S = \text{Spec}(k)$, recall that we attached an F-zip to a Dieudonne space over k (Proposition 3.2). One can check that the F-zip $\underline{H}_{\text{dR}}^1(A/k)$ coincides with the F-zip attached to $\mathbb{D}(A[p])$.

5 Additional structure

From now on, k will denote an algebraic closure of $\mathbb{F}_p$. It is natural to consider F-zips endowed with additional structure. For example, let (A, λ) be a principally polarized abelian variety over k of dimension g . Let $\underline{H}_{\text{dR}}^1(A/k) = (M, C^\bullet, D_\bullet, \iota_\bullet)$ be the attached F-zip. In particular, we have σ -linear isomorphisms $\iota_0 : M/C^1 \rightarrow D_0$ and $\iota_1 : C^1 \rightarrow M/D_0$. The polarization λ induces on M a perfect pairing $\langle -, - \rangle : M \times M \rightarrow k$ which satisfies the following conditions.

- (i) C_1 and D_0 are totally isotropic.
- (ii) One has $\langle \iota_0 x, \iota_1 y \rangle = \sigma \langle x, y \rangle$ for all $x \in M$ and all $y \in C^1$ (note that the expression $\langle \iota_0 x, \iota_1 y \rangle$ is well-defined because D_0 is totally isotropic).

More generally, we can define F-zips with G -structure for an arbitrary algebraic group G over $\mathbb{F}_p$. Denote by $\text{Rep}(G)$ the category of algebraic representations of G over $\mathbb{F}_p$. Since we saw that the category of F-zips over a scheme S is a tensor category, we may define the following notion.

Definition 5.1. *A G -zip functor over an $\mathbb{F}_p$ -scheme S is an exact $\mathbb{F}_p$ -linear tensor functor $\text{Rep}(G) \rightarrow \mathbf{F}\text{-Zip}(S)$.*

We denote by $G\text{-ZipFun}(S)$ the category of G -zip functors over S . Our goal in this section is to explain a more down-to-earth definition of F-zips with G -structure over S . For this, we need to understand how to generalize the notion of type.

First of all, if $G = GL_n$, the category $G\text{-ZipFun}(S)$ is equivalent to the category $\mathbf{F}\text{-Zip}^n(S)$ of F-zips $\underline{M} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ where $\mathcal{M}$ is a locally free $\mathcal{O}_S$ -module of rank n . If such an F-zip $\underline{M}$ has type η , then it must satisfy

$$\sum_{i \in \mathbb{Z}} \eta(i) = n.$$

Giving such a function is the same as giving a conjugacy class of cocharacters of GL_n . The bijection is given as follows. If η is a function as above, the corresponding conjugacy class of cocharacters is given by considering a decomposition

$$\mathbb{F}_p^n = \bigoplus_{i \in \mathbb{Z}} V_i$$

where V_i has dimension $\eta(i)$ and letting $z \in \mathbb{G}_m$ act on V_i by z^i . Hence, it seems natural that the generalization of the notion of type for G -zip functors is a conjugacy class of cocharacters of G .

Definition 5.2. *Let $\mu : \mathbb{G}_{m,k} \rightarrow G_k$ be a cocharacter. We say that a G -zip functor $z : \text{Rep}(G) \rightarrow \mathbf{F}\text{-Zip}(S)$ has type μ if for all representations $(V, \rho) \in \text{Rep}(G)$, the F-zip $z(V, \rho)$ has type $\rho \circ \mu$. Denote by $G\text{-ZipFun}^\mu(S)$ the full subcategory of G -zip functors of type μ .*

Now, we explain an equivalent definition of G -zips. Fix a cocharacter $\mu : \mathbb{G}_{m,k} \rightarrow G_k$. One obtains naturally a pair of opposite parabolic subgroups (P_-, P_+) in G_k and a common Levi subgroup $L := P_- \cap P_+ = \text{Cent}(\mu)$. The group $P_+(k)$ consists of those elements $g \in G(k)$ such that the limit

$$\lim_{t \rightarrow 0} \mu(t) g \mu(t)^{-1}$$

exists, i.e. such that the map $\mathbb{G}_{m,k} \rightarrow G_k$, $t \mapsto \mu(t) g \mu(t)^{-1}$ extends to a morphism of varieties $\mathbb{A}_k^1 \rightarrow G_k$. The Lie algebra of the parabolic P_+ (resp. P_-) is given by

$$\text{Lie}(P_+) = \bigoplus_{n \geq 0} \text{Lie}(G)_n \quad (\text{resp. } \text{Lie}(P_-) = \bigoplus_{n \leq 0} \text{Lie}(G)_n)$$

where $\text{Lie}(G)_n$ is the subspace where $z \in \mathbb{G}_m$ acts by z^n via the cocharacter μ . We set $P := P^-$, $Q := (P^+)^{(p)}$ and $M := L^{(p)}$, so that M is a Levi subgroup of Q . We denote by U and V the unipotent radicals of P and Q , respectively. For a k -scheme S , one defines:

Definition 5.3. A G -zip of type μ over S is a tuple $\underline{I} = (I, I_P, I_Q, \iota)$ where

- (i) I is a G -torsor over S ,
- (ii) $I_P \subset I$ is a P -torsor over S ,
- (iii) $I_Q \subset I$ is a Q -torsor over S ,
- (iv) $\iota : (I_P/U)^{(p)} \rightarrow I_Q/V$ is an isomorphism of M -torsors.

In the case $G = GL_n$, one recovers the usual notion of F-zip. Denote by $G\text{-Zip}^\mu(S)$ the category of G -zips of type μ . By a result of Pink-Wedhorn-Ziegler ([4, §1.4]), there is an equivalence of categories

$$G\text{-ZipFun}^\mu(S) \simeq G\text{-Zip}^\mu(S).$$

6 The stack of G -zips

It is convenient to use the language of stacks to study F-zips and G -zips. Roughly speaking, a stack is an object that generalizes the notion of scheme by allowing automorphisms of points. First, recall that a groupoid is a category in which every map is an isomorphism. A category fibred in groupoids over the category of k -schemes is a family of groupoids $\mathcal{X}(S)$ for each k -scheme S , such that if $\varphi : S \rightarrow T$ is a map of k -schemes, there is a functor $\varphi^* : \mathcal{X}(T) \rightarrow \mathcal{X}(S)$. This is called a base change functor and is denoted by $(-)_S$. Furthermore, if $\varphi : S \rightarrow T$ and $\psi : T \rightarrow U$ are maps of k -schemes, there is an isomorphism of functors $(\psi \circ \varphi)^* \simeq \varphi^* \circ \psi^*$ (and these isomorphisms satisfy a cocycle relation). A stack over k is a particular kind of category fibred in groupoids over the category of k -schemes.

Specifically, one requires two conditions to hold:

- (1) For all k -schemes S and all $x, y \in \mathcal{X}(S)$, the functor from S -schemes to sets which takes T to $\text{Hom}_{\mathcal{X}(T)}(x_T, y_T)$ is a sheaf for the étale topology.
- (2) All descent data are effective.

Roughly speaking, the second condition means that if $(T_i \rightarrow S)_{i \in I}$ is an étale covering, we may glue objects $x_i \in \mathcal{X}(T_i)$ to obtain an object $x \in \mathcal{X}(S)$. Specifically, write $V_{ij} := V_i \times_S V_j$. Then if $f_{ij} : (x_i)_{V_{ij}} \rightarrow (x_j)_{V_{ij}}$ are isomorphisms satisfying the usual cocycle relation, then there exists an object $x \in \mathcal{X}(S)$ such that $x_i = x_{V_i}$.

For example, a k -scheme X may be viewed as a stack over k . The groupoid $X(S)$ is simply the set $\text{Hom}_k(S, X)$, viewed as a category where the only maps are the identities of objects.

For each k -scheme S , consider the category $\mathbb{F}\text{-Zip}(S)$ whose objects are F-zips over S and whose morphisms are isomorphisms of F-zips. Clearly, this is a groupoid. If $f : T \rightarrow S$ is a map of k -schemes, then there is a base change functor $f^* : \mathbb{F}\text{-Zip}(S) \rightarrow \mathbb{F}\text{-Zip}(T)$. Indeed, let $\underline{M} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ be an F-zip over S . Then $f^*\underline{M} = (f^*\mathcal{M}, f^*\mathcal{C}^\bullet, f^*\mathcal{D}_\bullet, f^*\iota_\bullet)$ is its pull-back to T .

Definition 6.1. The above construction gives rise to a stack over k . We denote it by $\mathbb{F}\text{-Zip}$ and call it the stack of F-zips. Similarly, if $\eta : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ is a function with finite support, then the categories $\mathbb{F}\text{-Zip}^\eta(S)$ give rise to a stack over k , that we denote by $\mathbb{F}\text{-Zip}^\eta$. More generally, if G is a connected $\mathbb{F}_p$ -reductive group and $\mu : \mathbb{G}_{m,k} \rightarrow G_k$ is a cocharacter, the categories $G\text{-Zip}^\mu(S)$ give rise to a stack $G\text{-Zip}^\mu$ over k .

This turns out to be an algebraic stack. In our case, this means that there is a smooth surjective morphism from a scheme to this stack. For an algebraic stack $\mathcal{X}$, it is possible to define an underlying topological space by taking the equivalence classes of pairs (K, x) where $k \subset K$ is a field extension and $x \in \mathcal{X}(K)$. Two pairs (K, x) and (K', x') are equivalent if there exists a common field extension L of K and K' such that $x_L \simeq x'_L$. The set of equivalence classes is denoted by $|\mathcal{X}|$. This set is endowed with a topology, as follows. Say that a map of stacks $\mathcal{Y} \rightarrow \mathcal{X}$ is an open immersion if the map $\mathcal{Y} \times_{\mathcal{X}} X \rightarrow X$ is an open immersion of schemes for any scheme X mapping to $\mathcal{X}$. In this case, $|\mathcal{Y}|$ is naturally a subset of $|\mathcal{X}|$. Subsets of this kind form a topology, called the Zariski topology of $|\mathcal{X}|$. Similarly, one can define a closed substack $\mathcal{Y} \rightarrow \mathcal{X}$ as a map of stacks that becomes a closed immersion (of schemes) after base change to a scheme $X \rightarrow \mathcal{X}$.

For a function $\eta : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support, the substack $\mathbf{F}\text{-Zip}^\eta \subset \mathbf{F}\text{-Zip}$ is both open and closed. The stack $\mathbf{F}\text{-Zip}$ decomposes as a disjoint union

$$\mathbf{F}\text{-Zip} = \bigsqcup_{\eta} \mathbf{F}\text{-Zip}^\eta$$

and the substacks $\mathbf{F}\text{-Zip}^\eta$ are the connected components of $\mathbf{F}\text{-Zip}$. In particular, this implies that an F-zip over a connected scheme S has a type, because the corresponding map $S \rightarrow \mathbf{F}\text{-Zip}$ must factor through a certain component $\mathbf{F}\text{-Zip}^\eta$.

7 Representation as a quotient stack

A nice property of stacks is the existence of quotients. If H is a smooth k -algebraic group acting on the left on a k -scheme X , then the quotient stack $\mathcal{X} := [H \backslash X]$ is defined as follows. For any k -scheme S , the groupoid $\mathcal{X}(S)$ is the category of pairs (T, α) where T is an H -torsor on S and $\alpha : T \rightarrow X \times_k S$ is an $H \times_k S$ -equivariant map. It is clear that $\mathcal{X}(S)$ is a groupoid, and one can check that $\mathcal{X}$ is a stack over k . For example, when $X = \text{Spec}(k)$ is endowed with the trivial action of H , the quotient stack $B(H) = [H \backslash \text{Spec}(k)]$ is the classifying stack of H . For a k -scheme S , a morphism of stacks $S \rightarrow B(H)$ is essentially the same as an H -torsor over S .

Fix a connected reductive $\mathbb{F}_p$ -group G and a cocharacter $\mu : \mathbb{G}_{m,k} \rightarrow G_k$. We will see that the k -stack $G\text{-Zip}^\mu$ can be written as a quotient stack. Let P, Q, L, M, U, V be the attached groups, as defined in §5. The Frobenius restricts to a map $\varphi : L \rightarrow M$. The isomorphisms $L \simeq P/U$ and $M \simeq Q/V$ yield natural maps $P \rightarrow L$ and $Q \rightarrow M$ which we both denote by $x \mapsto \bar{x}$. Define the zip group E as:

$$E := \{(a, b) \in P \times Q \mid \varphi(\bar{a}) = \bar{b}\}.$$

The group E acts on G by the rule $(a, b) \cdot g := agb^{-1}$.

Theorem 7.1 ([4, Th. 1.5]). *There is an isomorphism $G\text{-Zip}^\mu \simeq [E \backslash G]$.*

In particular, the underlying topological space $|G\text{-Zip}^\mu|$ coincides with the set of E -orbits in G . Each such orbit is locally closed for the Zariski topology of G . We now give a parametrization of these E -orbits. Fix a Borel pair (B, T) satisfying $B \subset P$ and suppose for simplicity that (B, T) are defined over $\mathbb{F}_p$. After possibly changing μ to a conjugate cocharacter, it is always possible to find such a Borel pair. Denote by Φ the set of T -roots and Δ the set of simple roots. Recall that there is a bijection between subsets of Δ and conjugacy classes of parabolic subgroups of G_k . We normalize this bijection such that Borel subgroups correspond to the empty set. Let $I, J \subset \Delta$ be the types of P, Q respectively.

Since $B \subset P$, the set I consists of the simple roots of L . Write $W = N(T)/T$ for the Weyl group of T , it is a Coxeter group. There is a length function $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$. Write w_0 for the longest element in W . For a subset $K \subset \Delta$, let $w_{0,K}$ be the longest element of the subgroup $W_K \subset W$ generated by $\{s_\alpha \mid \alpha \in K\}$. Also define W^K as the set of elements $w \in W$ which are of minimal length in the coset wW_K .

For $w \in W$, choose a representative $\dot{w} \in N_G(T)$, such that $(w_1w_2)^\cdot = \dot{w}_1\dot{w}_2$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, see [13], Exp. XXIII, §6). Define $z := w_0w_{0,J}$.

For $w \in W$, define G_w as the E -orbit of $\dot{w}z^{-1}$. The E -orbits in G form a stratification of G by locally closed subsets.

Theorem 7.2 ([3, Th. 11.3]). *The map $w \mapsto G_w$ induces a bijection from W^J onto the set of E -orbits in G . Furthermore, for $w \in W^J$, one has*

$$\dim(G_w) = \ell(w) + \dim(P).$$

Endow G_w with the reduced subscheme structure. Then the quotient stack $\mathcal{X}_w = [E \backslash G_w]$ is a locally closed substack of $\mathcal{X} = G\text{-Zip}^\mu$. We call $\mathcal{X}_w$ a zip stratum. This gives a stratification of $\mathcal{X}$. Note that the underlying topological space of $\mathcal{X}_w$ is a single point.

8 Vector bundles on $G\text{-Zip}^\mu$

It is possible to define a notion of vector bundles for algebraic stacks. If $\mathcal{X}$ is an algebraic stack, one could define a vector bundle over $\mathcal{X}$ as a family of vector bundles $\mathcal{V} = (\mathcal{V}_S)_S$ for each scheme S and each morphism of stacks $S \rightarrow \mathcal{X}$. Furthermore, this family should be compatible in an obvious sense. The space of global sections of $\mathcal{V}$ over $\mathcal{X}$ is then defined as an inverse limit of the spaces $H^0(S, \mathcal{V}_S)$.

Let G be a smooth algebraic group over k acting on a k -variety X . Let $\mathcal{X}$ be the quotient stack $[G \backslash X]$. Then there is a natural way to attach a vector bundle on $\mathcal{X}$ to an algebraic representation $\rho : G \rightarrow GL(V)$. Specifically, if $S \rightarrow \mathcal{X}$ is a map from a scheme S , then by definition of the quotient stack, we have a natural G -torsor on S . Applying the representation ρ , we obtain a $GL(V)$ -torsor on S , hence a vector bundle of rank $\dim(V)$. This construction is functorial in S , so we obtain a vector bundle $\mathcal{V}(\rho)$ on the stack $\mathcal{X}$. Explicitly, the space of global sections $H^0(\mathcal{X}, \mathcal{V}(\rho))$ is identified with

$$H^0(\mathcal{X}, \mathcal{V}(\rho)) = \{f : X \rightarrow V, f(g \cdot x) = \rho(g)f(x), \forall g \in G, \forall x \in X\}.$$

Recall that the stack of G -zips of type μ is isomorphic to a quotient stack $[E \backslash G]$, as explained earlier. Hence, the previous construction attaches to each algebraic representation $\rho : E \rightarrow GL(V)$ a vector bundle $\mathcal{V}(\rho)$ on $G\text{-Zip}^\mu$. Furthermore, $\mathcal{V}(\rho)$ is a line bundle if and only if ρ is a character of E .

For the time being, we consider only line bundles. There are natural identifications between characters of E , P and L via the first projection $E \rightarrow P$ and the Levi projection $P \rightarrow L$. Indeed, all these groups coincide up to a unipotent group, which has no nontrivial characters. Hence, we parametrize line bundles on $G\text{-Zip}^\mu$ by characters of L : If $\lambda \in X^*(L)$, we denote by $\mathcal{V}(\lambda)$ the line bundle attached to the character $E \rightarrow \mathbb{G}_m, (a, b) \mapsto \lambda(\bar{a})$.

9 Hasse invariants

One interesting feature of the stack of G -zips is the existence (in many cases, but not always) of Hasse invariants for zip strata. Let us start with a definition of what we mean

by a Hasse invariant. Let $\mathcal{X}$ be an algebraic stack. We may thus consider its underlying topological space $|\mathcal{X}|$. Let $\mathcal{Y} \subset \mathcal{X}$ be a locally closed subset, and denote by $\overline{\mathcal{Y}}$ its Zariski closure. Endow both $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ with the reduced substack structure. Finally, let $\mathcal{L}$ be a line bundle over $\mathcal{X}$.

Definition 9.1. *A Hasse invariant for $\mathcal{Y}$ with respect to $\mathcal{L}$ is a section $h \in H^0(\overline{\mathcal{Y}}, \mathcal{L}^n)$ (some $n \geq 1$) such that the non-vanishing locus of h is exactly $\mathcal{Y}$.*

Recall that any character $\lambda \in X^*(L)$ gives rise to a line bundle $\mathcal{V}(\lambda)$ on the stack $\mathcal{X} = G\text{-Zip}^\mu$. Taking $\mathcal{Y}$ to be a single zip stratum $\mathcal{X}_w \subset \mathcal{X}$ (for some $w \in W^J$) in Definition 9.1, we have the notion of Hasse invariants for $\mathcal{X}_w$ with respect to $\mathcal{L}(\lambda)$. It is possible to give a combinatorial criterion for the existence of such Hasse invariants. For an element $w \in W$, we write E_w for the set of positive roots α satisfying $ws_\alpha < w$ and $\ell(ws_\alpha) = \ell(w) - 1$. Write σ for the action of Frobenius on W and $X^*(T)$. For $w \in W$ and an integer $n \geq 1$, let $w^{(n)}$ be the product $\sigma^n(w)\sigma^{n-1}(w)\dots\sigma(w)$ and set by convention $w^{(0)} = 1$. It is easy to see that there exists $r \geq 1$ such that $w^{(r)} = 1$. Furthermore, the set of integers $r \geq 1$ such that $w^{(r)} = 1$ is stable under addition. Hence we can find $r \geq 1$ such that $w^{(r)} = 1$ for all $w \in W$. We fix such an integer $r \geq 1$. We also fix an integer $m \geq 1$ such that T is split over $\mathbb{F}_{p^m}$.

Proposition 9.2 ([6, Prop. 3.2.1]). *Let $w \in W^J$ and $\lambda \in X^*(L)$. The following assertions are equivalent:*

- (i) *There is a Hasse invariant for $\mathcal{X}_w$ with respect to $\mathcal{L}(\lambda)$.*
- (ii) *For all $\alpha \in E_w$, one has:*

$$\sum_{i=0}^{rm-1} \langle (zw^{-1})^{(i)} \sigma^i(\lambda), w\alpha^\vee \rangle p^i > 0. \quad (9.0.1)$$

First, we want to mention negative results. The above proposition can provide a counter-example for the principal purity of the stratification $(\mathcal{X}_w)_w$. Principal purity means that every stratum admits a Hasse invariant (for some $\lambda \in X^*(L)$). The easiest counter-example that we could find is in the case of $G = Sp(6)$ for a cocharacter μ that corresponds to the middle point of the Dynkin diagram. For the prime number $p = 2$, there exists a stratum $\mathcal{X}_w$ which does not admit Hasse invariants (for any $\lambda \in X^*(L)$).

To obtain a positive results for the existence of Hasse invariants, it is of course very cumbersome to check that condition (ii) is satisfied in general. Hence we want to mention a result which has a much easier statement.

Theorem 9.3. *Assume that $\lambda \in X^*(L)$ satisfies the following conditions:*

- (i) *One has $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta \setminus I$.*
- (ii) *For all $\alpha \in \Phi$ such that $\langle \lambda, \alpha^\vee \rangle \neq 0$, for all $w \in W$ and all $j \in \mathbb{Z}$ we have*

$$\left| \frac{\langle \lambda, w\sigma^j(\alpha)^\vee \rangle}{\langle \lambda, \alpha^\vee \rangle} \right| \leq p - 1.$$

Then $\mathcal{L}(\lambda)$ admits Hasse invariants on all zip strata.

Theorem 9.3 is an elementary consequence of Proposition 9.2. One checks that the expression (9.0.1) is positive as follows: View this expression as a polynomial in p . The leading coefficient is

$$\langle (zw^{-1})^{(rm-1)}\sigma^{rm-1}(\lambda), w\alpha^\vee \rangle = \langle wz^{-1}\sigma^{-1}(\lambda), w\alpha^\vee \rangle = \langle \lambda, \sigma(z\alpha)^\vee \rangle.$$

We claim that this leading coefficient is positive. First, since $w \in W^J$ and $\alpha \in E_w$, we have $\alpha \notin J$. Since $z = w_0w_{0,J}$, we deduce easily that $z\alpha$ is a negative root not contained in M . It follows that $\sigma(z\alpha)$ is a negative root, not contained in L . Hence one can write $-\sigma(z\alpha)^\vee = \sum_j \alpha_j^\vee$ for $\alpha_j \in \Delta$, with at least one α_j in $\Delta \setminus I$. It follows from Condition (i) that $\langle \lambda, \sigma(z\alpha)^\vee \rangle > 0$, hence the claim.

Divide the expression (9.0.1) by this leading coefficient. Then Condition (ii) implies that the coefficients of this monic polynomial are $\leq p-1$. The result then follows from the inequality

$$(p-1) \sum_{i=0}^{r-2} p^i = p^{r-1} - 1 < p^{r-1}.$$

10 Ekedahl-Oort strata

Consider an abelian variety A of dimension $g \geq 1$ over k . The p -torsion $A[p]$ is a finite commutative p -torsion group scheme over k . Not all finite commutative p -torsion group scheme over k appear in this way. Those which do are exactly those whose Dieudonne module $\mathbb{D}(A[p])$ satisfies $\text{im}(F) = \text{Ker}(V)$ and $\text{im}(V) = \text{Ker}(F)$ (we say that $A[p]$ is a BT_1).

If $\mathcal{A} \rightarrow S$ is an abelian scheme over a base scheme of characteristic p , then S is naturally decomposed as a (set-theoretic) disjoint union

$$S = \bigsqcup_{\gamma} S_{\gamma}$$

where γ varies in the set of isomorphism classes of BT_1 's. The subset S_{γ} is the set of points $s \in S$ such that $\mathcal{A}_s[p]$ is in γ . By a theorem of Oort, this decomposition is locally closed. However, in general it may not be a stratification of S in the sense that the closure of S_{γ} may not be a union of $S_{\gamma'}$ for certain γ' .

As we explained, we may attach to $\mathcal{A} \rightarrow S$ an F-zip $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ over S whose type η has support in $\{0, 1\}$ and satisfies $\eta(0) = \eta(1) = g$. This gives rise to a morphism of stacks

$$\zeta : S \longrightarrow \mathbf{F}\text{-Zip}^\eta.$$

The strata S_{γ} of S coincide with the fibers of this morphism. The stack $\mathbf{F}\text{-Zip}^\eta$ coincides with the stack $G\text{-Zip}^\mu$ for the group $G = GL_{2g}$ and the cocharacter

$$\mu : z \mapsto \begin{pmatrix} z\text{Id}_g & \\ & \text{Id}_g \end{pmatrix}.$$

In particular, the Levi subgroup $L \subset G$ attached to μ is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \text{where } A, D \in GL_g. \quad (10.0.1)$$

Thus $X^*(L)$ is a free $\mathbb{Z}$ -module of rank 2. Consider the character $\lambda_\omega \in X^*(L)$ given by mapping the matrix (10.0.1) to $\det(A)^{-1}$. We have an associated line bundle $\mathcal{V}(\lambda_\omega)$ over $\mathcal{X} = \mathbf{F}\text{-Zip}^\eta$.

The Hodge vector bundle Ω of the abelian scheme $\mathcal{A} \rightarrow S$ is defined as the pullback along the unit section $S \rightarrow \mathcal{A}$ of the sheaf of relative differentials $\Omega_{\mathcal{A}/S}^1$. It is a rank g vector bundle on S . Denote by $\omega = \wedge^g \Omega$ its determinant. Then one has the following equation.

$$\zeta^* \mathcal{V}(\lambda_\omega) = \omega.$$

It is easy to check that the character λ_ω satisfies the conditions (i) and (ii) of Theorem 9.3 (for any value of the prime p). The first one is immediate, and for the second one, note that λ_ω is minuscule, hence for all $\alpha \in \Phi$ such that $\langle \lambda_\omega, \alpha^\vee \rangle \neq 0$, the quotient

$$\left| \frac{\langle \lambda_\omega, w\alpha^\vee \rangle}{\langle \lambda_\omega, \alpha^\vee \rangle} \right|$$

only takes the value 0 or 1 for any $w \in W$, hence is always $\leq p - 1$, even for $p = 2$. Thus we may apply the theorem to the line bundle $\mathcal{V}(\lambda_\omega)$. By pulling back to S , we deduce:

Proposition 10.1. *For each isomorphism class γ of BT_1 's over k , there exists $n \geq 1$ and a section $\text{Ha}_\gamma \in H^0(\overline{S}_\gamma, \omega^n)$ over the Zariski closure $\overline{S}_\gamma$ which satisfies*

$$\{s \in \overline{S}_\gamma \mid \text{Ha}_\gamma(s) \neq 0\} = S_\gamma.$$

11 Sketch of proof

We sketch the proof of Proposition 9.2. It relies heavily on the stack of G -zip flags. It is a stack $\mathcal{Y}$ with a natural projection map $\pi : \mathcal{Y} \rightarrow \mathcal{X}$. It carries a stratification $(\mathcal{Y}_w)_{w \in W}$ indexed by the whole Weyl group. Specifically, we give the following definition.

Definition 11.1. *A G -zip flag of type μ over a k -scheme S is a pair $\hat{I} = (\underline{I}, J)$ where $\underline{I} = (I, I_P, I_Q, \iota)$ is a G -zip of type μ over S , and $J \subset I_P$ is a B -torsor.*

We denote by $G\text{-ZipFlag}^\mu(S)$ the category of G -zip flags over S . By similar arguments as for G -zips, we obtain a stack $\mathcal{Y} := G\text{-ZipFlag}^\mu$ over k , which we call the stack of G -zip flags of type μ . There is a natural projection $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ given by forgetting the B -torsor. To stratify $\mathcal{Y}$, we need the following result. Define a subgroup $E' \subset E$ by $E' := E \cap (B \times G)$. By adapting the proof of Theorem 7.1, one can prove the following.

Theorem 11.2. *There is a commutative diagram, where the vertical maps are isomorphisms and the lower horizontal map is the natural projection:*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ \simeq \downarrow & & \downarrow \simeq \\ [E' \backslash G] & \xrightarrow{\pi} & [E \backslash G]. \end{array}$$

We explain now how $\mathcal{Y}$ admits a natural stratification. It is based on the observation that the group E' is contained in the product $B \times {}^z B$, where ${}^z B := zBz^{-1}$ (this is an easy verification). In particular, there is a natural projection map

$$\psi : \mathcal{Y} \rightarrow [B \backslash G / {}^z B] \simeq [B \backslash G / B]$$

By the Bruhat decomposition of G , the latter stack has a stratification $(\mathcal{B}_w)_{w \in W}$ parametrized by the Weyl group W . The strata $(\mathcal{Y}_w)_{w \in W}$ of $\mathcal{Y}$ are defined as the fibers of the map ψ . Now we give the main steps of the proof of Proposition 9.2.

- (1) First, one shows that for any $\lambda \in X^*(L)$, there exists $n \geq 1$ such that $\mathcal{V}(\lambda)^n$ admits a nonzero section $h_{\lambda,w}$ on the strata $\mathcal{X}_w$ of $\mathcal{X}$ for all $w \in W^J$. Furthermore, these sections are unique up to nonzero scalar. To prove Proposition 9.2, we need to understand for which λ the section $h_{\lambda,w}$ extends to a Hasse invariant of $\mathcal{X}_w$.
- (2) In general, for $w \in W$, the image of $\mathcal{Y}_w$ by π is a union of several strata $\mathcal{X}_{w'}$. However, when $w \in W^J$, one has $\pi(\mathcal{Y}_w) = \mathcal{X}_w$. Furthermore, the map $\pi : \mathcal{Y}_w \rightarrow \mathcal{X}_w$ is finite etale.
- (3) One shows that $h_{\lambda,w}$ extends to a Hasse invariant if and only if $\pi^*h_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{Y}_w$. One implication is clear. The other is the following lemma:

Lemma 11.3. *Let $f : X \rightarrow Y$ a proper surjective morphism of integral schemes of finite-type over k . Let $\mathcal{L}$ be a line bundle on Y . Let $U \subset Y$ be a normal open subset and $h \in \mathcal{L}(U)$ a non-vanishing section over U . Assume that the section $f^*(h) \in H^0(f^{-1}(U), f^*\mathcal{L})$ extends to X with non-vanishing locus $f^{-1}(U)$. Then there exists $d \geq 1$ such that h^d extends to Y , with non-vanishing locus U .*

- (4) One shows that the pull-back $\pi^*h_{\lambda,w}$ coincides with $\psi^*f_{\lambda,w}$ for a certain function $f_{\lambda,w}$ on the stratum $\mathcal{B}_w$ of $[B \backslash G/B]$ parametrized by w . Similarly, $\psi^*f_{\lambda,w} = \pi^*h_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{Y}_w$ if and only if $f_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{B}_w$.
- (5) Finally, the last part is just a computation. It is based on Chevalley's formula, which makes it possible to compute the divisor of the section $f_{\lambda,w}$. The result is that it extends to a Hasse invariant if and only if Condition (ii) of Proposition 9.2 is satisfied, because the expressions that appear (for α varying in E_w) are the multiplicities of the divisor of $f_{\lambda,w}$. This terminates the proof.

12 Global sections of vector bundles

Recall that any algebraic representation $\rho : E \rightarrow GL(V)$ gives rise to a vector bundle $\mathcal{V}(\rho)$ over $\mathcal{X} = G\text{-Zip}^\mu$. If $\rho : L \rightarrow GL(V)$ is a representation of L , we may view it as a representation of E via the map $E \rightarrow L$ defined as the composition of the first projection $E \rightarrow P$ and the Levi projection $P \rightarrow L$.

So far, we have only considered characters of L , which is the rank 1 case, and constructed Hasse invariants for those vector bundles. In what follows, we want to study higher rank vector bundles. Of particular interest are the vector bundles attached to L -dominant characters by induction. For a character $\lambda \in X^*(T)$, view λ as a character $B \rightarrow \mathbb{G}_m$ and consider the induced representation

$$V(\lambda) = \text{Ind}_B^P(\lambda).$$

It is a representation of P where the unipotent radical of P acts trivially, so we may view it as a representation of L . Note that if λ is not an L -dominant character, we have $V(\lambda) = 0$. We denote by $\mathcal{V}(\lambda)$ the vector bundle over $\mathcal{X}$ attached to $V(\lambda)$. This provides an interesting family of vector bundles $(\mathcal{V}(\lambda))_\lambda$ on $\mathcal{X}$ indexed by the L -dominant characters $\lambda \in X^*(T)$.

We end this survey with a result that determines the space of global sections of $\mathcal{V}(\lambda)$ over $\mathcal{X}$. To simplify, we assume that μ is defined over $\mathbb{F}_p$. We choose again a Borel pair (B, T) defined over $\mathbb{F}_p$ and we assume also that T is split over $\mathbb{F}_p$. For a character $\lambda \in X^*(T)$, our goal is to determine the space $H^0(\mathcal{X}, \mathcal{V}(\lambda))$.

Denote by $\mathcal{U} \subset \mathcal{X}$ the unique open zip stratum. The first step is to determine the space $H^0(\mathcal{U}, \mathcal{V}(\lambda))$. This is elementary, and can be done for an arbitrary representation $\rho : L \rightarrow GL(V)$. Specifically, one has the following result.

Lemma 12.1. *For any representation $\rho : L \rightarrow GL(V)$, there is an isomorphism*

$$H^0(\mathcal{U}, \mathcal{V}(\rho)) \simeq V^{L(\mathbb{F}_p)}.$$

This is almost a tautology, given that the stack $\mathcal{U}$ can be seen to be isomorphic to $[1/L(\mathbb{F}_p)]$. In particular, this shows that $H^0(\mathcal{X}, \mathcal{V}(\rho))$ is a subspace of the $L(\mathbb{F}_p)$ -invariants of V . To determine exactly which subspace demands some work.

First, we introduce some notation. For any representation $\rho : L \rightarrow GL(V)$, we may decompose V with respect to T -eigenspaces.

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi.$$

Define a subspace $V_{\leq 0} \subset V$ as follows. It is the direct sum of the T -eigenspaces V_χ for the characters $\chi \in X^*(T)$ which satisfy the condition

$$\langle \chi, \alpha^\vee \rangle \leq 0 \quad \text{for all } \alpha \in \Delta \setminus I.$$

Note that $V_{\leq 0}$ is stable under the action of T , but it is not a sub- L -representation of V . From now on we consider the L -representation $V(\lambda)$ defined previously, attached to a character $\lambda \in X^*(T)$. One has the following.

Theorem 12.2. *There is a commutative diagram where the vertical maps are the natural inclusions, and the horizontal maps are isomorphisms:*

$$\begin{array}{ccc} H^0(\mathcal{U}, \mathcal{V}(\lambda)) & \xrightarrow{\simeq} & V(\lambda)^{L(\mathbb{F}_p)} \\ \uparrow & & \uparrow \\ H^0(\mathcal{X}, \mathcal{V}(\lambda)) & \xrightarrow{\simeq} & V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbb{F}_p)} \end{array}$$

In a recent paper [17], we show that the above theorem also holds for an arbitrary L -representation (V, ρ) (not necessarily of the form $V(\lambda)$). Furthermore, we also determine in *loc. cit.* the space $H^0(\mathcal{X}, \mathcal{V}(\rho))$ for an arbitrary P -representation (V, ρ) , and without assuming that L is defined over $\mathbb{F}_p$. It is a difficult problem in representation theory to determine for which $\lambda \in X^*(T)$, the intersection $V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbb{F}_p)}$ is non-zero. The set

$$C_{\text{zip}} := \{ \lambda \in X^*(T) \mid V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbb{F}_p)} \neq 0 \}$$

is an additive submonoid (i.e. a cone) of $X^*(T)$. The paper [9] contains some partial results on this set, but it remains quite mysterious.

Let us mention an even more difficult problem. Instead of considering one λ at a time, it is interesting to put them all together by forming the direct sum

$$R_{\text{zip}} := \bigoplus_{\lambda \in X^*(T)} H^0(\mathcal{X}, \mathcal{V}(\lambda)).$$

This group inherits a natural structure of graded k -algebra, and one can ask what the isomorphism class of R_{zip} is. The set C_{zip} is then simply the grading monoid of this graded algebra, so the question of determining R_{zip} is a refinement of the determination of C_{zip} . It is not even clear whether R_{zip} is a finite-type k -algebra. Here are some partial results:

Proposition 12.3. *One has the following:*

- (i) *The algebra R_{zip} is isomorphic to a subalgebra of $k[G]$. In particular, it is integral.*
- (ii) *The field of fractions of R_{zip} is isomorphic to the function field of $R_u(B \cap L)$. In particular, the scheme $\text{Spec}(R_{\text{zip}})$ is birational to an affine space.*
- (iii) *If $\text{Pic}(G) = 0$, then R_{zip} is a UFD.*

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