Griffiths-Schmid conditions for automorphic forms via characteristic p

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Abstract

We establish vanishing results for spaces of automorphic forms in characteristic 0 and characteristic p. We prove that for Hodge-type Shimura varieties, the weight of any nonzero automorphic form in characteristic 0 satisfies the Griffiths–Schmid conditions, by purely algebraic, characteristic p methods. We state a conjecture for general Hodge-type Shimura varieties regarding the vanishing of the space of automorphic forms in characteristic p in terms of the weight. We verify this conjecture for unitary PEL Shimura varieties of signature $(n-1,1)$ at a split prime.

Introduction

In this paper, we establish vanishing results for spaces of automorphic forms in both characteristic 0 and characteristic p. Let $(\mathbf{G},\mathbf{X})$ be a Shimura datum, where $\mathbf{G}$ is a connected reductive $\mathbb{Q}$-group. For a compact open subset $K\subset \mathbf{G}(\mathbb{A}_f)$, we have a Shimura variety $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ defined over a number field $\mathbf{E}$. Let $\mathbf{P}\subset \mathbf{G}_{\mathbf{E}}$ be the parabolic subgroup attached to the Shimura datum (see 1.1.2). Choose a Borel subgroup $\mathbf{B}$ and a maximal torus $\mathbf{T}$ such that $\mathbf{T}\subset\mathbf{B}\subset\mathbf{P}$. Then, any algebraic $\mathbf{P}$-representation (V,ρ) naturally gives rise to a vector bundle $\mathcal{V}(\rho)$ on $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$. Let $\mathbf{L}\subset\mathbf{P}$ denote the unique Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$. Write Φ for the $\mathbf{T}$-roots of $\mathbf{G}$, and Φ_+,Δ respectively for the positive roots and the simple roots with respect to $\mathbf{B}$. Let $I\subset\Delta$ be the subset of simple roots contained in $\mathbf{L}$. For $\lambda\in X^*(\mathbf{T})$, we consider the $\mathbf{P}$-representation $V_I(\lambda)=\mathrm{Ind}_{\mathbf{B}}^{\mathbf{P}}(\lambda)$ and denote by $\mathcal{V}_I(\lambda)$ the associated vector bundle on $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$. We call $\mathcal{V}_I(\lambda)$ the automorphic vector bundle attached to the weight λ. The global sections of $\mathcal{V}_I(\lambda)$ over $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ will be called automorphic forms of weight λ and level K.

We now restrict to the case when $(\mathbf{G},\mathbf{X})$ is of Hodge-type and K is of the form $K=K_pK^p$ with $K_p\subset\mathbf{G}(\mathbb{Q}_p)$ hyperspecial and $K^p\subset\mathbf{G}(\mathbb{A}_f^p)$ is compact open (we say that p is a prime of good reduction). Let $v|p$ be a place of $\mathbf{E}$ and write $\mathbf{E}_v$ for the completion of $\mathbf{E}$ at v. By results of Kisin ([Kis10]) and Vasiu ([Vas99]), the variety $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ admits a smooth canonical model $\mathscr{S}_K$ over $\mathcal{O}_{\mathbf{E}_v}$. The vector bundle $\mathcal{V}_I(\lambda)$ over $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ extends naturally to $\mathscr{S}_K$. For any $\mathcal{O}_{\mathbf{E}_v}$-algebra F that is a field, we investigate which weights λ admit nonzero automorphic forms with coefficients in F. In other words, we study the following set:

$$C_K(F):=\{\lambda\in X^*(\mathbf{T})\mid H^0(\mathscr{S}_K\otimes_{\mathcal{O}_{\mathbf{E}_v}}F,\mathcal{V}_I(\lambda))\neq 0\}.$$

It is a subcone (i.e additive submonoid) of $X^*(\mathbf{T})$. It is contained in the set $X^*_{+,I}(\mathbf{T})$ of $\mathbf{L}$-dominant characters, because $\mathcal{V}_I(\lambda)=0$ for non $\mathbf{L}$-dominant λ (by $\mathbf{L}$-dominant, we mean that it satisfies $\langle\lambda,\alpha^\vee\rangle\geq 0$ for all $\alpha\in I$). It suffices to consider the cases $F=\mathbb{C}$ and $F=\overline{\mathbb{F}}_p$. Indeed, note that if $F\subset F'$ then by flat base change, we have

$$H^0(\mathscr{S}_K\otimes_{\mathcal{O}_{\mathbf{E}_v}}F',\mathcal{V}_I(\lambda))=H^0(\mathscr{S}_K\otimes_{\mathcal{O}_{\mathbf{E}_v}}F,\mathcal{V}_I(\lambda))\otimes_F F',$$

therefore $C_K(F) = C_K(F')$. By [MS11], there exists a smooth, toroidal compactification $\mathscr{S}_K^\Sigma$ of $\mathscr{S}_K$, where Σ is a sufficiently fine cone decomposition. The vector bundle $\mathcal{V}_I(\lambda)$ over $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ extends naturally to the toroidal compactification $\mathscr{S}_K^\Sigma$. By results of Lan–Stroh in [LS18], the Koecher principle holds, i.e there is an identification $H^0(\mathscr{S}_K \otimes_R R, \mathcal{V}_I(\lambda)) = H^0(\mathscr{S}_K^\Sigma \otimes_R R, \mathcal{V}_I(\lambda))$ for all $\mathcal{O}_{\mathbf{E}_v}$-algebra R and all $\lambda \in X^*(\mathbf{T})$, except when $\dim(\mathrm{Sh}(\mathbf{G},\mathbf{X})_K) = 1$ and $\mathscr{S}_K^\Sigma \setminus \mathscr{S}_K \neq \emptyset$. We assume henceforth that $\dim(\mathrm{Sh}(\mathbf{G},\mathbf{X})) > 1$ or that $\mathscr{S}_K$ is proper, so that the Koecher principle holds.

In general, the set $C_K(F)$ highly depends on the choice of the level K (even in the case of the modular curve). For a subcone $C \subset X^*(\mathbf{T})$, define its saturated cone $\langle C \rangle$ as the set of $\lambda \in X^*(\mathbf{T})$ such that some positive multiple of λ lies in C. The saturated cone $\langle C_K(F) \rangle$ is then independent of the level K ([Kos19, Corollary 1.5.3]). Hence, it should be possible to give an expression for the saturated cone in terms of the root data of $\mathbf{G}$. Indeed, it is known (at least for $F = \mathbb{C}$) that the cohomology of the Shimura variety $\mathscr{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_v}} F$ can be expressed in terms of automorphic representations, and the theory of automorphic representations is to a large extent controlled by the root datum of the reductive group $\mathbf{G}$.

We first consider the case $F = \mathbb{C}$. Griffiths–Schmid considered in [GS69] the following set of characters:

$$C_{\mathrm{GS}} = \left\{ \lambda \in X^*(\mathbf{T}) \;\middle|\; \begin{array}{ll} \langle \lambda, \alpha^\vee \rangle \geq 0 & \text{for } \alpha \in I, \\ \langle \lambda, \alpha^\vee \rangle \leq 0 & \text{for } \alpha \in \Phi_+ \setminus \Phi_{\mathbf{L},+} \end{array} \right\}.$$

Here $\Phi_{\mathbf{L},+}$ denotes the positive $\mathbf{T}$-roots in $\mathbf{L}$. We call this cone the Griffiths-Schmid cone. The following seems to be known to experts, but as far as we know there is no reference where this result is explicitly stated.

Theorem 1. *Let $(\mathbf{G},\mathbf{X})$ be any Hodge-type Shimura datum. Let $\lambda \in X^*(\mathbf{T})$ be a character and assume that $\lambda \notin C_{\mathrm{GS}}$. Then we have $H^0(\mathrm{Sh}(\mathbf{G},\mathbf{X})_K, \mathcal{V}_I(\lambda)) = 0$.*

In other words, this theorem amounts to the inclusion $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$. We note that the equality $\langle C_K(\mathbb{C}) \rangle = C_{\mathrm{GS}}$ is expected in general. It seems possible to show the above theorem using the theory of Lie algebra cohomology. In this paper, we give a proof based on purely characteristic p methods, which is a novel aspect of our approach. For an automorphic form f in characteristic zero, we may consider the reduction of f modulo v for all except finitely many places v of $\mathbf{E}$. Then, our approach is to use the geometric structure (namely the Ekedahl–Oort stratification) of the special fiber at v to extract information about the weight of f.

In our proof of Theorem 1, only weak information at each prime is sufficient to obtain the result because we are able to reduce f at infinitely many places. On the other hand, a more difficult question is to fix a prime p (of good reduction) and study the cone $C_K(\overline{\mathbb{F}}_p)$. Similarly to the characteristic zero case, the saturated cone $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ is independent of K and we expect that it can be expressed in terms of root data. However, it also depends in general on the prime p. We have conjectured the following ([GK18, Conjecture C]):

Conjecture 1. *We have $\langle C_K(\overline{\mathbb{F}}_p) \rangle = \langle C_{\mathrm{zip}} \rangle$.*

Here, the cone C_{zip} is an entirely group-theoretical object defined using the stack of G-zips defined by Moonen–Wedhorn ([MW04]) and Pink–Wedhorn–Ziegler ([PWZ11, PWZ15]). Specifically, write $S_K := \mathscr{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_v}} \overline{\mathbb{F}}_p$. Since K_p is hyperspecial, $\mathbf{G}$ admits a $\mathbb{Z}_p$-reductive model $\mathcal{G}$. Set $G := \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, and write similarly T, L for the reduction of $\mathbf{T}, \mathbf{L}$ respectively. By results of Zhang ([Zha18]) there exists a smooth map $\zeta \colon S_K \to G\text{-}\mathtt{Zip}^\mu$ where $G\text{-}\mathtt{Zip}^\mu$ is the stack of G-zips of type μ (here μ is a cocharacter of $G_{\overline{\mathbb{F}}_p}$ whose centralizer is L). The map ζ is also surjective by [SYZ19, Corollary 3.5.3(1)]. The automorphic vector bundles $\mathcal{V}_I(\lambda)$ also exist on the stack $G\text{-}\mathtt{Zip}^\mu$ (see [IK21a, §2.4]), compatibly with the map ζ. We defined C_{zip} ([Kos19, (1.2.3)]) as the set of $\lambda \in X^*(\mathbf{T})$ such that $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0$. The space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ can be interpreted in terms of representation theory of reductive groups ([Kos19, Theorem 3.7.2], [IK21a, Theorem 1]).

Conjecture 1 was proved in [GK18, Theorem D] for Hilbert–Blumenthal Shimura varieties, Siegel threefolds and Picard surfaces (at split primes). The Hilbert–Blumenthal case was also treated independently by Diamond–Kassaei in [DK17, Corollary 1.3] using different methods and a different formulation. In the preprint [GK22], it is proved in the cases $G = \mathrm{GSp}(6)$, $\mathrm{GU}(r,s)$ for $r+s \leq 4$ (except when $r=s=2$ and p is inert). The set C_{zip} is much more tractable than $\langle C_K(\overline{\mathbb{F}}_p)\rangle$, but is still difficult to determine in general. We can use Conjecture 1 in order to gain intuition about the cone $\langle C_K(\overline{\mathbb{F}}_p)\rangle$. Conversely, facts pertaining to automorphic forms and their weights should have an equivalent group-theoretical statement on the level of the stack $G\text{-}\mathtt{Zip}^\mu$. For example, using reduction modulo p, one shows easily that $C_K(\mathbb{C}) \subset C_K(\overline{\mathbb{F}}_p)$ (see [Kos19, Proposition 1.8.3]), hence also $\langle C_K(\mathbb{C})\rangle \subset \langle C_K(\overline{\mathbb{F}}_p)\rangle$. Since it is expected that $\langle C_K(\mathbb{C})\rangle = C_{\mathrm{GS}}$ in general, one should expect an inclusion $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$. This fact is highly nontrivial, and was indeed proved in general in the recent preprint [IK22, Theorem Theorem 6.4.2], as a sanity check for Conjecture 1 to hold.

In this paper, our second goal is to seek an upper bound approximation of $\langle C_K(\overline{\mathbb{F}}_p)\rangle$. To gain intuition, we first consider the cone C_{zip} and determine an upper bound for it. We define in section 3.2 the unipotent-invariance cone $C_{\mathrm{unip}} \subset X^*(\mathbf{T})$ and show that $C_{\mathrm{zip}} \subset C_{\mathrm{unip}}$. When G is split over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ and P is defined over $\mathbb{F}_p$, we can give concrete equations for an upper bound of C_{zip}. Let $W_L = W(L,T)$ be the Weyl group of L. Note that $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ acts naturally on the set $\Phi_+ \setminus \Phi_{L,+}$. Let $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$ be an orbit under the action of $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ and let $S \subset \mathcal{O}$ be any subset. Set

$$\Gamma_{\mathcal{O},S,p}(\lambda) := \sum_{\alpha \in \mathcal{O}\setminus S} \langle \lambda, \alpha^\vee \rangle \ + \ \frac{1}{p}\sum_{\alpha \in S} \langle \lambda, \alpha^\vee \rangle$$

Define $C_{\mathcal{O}}$ as the set of $\lambda \in X^*(\mathbf{T})$ such that $\Gamma_{\mathcal{O},S}(\lambda) \leq 0$ for all subsets $S \subset \mathcal{O}$. Then we have

$$C_{\mathrm{zip}} \subset \bigcap_{\substack{\text{orbits} \\ \mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}}} C_{\mathcal{O}}.$$

Only certain choices of $(\mathcal{O}, S)$ will contribute non-trivially to the above intersection, but for a general group it is unclear to us how to determine the important pairs $(\mathcal{O}, S)$. By Conjecture 1, we can expect the following:

Conjecture 2. *Let S_K be the special fiber of a Hodge-type Shimura variety at a prime p of good reduction which splits in $\mathbf{E}$. Furthermore, assume that the attached reductive $\mathbb{F}_p$-group G is split over $\mathbb{F}_{p^2}$. Then if $f \in H^0(S_K, \mathcal{V}_I(\lambda))$ is a nonzero automorphic form of weight $\lambda \in X^*(\mathbf{T})$, we have $\Gamma_{\mathcal{O},S,p}(\lambda) \leq 0$ for all $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$-orbit $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$ and all subsets $S \subset \mathcal{O}$.*

We now consider the case of Shimura varieties attached to a unitary similitude group $\mathbf{G}$ such that $\mathbf{G}_{\mathbb{R}} \simeq \mathrm{GU}(n-1,1)$. We choose a split prime p of good reduction. In this case $G \simeq \mathrm{GL}_{n-1,\mathbb{F}_p} \times \mathbb{G}_{\mathrm{m},\mathbb{F}_p}$. We parametrize weights by n-tuples $(k_1,\ldots,k_n) \in \mathbb{Z}$. We prove Conjecture 2 in this case. More precisely, we have the following:

Theorem 2. *Let S_K be the good reduction special fiber of a unitary Shimura variety of signature $(n-1,1)$ at a split prime p. Let $f \in H^0(S_K, \mathcal{V}_I(\lambda))$ be a nonzero mod p automorphic* *form and write* $\lambda = (k_1, \dots, k_n) \in \mathbb{Z}^n$*. Then we have:*

$$\sum_{i=1}^{j}(k_i - k_n) + \frac{1}{p}\sum_{i=j+1}^{n-1}(k_i - k_n) \leq 0 \quad \textit{for all } j = 1, \dots, n-1.$$

The inequalities appearing in the statement of the theorem are of the form $\Gamma_{\mathcal{O},S}(\lambda) \leq 0$, as in Conjecture 2. In this case, the set $\Phi_+ \setminus \Phi_{L,+}$ consists of a single orbit under the group W_L. Furthermore, we only need consider the sets $S \subset \Phi_+ \setminus \Phi_{L,+}$ which satisfy the property that if $w \in S$, then any $w' \geq w$ is also in S, because one sees easily that the other sets do not contribute. This gives the n inequalities in Theorem 2. It is compatible with Theorem 1 in the following sense. In our convention of positivity, we have

$$C_{\mathrm{GS}} = \{\lambda = (k_1, \dots, k_n) \in \mathbb{Z}^n \ \mid \ k_n \geq k_1 \geq \cdots \geq k_{n-1}\}.$$

Note that C_{GS} is the set of L-dominant characters $\lambda \in X^*_{+,L}(T)$ satisfying the condition $k_1 \leq k_n$. If we let p go to infinity in the inequality corresponding to $j = 1$ in Theorem 2, we deduce that the weight $\lambda = (k_1, \dots, k_n)$ of any characteristic zero automorphic form satisfies $k_1 \leq k_n$, hence lies in C_{GS}.

We briefly explain the proof of Theorem 2. First, we consider the flag space of S_K, which is a P/B-fibration $\pi_K\colon \mathrm{Flag}(S_K) \to S_K$. It carries a family of line bundles $\mathcal{V}_{\mathrm{flag}}(\lambda)$ for $\lambda \in X^*(\mathbf{T})$ such that $\pi_{K,*}(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. Furthermore, it carries a stratification $(\mathrm{Flag}(S_K)_w)_{w\in W}$ defined as the fibers of a natural map $\psi_K\colon \mathrm{Flag}(S_K) \to [B\backslash G/B]$. For each $w \in W$, we define a cone $C_{K,w} \subset X^*(\mathbf{T})$ as the set of λ such that the line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ admits nonzero sections on the Zariski closure $\overline{\mathrm{Flag}(S_K)}_w$. There is a natural subcone $C_{\mathrm{Hasse},w} \subset C_{K,w}$ given by the weights of sections which arise by pullback from the stack $[B\backslash G/B]$ via ψ_K. We say that the stratum $\mathrm{Flag}(S_K)_w$ is Hasse-regular if $\langle C_{\mathrm{Hasse},w}\rangle = \langle C_{K,w}\rangle$. Let w_0 and $w_{0,L}$ be the longest elements in W and W_L respectively. Set $z = w_{0,L}w_0$. The projection π_K restricts to a map $\pi_K\colon \overline{\mathrm{Flag}(S_K)}_z \to S_K$ which is finite etale on the open subset $\mathrm{Flag}(S_K)_z$. The proof of Theorem 2 uses the following result as a starting point:

Theorem 3. *Let* S_K *be the good reduction special fiber of a unitary Shimura variety of signature* $(n-1, 1)$ *at a split prime. For any* $w \in W$ *such that* $w \leq z$*, the stratum* $\mathrm{Flag}(S_K)_w$ *is Hasse-regular.*

We conjecture that the above also generalizes for all Hodge-type Shimura varieties when G is split over $\mathbb{F}_p$. Concretely, this theorem implies the following: Let f be any nonzero section of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $\overline{\mathrm{Flag}(S_K)}_z$ for $\lambda = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Then λ satisfies $k_i - k_n \leq 0$ for all $i = 1, \dots, n-1$. In particular, let f be any nonzero automorphic form in characteristic p, of weight λ. We may view f as a global section of the line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $\mathrm{Flag}(S_K)$, using the relation $\pi_{K,*}(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. If $\lambda \notin C_{\mathrm{GS}}$, then the restriction of f to the stratum $\mathrm{Flag}(S_K)_z$ is zero. We expect this result to generalize to all Hodge-type cases at split primes of good reduction.

We prove Theorem 2 as a consequence of Theorem 3, by using a suitable sequence of elements $w_1, \dots, w_N$ in W starting at $w_1 = w_0$ and ending at $w_N = z$. For each $1 \leq i \leq N-1$, w_{i+1} is a lower neighbour of w_i with respect to the Bruhat order on W. Furthermore, the flag stratum corresponding to w_{i+1} is cut out inside the Zariski closure of $\mathrm{Flag}(S_K)_{w_i}$ by a certain partial Hasse invariant Ha_i. It then follows easily that the weight of any nonzero global section of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ is the sum of the weights of Ha_i and of an element of C_{GS}, which proves the result.

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1 Weights of automorphic forms

1.1 Automorphic forms on Shimura varieties

1.1.1 Shimura varieties

Let $(\mathbf{G},\mathbf{X})$ be a Shimura datum of Hodge-type [Del79, 2.1.1]. In particular, $\mathbf{G}$ is a connected, reductive group over $\mathbb{Q}$. Furthermore, $\mathbf{X}$ gives rise to a well-defined $\mathbf{G}(\overline{\mathbb{Q}})$-conjugacy class of cocharacters $\{\mu\}$ of $\mathbf{G}_{\overline{\mathbb{Q}}}$. Let $\mathbf{E}=\mathbf{E}(\mathbf{G},\mathbf{X})$ be the reflex field of $(\mathbf{G},\mathbf{X})$ (i.e. the field of definition of $\{\mu\}$) and $\mathcal{O}_\mathbf{E}$ its ring of integers. If $K\subset\mathbf{G}(\mathbb{A}_f)$ is an open compact subgroup, write $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ for Deligne's canonical model at level K over $\mathbf{E}$ (see [Del79]). When K is small enough, $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ is a smooth, quasi-projective scheme over $\mathbf{E}$. Fix a finite set of "bad" primes S and a compact open subgroup $K\subset\mathbf{G}(\mathbb{A}_f)$ of the form

$$K=K_S\times K^S$$

where $K_S\subset\mathbf{G}(\mathbb{Q}_S)$ and $K^S=\mathbf{G}(\widehat{\mathbb{Z}}^S)$, where $\mathbb{Q}_S=\prod_{p\in S}\mathbb{Q}_p$ and $\widehat{\mathbb{Z}}^S=\prod_{p\notin S}\mathbb{Z}_p$. For all $p\notin S$, the Shimura variety $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ has good reduction at all primes above p. In particular, for each $p\notin S$, the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified, so there exists a reductive $\mathbb{Z}_p$-model $\mathcal{G}$, such that $G:=\mathcal{G}\otimes_{\mathbb{Z}_p}\mathbb{F}_p$ is connected. For any place v above p in $\mathbf{E}$, Kisin ([Kis10]) and Vasiu ([Vas99]) constructed a smooth canonical model $\mathscr{S}_K$ of $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ over $\mathcal{O}_{\mathbf{E},v}$. By glueing, we obtain a smooth $\mathcal{O}_\mathbf{E}\left[\frac{1}{N}\right]$-model, that we will abusively continue to denote by $\mathscr{S}_K$, where $N\geq 1$ is an integer divisible by all the primes in S. We denote its mod p reduction by $S_{K,p}:=\mathscr{S}_K\otimes_{\mathcal{O}_{\mathbf{E},v}}\overline{\mathbb{F}}_p$ (we simply write S_K when the choice of p is fixed). We will have to extend the ring of definition so that all objects we consider are defined over that ring. Therefore, we let R be a ring of the form $\mathcal{O}_{\mathbf{E}'}\left[\frac{1}{N'}\right]$ for a number field $\mathbf{E}\subset\mathbf{E}'$ and an integer N' divisible by N. We will freely change R to a suitable extention by modifying $\mathbf{E}'$ and N'.

1.1.2 Automorphic vector bundles

A cocharacter $\mu\in\{\mu\}$ induces a decomposition of $\mathfrak{g}:=\mathrm{Lie}(\mathbf{G}_\mathbb{C})$ as $\mathfrak{g}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$, where $\mathfrak{g}_n$ is the subspace where $\mathbb{G}_{\mathrm{m},\mathbb{C}}$ acts on $\mathfrak{g}$ by $x\mapsto x^n$ via μ. It gives rise to an opposite pair of parabolic subgroups $\mathbf{P}_\pm(\mu)$ such that $\mathrm{Lie}(\mathbf{P}_+(\mu))$ (resp. $\mathrm{Lie}(\mathbf{P}_-(\mu))$ is the direct sum of $\mathfrak{g}_n$ for $n\geq 0$ (resp. $n\leq 0$). We set $\mathbf{P}=\mathbf{P}_-(\mu)$. Let $(\mathbf{B},\mathbf{T})$ be a Borel pair of $\mathbf{G}_\mathbb{C}$ such that $\mathbf{B}\subset\mathbf{P}$ and such that $\mu\colon\mathbb{G}_{\mathrm{m},\mathbb{C}}\to\mathbf{G}_\mathbb{C}$ factors through $\mathbf{T}$. As usual, $X^*(\mathbf{T})$ denotes the group of characters of $\mathbf{T}$. Let $\mathbf{B}^+$ be the opposite Borel subgroup (i.e the unique Borel subgroup such that $\mathbf{B}^+\cap\mathbf{B}=\mathbf{T}$). Let $\Phi\subset X^*(\mathbf{T})$ be the set of $\mathbf{T}$-roots of G and $\Phi_+\subset\Phi$ the system of positive roots with respect to $\mathbf{B}^+$ (i.e. $\alpha\in\Phi_+$ whenever the α-root group U_α is contained in $\mathbf{B}^+$). We use this convention to match those of the previous publications [GK19a, Kos19]. Let $\Delta\subset\Phi_+$ be the set of simple roots. Let $I\subset\Delta$ denote the set of simple roots of the unique Levi subgroup $\mathbf{L}\subset\mathbf{P}$ containing $\mathbf{T}$ (note that $\mathbf{L}$ is the centralizer of μ).

We may assume that there exists a reductive, smooth group scheme $\mathcal{G}$ over $\mathbb{Z}[\frac{1}{N'}]$ such that $\mathcal{G}\otimes_{\mathbb{Z}[1/N']}\mathbb{Q}\simeq\mathbf{G}$ and that μ extends to a cocharacter of $\mathcal{G}\otimes_{\mathbb{Z}[1/N']}R$. In particular, we obtain a parabolic subgroup $\mathcal{P} \subset \mathcal{G} \otimes_{\mathbb{Z}[1/N']} R$ that extends $\mathbf{P}$. The R-scheme $\mathscr{S}_K$ carries a universal $\mathcal{P}$-torsor afforded by the Hodge filtration. This torsor yields a natural functor

$$\mathcal{V}\colon \operatorname{Rep}_R(\mathcal{P}) \longrightarrow \mathfrak{VB}(\mathscr{S}_K) \tag{1.1.1}$$

where $\operatorname{Rep}_R(\mathcal{P})$ denotes the category of algebraic R-representations of $\mathcal{P}$, and $\mathfrak{VB}(\mathscr{S}_K)$ is the category of vector bundles on $\mathscr{S}_K$. Furthermore, the functor $\mathcal{V}$ commutes in an obvious sense with change of level. The vector bundles of the form $\mathcal{V}(\rho)$ for $\rho \in \operatorname{Rep}_R(\mathcal{P})$ are called *automorphic vector bundles* in [Mil90, III. Remark 2.3].

Let $\lambda \in X^*(\mathbf{T})$ be an $\mathbf{L}$-dominant character, by which we mean that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in I$. Set $\mathbf{V}_I(\lambda) = H^0(\mathbf{P}/\mathbf{B}, \mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda$ is the line bundle on $\mathbf{P}/\mathbf{B}$ attached to λ. It is the unique irreducible representation of $\mathbf{P}$ over $\overline{\mathbb{Q}}$ of highest weight λ. After possibly extending R, we may assume that $\mathbf{V}_I(\lambda)$ admits a natural model over R, namely $\mathbf{V}_I(\lambda)_R := H^0(\mathcal{P}/\mathcal{B}, \mathcal{L}_\lambda)$, where $\mathcal{B}$ is a $\mathbb{Z}[1/N']$-Borel subgroup of $\mathcal{G}$ extending $\mathbf{B}$. We denote by $\mathcal{V}_I(\lambda)$ the vector bundle on $\mathscr{S}_K$ attached to the $\mathcal{P}$-representation $\mathbf{V}_I(\lambda)_R$.

1.1.3 The stack of G-zips

Let p be a prime number and q a p-power. Fix an algebraic closure k of $\mathbb{F}_q$. For a k-scheme X, we denote by $X^{(q)}$ its q-th power Frobenius twist and by $\varphi\colon X \to X^{(q)}$ its relative Frobenius. Let $\sigma \in \operatorname{Gal}(k/\mathbb{F}_q)$ be the automorphism $x \mapsto x^q$. If G is a connected, reductive group over $\mathbb{F}_q$ and $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ is a cocharacter, we call the pair (G,μ) a cocharacter datum over $\mathbb{F}_q$. In the context of Shimura varieties, we always take $q = p$, and G will be the reduction modulo p of $\mathbf{G}$ at a prime of good reduction. To the pair (G,μ), we can attach (functorially) a finite smooth stack $G\text{-}\mathtt{Zip}^\mu$ called the stack of G-zips of type μ. It was introduced by Moonen–Wedhorn and Pink–Wedhorn–Ziegler in [MW04, PWZ11, PWZ15]. As in section 1.1.2, μ gives rise to two opposite parabolic subgroups $P_\pm(\mu) \subset G_k$. We set $P := P_-(\mu)$ and $Q := P_+(\mu)^{(q)}$. Let $L := \operatorname{Cent}(\mu)$ be the centralizer of μ, it is a Levi subgroup of P. Put $M := L^{(q)}$, which is a Levi subgroup of Q. We have a Frobenius map $\varphi\colon L \to M$. The tuple $\mathcal{Z} := (G,P,Q,L,M,\varphi)$ is called the zip datum attached to (G,μ).

Let $\theta^P_L\colon P \to L$ be the projection onto the Levi subgroup L modulo the unipotent radical $R_{\mathrm{u}}(P)$. Define $\theta^Q_M\colon Q \to M$ similarly. The zip group of $\mathcal{Z}$ is defined by

$$E := \{(x,y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}.$$

Let E act on G by the rule $(x,y)\cdot g := xgy^{-1}$. The stack of G-zips $G\text{-}\mathtt{Zip}^\mu$ can be defined as the quotient stack

$$G\text{-}\mathtt{Zip}^\mu := [E\backslash G_k]\,.$$

To any E-representation (V,ρ), one attaches a vector bundle $\mathcal{V}(\rho)$ on $G\text{-}\mathtt{Zip}^\mu$, as explained in [IK21a, §2.4.2] using the associated sheaf construction ([Jan03, I.5.8]). In particular, a P-representation (V,ρ) gives rise to an E-representation via the first projection $\mathrm{pr}_1\colon E \to P$, thus to a vector bundle $\mathcal{V}(\rho)$. Choose a Borel pair (B,T) of G_k such that $B \subset P$ and such that μ factors through T. For $\lambda \in X^*(T)$, define $V_I(\lambda)$ as the P-representation $\operatorname{Ind}^P_B(\lambda) = H^0(P/B, \mathcal{L}_\lambda)$ similarly to section 1.1.2. The vector bundle on $G\text{-}\mathtt{Zip}^\mu$ attached to $V_I(\lambda)$ is denoted again by $\mathcal{V}_I(\lambda)$.

We now explain the connection with Shimura varieties. We return to the setting of section 1.1.1. Fix a prime $p \notin S$ of good reduction and let $G := \mathcal{G} \otimes_{\mathbb{Z}[\frac{1}{N'}]} \mathbb{F}_p$. Write again μ for the cocharacter of $G_{\overline{\mathbb{F}}_p}$ obtained by reduction mod p. We obtain a cocharacter datum (G,μ) over $\mathbb{F}_p$, and hence a zip datum (G,P,L,Q,M,φ) and a stack of G-zips $G\text{-}\mathtt{Zip}^\mu$. Write $S_{K,p} := \mathscr{S} \otimes_R \overline{\mathbb{F}}_p$. Zhang ([Zha18, 4.1]) constructed a smooth morphism

$$\zeta\colon S_{K,p} \to G\text{-}\mathtt{Zip}^\mu\,.$$

This map is also surjective by [SYZ19, Corollary 3.5.3(1)]. Furthermore, the automorphic vector bundle $\mathcal{V}_I(\lambda)$ defined on $S_{K,p}$ using the functor (1.1.1) coincides with the pullback via ζ of the vector bundle $\mathcal{V}_I(\lambda)$ defined on $G\text{-}\mathtt{Zip}^\mu$.

1.1.4 Toroidal compactification

By [MS11, Theorem 1], there is a sufficiently fine cone decomposition Σ and a toroidal compactification $\mathscr{S}_K^\Sigma$ of $\mathscr{S}_K$ over $\mathcal{O}_{\mathbf{E},v}$. Again, by glueing we may assume that there exists a toroidal compactification of $\mathscr{S}_K$ over the ring R, that we denote again by $\mathscr{S}_K^\Sigma$. Furthermore, the family $(\mathcal{V}_I(\lambda))_{\lambda\in X^*(\mathbf{T})}$ admits a canonical extension $(\mathcal{V}_I^\Sigma(\lambda))_{\lambda\in X^*(\mathbf{T})}$ to $\mathscr{S}_K^\Sigma$. For a prime p, set $S_{K,p}^\Sigma := \mathscr{S}_K^\Sigma \otimes_R \overline{\mathbb{F}}_p$. By [GK19a, Theorem 6.2.1], the map $\zeta\colon S_{K,p} \to G\text{-}\mathtt{Zip}^\mu$ extends naturally to a map

$$\zeta^\Sigma\colon S_{K,p}^\Sigma \to G\text{-}\mathtt{Zip}^\mu .$$

Furthermore, by [And21, Theorem 1.2], the map ζ^Σ is smooth. Since ζ is surjective, ζ^Σ is also surjective. Moreover, [WZ, Proposition 6.20] shows that any connected component $S^\circ \subset S_{K,p}^\Sigma$ intersects the unique zero-dimensional stratum. Since the map $\zeta^\Sigma\colon S^\circ \to G\text{-}\mathtt{Zip}^\mu$ is smooth, its image is open, hence surjective. Therefore, the restriction of $\zeta^\Sigma\colon S_{K,p}^\Sigma \to G\text{-}\mathtt{Zip}^\mu$ to any connected component is also surjective.

By construction, the pullback of $\mathcal{V}_I(\lambda)$ via ζ coincides with the canonical extension $\mathcal{V}_I^\Sigma(\lambda)$. We have the following Koecher principle:

Theorem 1.1.1 ([LS18, Theorem 2.5.11]). *Let F be a field which is an R-algebra. The natural map*

$$H^0(\mathscr{S}_K^\Sigma \otimes_R F, \mathcal{V}_I^\Sigma(\lambda)) \to H^0(\mathscr{S}_K \otimes_R F, \mathcal{V}_I(\lambda))$$

is a bijection, except when $\dim(\mathscr{S}_K) = 1$ *and* $\mathscr{S}_K^\Sigma \setminus \mathscr{S}_K \neq \emptyset$.

We will only consider Shimura varieties satisfying the condition $\dim(\mathscr{S}_K) > 1$ or $\mathscr{S}_K^\Sigma \setminus \mathscr{S}_K \neq \emptyset$.

1.2 Weight cones of automorphic forms

1.2.1 Griffiths–Schmid conditions

The motivation of this paper is to study the possible weights of automorphic forms over various fields. Specifically, for any field F which is an R-algebra, define

$$C_K(F) := \{\lambda \in X^*(\mathbf{T}) \mid H^0(\mathscr{S}_K \otimes_R F, \mathcal{V}_I(\lambda)) \neq 0\}.$$

By the Koecher principle (Theorem 1.1.1), we may replace the pair $(\mathscr{S}_K \otimes_R F, \mathcal{V}_I(\lambda))$ with the pair $(\mathscr{S}_K^\Sigma \otimes_R F, \mathcal{V}_I^\Sigma(\lambda))$ in the definition of $C_K(F)$.

As explained in the introduction, there are two main cases to consider, namely $F = \mathbb{C}$ and $F = \overline{\mathbb{F}}_p$ for a prime number p of good reduction. We first consider the case $F = \mathbb{C}$. The space $H^0(\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}), \mathcal{V}_I(\lambda))$ is the space of classical, characteristic zero automorphic forms of weight λ and level K. Therefore, the set $C_K(\mathbb{C})$ is the set of possible weights of nonzero automorphic forms in characteristic 0. It is a subcone of $X^*(\mathbf{T})$ (by "cone", we mean an additive monoid containing zero). Write $\Phi_{\mathbf{L},+}$ for the set of positive $\mathbf{T}$-roots of $\mathbf{L}$. The Griffiths–Schmid cone C_{GS} is defined as follows.

$$C_{\mathrm{GS}} = \left\{ \lambda \in X^*(\mathbf{T}) \;\middle|\; \begin{array}{ll} \langle \lambda, \alpha^\vee \rangle \geq 0 & \text{for } \alpha \in I, \\ \langle \lambda, \alpha^\vee \rangle \leq 0 & \text{for } \alpha \in \Phi_+ \setminus \Phi_{\mathbf{L},+} \end{array} \right\}.$$

The conditions defining the cone C_{GS} were first introduced by Griffiths–Schmid in [GS69]. It is expected that $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$ for general Shimura varieties, although we are not aware of any reference where this statement is proved. We show this containment in the case of general Hodge-type Shimura varieties. More generally, we may consider any projective R-scheme X endowed with the following structure:

Assumption 1.2.1.
(1) *There is a connected, reductive $\mathbb{Z}[1/N]$-group $\mathcal{G}$ and a cocharacter $\mu\colon \mathbb{G}_{m,R} \to \mathcal{G}_R$ satisfying the following condition: For p sufficiently large, there exists a smooth map $\zeta_p\colon X_p \to G_p\text{-}\mathtt{Zip}^{\mu}$, where $X_p := X \otimes_R \overline{\mathbb{F}}_p$ and $G_p := \mathcal{G} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p$. Furthermore, ζ_p is surjective on each connected component of X_p.*
(2) *There is a family of vector bundles $(\mathcal{V}_I(\lambda))_{\lambda \in X^*(\mathbf{T})}$ on X such that the restriction of $\mathcal{V}_I(\lambda)$ to X_p coincides with the pullback via ζ of the vector bundle $\mathcal{V}_I(\lambda)$ on $G_p\text{-}\mathtt{Zip}^{\mu}$.*

As we explained in section 1.1.4, the scheme $\mathscr{S}_K^{\Sigma}$ satisfies Assumption 1.2.1. For such a scheme X, define similarly $C_X(F)$ as the set of $\lambda \in X^*(\mathbf{T})$ such that $H^0(X \otimes_R F, \mathcal{V}_I(\lambda)) \neq 0$. In Theorem 2.6.4 below, we prove the following:

Theorem 1.2.2. *We have $C_X(\mathbb{C}) \subset C_{\mathrm{GS}}$.*

In particular, we may take X to be $\mathscr{S}_K$, which implies that $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$. As the setting suggests, our proof relies entirely on characteristic p methods rather than studying the space $H^0(\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}), \mathcal{V}_I(\lambda))$ directly via the theory of automorphic representations or Lie algebra cohomology.

1.2.2 The zip cone

We now consider the case $F = \overline{\mathbb{F}}_p$. In our approach, the proof of Theorem 1.2.2 relies on the study of $C_X(\overline{\mathbb{F}}_p)$ for various prime numbers p. In [GK18, GK22], the authors started a vast project to investigate the set $C_K(\overline{\mathbb{F}}_p)$ using the stack of G-zips. For a cocharacter datum (G, μ) over $\mathbb{F}_q$, we defined the zip cone of (G, μ) in [Kos19, §1.2] and [IK22, §3] as

$$C_{\mathrm{zip}} := \{\lambda \in X^*(T) \mid H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) \neq 0\}.$$

This cone can be seen as a group-theoretical version of the set $C_K(\overline{\mathbb{F}}_p)$ in the case of Shimura varieties. To emphasize the analogy between S_K and $G\text{-}\mathtt{Zip}^{\mu}$, we call $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda))$ the space of automorphic forms of weight λ on $G\text{-}\mathtt{Zip}^{\mu}$. Since $V_I(\lambda) = 0$ when λ is not L-dominant, C_{zip} is a subset of the set of L-dominant characters $X^*_{+,I}(T)$. One can see that C_{zip} is a subcone of $X^*(T)$ ([Kos19, Lemma 1.4.1]). For a cone $C \subset X^*(T)$, define the saturated cone $\langle C \rangle$ as:

$$\langle C \rangle := \{\lambda \in X^*(T) \mid \exists N \geq 1, N\lambda \in C\}.$$

We say that C is saturated in $X^*(T)$ if $\langle C \rangle = C$. We explain the main conjecture that motivates the series of papers [GK18, IK22, GK22]. Consider the special fiber S_K of good reduction of a Hodge-type Shimura variety (such that $\dim(S_K) > 1$ or $S_K = S_K^{\Sigma}$), and its associated map $\zeta\colon S_K \to G\text{-}\mathtt{Zip}^{\mu}$. Since ζ is surjective, we have a natural inclusion

$$H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) \subset H^0(S_K, \mathcal{V}_I(\lambda)).$$

In particular, we deduce $C_{\mathrm{zip}} \subset C_K(\overline{\mathbb{F}}_p)$.

Conjecture 1.2.3. *One has $\langle C_K(\overline{\mathbb{F}}_p) \rangle = \langle C_{\mathrm{zip}} \rangle$.*

It was noted in [Kos19, Corollary 1.5.3] that the set $\langle C_K(\overline{\mathbb{F}}_p)\rangle$ is independent of the level (because the change of level maps are finite etale). Therefore, the above conjecture is indeed reasonable. However, note that the set $C_K(\overline{\mathbb{F}}_p)$ highly depends on the choice of the level K.

More generally, we expect Conjecture 1.2.3 to hold for any scheme X endowed with a map $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$ satisfying the conditions of [GK18, Conjecture 2.1.6]. In particular, it should hold when X is proper and irreducible and ζ is smooth, surjective (it may also be possible to remove the assumption that ζ is smooth). As explained in the introduction, we have $C_K(\mathbb{C}) \subset C_K(\overline{\mathbb{F}}_p)$. Furthermore, it is expected that $\langle C_K(\mathbb{C})\rangle = C_{\mathrm{GS}}$. Hence, Conjecture 1.2.3 predicts the containment $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$ (which is a purely group-theoretical statement). In [IK22, Theorem 6.4.2], we prove $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$ for any arbitrary pair (G,μ), which gives evidence for Conjecture 1.2.3.

2 Automorphic forms in characteristic p

We first work a fixed prime p in sections 2.2, 2.3, 2.5. In section 2.6, we consider objects in families and let p go to infinity.

2.1 Notation

For now, fix a cocharacter datum (G,μ) over $\mathbb{F}_q$, i.e G is a connected, reductive group over $\mathbb{F}_q$ and $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ is a cocharacter, where k is an algebraic closure of $\mathbb{F}_q$. Let (G,P,Q,L,M,φ) be the attached zip datum. For simplicity, assume that there is an $\mathbb{F}_q$-Borel pair (B,T) such that μ factors through T and $B \subset P$ (this can always be achieved after possibly changing μ to a conjugate cocharacter). Then, the group $\mathrm{Gal}(k/\mathbb{F}_q)$ acts naturally on $X^*(T)$. Let $W = W(G_k,T)$ be the Weyl group of G_k. Similarly, $\mathrm{Gal}(k/\mathbb{F}_q)$ acts on W and the actions of $\mathrm{Gal}(k/\mathbb{F}_q)$ and W on $X^*(T)$ and $X_*(T)$ are compatible in a natural sense. For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. The system $(W, \{s_\alpha \mid \alpha \in \Delta\})$ is a Coxeter system. We write $\ell\colon W \to \mathbb{N}$ for the length function, and $\leq$ for the Bruhat order on W. Let w_0 denote the longest element of W. For a subset $K \subset \Delta$, let W_K denote the subgroup of W generated by $\{s_\alpha \mid \alpha \in K\}$. Write $w_{0,K}$ for the longest element in W_K. Let ${}^KW$ (resp. W^K) denote the subset of elements $w \in W$ which have minimal length in the coset $W_K w$ (resp. wW_K). Then ${}^KW$ (resp. W^K) is a set of representatives of $W_K\backslash W$ (resp. W/W_K). The map $g \mapsto g^{-1}$ induces a bijection ${}^KW \to W^K$. The longest element in the set ${}^KW$ (resp. W^K) is $w_{0,K}w_0$ (resp. $w_0w_{0,K}$). For any parabolic $P' \subset G_k$ containing B, write $I_{P'} \subset \Delta$ for the type of P', i.e. the subset of simple roots of the unique Levi subgroup of P' containing T. For an arbitrary parabolic $P' \subset G_k$, let $I_{P'}$ be the type of the unique conjugate of P' containing B. Put $I := I_P$ and $J := I_Q$. We set

$$z = \sigma(w_{0,I})w_0 = w_0w_{0,J}.$$

The triple (B,T,z) is a W-frame, in the terminology of [GK19b, Definition 2.3.1] (we will simply call such a triple a frame). In sections 2.2, 2.3, 2.5, we let X be a projective scheme over $k = \overline{\mathbb{F}}_p$ endowed with a map $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$ satisfying:

Assumption 2.1.1.
(1) *ζ is smooth.*
(2) *The restriction of ζ to any connected component of X is surjective.*

For $\lambda \in X^*(T)$, we write again $\mathcal{V}_I(\lambda)$ for the pullback via ζ of $\mathcal{V}_I(\lambda)$. Write C_X for the set of $\lambda \in X^*(T)$ such that $H^0(X, \mathcal{V}_I(\lambda)) \neq 0$.

2.2 The flag space

The rank of the vector bundle $\mathcal{V}_I(\lambda)$ equals the dimension of the representation $V_I(\lambda)$, which can be very large. For this reason, it convenient to consider line bundles on the flag space of X and of $G\text{-}\mathtt{Zip}^\mu$ instead. We recall the definitions below.

2.2.1 The stack of zip flags

The stack of zip flags ([GK19a, Definition 2.1.1]) is defined as

$$G\text{-}\mathtt{ZipFlag}^\mu = [E\backslash(G_k \times P/B)]$$

where the group E acts on the variety $G_k \times (P/B)$ by the rule $(a,b)\cdot(g,hB) := (agb^{-1}, ahB)$ for all $(a,b) \in E$ and all $(g,hB) \in G_k \times P/B$. The first projection $G_k \times P/B \to G_k$ is E-equivariant, and yields a natural morphism of stacks

$$\pi\colon\ G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$$

whose fibers are isomorphic to P/B. Set $E' := E \cap (B \times G_k)$. The injective map $G_k \to G_k \times P/B;\ g \mapsto (g, B)$ induces an isomorphism of stacks $[E'\backslash G_k] \simeq G\text{-}\mathtt{ZipFlag}^\mu$ (see [GK19a, (2.1.5)]).

2.2.2 Line bundles $\mathcal{V}_{\mathrm{flag}}(\lambda)$

To any character $\lambda \in X^*(T)$, we can naturally attach a line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $G\text{-}\mathtt{ZipFlag}^\mu$. Indeed, we may view λ as a character of E' via the first projection $E' \to B$ and use the associated sheaf construction for the quotient stack $[E'\backslash G_k]$. We have by [IK21b, Proposition 3.2.1]:

$$\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda).$$

In particular, we have an identification

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathrm{flag}}(\lambda)). \tag{2.2.1}$$

The line bundles $\mathcal{V}_{\mathrm{flag}}(\lambda)$ satisfy the following identity:

$$\mathcal{V}_{\mathrm{flag}}(\lambda + \lambda') = \mathcal{V}_{\mathrm{flag}}(\lambda) \otimes \mathcal{V}_{\mathrm{flag}}(\lambda'), \quad \forall \lambda, \lambda' \in X^*(T).$$

In particular, this identity combined with the identification (2.2.1) shows that C_{zip} is stable by sum, hence is indeed a subcone of $X^*(T)$.

2.2.3 Flag stratification

Another important feature of $G\text{-}\mathtt{ZipFlag}^\mu$ is that it carries a locally closed stratification $(\mathcal{F}_w)_{w\in W}$. First, define the Schubert stack as the quotient stack

$$\mathrm{Sbt} := [B\backslash G_k/B].$$

The underlying topological space of Sbt is homeomorphic to W, endowed with the topology induced by the Bruhat order on W. This follows easily from the Bruhat decomposition of G. There is a smooth, surjective map of stacks

$$\psi\colon\ G\text{-}\mathtt{ZipFlag}^\mu \to \mathrm{Sbt}\,. \tag{2.2.2}$$

It is defined as follows: Since the group E' is contained in $B \times {}^zB$, we have a natural projection map $[E'\backslash G_k] \to [B\backslash G_k/{}^zB]$. Composing this map with the isomorphism $[B\backslash G_k/{}^zB] \to [B\backslash G_k/B]$ induced by $G_k \to G_k$; $g \mapsto gz$, we obtain the map ψ in (2.2.2). For $w \in W$, put $\mathrm{Sbt}_w := [B\backslash BwB/B]$, it is a locally closed substack of Sbt. The flag strata of $G\text{-}\mathtt{ZipFlag}^\mu$ are defined as the fibers of ψ. Specifically, for $w \in W$ put:

$$F_w := B(wz^{-1})^zB = BwBz^{-1}.$$

Then F_w is locally closed in G_k of dimension $\dim(F_w) = \ell(w) + \dim(B)$. Via the isomorphism $G\text{-}\mathtt{ZipFlag}^\mu \simeq [E'\backslash G_k]$, the flag strata of $G\text{-}\mathtt{ZipFlag}^\mu$ are the locally closed substacks

$$\mathcal{F}_w := [E'\backslash F_w], \quad w \in W.$$

The set $F_{w_0} \subset G_k$ is open in G_k and similarly the stratum $\mathcal{F}_{w_0}$ is open in $G\text{-}\mathtt{ZipFlag}^\mu$. The Zariski closure $\overline{F}_w$ is normal by [RR85, Theorem 3] and coincides with $\bigcup_{w' \leq w} F_{w'}$.

2.2.4 The flag space of X

Define the flag space $Y := \mathrm{Flag}(X)$ of X as the fiber product

$$\begin{array}{ccc} \mathrm{Flag}(X) & \xrightarrow{\zeta_{\mathrm{flag}}} & G\text{-}\mathtt{ZipFlag}^\mu \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\zeta]{} & G\text{-}\mathtt{Zip}^\mu . \end{array}$$

For $w \in W$, put $Y_w := \zeta_{\mathrm{flag}}^{-1}(\mathcal{F}_w)$. We obtain on Y a similar stratification by locally closed, smooth subschemes. For $\lambda \in X^*(T)$, we denote again by $\mathcal{V}_{\mathrm{flag}}(\lambda)$ the pullback of the line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ via ζ_{flag}. Similarly to $G\text{-}\mathtt{Zip}^\mu$, we have the formula $\pi_{X,*}(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. In particular, we have an identification

$$H^0(X, \mathcal{V}_I(\lambda)) = H^0(Y, \mathcal{V}_{\mathrm{flag}}(\lambda)). \tag{2.2.3}$$

2.3 Hasse cones of flag strata

To a pair of characters $(\lambda, \nu) \in X^*(T) \times X^*(T)$, we can attach a line bundle $\mathcal{V}_{\mathrm{Sbt}}(\lambda, \nu)$ on the stack Sbt, as in [GK19a, I.2.2] (where it was denoted by $\mathcal{L}_{\mathrm{Sbt}}(\lambda, \nu)$). For each $w \in W$, the space $H^0(\mathrm{Sbt}_w, \mathcal{V}_{\mathrm{Sbt}}(\lambda, \nu))$ has dimension ≤ 1 and is nonzero if and only if $\nu = -w^{-1}\lambda$ (*loc. cit.*, Theorem 2.2.1). For each $w \in W$ and $\lambda \in X^*(T)$, denote by $f_{w,\lambda}$ a nonzero element of the one-dimensional space $H^0(\mathrm{Sbt}_w, \mathcal{V}_{\mathrm{Sbt}}(\lambda, -w^{-1}\lambda))$. Put

$$E_w := \{\alpha \in \Phi_+ \mid ws_\alpha < w \text{ and } \ell(ws_\alpha) = \ell(w) - 1\}. \tag{2.3.1}$$

Elements $w' \in W$ such that $w' < w$ and $\ell(w') = \ell(w) - 1$ will be called lower neighbours of w. They correspond bijectively to the set E_w by the map $\alpha \mapsto ws_\alpha$. Define $X^*_{+,w}(T) \subset X^*(T)$ as the subset of $\chi \in X^*(T)$ such that $\langle \chi, \alpha^\vee \rangle \geq 0$ for all $\alpha \in E_w$. Let $\chi \in X^*(T)$. By *loc. cit.*, Theorem 2.2.1, the multiplicity of $\mathrm{div}(f_{w,-w\chi})$ along Sbt_{ws_α} is precisely $\langle \chi, \alpha^\vee \rangle$ for all $\alpha \in E_w$. Hence $f_{w,-w\chi}$ extends to the Zariski closure $\overline{\mathrm{Sbt}}_w$ if and only if $\chi \in X^*_{+,w}(T)$. For any $\lambda, \nu \in X^*(T)$, one has the formula

$$\psi^*(\mathcal{V}_{\mathrm{Sbt}}(\lambda, \nu)) = \mathcal{V}_{\mathrm{flag}}(\lambda + qw_{0,I}w_0\sigma^{-1}(\nu))$$

by [GK19a, Lemma 3.1.1 (b)] (note that *loc. cit.* contains a typo; it should be σ^{-1} instead of σ). In particular, the pullback $\psi^*(\mathcal{V}_{\mathrm{Sbt}}(\lambda, -w^{-1}\lambda))$ coincides with $\mathcal{V}_{\mathrm{flag}}(\lambda - qw_{0,I}w_0\sigma^{-1}(w^{-1}\lambda))$. Define a map

$$h_w \colon X^*(T) \to X^*(T), \quad \chi \mapsto -w\chi + qw_{0,I}w_0\sigma^{-1}(\chi).$$

Hence $\psi^*(\mathcal{V}_{\mathrm{Sbt}}(-w\lambda, \lambda)) = \mathcal{V}_{\mathrm{flag}}(h_w(\lambda))$. Note that for any $w \in W$, the map $h_w \colon X^*(T) \to X^*(T)$ induces an automorphism of $X^*(T)_{\mathbb{Q}}$ (because $h_w \otimes \mathbb{F}_p$ is clearly an automorphism of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{F}_p$). For each $\chi \in X^*(T)$, define

$$\mathrm{Ha}_{w,\chi} := \psi^*(f_{w,-w\chi}).$$

By the above discussion, $\mathrm{Ha}_{w,\chi}$ is a section over the stratum $\mathcal{F}_w$ of the line bundle $\mathcal{V}_{\mathrm{flag}}(h_w(\chi))$ and $\mathrm{Ha}_{w,\chi}$ extends to $\overline{\mathcal{F}}_w$ if and only if $\chi \in X^*_{+,w}(T)$. The multiplicity of $\mathrm{div}(\mathrm{Ha}_{w,\chi})$ along $\overline{\mathcal{F}}_{ws_\alpha}$ is precisely $\langle \chi, \alpha^\vee \rangle$ for all $\alpha \in E_w$. Define the Hasse cone $C_{\mathrm{Hasse},w}$ by

$$C_{\mathrm{Hasse},w} := h_w(X^*_{+,w}(T)).$$

Concretely, $C_{\mathrm{Hasse},w}$ is the set of all possible weights $\lambda \in X^*(T)$ of nonzero sections over $\overline{\mathcal{F}}_w$ which arise by pullback from $\overline{\mathrm{Sbt}}_w$.

2.4 Regularity of strata

In general, there exist many sections on $\overline{\mathcal{F}}_w$ that do not arise by pullback from $\overline{\mathrm{Sbt}}_w$. For $w \in W$, define the cones $C_{\mathrm{flag},w}$ and $C_{Y,w}$ as follows:

$$C_{\mathrm{flag},w} := \{\lambda \in X^*(T) \;\mid\; H^0(\overline{\mathcal{F}}_w, \mathcal{V}_{\mathrm{flag}}(\lambda)) \neq 0\}$$
$$C_{Y,w} := \left\{\lambda \in X^*(T) \;\middle|\; H^0(\overline{Y}_w, \mathcal{V}_{\mathrm{flag}}(\lambda)) \neq 0\right\}.$$

In particular, via the identification (2.2.3), the cone C_{Y,w_0} is the set of $\lambda \in X^*(T)$ such that $\mathcal{V}_I(\lambda)$ admits nonzero sections over X, hence we have an equality $C_{Y,w_0} = C_X$ and similarly $C_{\mathrm{flag},w_0} = C_{\mathrm{zip}}$. For any $w \in W$, we clearly have

$$C_{\mathrm{Hasse},w} \subset C_{\mathrm{flag},w} \subset C_{Y,w}.$$

Definition 2.4.1. *Let $w \in W$.*
(a) *We say that Y_w is Hasse-regular if $\langle C_{Y,w} \rangle = \langle C_{\mathrm{Hasse},w} \rangle$.*
(b) *We say that Y_w is flag-regular if $\langle C_{Y,w} \rangle = \langle C_{\mathrm{flag},w} \rangle$.*

A Hasse-regular stratum is obviously flag-regular. Assumptions 2.1.1 are made so that the following easy lemma holds:

Lemma 2.4.2 ([GK18, Proposition 3.2.1])**.** *If $\ell(w) = 1$, then Y_w is Hasse-regular.*

Since $C_{Y,w_0} = C_X$ and $C_{\mathcal{Y},w_0} = C_{\mathrm{zip}}$, Conjecture 1.2.3 asserts that the maximal flag stratum Y_{w_0} is always flag-regular. It is not Hasse-regular in general (but it is conjecturally Hasse-regular for Hasse-type zip data, see [IK22]). In the case of Hilbert–Blumenthal Shimura varieties attached to a totally real extension $\mathbf{F}/\mathbb{Q}$, a sufficent condition for the Hasse-regularity of strata is given in [GK18, Theorem 4.2.3]. When p is split in $\mathbf{F}$, all strata are Hasse-regular. For a general prime p, the criterion involves the parity of "jumps" in the orbit under the Galois action. A more elegant proof, using the notion of "intersection cone" (introduced in [GK22]) can be found in the unpublished note [Kos22].

Let $w \in W$ with $\ell(w) = 1$, and write $w = s_\beta$ with $\beta \in \Delta$. One checks readily:

$$\langle C_{\mathrm{Hasse},w} \rangle = \{\lambda \in X^*(T) \;\mid\; \langle h_w^{-1}(\lambda), \beta^\vee \rangle \geq 0\}.$$

We deduce:

Proposition 2.4.3. *Let $f \in H^0(Y, \mathcal{V}_{\mathrm{flag}}(\lambda))$ such that the restriction of f to the stratum Y_w is not identically zero, where $w = s_\beta$ ($\beta \in \Delta$). Then we have $\langle h_{s_\beta}^{-1}(\lambda), \beta^\vee \rangle \geq 0$.*

2.5 Upper bounds for strata cones

2.5.1 Intersection cones

We recall the notion of intersection cone introduced in [GK22], which will be used in section 4. We give a simplified version of the one appearing in *loc. cit.* which suffices for our purpose.

Definition 2.5.1. *For each $w \in W$, let $\mathbb{E}_w \subset E_w$ be a subset (possibly empty) and let $\{\chi_\alpha\}_{\alpha\in\mathbb{E}_w}$ be a family of characters satisfying the conditions:*
(a) $\langle \chi_\alpha, \alpha^\vee \rangle > 0$,
(b) $\langle \chi_\alpha, \beta^\vee \rangle = 0$ *for all* $\beta \in E_w \setminus \{\alpha\}$.
We call $\mathbb{E} = (\mathbb{E}_w)_{w\in W}$ a separating system.

We fix such a system $\mathbb{E}$ and define the intersection cones $(C_w^{+,\mathbb{E}})_{w\in W}$ of $\mathbb{E}$ as follows. First, set

$$\Gamma_w := \sum_{\alpha\in\mathbb{E}_w} \mathbb{Z}_{\geq 0}\chi_\alpha$$
$$C^{\mathbb{E}}_{\mathrm{Hasse},w} := h_w(\Gamma_w).$$

Note that $\chi_\alpha \in X^*_{+,w}(T)$, therefore $\Gamma_w \subset C_{\mathrm{Hasse},w}$, but Γ_w can be much smaller (for example, if we choose $\mathbb{E}_w$ to be a singleton, Γ_w is a half-line in $X^*(T)$.

Definition 2.5.2. *For $\ell(w) = 1$, set $C_w^{+,\mathbb{E}} := C_{\mathrm{Hasse},w}$. For $\ell(w) \geq 2$, define inductively*

$$C_w^{+,\mathbb{E}} := C^{\mathbb{E}}_{\mathrm{Hasse},w} + \bigcap_{\alpha\in\mathbb{E}_w} C^{+,\mathbb{E}}_{ws_\alpha}.$$

In the case $\mathbb{E}_w = \emptyset$, we define by convention $\bigcap_{\alpha\in\mathbb{E}_w} C^{+,\mathbb{E}}_{ws_\alpha} = X^*(T)$.

The intersection cones provide upper bounds for the strata cones $C_{Y,w}$. Specifically, by [GK22, Theorem 2.3.9], we have:

Theorem 2.5.3. *Let $\mathbb{E}$ be a separating system. For each $w \in W$, we have*

$$C_{Y,w} \subset \langle C_w^{+,\mathbb{E}} \rangle.$$

2.5.2 Upper bound by degree

In general, we do not know a way to construct nontrivial separating systems $\mathbb{E}$ for arbitrary reductive groups. For a given $w \in W$ and $\alpha \in E_w$, there may not always exist a character χ_α satisfying the conditions explained in section 2.5.1. Here, we explain a more straightforward method to produce an upper bound for $C_{Y,w}$. The advantage of this method is that it applies in general. However, it only gives a rather coarse upper bound (but it will be sufficient for our purpose).

Since $h_w\colon X^*(T)_\mathbb{Q} \to X^*(T)_\mathbb{Q}$ is an automorphism, there exists $N \geq 1$ such that $NX^*(T) \subset h_w(X^*(T))$. We fix such an integer. For $\lambda \in X^*(T)$, let $\chi_{w,\lambda} := h_w^{-1}(N\lambda)$ and write $\mathrm{Ha}_w^\lambda := \mathrm{Ha}_{w,\chi_{w,\lambda}}$ for the associated Hasse section on $\mathcal{F}_w$ and Y_w, with weight $N\lambda$. Since the map $\zeta_{\mathrm{flag}}\colon Y \to G\text{-}\mathtt{ZipFlag}^\mu$ is smooth and surjective, the multiplicities of sections do not change under pullback. Hence, the divisor of Ha_w^λ over Y_w is given by:

$$\mathrm{div}(\mathrm{Ha}_w^\lambda) = \sum_{\alpha\in E_w} \langle \chi_{w,\lambda}, \alpha^\vee \rangle [\overline{Y}_{ws_\alpha}].$$

Define

$$\deg(w,\lambda) := \frac{1}{N}\deg(\operatorname{div}(\mathrm{Ha}_w^{\lambda})) = \sum_{\alpha\in E_w} \langle h_w^{-1}(\lambda), \alpha^\vee\rangle.$$

We write $\deg_q(w,\lambda)$ when we want to emphasize that the degree depends on the prime power q (since the map h_w itself depends on q). Since $\mathrm{Ha}_{w,\lambda+\lambda'} = \mathrm{Ha}_{w,\lambda}\cdot \mathrm{Ha}_{w,\lambda'}$, we have

$$\deg(w,\lambda+\lambda') = \deg(w,\lambda) + \deg(w,\lambda').$$

Lemma 2.5.4. *Let $w \in W$ of length ≥ 1. Suppose that the space $H^0(\overline{Y}_w, \mathcal{V}_{\mathrm{flag}}(\lambda))$ is nonzero. Then we have* $\deg(w,\lambda) \geq 0$.

Proof. Let f be a nonzero section on $\overline{Y}_w$ of weight λ. Then $f^N/\mathrm{Ha}_w^{\lambda}$ is a rational section of $\mathcal{O}_Y$ over Y_w. Since $\overline{Y}_w$ is projective and normal, we have $\deg(\operatorname{div}(f^N/\mathrm{Ha}_w^{\lambda})) = 0$, hence $\deg(\operatorname{div}(f)) = \frac{1}{N}\deg(\operatorname{div}(\mathrm{Ha}_w^{\lambda})) = \deg(w,\lambda)$. Since $\operatorname{div}(f)$ is effective, the result follows. □

Define $C_w^{\deg} := \{\lambda \in X^*(T) \mid \deg(w,\lambda) \geq 0\}$. As a consequence, we deduce:

Corollary 2.5.5. *We have $\langle C_{Y,w}\rangle \subset C_w^{\deg}$.*

In other words, if $\deg(w,\lambda) < 0$, then the space $H^0(\overline{Y}_w, \mathcal{V}_{\mathrm{flag}}(\lambda))$ is zero. We will apply this result when $p = q$ tends to infinity. Therefore, we need to know the behaviour of the function $\deg_q(w,\lambda)$ as q varies. By [GK19a, Lemma 3.1.3], $h_w^{-1}(\lambda)$ is an expression of the form $-\frac{1}{q^m-1}\sum_{i=0}^{m-1} q^i u_i \sigma^i(\lambda)$ for certain elements $u_i \in W$ independent of q. Furthermore, for $i = m-1$, the element $u_{m-1}\sigma^{m-1}(\lambda)$ equals $\sigma(w_0 w_{0,I}\lambda)$. We deduce:

Proposition 2.5.6. *There exists an integer $m \geq 1$ such that*

$$\deg_q(w,\lambda) = \frac{1}{q^m-1}\left(q^{m-1}\sum_{\alpha\in E_w}\langle \sigma(w_{0,I}w_0\lambda), \alpha^\vee\rangle \ + \ \text{lower terms}\right)$$

2.6 Vanishing in families

In this section we take X to be a scheme over R satisfying Assumption 1.2.1 (for example $X = \mathscr{S}_K$). By flat base change along the map $\operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(R)$, we have $H^0(X\otimes_R \mathbb{C}, \mathcal{V}_I(\lambda)) = H^0(X, \mathcal{V}_I(\lambda)) \otimes_R \mathbb{C}$. Hence, for $\lambda \in C_X(\mathbb{C})$ the space $H^0(X, \mathcal{V}_I(\lambda))$ is also nonzero. Therefore, we can apply the proof of [Kos19, Proposition 1.8.3] to show that the space $H^0(X\otimes_R \overline{\mathbb{F}}_p, \mathcal{V}_I(\lambda))$ is also nonzero for all p. In particular, we deduce:

$$C_X(\mathbb{C}) \subset C_X(\overline{\mathbb{F}}_p) \tag{2.6.1}$$

for all primes where X_p is defined. The main goal of this section is to show $C_X(\mathbb{C}) \subset C_{\mathrm{GS}}$. We may interpret this as a vanishing result (the space $H^0(X\otimes_R, \mathcal{V}_I(\lambda))$ vanishes for λ outside of C_{GS}). We will see later some stronger forms of vanishing results at fixed prime p.

Let f be a nonzero section of $\mathcal{V}_I(\lambda)$ over X. We will show that the weight λ is in C_{GS} by exploiting the fact that f gives rise to a family $(f_p)_p$, where f_p is the reduction of f to the subscheme $X_p = X\otimes_R \overline{\mathbb{F}}_p$. For sufficently large p, we have by assumption a map $\zeta_p\colon X_p \to G_p\text{-}\mathtt{Zip}^{\mu}$. Denote by Y_p the flag space of X_p as in 2.2.4.

Theorem 2.6.1. *For sufficiently large p, the section f_p restricts to a nonzero section on each flag stratum $Y_{p,w}$ (for $w \in W$).*

Proof. Clearly, it suffices to show that f_p restricts to a nonzero section on the zero-dimensional stratum for sufficiently large p. For this, we will prove by decreasing induction that for each $0 \leq i \leq \ell(w_0)$, there exists an element w_i of of length i in W such that f_p is not identically zero on S_{w_i} for sufficiently large p. The result is clear for $i = \ell(w_0)$. Suppose that f_p is nonzero on S_{w_i} for large p. For a contradiction, assume that f_p is zero on each stratum in the closure of S_{w_i} for infinitely many primes p. Choose any character $\chi \in X^*(T)$ such that $\langle \chi, \alpha^\vee \rangle \geq 0$ for all $\alpha \in E_{w_i}$ and $\langle \chi, \alpha_0^\vee \rangle > 0$ for at least one $\alpha_0 \in E_{w_i}$. The multiplicities of the divisor of $\mathrm{Ha}_{w_i,\chi}$ are the numbers $\langle \chi, \alpha^\vee \rangle$ (for $\alpha \in E_{w_i}$). Hence, by assumption we can find an integer m (independent of p) such that for infinitely many primes p, the section f_p^m is divisible by $\mathrm{Ha}_{w_i,\chi}$. Thus, we deduce that for infinitely many primes p,

$$\deg_p(w_i, m\lambda - h_{w_i,p}(\chi)) = m \deg_p(w_i, \lambda) - \sum_{\alpha \in E_w} \langle \chi, \alpha^\vee \rangle \geq 0$$

When p tends to infinity, the expression $\deg_p(w_i, \lambda)$ tends to zero by Proposition 2.5.6. Since $\langle \chi, \alpha_0^\vee \rangle > 0$ for at least one $\alpha_0 \in E_{w_i}$, we have a contradiction. The result follows. □

Remark 2.6.2. In this remark, we consider the case $X = \mathscr{S}_K$. Theorem 2.6.1 is related to Deuring's theorem regarding the superspecial reduction of abelian varieties. Indeed, assume the following result: any CM abelian variety over $\overline{\mathbb{Q}}$ has superspecial reduction for infinitely many primes p. Then a slightly weaker variant of Theorem 2.6.1 would follow immediately (at least for the Siegel-type Shimura variety $\mathcal{A}_g$) as follows: Since CM points are dense, we may choose a CM point $x \in \mathscr{S}_K(\overline{\mathbb{Q}})$ such that $f(x) \neq 0$. Then, for all p sufficently large, we must have $f_p(x_p) \neq 0$ where x_p denotes the specialization of x (which is well-defined for large p). Since x_p lies in the zero-dimensional stratum for infinitely many primes, f_p is nonzero on the zero-dimensional stratum (hence on all strata) for inifitely many primes p. This is slightly weaker than the content of Theorem 2.6.1, which states the same result for sufficiently large p.

Proposition 2.6.3. *Let $f \in H^0(X, \mathcal{V}_I(\lambda))$. Suppose that for infinitely many primes p, the section f_p (viewed as a section of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on the flag space Y_p) restricts to a nonzero section on each flag stratum of Y_p of length one. Then $\lambda \in C_{\mathrm{GS}}$.*

Proof. By Proposition 2.4.3, we have $\langle h_{s_\beta,p}^{-1}(\lambda), \beta^\vee \rangle \geq 0$ for all $\beta \in \Delta$ and infinitely many primes p. Looking at the leading term, we obtain $\langle \sigma(w_{0,I} w_0 \lambda), \beta^\vee \rangle \geq 0$ for all $\beta \in \Delta$. Since $\sigma(\Delta) = \Delta$, we deduce that $w_{0,I} w_0 \lambda$ is a dominant character. In other words, $\lambda \in C_{\mathrm{GS}}$. □

We deduce immediately from Theorem 2.6.1 and Proposition 2.6.3 our main result of this section:

Theorem 2.6.4. *We have $C_X(\mathbb{C}) \subset C_{\mathrm{GS}}$.*

In particular for $X = \mathscr{S}_K$, we obtain $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$. We now explain a slighty more precise result.

Definition 2.6.5. *We say that a family of cones $(C_p)_p$ (defined for sufficiently large primes p) is asymptotic to C_{GS} if*

$$\bigcap_{p \geq N} C_p = C_{\mathrm{GS}}$$

for any $N \geq 1$. We say that $(C_p)_p$ is asymptotically contained in C_{GS} if $\bigcap_{p \geq N} C_p \subset C_{\mathrm{GS}}$ for all $N \geq 1$.

The proof of Theorem 2.6.4 actually shows the following:

Corollary 2.6.6. *The family of cones $(C_{X,p})_p$ is asymptotically contained in C_{GS}.*

Proof. Let $\lambda \in \bigcap_{p\geq N} C_{X,p}$. For sufficiently large p, there exists a nonzero form f_p over Y_p of weight λ. Then, we may apply the proof of Theorem 2.6.1 to the family $(f_p)_p$ (even if this family does not arise by reduction from a characteristic zero section). It shows that $\lambda \in C_{\mathrm{GS}}$. The result follows. □

However, we were not able to show in general that the family of saturated cones $(\langle C_{K,p}\rangle)_p$ is asymptotically contained in C_{GS}. Corollary 2.6.6 is slightly more precise than Theorem 2.5.3, since it implies $C_X(\mathbb{C}) \subset \bigcap_p C_{X,p} \subset C_{\mathrm{GS}}$ using (2.6.1). The proof of Theorem 2.6.4 explained above crucially uses the fact that we have a family of schemes $(X_p)_p$ for almost all prime numbers p. However, the proof gives no information about the set $C_X(\overline{\mathbb{F}}_p)$ for a fixed prime number p. Eventually, we are interested in vanishing results for automorphic forms in both characteristics. Therefore, a more desirable method of proof of Theorem 2.6.4 is the following: Assume that for each p, we can show that any weight $\lambda \in C_{K,p} := C_K(\overline{\mathbb{F}}_p)$ satisfies certain inequalities

$$\gamma_i(p,\lambda) \leq 0, \quad i = 1, \dots N. \tag{2.6.2}$$

where $\gamma_i(p,\lambda)$ is an algebraic expression involving p and which is linear in λ. Denote by $C_{\gamma,p}$ the cone of $\lambda \in X^*_{+,I}(T)$ satisfying the inequalities (2.6.2). By assumption, we have $C_{K,p} \subset C_{\gamma,p}$ (note that since $C_{\gamma,p}$ is defined by inequalities, it is obviously saturated, hence we also have $\langle C_{K,p}\rangle \subset C_{\gamma,p}$). We deduce:

$$C_K(\mathbb{C}) \subset \bigcap_{p>>0} C_{K,p} \subset \bigcap_{p>>0} C_{\gamma,p}.$$

Therefore, if we can choose $(\gamma_i)_{1\leq i\leq N}$ such that $\bigcap_{p>>0} C_{\gamma,p} = C_{\mathrm{GS}}$, we obtain the desired containment $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$. We call such a family $(\gamma_i)_{i=1,\dots,N}$ a GS-approximation of the family $(C_{K,p})_p$. This method of proof gives much more control and information on the weights of automorphic forms in all characteristics. We will implement such a strategy in the next section. In general, it is a difficult problem to give an upper bound for the cone $C_{K,p}$ at a fixed prime p, let alone construct a GS-approximation for the family $(C_{K,p})_p$. We will do this for unitary Shimura varieties of signature $(n-1,1)$.

3 Vanishing results for G-$\mathtt{Zip}^{\mu}$

We investigate the strategy explained in section 2.6. Recall that we work at a fixed prime number p and want to show that there exists certain suitable algebraic expressions $(\gamma_i)_{i=1,\dots,N}$ satisfying $C_{K,p} \subset C_{\gamma,p}$. However, we also keep in mind that when p varies, we want the condition $\bigcap_{p>>0} C_{\gamma,p} = C_{\mathrm{GS}}$ to be satisfied.

Write $C_{\mathrm{zip},p}$ for the zip cone of (G_p,μ_p). Since $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip},p}\rangle \subset \langle C_{K,p}\rangle$, the family $(\gamma_i)_{i=1,\dots,N}$ would also be a GS-approximation of the family $(C_{\mathrm{zip},p})_p$. For this reason, we first seek a GS-approximation of the family $(C_{\mathrm{zip},p})_p$ to gain intuition, which is a more tractable, group-theoretical object. We will give a natural and explicit GS-approximation of $(C_{\mathrm{zip},p})_p$ in certain cases (including all cases when G is split over $\mathbb{F}_p$). In the unitary split case of signature $(n-1,1)$, we show in section 4 that this also provides a GS-approximation of the Shimura cone family $(C_{K,p})_p$.

3.1 Group-theoretical preliminaries

Let (G,μ) be a cocharacter datum over $\mathbb{F}_q$ (as usual, we take $q=p$ for Shimura varieties). Let $\mathcal{Z}=(G,P,Q,L,M,\varphi)$ be the attached zip datum (see section 1.1.3). Choose a frame (B,T,z), with (B,T) defined over $\mathbb{F}_q$ as in section 2.1 and $z=\sigma(w_{0,I})w_0$. Define $B_M:=B\cap M$. We first explain that we can naturally inject the space of global sections $H^0(G\text{-}\mathtt{Zip}^\mu,\mathcal{V}_I(\lambda))$ into a space of regular maps $B_M\to\mathbb{A}^1$ which are eigenfunctions for a certain action of T on B_M. We recall some results from [IK22]. Recall that $H^0(G\text{-}\mathtt{Zip}^\mu,\mathcal{V}_I(\lambda))$ identifies with $H^0(G\text{-}\mathtt{ZipFlag}^\mu,\mathcal{V}_{\mathrm{flag}}(\lambda))$ by (2.2.1). Furthermore, using the isomorphism $G\text{-}\mathtt{ZipFlag}^\mu\simeq[E'\backslash G_k]$ (see section 2.2), an element of the space $H^0(G\text{-}\mathtt{ZipFlag}^\mu,\mathcal{V}_{\mathrm{flag}}(\lambda))$ can be viewed as a function $f\colon G_k\to\mathbb{A}^1_k$ satisfying

$$f(agb^{-1})=\lambda(a)f(g),\quad \forall(a,b)\in E',\ \forall g\in G_k. \tag{3.1.1}$$

Recall that $G\text{-}\mathtt{ZipFlag}^\mu$ admits a unique open stratum $\mathcal{U}_{\max}=\mathcal{F}_{w_0}$. Write also $U_{\max}:=F_{w_0}=Bw_0Bz^{-1}$ (the $B\times{}^zB$-orbit of $w_0z^{-1}=\sigma(w_{0,I})^{-1}$).

Lemma 3.1.1 ([IK22, Lemma 4.2.1]). *The map $B_M\to U_{\max}$, $b\mapsto\sigma(w_{0,I})b^{-1}$ induces an isomorphism $[B_M/T]\simeq\mathcal{U}_{\max}$, where T acts on B_M on the right by the action $B_M\times T\to B_M$ given by $(b,t)\mapsto\varphi(t)^{-1}b\sigma(w_{0,I})t\sigma(w_{0,I})^{-1}$.*

For $\lambda\in X^*(T)$, let $S(\lambda)$ denote the space of functions $h\colon B_M\to\mathbb{A}^1$ satisfying

$$h(\varphi(t)^{-1}b\sigma(w_{0,I})t\sigma(w_{0,I})^{-1})=\lambda(t)^{-1}h(b),\quad \forall t\in T,\ \forall b\in B_M.$$

Corollary 3.1.2. *The isomorphism from Lemma 3.1.1 induces an isomorphism*

$$\vartheta\colon H^0(\mathcal{U}_{\max},\mathcal{V}_{\mathrm{flag}}(\lambda))\to S(\lambda).$$

We describe explicitly this isomorphism. Let $f\in H^0(\mathcal{U}_{\max},\mathcal{V}_{\mathrm{flag}}(\lambda))$, viewed as a function $f\colon U_{\max}\to\mathbb{A}^1$ satisfying (3.1.1). The corresponding element $\vartheta(f)\in S(\lambda)$ is the function $B_M\to\mathbb{A}^1$; $b\mapsto f(\sigma(w_{0,I})b^{-1})$. Conversely, if $h\colon B_M\to\mathbb{A}^1$ is an element of $S(\lambda)$, the function $f=\vartheta^{-1}(h)$ is given by

$$f(b_1\sigma(w_{0,I})b_2^{-1})=\lambda(b_1)h(\varphi(\theta^P_L(b_1))^{-1}\theta^Q_M(b_2)),\quad (b_1,b_2)\in B\times{}^zB,$$

where the functions θ^P_L and θ^Q_M were defined in section 1.1.3. By the property of h, the function f is well-defined.

Given a section of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ over $G\text{-}\mathtt{ZipFlag}^\mu$, we can restrict it to the open substack $\mathcal{U}_{\max}$, and then apply ϑ to obtain an element of $S(\lambda)$. Hence, we may view $H^0(G\text{-}\mathtt{Zip}^\mu,\mathcal{V}_I(\lambda))$ as a subspace of $S(\lambda)$. In general, it is difficult to determine the image of this map. On the other hand, by the previous discussion, a section $f\in H^0(G\text{-}\mathtt{Zip}^\mu,\mathcal{V}_I(\lambda))$ can be viewed as a regular function $f\colon G\to\mathbb{A}^1$ satisfying condition (3.1.1). In particular, f is equivariant under the action of the unipotent subgroup $U\times V\subset E'$. Denote by S_{unip} the space of such functions:

$$S_{\mathrm{unip}}:=\left\{f\colon G\to\mathbb{A}^1\ \middle|\ f(ugv)=f(g),\ u\in U,\ v\in V\right\}.$$

Hence, we may also view $H^0(G\text{-}\mathtt{Zip}^\mu,\mathcal{V}_I(\lambda))$ as a subspace of S_{unip}. The reason for introducing this space is the following: We will see in the next section that elements of S_{unip} can be conveniently decomposed with respect to the action of $T\times T$ on G.

Finally, for $f\in S_{\mathrm{unip}}$, we define $\widetilde{f}\colon B_M\to\mathbb{A}^1$ by $\widetilde{f}(b)=f(w_{0,M}b^{-1})$. We write again ϑ for the map $S_{\mathrm{unip}}\to k[B_M]$, $f\mapsto\widetilde{f}$ (this map is not injective in general). By construction, the following diagram is clearly commutative.

$$\begin{array}{ccc} H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) & \xhookrightarrow{\vartheta} & S(\lambda) \\ \downarrow & & \downarrow \\ S_{\mathrm{unip}} & \xrightarrow[\vartheta]{} & k[B_M] \end{array}$$

In particular, we deduce that if we view a nonzero element $f \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\lambda))$ as an element of S_{unip} and then apply $\vartheta\colon S_{\mathrm{unip}} \to k[B_M]$, the result is nonzero. This will imply that when we decompose f in S_{unip} as a sum of $T \times T$-eigenvectors, at least one of the components of f will map via ϑ to a nonzero element of $k[B_M]$.

Next, we choose coordinates for the Borel subgroup B_M of M. This can be accomplished using the following result. For $\alpha \in \Phi$, let U_α be the corresponding α-root group. Recall that by our convention, $\alpha \in \Phi_+$ when U_α is contained in the opposite Borel B^+ to B.

Proposition 3.1.3 ([ABD+66, XXII, Proposition 5.5.1])**.** *Let G be a reductive group over k and let (B, T) be a Borel pair. Choose a total order on Φ_-. The k-morphism*

$$\gamma\colon T \times \prod_{\alpha \in \Phi_-} U_\alpha \to G \tag{3.1.2}$$

defined by taking the product with respect to the chosen order is a closed immersion with image B.

We apply Proposition 3.1.3 to (M, B_M). Choose an order on $\Phi_{M,-}$ and consider the corresponding map γ as in (3.1.2), with image B_M. For a function $h\colon B_M \to \mathbb{A}^1$, put $P_h := h \circ \gamma$. For all $\alpha \in \Phi$, choose an isomorphism $u_\alpha\colon \mathbb{G}_{\mathrm{a}} \to U_\alpha$ so that $(u_\alpha)_{\alpha \in \Phi}$ is a realization in the sense of [Spr98, 8.1.4]. In particular, we have

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad \forall x \in \mathbb{G}_{\mathrm{a}},\ \forall t \in T.$$

Via the isomorphism $u_\alpha\colon \mathbb{G}_a \to U_\alpha$, we can view P_h as a polynomial $P_h \in k[T][(x_\alpha)_{\alpha \in \Phi_{M,-}}]$, where the x_α are indeterminates indexed by $\Phi_{M,-}$. For $m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi_{M,-}}$ and $\xi \in X^*(T)$, denote by $P_{m,\xi}$ the monomial

$$P_{m,\xi} = \lambda(t) \prod_{\alpha \in \Phi_{M,-}} x_\alpha^{m_\alpha} \ \in \ k[T][(x_\alpha)_{\alpha \in \Phi_{M,-}}].$$

We can write any element P of $k[T][(x_\alpha)_{\alpha \in \Phi_{M,-}}]$ as a sum of monomials

$$P = \sum_{i=1}^{N} c_i P_{m_i, \xi_i} \tag{3.1.3}$$

where for all $1 \leq i \leq N$, we have $m_i \in \mathbb{N}^{\Phi_{M,-}}$, $\xi_i \in X^*(T)$ and $c_i \in k$. Furthermore, we may assume that the (m_i, ξ_i) are pairwise distinct. Under this assumption, the expression (3.1.3) is uniquely determined up to permutation of the indices. For $P \in k[T][(x_\alpha)_{\alpha \in \Phi_{M,-}}]$, define $h_P\colon B_L \to \mathbb{A}^1$ as the function $P \circ \gamma^{-1}$. For $m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi_{L,-}}$ and $\xi \in X^*(T)$, define $h_{m,\xi} := h_{P_{m,\lambda}}$.

Decompose $k[B_M]$ with respect to the action of $T \times T$ on B_M:

$$k[B_M] = \bigoplus_{(\chi_1, \chi_2)} k[B_M]_{\chi_1, \chi_2}$$

where $k[B_M]_{\chi_1,\chi_2}$ is the set of functions $h\colon B_M \to \mathbb{A}^1$ satisfying $h(t_1bt_2^{-1}) = \chi_1(t_1)\chi_2(t_2)h(b)$ for characters $\chi_1, \chi_2 \in X^*(T)$. Put $\lambda(\chi_1,\chi_2) = q\sigma^{-1}\chi_1 + \sigma(w_{0,I})\chi_2$. Then we have:

$$S(\lambda) = \bigoplus_{\lambda(\chi_1,\chi_2)=\lambda} k[B_M]_{\chi_1,\chi_2}.$$

It is clear that functions of the form $h_{m,\xi}$ are $T\times T$-eigenfunctions. Lemma 3.1.4 determines exactly its weight (χ_1,χ_2). The proof of the lemma is similar to that of [IK22, Lemma 4.3.3].

Lemma 3.1.4. *Let $(m,\xi) \in \mathbb{N}^{\Phi_{M,-}} \times X^*(T)$. Then $h_{m,\xi}$ lies in $k[B_M]_{\chi_1,\chi_2}$ for*

$$\chi_1 = \xi, \quad \textit{and} \quad \chi_2 = -\xi + \sum_{\alpha\in\Phi_{M,-}} m_\alpha \alpha.$$

For $(m,\xi) \in \mathbb{N}^{\Phi_{M,-}} \times X^*(T)$, define the weight $\omega(m,\xi)$ by

$$\omega(m,\xi) := q\sigma^{-1}(\xi) - w_{0,M}\xi + \sum_{\alpha\in\Phi_{M,-}} m_\alpha(w_{0,M}\alpha) \in X^*(T).$$

It follows immediately from Lemma 3.1.4 that the function $h_{m,\xi}\colon B_M \to \mathbb{A}^1$ lies in $S(\omega(m,\xi))$.

3.2 Unipotent-invariance cone

3.2.1 Regular maps invariant under a unipotent subgroup

To give an upper bound on the cone C_{zip}, we will view sections over $G\text{-}\mathtt{Zip}^\mu$ as regular functions $f\colon G \to \mathbb{A}^1$ and use their invariance under the action of the unitary group $U\times V$. We will show that this invariance condition forces the weight of f to be constrained to a certain region of $X^*(T)$. We will therefore call this the "unipotent-invariance cone".

Let (G,μ) be a cocharacter datum over $\mathbb{F}_q$. Let $\mathcal{Z}_\mu = (G,P,Q,L,M,\varphi)$ be the attached zip datum. To simplify, we restrict ourselves to the case when P is defined over $\mathbb{F}_q$. Recall that $U = R_{\mathrm{u}}(P)$ and $V = R_{\mathrm{u}}(Q)$. The key fact is the following easy lemma:

Lemma 3.2.1. *Let $f\colon G \to \mathbb{A}^1$ be a regular function satisfying $f(gu) = f(g)$ for all $g \in G$ and all u in the unipotent radical U' of a standard parabolic $P' \subset G$. Let $I' \subset \Delta$ be the type of P'. Then:*
(1) We may decompose f uniquely as $f = \sum_\chi f_\chi$ where $\chi \in X^*(T)$, such that f_χ is also U'-equivariant and satisfies furthermore $f_\chi(gt) = \chi(t)^{-1}f_\chi(g)$ for all $g \in G$, $t \in T$.
(2) For all χ such that $f_\chi \neq 0$, we have $\langle \chi, \alpha^\vee\rangle \leq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,I'}$.

Proof. Consider the space W of all U'-equivariant functions $h\colon G \to \mathbb{A}^1$. Since U' is normal in P', it is clear that any right-translate of h by an element of P' is again U'-equivariant. Hence the space W is a P'-representation. In particular, it decomposes with respect to the action of T. This shows (1). For the second assertion, by (1) we may assume $f = f_\chi$. Let $\phi_\alpha\colon \mathrm{SL}_2 \to G$ denote the map attached to α, as in [Spr98, 9.2.2]. It satisfies

$$\phi_\alpha\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = u_\alpha(x), \quad \phi_\alpha\left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right) = u_{-\alpha}(x).$$

For a fixed element $g_0 \in G$ and $\alpha \in \Phi_+ \setminus \Phi_{+,I'}$, consider the map

$$f_\alpha\colon \mathrm{SL}_2 \to \mathbb{A}^1, \quad A \mapsto f\left(g_0\phi_\alpha(A)\right).$$

Let $V(m) := \mathrm{Ind}_{B_0}^{\mathrm{SL}_2}(\chi_m)$ where B_0 is the lower Borel subgroup of SL_2 and χ_m is the character $\mathrm{diag}(x,x^{-1}) \mapsto x^m$. It is immediate that f_α lies in the SL_2-representation $V(-\langle\chi,\alpha^\vee\rangle)$. We can clearly choose g_0 such that f_α is nonzero. In particular $V(-\langle\chi,\alpha^\vee\rangle) \neq 0$ hence $\langle\chi,\alpha^\vee\rangle \leq 0$. □

Corollary 3.2.2. *Let $f\colon G \to \mathbb{A}^1$ be a regular map satisfying $f(ugv) = f(g)$ for all $g \in G$ and all $(u,v) \in U \times V$. Then:*

(1) *We may decompose f as*

$$f = \sum_{(\chi_1,\chi_2)} f_{\chi_1,\chi_2}$$

where $\chi_1, \chi_2 \in X^*(T)$ and f_{χ_1,χ_2} satisfies $f_{\chi_1,\chi_2}(t_1 g t_2) = \chi_1(t_1)\chi_2(t_2)^{-1} f_{\chi_1,\chi_2}(g)$ for all $g \in G$, $t_1, t_2 \in T$, as well as $f_{\chi_1,\chi_2}(ugv) = f_{\chi_1,\chi_2}(g)$ for all $g \in G$ and $(u,v) \in U \times V$.

(2) *For all (χ_1,χ_2) such that $f_{\chi_1,\chi_2} \neq 0$, we have $\langle \chi_1, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,L}$ and $\langle \chi_2, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,M}$*

Proof. The first assertion is proved as in Lemma 3.2.1, noting that the space of $U \times V$-invariant regular functions is stable by the action of $P \times Q$. For the second assertion, apply the lemma to the functions $g \mapsto f(g)$ (resp. $g \mapsto f(w_0 g^{-1} w_0)$) to obtain the inequality satisfied by χ_2 (resp. χ_1). □

Corollary 3.2.3. *The space S_{unip} decomposes as follows:*

$$S_{\mathrm{unip}} = \bigoplus_{(\chi_1,\chi_2)} S_{\mathrm{unip}}(\chi_1,\chi_2)$$

where $S_{\mathrm{unip}}(\chi_1,\chi_2)$ is the subspace of functions $f \in S_{\mathrm{unip}}$ satisfying $f(t_1 g t_2^{-1}) = \chi_1(t_1)\chi_2(t_2) f(g)$. Furthermore, any (χ_1,χ_2) such that $S_{\mathrm{unip}}(\chi_1,\chi_2) \neq 0$ satisfies $\langle \chi_1, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,L}$ and $\langle \chi_2, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,M}$.

We note that the map $\vartheta\colon S_{\mathrm{unip}} \to k[B_M]$ is not $T \times T$-equivariant. It maps $S_{\mathrm{unip}}(\chi_1,\chi_2)$ to the weight space $k[B_M]_{\chi_2, w_{0,M}\chi_1}$.

3.2.2 Unipotent-invariance cone

We now start with a nonzero section $f \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ for some $\lambda \in X^*_{+,I}(T)$. Our goal is to show that λ satisfies certain constraints. First, view f as an element of S_{unip}. By Corollary 3.2.3, we may decompose f as

$$f = \sum_{\chi_1,\chi_2} f_{\chi_1,\chi_2}$$

where $f_{\chi_1,\chi_2} \in S_{\mathrm{unip}}(\chi_1,\chi_2)$. Furthermore, we have $\langle \chi_1, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,L}$ and $\langle \chi_2, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Phi_+ \setminus \Phi_{+,M}$ whenever (χ_1,χ_2) appears. Next, we apply $\vartheta\colon S_{\mathrm{unip}} \to k[B_M]$. By the discussion in section 3.1, there exists (χ_1,χ_2) such that $\vartheta(f_{\chi_1,\chi_2})$ is a nonzero element $h \in k[B_M]$. Recall also that the weight of h with respect to the $T \times T$-action on $k[B_M]$ is $(\chi_2, w_{0,M}\chi_1)$. We can decompose h as a sum of monomials of the form $h_{m,\xi}$ for $(m,\xi) \in \mathbb{N}^{\Phi_{M,-}} \times X^*(T)$ as in section 3.1. Since $\vartheta(f) \in S(\lambda)$, we have simultaneously:

$$\begin{cases} \lambda & = \lambda(\chi_2, w_{0,M}\chi_1) = q\sigma^{-1}\chi_2 + \chi_1 \\ \chi_1 & = -w_{0,M}\xi + \sum_{\alpha \in \Phi_{M,-}} m_\alpha (w_{0,M}\alpha) \\ \chi_2 & = \xi. \end{cases}$$

Note that in the above sum $w_{0,M}\alpha$ lies in $\Phi_{M,+}$. Therefore, putting everything together, we deduce that λ satisfies the following condition: There exists a character $\chi_2 \in X^*(T)$ such that

$$\begin{cases} \lambda - q\sigma^{-1}(\chi_2) + w_{0,M}\chi_2 \text{ is a sum of positive roots of } M \\ \langle \chi_2, \alpha^\vee \rangle \leq \min\left(0, \frac{1}{q}\langle \sigma(\lambda), \alpha^\vee \rangle\right) \quad \text{for all } \alpha \in \Phi_+ \setminus \Phi_{+,M}. \end{cases}$$

Definition 3.2.4. *Let $C_{\text{unip}} \subset X^*(T)$ be the set of $\lambda \in X^*(T)$ such that there exists a character $\chi_2 \in X^*(T)$ satisfying the condition above. We call C_{unip} the unipotent-invariance cone.*

It is clear that C_{unip} is a saturated subcone of $X^*(T)$. We have shown that if $f \in H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda))$ is nonzero, then the weight of f lies in C_{unip}. Hence:

Theorem 3.2.5. *We have $\langle C_{\text{zip}} \rangle \subset C_{\text{unip}}$.*

The saturated cone $\langle C_{\text{unip}} \rangle$ has a similar description as C_{unip}, except that we allow a linear combination of positive roots of M with non-negative rational coefficients.

3.3 The split case

We simplify the situation by making the following assumptions:
(1) P is defined over $\mathbb{F}_q$. In particular, we have $L = M$.
(2) The group G is split over $\mathbb{F}_{q^2}$.
In particular, both conditions are satisfied if G is split over $\mathbb{F}_q$. For characters $\chi_2, \lambda \in X^*(T)$, write $\gamma = \lambda - q\sigma^{-1}(\chi_2) + w_{0,L}\chi_2$. We wish to express χ_2 in terms of λ and γ. Using the above assumptions, we find:

$$\chi_2 = -\frac{1}{q^2-1}(w_{0,I}(\gamma - \lambda) + q\sigma(\gamma - \lambda)).$$

For characters λ_1, λ_2, write $\lambda_1 \leq_L \lambda_2$ if for all roots $\alpha \in \Phi_+ \setminus \Phi_{L,+}$, we have $\langle \lambda_1 - \lambda_2, \alpha^\vee \rangle \leq 0$. Under the assumptions (1)-(2), we deduce that any weight $\lambda \in C_{\text{unip}}$ satisfies : There exists a character $\gamma \in X^*(T)$ which is a sum of positive roots of L such that

$$\begin{cases} w_{0,I}\lambda + q\sigma(\lambda) \leq_L w_{0,I}\gamma + q\sigma(\gamma) \\ w_{0,I}\lambda + \frac{1}{q}\sigma(\lambda) \leq_L w_{0,I}\gamma + q\sigma(\gamma) \end{cases}$$

In particular, assume that $\alpha_1, \dots, \alpha_m \in \Phi_+ \setminus \Phi_{L,+}$ and that $\alpha_1^\vee + \dots + \alpha_m^\vee = \delta$ is a cocharacter in $X_*(L)$. Since $w_{0,I}\gamma + q\sigma(\gamma)$ is again a sum of roots of L, it is orthogonal to δ. Let $\{1, \dots, m\} = S_1 \sqcup S_2$ be any partition of $\{1, \dots, m\}$. We obtain

$$\sum_{i \in S_1} \langle w_{0,I}\lambda + q\sigma(\lambda), \alpha_i^\vee \rangle + \sum_{i \in S_2} \langle w_{0,I}\lambda + \frac{1}{q}\sigma(\lambda), \alpha_i^\vee \rangle \leq 0,$$

$$\text{hence} \quad \langle \lambda, \delta \rangle + q\sum_{i \in S_1} \langle \sigma(\lambda), \alpha_i^\vee \rangle + \frac{1}{q}\sum_{i \in S_2} \langle \sigma(\lambda), \alpha_i^\vee \rangle \leq 0$$

If we assume that $\{\alpha_1, \dots, \alpha_m\}$ is stable by σ, then we can replace $\sigma(\lambda)$ by λ in the above formula (using the partition $\sigma(S_1) \sqcup \sigma(S_2)$). In this case, we obtain

$$(q+1)\sum_{i \in S_1} \langle \lambda, \alpha_i^\vee \rangle + \left(\frac{1}{q} + 1\right)\sum_{i \in S_2} \langle \lambda, \alpha_i^\vee \rangle \leq 0$$

$$\sum_{i \in S_1} \langle \lambda, \alpha_i^\vee \rangle + \frac{1}{q}\sum_{i \in S_2} \langle \lambda, \alpha_i^\vee \rangle \leq 0$$

where we divided the equation by $q+1$. Consider the action of $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ on Φ. Note that this action preserves positivity of roots outside of L. In particular, the set

$\Phi_+ \setminus \Phi_{L,+}$ is stable under $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. It is not always the case that $\Phi_+ \setminus \Phi_{L,+}$ consists of a single orbit. Let $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$ be an orbit. Define $\delta_{\mathcal{O}}$ as the sum of all coroots in $\mathcal{O}$:

$$\delta_{\mathcal{O}} := \sum_{\alpha \in \mathcal{O}} \alpha^\vee.$$

For any root $\beta \in \Delta_L$, the reflection s_β satisfies $s_\beta(\delta_{\mathcal{O}}) = \delta_{\mathcal{O}}$, hence $\langle \beta, \delta_{\mathcal{O}} \rangle = 0$. It follows that $\delta_{\mathcal{O}} \in X_*(L)$. Moreover, it is clear that $\sigma(\delta_{\mathcal{O}}) = \delta_{\mathcal{O}}$. The above discussion applies to $\delta_{\mathcal{O}}$ and shows that C_{unip} satisfies all the inequalities of the type

$$\Gamma_{\mathcal{O},S}(\lambda) := \sum_{\alpha \in \mathcal{O} \setminus S} \langle \lambda, \alpha^\vee \rangle \ + \ \frac{1}{q} \sum_{\alpha \in S} \langle \lambda, \alpha^\vee \rangle \ \leq \ 0 \qquad (3.3.1)$$

for any subset $S \subset \mathcal{O}$. Denote by $C_{\mathcal{O}} \subset X^*(T)$ the cone of λ satisfying the inequalities (3.3.1) for all subset $S \subset \mathcal{O}$. Note that we could have defined $\delta_{\mathcal{O}}$ similarly when $\mathcal{O}$ is a union of $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-orbits. In particular, we may speak of the cone $C_{\Phi_+ \setminus \Phi_{L,+}}$. However, note that if $\mathcal{O} = \mathcal{O}_1 \sqcup \mathcal{O}_2$ and $S \subset \mathcal{O}$ is any subset, we have

$$\Gamma_{\mathcal{O},S}(\lambda) = \Gamma_{\mathcal{O}_1, S \cap \mathcal{O}_1}(\lambda) + \Gamma_{\mathcal{O}_2, S \cap \mathcal{O}_1}(\lambda).$$

Hence we deduce that $C_{\mathcal{O}_1} \cap C_{\mathcal{O}_2} \subset C_{\mathcal{O}}$ and thus we can reduce to considering the cones $C_{\mathcal{O}}$ when $\mathcal{O}$ is an $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-orbit in $\Phi_+ \setminus \Phi_{L,+}$. We define the orbit cone C_{orb} as follows:

$$C_{\mathrm{orb}} := \bigcap_{\substack{\text{orbits} \\ \mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}}} C_{\mathcal{O}}$$

where the intersection is taken over all $W_L \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-orbits $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$. By the above discussion, we have inclusions

$$C_{\mathrm{zip}} \ \subset \ C_{\mathrm{unip}} \ \subset \ C_{\mathrm{orb}} \ \subset \ C_{\Phi_+ \setminus \Phi_{L,+}}. \qquad (3.3.2)$$

All inclusions above are in general strict. We illustrate the difference between C_{zip}, C_{orb} and C_{unip} in section 3.5 in the case $G = \mathrm{Sp}(6)_{\mathbb{F}_q}$. We were not able to determine C_{unip} in general (or even under assumptions (1)-(2)), but it will be sufficient for our purposes to work with the cone C_{orb} since it already provides a sharp approximation. In certain cases, the inclusion $C_{\mathrm{zip}} \subset C_{\Phi_+ \setminus \Phi_{L,+}}$ will be enough for our purpose, as in the proof of Theorem 3.4.1 in section 3.4. However, in the case $G = \mathrm{Sp}_{2n,\mathbb{F}_q}$, the set $\Phi_+ \setminus \Phi_{L,+}$ contains two orbits, and the cone $C_{\Phi_+ \setminus \Phi_{L,+}}$ is strictly coarser than $C_{\mathcal{O}}$, where $\mathcal{O}$ is the orbit of the unique simple root outside of L. When we want to emphasize the dependance of $C_{\mathcal{O}}$ on the prime power q, it will be convenient to write $C_{\mathcal{O},q}$. Similarly, we write $\Gamma_{\mathcal{O},S,q}$ for the function $\Gamma_{\mathcal{O},S}$.

The number of inequalities defining the cone $C_{\mathcal{O}}$ is the cardinality of the powerset of $\mathcal{O}$, which can be quite large. However, we are eventually interested in the cone C_{zip}, which is contained in the L-dominant cone $X^*_{+,I}(T)$. Therefore, it is sufficient to consider the intersection $C_{\mathcal{O}} \cap X^*_{+,I}(T)$. Looking at concrete examples, we see that this intersection is cut out in $X^*_{+,I}(T)$ by inequalities $\Gamma_{\mathcal{O},S}(\lambda) \leq 0$ for a rather small number of subsets $S \subset \mathcal{O}$ (the other subsets do not contribute to this intersection). The following notion seems to be relevant:

Definition 3.3.1. *A subset $S \subset \Phi_+ \setminus \Phi_{L,+}$ is L-minimal if it satisfies the following condition: For any $\alpha \in S$ and any $\beta \in \Delta_L$ such that $\alpha - \beta \in \Phi_+$, we have $\alpha - \beta \in S$. Denote by $\mathrm{Min}(\Phi_+ \setminus \Phi_{L,+})$ the set of all L-minimal subsets of $\Phi_+ \setminus \Phi_{L,+}$.*

For $w \in W^I$, define a subset

$$\mathrm{Min}(w) := \{\alpha \in \Phi_+ \setminus \Phi_{L,+} \mid \ell(ws_\alpha) < \ell(w)\}.$$

Then one can show that $\mathrm{Min}(w)$ is a L-minimal subset, and the map $w \mapsto \mathrm{Min}(w)$ induces a bijection $W^I \to \mathrm{Min}(\Phi_+ \setminus \Phi_{L,+})$. For an orbit $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$, define $\mathrm{Min}(\mathcal{O})$ as the set of L-minimal subsets contained in $\mathcal{O}$, i.e:

$$\mathrm{Min}(\mathcal{O}) = \mathrm{Min}(\Phi_+ \setminus \Phi_{L,+}) \cap \mathcal{P}(\mathcal{O}).$$

Then, we expect the following to hold:

$$C_{\mathcal{O}} \cap X^*_{+,I}(T) = \{\lambda \in X^*_{+,I}(T) \mid \Gamma_{\mathcal{O},S}(\lambda) \leq 0 \text{ for all } S \in \mathrm{Min}(\mathcal{O})\}.$$

In particular, only a small number of subsets S contribute nontrivially. The above can be easily checked this in the cases $G = \mathrm{GL}_{n,\mathbb{F}_q}$ and $G = \mathrm{Sp}(2n)_{\mathbb{F}_q}$ considered in sections 3.5 and 4, but we have not proved it in general. It will be convenient to define the following set, which we call the L-minimal cone:

$$C_{L-\mathrm{Min}} = \{\lambda \in X^*(T) \mid \Gamma_{\mathcal{O},S}(\lambda) \leq 0 \text{ for all orbits } \mathcal{O} \text{ and all } S \in \mathrm{Min}(\mathcal{O})\}. \tag{3.3.3}$$

Hence, at least in the cases considered in sections 3.5 and 4, we have $C_{\mathrm{orb}} \cap X^*_{+,I}(T) = C_{L-\mathrm{Min}} \cap X^*_{+,I}(T)$

3.4 Asymptotic zip cone

We apply the results of the previous section to study the asymptotic behaviour of the cone $C_{\mathrm{zip},p}$ in families. We may work in a more general setting, independently of the theory of Shimura varieties. We let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group endowed with a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m}} \to \mathbf{G}_{\overline{\mathbb{Q}}}$. There exists an integer $N \geq 1$ such that $\mathbf{G}$ admits a reductive $\mathbb{Z}[\frac{1}{N}]$-model $\mathcal{G}$. Furthermore, there is a number field E such that μ extends to a cocharacter of $\mathcal{G}_R$ where $R = \mathcal{O}_E[\frac{1}{N}]$. From this, we obtain a zip datum (G_p, μ_p) for all primes p not dividing N. We choose a Borel pair $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$ and we may assume that it has a model $(\mathcal{B}, \mathcal{T})$ in $\mathcal{G}$. We obtain compatible Borel pairs (B_p, T_p) for all G_p, and we may identify their character groups and their root data. Write $C_{\mathrm{zip},p}$ for the zip cone of the zip datum attached to (G_p, μ_p). We may view all the cones $C_{\mathrm{zip},p}$ inside the same character group $X^*(\mathbf{T})$. We have the following:

Theorem 3.4.1. *The family $(\langle C_{\mathrm{zip},p}\rangle)_p$ is asymptotic to C_{GS}.*

Proof. Since μ is defined over the number field $\mathbf{E}$, the cocharacter μ_p will be defined over $\mathbb{F}_p$ for any p which is split in $\mathbf{E}$. Similarly, choose a number field $\mathbf{F}$ such that $\mathbf{G}_{\mathbf{F}}$ is split. Then for any prime p split in $\mathbf{F}$, the group G_p is split over $\mathbb{F}_p$. In particular, there are infinitely many such primes. It suffices to show for all $N \geq 1$:

$$\bigcap_{\substack{G_p \text{ split} \\ p \geq N}} \langle C_{\mathrm{zip},p}\rangle = C_{\mathrm{GS}}.$$

For such primes, we may apply the results of the previous section. Define the cone $C_{\Phi_+ \setminus \Phi_{L,+},p}$ as in section 3.3 (where we take $q = p$). By the inclusions (3.3.2), it suffices to show that the intersection of all the cones $C_{\Phi_+ \setminus \Phi_{L,+},p}$ (for p such that G_p is split) coincides with

C_{GS}. Let λ be a character in all the $C_{\Phi_+\setminus\Phi_{L,+},p}$. For $\beta \in \Phi_+ \setminus \Phi_{L,+}$, consider the subset $S = \Phi_+ \setminus (\Phi_{L,+} \cup \{\beta\})$. By assumption, λ satisfies

$$\Gamma_{\Phi_+\setminus\Phi_{L,+},S,p}(\lambda) = \langle \lambda, \beta^\vee \rangle + \frac{1}{p} \sum_{\substack{\alpha\in\Phi_+\setminus\Phi_{L,+} \\ \alpha\neq\beta}} \langle \lambda, \alpha^\vee \rangle \leq 0.$$

Passing to the limit on p, we obtain $\langle \lambda, \beta^\vee \rangle \leq 0$ for any $\beta \in \Phi_+ \setminus \Phi_{L,+}$. Since $C_{\mathrm{zip}} \subset X^*_{+,I}(T)$ and $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip},p} \rangle$ for all p, we deduce that the intersection of the cones $\langle C_{\mathrm{zip},p} \rangle$ coincides with C_{GS}. □

The proof shows that the family $(C_{\Phi_+\setminus\Phi_{L,+},p} \cap X^*_{+,I}(T))_p$ is a GS-approximation of the family $(C_{\mathrm{zip},p})_p$ (this is a slight abuse of terminology, since we have not defined $C_{\Phi_+\setminus\Phi_{L,+},p}$ for general p). Theorem 3.4.1 combined with Conjecture 1.2.3 indicates that we should expect a similar result for the Shimura cone family $(\langle C_{K,p} \rangle)_p$ (recall that $C_{K,p} := C_K(\overline{\mathbb{F}}_p)$). In particular, we expect that $C_{K,p}$ is contained in $C_{\Phi_+\setminus\Phi_{L,+},p}$ for all p where G_p is split.

3.5 GS-approximations for GL_n and $\mathrm{Sp}(2n)$

We give explicit equations for $C_{\mathcal{O}}$ and C_{orb} in the case of general linear groups and symplectic groups.

3.5.1 General linear groups

Set $G = \mathrm{GL}_{n,\mathbb{F}_q}$ (as usual, we take $q = p$ in the context of Shimura varieties). Consider the cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ by $\mu(x) = \mathrm{diag}(xI_r, I_s)$ with $r + s = n$. Write $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ for the attached zip datum. If $(u_1, \ldots, u_n)$ denotes the canonical basis of k^n, then P is the stabilizer of $V_P := \mathrm{Span}_k(u_{r+1}, \ldots, u_n)$ and Q is the stabilizer of $V_Q := \mathrm{Span}_k(u_1, \ldots, u_r)$. Let B denote the lower-triangular Borel and T the diagonal torus. The Levi subgroup $L = P \cap Q$ is isomorphic to $\mathrm{GL}_{r,\mathbb{F}_q} \times \mathrm{GL}_{s,\mathbb{F}_q}$. Identify $X^*(T) = \mathbb{Z}^n$ such that $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ corresponds to the character $\mathrm{diag}(x_1, \ldots, x_n) \mapsto \prod_{i=1}^n x_i^{a_i}$. The simple roots with respect to B are $\{\alpha_i\}_{1\leq i\leq n-1}$ where

$$\alpha_i = e_i - e_{i+1}$$

and $(e_i)_{1\leq i\leq n}$ denotes the canonical basis of $\mathbb{Z}^n$. For general (r, s), we do not know a description of C_{zip} or even $\langle C_{\mathrm{zip}} \rangle$. The cones $X^*_{+,I}(T)$ and C_{GS} are given by

$$\begin{aligned} X^*_{+,I}(T) &= \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \cdots \geq a_r \text{ and } a_{r+1} \geq \cdots \geq a_n\} \\ C_{\mathrm{GS}} &= \{(a_1, \ldots, a_n) \in X^*_{+,I}(T) \mid a_1 \leq a_n\}. \end{aligned}$$

In this case, the group W_L acts transitively on $\Phi_+ \setminus \Phi_{L,+}$.

First, we explicit the set $\mathrm{Min}(\Phi^+ \setminus \Phi^+_L)$. This set is in bijection with the set of finite decreasing sequences $x = (x_j)_{1\leq j\leq s}$ such that $r \geq x_1 \geq x_2 \geq \cdots \geq x_s \geq 0$. To each such sequence, we can attach the L-minimal subset

$$S_x := \{e_i - e_j \mid r + 1 - x_j \leq i \leq r,\ r + 1 \leq j \leq n\}.$$

Write simply $\Gamma_x(\lambda)$ for the function $\Gamma_{\Phi^+\setminus\Phi^+_L,S_x}(\lambda)$. If we write $\lambda = (a_1, \ldots, a_n)$, we have:

$$\Gamma_x(\lambda) = \sum_{j=r+1}^{n} \left(\sum_{i=1}^{r-x_{j-r}} (a_i - a_j) + \frac{1}{q} \sum_{i=r-x_{j-r}+1}^{r} (a_i - a_j) \right).$$

Hence, the L-minimal cone ((3.3.3)) is given as follows:

$$C_{L-\mathrm{Min}} = \{\lambda \in X^*(T) \mid \Gamma_x(\lambda) \leq 0, \text{ for all decreasing sequences } x\}.$$

On this example, the expected equality $C_{L-\mathrm{Min}} \cap X^*_{+,I}(T) = C_{\mathrm{orb}} \cap X^*_{+,I}(T)$ (see end of section 3.3) is a straightforward computation. When $(r,s) = (n-1,1)$, we obtain that $C_{L-\mathrm{Min}}$ is given by the following equations

$$\sum_{i=1}^{k}(a_i - a_n) + \frac{1}{q}\sum_{i=k+1}^{n-1}(a_i - a_n) \leq 0 \quad \text{for all } 0 \leq k \leq n-1. \tag{3.5.1}$$

Furthermore, one can check that the intersection $C_{L-\mathrm{Min}} \cap X^*_{+,I}(T)$ coincides with the $\lambda = (a_1, \dots, a_n) \in X^*_{+,I}(T)$ satisfying the inequalities (3.5.1) for $k = 1, \dots, n-1$ (the inequality for $k = 0$ can be omitted).

Consider the case $(r,s) = (2,2)$. The L-dominant cone $X^*_{+,I}(T)$ is the set of $\lambda = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ such that $a_1 \geq a_2$ and $a_3 \geq a_4$. The set ${}^IW$ has cardinality $\frac{4!}{2!2!} = 6$. When intersecting with $X^*_{+,I}(T)$, three of the corresponding 6 equations become redundant. Specifically, $C_{\mathrm{orb}} \cap X^*_{+,I}(T)$ is the set of $\lambda = (a_1, a_2, a_3, a_4) \in X^*_{+,I}(T)$ satisfying

$$\begin{cases} 2qa_1 + 2a_2 - (q+1)a_3 - (q+1)a_4 \leq 0 \\ (q+1)a_1 + 2a_2 - 2a_3 - (q+1)a_4 \leq 0 \\ (q+1)a_1 + (q+1)a_2 - 2a_3 - 2qa_4 \leq 0 \end{cases}$$

In the case of a unitary group of signature $(2,2)$ at a split prime of good reduction, Conjecture 1.2.3 holds by [GK22, Theorem 4.2.8]. Furthermore, this case is of Hasse-type ([IK22, Definition 5.1.6]), hence we have $\langle C_{\mathrm{zip}} \rangle = \langle C_{\mathrm{Hasse}} \rangle$ by *loc. cit.*, Theorem 5.3.1. Therefore, if X denotes any $\overline{\mathbb{F}}_q$-scheme satisfying Assumption 2.1.1 (for example, the corresponding unitary Shimura variety), we have:

$$\langle C_X \rangle = \langle C_{\mathrm{zip}} \rangle = \langle C_{\mathrm{Hasse}} \rangle = \{(a_1, a_1, a_3, a_4) \in X^*_{+,I}(T) \mid q(a_1 - a_4) + (a_2 - a_3) \leq 0\}.$$

We see on this example that the actual cones $\langle C_X \rangle$ and $\langle C_{\mathrm{zip}} \rangle$ have a much simpler expression than the approximation C_{orb}. However, for general groups we do not have an expression for either $\langle C_X \rangle$ or $\langle C_{\mathrm{zip}} \rangle$. Even worse, we could not prove that they are polyhedral cones.

3.5.2 Sympectic groups

We first give some notations for an arbitrary symplectic group. Let (V_0, ψ) be a non-degenerate symplectic space over $\mathbb{F}_q$ of dimension $2n$, for some integer $n \geq 1$. After choosing an appropriate basis $\mathcal{B}$ for V_0, we assume that ψ is given by the matrix

$$\begin{pmatrix} & -J \\ J & \end{pmatrix} \quad \text{where} \quad J := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Define G as follows:

$$G(R) = \{f \in \mathrm{GL}_{\mathbb{F}_q}(V_0 \otimes_{\mathbb{F}} R) \mid \psi_R(f(x), f(y)) = \psi_R(x,y), \ \forall x, y \in V_0 \otimes_{\mathbb{F}_q} R\}$$

for all $\mathbb{F}_q$-algebras R. Identify $V_0 = \mathbb{F}_q^{2n}$ via $\mathcal{B}$ and view G as a subgroup of $\mathrm{GL}_{2n,\mathbb{F}_q}$. Fix the $\mathbb{F}_q$-split maximal torus T given by diagonal matrices in G, i.e.

$$T(R) := \{\mathrm{diag}_{2n}(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) \mid x_1, \dots, x_n \in R^\times\}.$$

Define B as the Borel subgroup of G consisting of the lower-triangular matrices in G. For a tuple $(a_1,\dots,a_n)\in\mathbb{Z}^n$, define a character of T by mapping $\mathrm{diag}_{2n}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1})$ to $x_1^{a_1}\cdots x_n^{a_n}$. From this, we obtain an identification $X^*(T)=\mathbb{Z}^n$. Denoting by $(e_1,\dots,e_n)$ the standard basis of $\mathbb{Z}^n$, the T-roots of G and the B-positive roots are respectively

$$\Phi := \{\pm e_i \pm e_j \mid 1\le i\neq j\le n\}\cup\{\pm 2e_i \mid 1\le i\le n\},$$
$$\Phi_+ := \{e_i \pm e_j \mid 1\le i< j\le n\}\cup\{2e_i \mid 1\le i\le n\}$$

and the B-simple roots are $\Delta := \{\alpha_1,\dots,\alpha_{n-1},\beta\}$ where

$$\alpha_i := e_i - e_{i+1} \text{ for } i=1,...,n-1,$$
$$\beta := 2e_n.$$

The Weyl group $W := W(G,T)$ can be identified with the group of permutations $\sigma\in\mathfrak{S}_{2n}$ satisfying $\sigma(i)+\sigma(2n+1-i)=2n+1$ for all $1\le i\le 2n$. Define a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{F}_q}\to G$ by $z\mapsto \mathrm{diag}(zI_n, z^{-1}I_n)$. Write $\mathcal{Z} := (G,P,L,Q,M,\varphi)$ for the associated zip datum (since μ is defined over $\mathbb{F}_q$, we have $M=L$). Concretely, if we denote by $(u_i)_{i=1}^{2n}$ the canonical basis of k^{2n}, then P is the stabilizer of $V_{0,P}=\mathrm{Span}_k(u_{n+1},...,u_{2n})$ and Q is the stabilizer of $V_{0,Q}=\mathrm{Span}_k(u_1,...,u_n)$. The intersection $L:=P\cap Q$ is a common Levi subgroup and there is an isomorphism $\mathrm{GL}_{n,\mathbb{F}_q}\to L$, $A\mapsto\theta(A)$, where:

$$\theta(A) := \begin{pmatrix} A & \\ & J{}^tA^{-1}J\end{pmatrix}.$$

Under the identification $X^*(T)=\mathbb{Z}^n$, we have:

$$X^*_{+,I}(T) = \{(a_1,\dots,a_n)\in\mathbb{Z}^n \mid \ a_1\ge\cdots\ge a_n\}$$
$$C_{\mathrm{GS}} = \{(a_1,\dots,a_n)\in X^*_{+,I}(T) \mid \ a_1\le 0\}$$

We do not know the general form of the cone C_{zip} outside the case $n=2$ (see [Kos19]). For $n=3$, we determined $\langle C_{\mathrm{zip}}\rangle$ in *loc. cit.*. For $n\ge 4$, neither C_{zip} nor its saturation $\langle C_{\mathrm{zip}}\rangle$ are known. Some approximations by subcones (Hasse cone, highest weight cone) were constructed in *loc. cit.*. These notions were generalized to arbitrary groups in [IK22].

Next, we explicit the results of the previous section and give an upper bound on C_{zip}. There are two W_L-orbits in $\Phi_+\setminus\Phi_{L,+}$, given by

$$\mathcal{O}_1 = \{e_i \ \mid\ 1\le i\le n\}$$
$$\mathcal{O}_2 = \{e_i+e_j \ \mid\ 1\le i<j\le n\}.$$

It turns out that the cone $C_{\mathcal{O}_2}$ is coarser that $C_{\mathcal{O}_1}$, so we will only consider $C_{\mathcal{O}_1}$. One can prove that the cone $C_{\mathcal{O}_1}\cap X^*_{+,I}(T)$ is the set of $\lambda=(a_1,\dots,a_n)\in X^*_{+,I}(T)$ satisfying

$$\sum_{i=1}^{k} a_i + \frac{1}{q}\sum_{i=k+1}^{n} a_i \le 0 \quad \text{for all } 1\le k\le n-1.$$

Therefore, the cone $C_{\mathrm{orb}}\cap X^*_{+,I}(T)$ is also given by the above inequalities. Note the similarities between the cases $G=\mathrm{Sp}(2n)$ and $G=\mathrm{GL}_{n+1}$ of signature $(n,1)$. Namely, if we set $a_{n+1}=0$ in the latter, we recover the equations for $\mathrm{Sp}(2n)$. As explained in [GK22, §4.2.2], there is a correspondence between automorphic forms on the corresponding stacks of G-zips for these two groups. Even though the number of W_L-orbits are different for these groups, this correspondence persists for the approximation cones C_{orb}.

In the graph below, we consider the case $G = \mathrm{Sp}(6)$. We illustrate the approximations C_{orb}, C_{unip} of the cone $\langle C_{\mathrm{zip}} \rangle$. Note that $X^*(T) = \mathbb{Z}^3$ is 3-dimensional, so to simplify we represent a slice of the cones. Hence, each dot on the picture represents a half-line from the origin. For a cone $C \subset X^*(T)$, write $C^{+,I}$ for its intersection with $X^*_{+,I}(T)$. We have:

$$C^{+,I}_{\mathrm{orb}} = C^{+,I}_{L-\mathrm{Min}} = \{(a_1,a_2,a_3) \in X^*_{+,I}(T) \mid a_1 + \frac{1}{q}(a_2+a_3) \leq 0,\ a_1 + a_2 + \frac{1}{q}a_3 \leq 0\}$$
$$C^{+,I}_{\mathrm{unip}} = \{(a_1,a_2,a_3) \in X^*_{+,I}(T) \mid a_1 + \frac{1}{q}(a_2+a_3) \leq 0,\ qa_1 + q^2a_2 + a_3 \leq 0\}$$
$$C_{\mathrm{zip}} = \{(a_1,a_2,a_3) \in X^*_{+,I}(T) \mid q^2a_1 + a_2 + qa_3 \leq 0,\ qa_1 + q^2a_2 + a_3 \leq 0\}$$

As one sees on the figure below, the inclusions $C_{\mathrm{zip}} \subset C^{+,I}_{\mathrm{unip}} \subset C^{+,I}_{\mathrm{orb}}$ are strict.

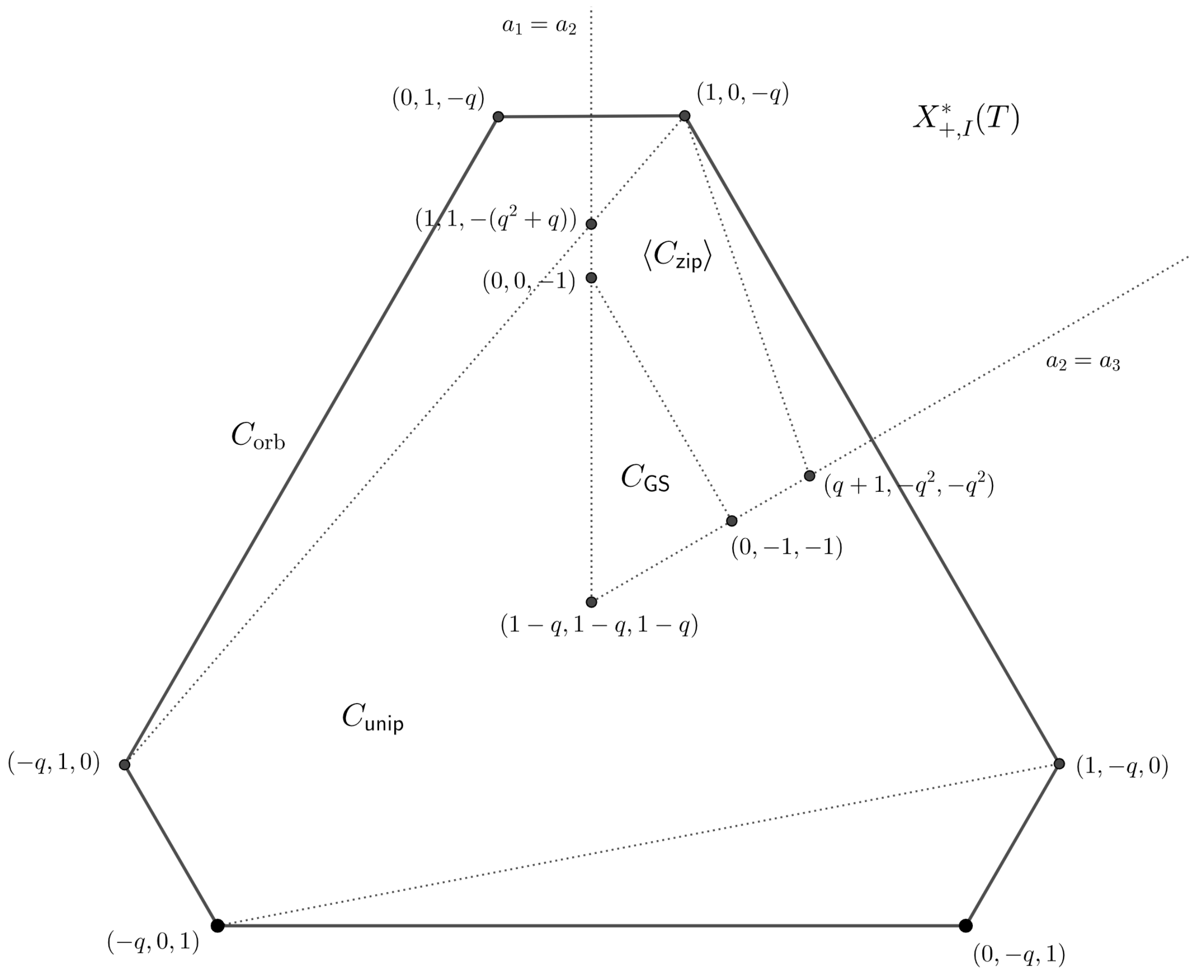

Figure 1: The case $\mathrm{Sp}(6)_{\mathbb{F}_q}$

4 Vanishing at a fixed prime for unitary Shimura varieties

In this section, we take $G = \mathrm{GL}_{n,\mathbb{F}_q}$ and consider the setting of section 3.5.1. In particular, $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ is the cocharacter $x \mapsto \mathrm{diag}(xI_r, I_s)$ for $r+s=n$ and $\mathcal{Z}_\mu = (G,P,L,Q,M,\varphi)$ is the attached zip datum. In section 4.4, we will specialize to the case $(r,s) = (n-1,1)$.

4.1 Partial Hasse invariants

We let $S_n = W$ be the group of permutations of $\{1, \ldots, n\}$. We start by recalling the following criterion for determining the set E_w for $w \in S_n$ (see (2.3.1)). For $1 \leq i \neq j \leq n$, we denote by $(i\ j) \in S_n$ the transposition exchanging i and j.

Proposition 4.1.1. *Let $1 \leq i < j \leq n$. Then $w \times (i\ j)$ is a lower neighbour of w if and only if the folllowing conditions hold*
(1) *$\sigma(i) > \sigma(j)$,*
(2) *There is no $i < k < j$ such that $\sigma(j) < \sigma(k) < \sigma(i)$.*

We may represent this criterion visually as follows: Consider the submatrix of w whose corners are $(i, \sigma(i))$ and $(j, \sigma(j))$. Condition (i) says that $(i, \sigma(i))$ is the lower left corner of this matrix, and $(j, \sigma(j))$ is the upper right corner. Condition (ii) says that all the coefficients of this submatrix are zero except for these two corners.

$$\begin{pmatrix} & & & \\ & \begin{array}{|cc|} \hline 0 & 1 \\ & \\ 1 & 0 \\ \hline \end{array} & \\ & & & \end{pmatrix}$$

Definition 4.1.2. *We say that $w \in W$ admits a system of partial Hasse invariants if the elements $\alpha^\vee$ for $\alpha \in E_w$ are linearly independent in $X_*(T)_{\mathbb{Q}}$.*

If w admits a system of partial Hasse invariants, then for each $\alpha \in E_w$, we can find $\chi \in X^*(T)$ satisfying Conditions (a) and (b) of Definition 2.5.1. This will be used to construct a separating system in section 4.4.3. Let us introduce some non-standard terminology: Let $w \in S_n$ be a permutation. A triplet (i, j, k) satisfying $i < j < k$ and $w(i) < w(j)$ and $w(k) < w(j)$ will be called a V-shape. Furthermore, if $w(i) < w(k)$, we call it a $\surd$-shape.

Lemma 4.1.3. *Assume that w has no $\surd$-shape. Then w admits a system of partial Hasse invariants.*

Proof. For a transposition $t = (i\ j)$, put $t_- := \min\{i, j\}$ and $t_+ := \max\{i, j\}$. Since w has no $\surd$-shape, it is clear that the map $E_w \to \{1, \ldots, n\}$, $t \mapsto t_+$ is injective. This implies that the elements $(\alpha^\vee)_{\alpha \in E_w}$ are linearly independent. $\square$

4.2 Auxilliary sequence

For $1 \leq d \leq n$, define the matrix

$$\Lambda_d := \left(\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{array}\right)$$

where the upper right block has size $d \times d$ and the lower left block has size $(n-d) \times (n-d)$. For example, $\Lambda_1 = w_0$ is the longest element of $W = S_n$ and $\Lambda_n = I_n$ is the identity element.

For two elements w, w' such that $w > w'$, we define a path from w to w' to be a sequence $w_1, \ldots, w_N$ satisfying the following conditions:

(a) $w_1 = w$ and $w_N = w'$.
(b) $w_1 > \cdots > w_N$ and $\ell(w_{i+1}) = \ell(w_i) - 1$ for each $i = 1, \ldots, N-1$.
For $1 \le d < n$, we construct a path from Λ_d to Λ_{d+1} as follows: We multiply Λ_d successively on the right by the transpositions $(n-d \;\; n-i+1)$ for $i = 1, \ldots, d$. In other words, we define $w_1^{(d)} = \Lambda_d$ and for $2 \le i \le d+1$,

$$w_i^{(d)} = \Lambda_d(n-d \;\; n)(n-d \;\; n-1)\ldots(n-d \;\; n-i+2).$$

Then $(w_1^{(d)}, \ldots, w_{d+1}^{(d)})$ is a path from Λ_d to Λ_{d+1}. At each step, the coefficient on the $n-d$-th column of the matrix moves up by one. Moreover, the last d coefficients are in increasing order at each step of the sequence.

Lemma 4.2.1. *Each element in the sequence $(w_1^{(d)}, \ldots, w_{d+1}^{(d)})$ admits a system of partial Hasse invariants.*

Proof. Every element in the sequence has no V-shape (in particular no $\sqrt{}$-shape), hence the result follows from Lemma 4.1.3. □

The number of lower neighbours of $w_i^{(d)}$ is exactly $n-1$ for all $1 \le d < n-1$. Furthermore, for $1 \le d < n-1$, the set $E_{w_i^{(d)}}$ can be partitioned into three subsets, namely:

$$\begin{aligned} &E_{w_i^{(d)}} = A \sqcup B \sqcup C \\ &A := \{(j \; j+1) \mid 1 \le j \le n-d-1\} \\ &B := \{(n-d-1 \;\; n+1-j) \mid 1 \le j \le i-1\} \\ &C := \{(n-d \;\; n-d+j) \mid 1 \le j \le d-i+1\}. \end{aligned}$$

For $d = n-1$, the number of lower neighbours of $w_i^{(n-1)}$ is $n-i$, and we have

$$E_{w_i^{(d)}} = \{(1 \; j) \mid 2 \le j \le n-i+1\}.$$

Next, we compute the weight of a Hasse invariant which cuts out the stratum $\overline{Y}_{w_{i+1}^{(d)}}$ in the stratum $\overline{Y}_{w_i^{(d)}}$ for $1 \le i \le d$ and $1 \le d < n-1$. By construction, we have $w_{i+1}^{(d)} = w_i^{(d)} s_{\alpha_i^{(d)}}$ for the root $\alpha_i^{(d)} := e_{n-d} - e_{n+1-i}$. Recall that for any $w \in W$, the Hasse section $\mathrm{Ha}_{w,\chi}$ is a section of $\mathcal{V}_{\mathrm{flag}}(h_w(\chi))$ whose divisor has multiplicity $\langle \chi, \alpha^\vee \rangle$ along $\mathcal{F}_{ws_\alpha}$ for each $\alpha \in E_w$ (see section 2.3). We call $h_w(\chi)$ the weight of $\mathrm{Ha}_{w,\chi}$. Consider the character

$$\chi_i^{(d)} := -e_{d-i+1}.$$

It satisfies

$$\begin{cases} \langle \chi_i^{(d)}, \alpha^\vee \rangle = 1 & \text{for } \alpha = \alpha_i^{(d)} \\ \langle \chi_i^{(d)}, \alpha^\vee \rangle = 0 & \text{for } \alpha \in E_{w_i^{(d)}} \setminus \{\alpha_i^{(d)}\}. \end{cases}$$

Therefore, the partial Hasse invariant $\mathrm{Ha}_i^{(d)}$ on $\overline{\mathcal{F}}_{w_i^{(d)}}$ cuts out with multiplicity one the the stratum $\overline{\mathcal{F}}_{w_{i+1}^{(d)}}$. Similarly, the pullback to Y is a section over $\overline{Y}_{w_i^{(d)}}$ which cuts out the stratum $\overline{Y}_{w_{i+1}^{(d)}}$. We denote the weight of $\mathrm{Ha}_i^{(d)}$ by $\mathrm{ha}_i^{(d)} := h_{w_i^{(d)}}(\chi_i^{(d)})$. We obtain:

$$\mathrm{ha}_i^{(d)} = e_{d-i+1} - q w_{0,I}(e_i).$$

Proposition 4.2.2. *Define $\lambda_{a,b} \in \mathbb{Z}^n$ by $\lambda_{a,b} := e_a - qe_b$ where $1 \le a, b \le n$. Then $\lambda_{a,b} \in C_{L\text{-Min}}$ if and only if $b \le r$.*

Proof. Assume $a \leq r$. Let $x = (x_j)_{1 \leq j \leq s}$ be a finite sequence such that $r \geq x_1 \geq x_2 \geq \cdots \geq x_s \geq 0$. We need to show that $\Gamma_x(\lambda_{a,b}) \leq 0$. Write $\lambda_{a,b} = (y_1, \ldots, y_n)$. We have:

$$\Gamma_x(\lambda_{a,b}) = \sum_{j=r+1}^{n} \left(\sum_{i=1}^{r-x_{j-r}} (y_i - y_j) + \frac{1}{q} \sum_{i=r-x_{j-r}+1}^{r} (y_i - y_j) \right).$$

Since $b \leq r$, the sum $\sum_{i=1}^{d}(y_i - y_j) + \frac{1}{q}\sum_{i=d+1}^{r}(y_i - y_j)$ is ≤ 0 for any $1 \leq d \leq r$. This shows that $\lambda_{a,b} \in C_{L-\mathrm{Min}}$. We leave the converse implication to the reader, as we will not use it. □

Corollary 4.2.3. *For any $d \leq \min(r, n-1)$ and any $1 \leq i \leq d$, one has $\mathrm{ha}_i^{(d)} \in C_{L-\mathrm{Min}}$.*

Proof. We have $\mathrm{ha}_i^{(d)} = e_{d-i+1} - q w_{0,I}(e_i)$. Since $i \leq d \leq r$, we have $w_{0,I}(e_i) \leq r$. The result follows from Proposition 4.2.2. □

Hence, when $(r, s) = (n-1, 1)$, we obtain a path from $\Lambda_1 = w_0$ to Λ_{n-1} such that each element of the sequence admits a system of partial Hasse invariants, and furthermore the weights $\mathrm{ha}_i^{(d)}$ (for all $1 \leq i \leq d \leq n-1$) all lie in $C_{L-\mathrm{Min}}$.

4.3 Hasse-regularity

In the case $(r, s) = (n-1, 1)$, we have $\Lambda_{n-1} = z$. Recall that for a general cocharacter datum (G, μ) over $\mathbb{F}_q$, the element z is defined by $z := \sigma(w_{0,I})w_0$ (see section 2.1). The last ingredient of our proof will be to show that the stratum Y_z is Hasse-regular (Definition 2.4.1). Before we show this, we collect in this section some expectations in the general case.

Let (G, μ) be a general cocharacter datum (G, μ) over $\mathbb{F}_q$ and (X, ζ) satisfying Assumption 2.1.1. In the terminology of [GK19a, Definition 2.4.2], $z := \sigma(w_{0,I})w_0$ is the cominimal element of maximal length. We recall some results from *loc. cit.* about the stratum $\mathcal{F}_z$. First, by [Kos18, Proposition 2.2.1] the projection map $\pi\colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ restricts to a finite etale map $\mathcal{F}_z \to \mathcal{U}_\mu$, where $\mathcal{U}_\mu$ is the open stratum of $G\text{-}\mathtt{Zip}^\mu$. On the Zariski closure, the map $\pi\colon \overline{\mathcal{F}}_z \to G\text{-}\mathtt{Zip}^\mu$ is not finite in general. Similar results hold for the stratum $Y_z \subset Y$ and the projection map $\pi_Y\colon Y_z \to X$. We conjecture the following in general:

Conjecture 4.3.1. *The flag stratum Y_z is Hasse-regular.*

For example, take $G = \mathrm{Res}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\mathrm{GL}_{2,\mathbb{F}_{q^m}})$ endowed with the parabolic $P = B$. This corresponds to the case of Hilbert–Blumenthal Shimura varieties. In this case, the flag space $Y = \mathrm{Flag}(X)$ coincides with X. Hence Y_z is simply the unique open stratum of X, and we have $\overline{Y}_z = X$. In particular, Conjecture 4.3.1 says in this case that $\langle C_X \rangle = C_{\mathrm{Hasse}}$, which was indeed proved in [GK18]. In the case when G is $\mathbb{F}_q$-split, the Hasse cone of z has a simple form:

$$C_{\mathrm{Hasse},z} = \{\lambda \in X^*(T) \mid \langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for all } \alpha \in \Phi^+ \setminus \Phi_{L,+}\}.$$

Furthermore, in this case we expect the following stronger version:

Conjecture 4.3.2. *Assume that G is $\mathbb{F}_q$-split. For any $w \in W$ such that $w \leq z$, the flag stratum Y_w is Hasse-regular.*

Conjecture 4.3.2 holds for Hilbert–Blumenthal Shimura varieties at a split prime p by [GK18]. Furthermore, it also holds for the groups $G = \mathrm{Sp}(4)_{\mathbb{F}_q}$ and $G = \mathrm{GL}_{3,\mathbb{F}_q}$ (in signature $(2,1)$) by *loc. cit.* (§5.2, Figure 1 and Figure 2). For $G = \mathrm{GL}_{4,\mathbb{F}_q}$ with a parabolic of type $(3,1)$, it follows from [GK22, §5.2]. We will generalize the result to the case $G = \mathrm{GL}_{n,\mathbb{F}_q}$ with a parabolic of type $(n-1,1)$ in the next section.

4.4 The unitary case of signature $(n-1,1)$ at split primes

We now return to the case $G = \mathrm{GL}_{n,\mathbb{F}_q}$ and we consider the case $(r,s) = (n-1,1)$. In this case, the element z coincides with Λ_{n-1}. We say that a permutation $w \in S_n$ is z-small if $w \leq z$. Similarly, a stratum Y_w paramatrized by such an element will be called z-small.

4.4.1 Hasse cones of z-small strata

For an integer $m \geq 1$, we consider the $m \times m$-matrix

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}$$

which we simply denote by $[m]$ (when no confusion arises from this notation). Similarly, for a tuple of positive integers $(m_1, \dots, m_k)$, we define

$$[m_1, \dots, m_k] := \begin{pmatrix} [m_1] & & \\ & \ddots & \\ & & [m_k] \end{pmatrix}.$$

By Proposition 4.1.1, the z-small elements of S_n are precisely the permutations of the form $[m_1, \dots, m_k]$ for positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n$. Note that any lower neighbour of a z-small element is again z-small. It is clear that a z-small element admits a system of partial Hasse invariants, because each block $[m_i]$ admits such a system.

We compute the Hasse cone $C_{\mathrm{Hasse},w}$ for each z-small element w. For $w = [m_1, \dots, m_k]$, we put $M_i(w) := \sum_{d=1}^{i} m_i$ for $1 \leq d \leq k$ and $M_0(w) := 0$. If the choice of w is clear, we simply write M_i instead of $M_i(w)$. The set E_w is given by

$$E_w = \bigsqcup_{i=1}^{k} E_w^{(i)}, \quad E_w^{(i)} := \{(M_{i-1}+1 \;\; M_{i-1}+j) \mid 1 < j \leq m_i\}.$$

We say that w' is an i-lower neighbour if it corresponds to an element of $E_w^{(i)}$, i.e if $w' = ws_\alpha$ for $\alpha \in E_w^{(i)}$. In other words, an i-lower neighbour of w amounts to a partition $m_i = a + b$ with $a, b \geq 1$. For $w \in S_n$ z-small, put $\gamma_w := w^{-1}z$. If $w = [m_1, \dots, m_k]$, we have:

$$\gamma_w = (1 \;\; M_{k-1}+1 \;\; M_{k-2}+1 \;\; \dots \;\; M_1+1).$$

In particular, γ_w is a k-cycle, so it has order k in S_n. The cone $\langle C_{\mathrm{Hasse},w} \rangle$ is defined by a number of $|E_w|$ inequalities. The inequality corresponding to $\alpha \in E_w$ is

$$\sum_{d=0}^{k-1} q^{k-1-d} \langle z^{-1}\lambda, \gamma_w^d \alpha^\vee \rangle \geq 0$$

where $\lambda \in \mathbb{Z}^n$. For $\alpha \in E_w$, write $C^\alpha_{\mathrm{Hasse},w}$ for the cone in $\mathbb{Z}^n$ defined by this condition. Therefore, $\langle C_{\mathrm{Hasse},w} \rangle = \bigcap_{\alpha \in E_w} C^\alpha_{\mathrm{Hasse},w}$. To simplify, we always write $z^{-1}\lambda = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Let f be a linear polynomial in the variables $x_1, \dots, \widehat{x_i}, \dots x_n$ (where $\widehat{x_i}$ means that we omit the variable x_i). We write $f(x_1, \dots, \widehat{x_i}, \dots x_n) \leq_i 0$ for the homogeneous inequality $f(x_1 - x_i, \dots, x_n - x_i) \leq 0$. If $w = [m_1, \dots, m_k]$ and $\alpha = (M_{i-1}+1 \;\; M_{i-1}+j)$ for $1 < j \leq m_i$, the corresponding inequality defining $C^\alpha_{\mathrm{Hasse},w}$ is given by

$$\sum_{d=1}^{i-1} q^{k-d} x_{M_{i-d}+1} + q^{k-i} x_1 + \sum_{d=i}^{k-1} q^{d-i} x_{M_d+1} \quad \leq_{M_{i-1}+j} 0.$$

4.4.2 Intersection cones

The goal of this section is to show the following result:

Proposition 4.4.1. *Let $w \in S_n$ be a z-small permutation of length $\ell(w) \geq 2$ and let $\alpha \in E_w$. There exist two lower neighbours w_1, w_2 of w (depending on α) such that*

$$C_{\mathrm{Hasse},w_1} \cap C_{\mathrm{Hasse},w_2} \subset C^{\alpha}_{\mathrm{Hasse},w}.$$

We write $w = [m_1, \ldots, m_k]$ and $\alpha = (M_i + 1 \ \ M_i + j)$ for $0 \leq i < k$ and $1 < j \leq m_{i+1}$. There are several cases to consider.

The case $j \geq 3$. In this case, we show that we may take w_1 and w_2 to be i-lower neighbours of w. Put:

$$w_1 := [m_1, \ldots, m_{i-1}, 1, m_i - 1, m_{i+1}, \ldots, m_k]$$
$$w_2 := [m_1, \ldots, m_{i-1}, j - 1, m_i - j + 1, m_{i+1}, \ldots, m_k]$$

In other words, w_1, w_2 are given respectively by partitioning m_i into $[1, m_i - 1]$ and $[j - 1, m_i - j + 1]$. Note that by assumption $j - 1 \geq 2$. Consider the roots:

$$\alpha_1 := (M_{i-1} + 2 \ \ M_{i-1} + j)$$
$$\alpha_2 := (M_{i-1} + 1 \ \ M_{i-1} + 2).$$

It suffices to show $C^{\alpha_1}_{\mathrm{Hasse},w_1} \cap C^{\alpha_2}_{\mathrm{Hasse},w_2} \subset C^{\alpha}_{\mathrm{Hasse},w}$. The equations satisfied by $C^{\alpha_1}_{\mathrm{Hasse},w_1}$ and $C^{\alpha_2}_{\mathrm{Hasse},w_2}$ are respectively:

$$(E_1): \quad q^k x_{M_{i-1}+2} + \sum_{d=1}^{i-1} q^{k-d} x_{M_{i-d}+1} + q^{k-i} x_1 + \sum_{d=i}^{k-1} q^{d-i} x_{M_d+1} \qquad \leq_{M_{i-1}+j} 0$$

$$(E_2): \quad \sum_{d=0}^{i-2} q^{k-d} x_{M_{i-1-d}+1} + q^{k-i+1} x_1 + \sum_{d=i}^{k-1} q^{d-i+1} x_{M_d+1} + x_{M_{i-1}+j} \qquad \leq_{M_{i-1}+2} 0.$$

Equation (E_1) is very similar to the one defining $C^{\alpha}_{\mathrm{Hasse},w}$, except for the presence of the leading term $q^k x_{M_{i-1}+2}$. We can remove this term by using a linear combination with the second inequality (recall that the variable $x_{M_{i-1}+2}$ appears in (E_2) by definition of the symbol $\leq_{M_{i-1}+2}$). Specifically, put $\delta := \frac{q^k}{\sum_{j=0}^{k} q^j} = \frac{q^k(q-1)}{q^{k+1}-1}$. Since δ is positive, we may form the inequality $(E_1) + \delta(E_2)$. Dividing throughout by $1 + \delta q$, we obtain precisely the inequality for $C^{\alpha}_{\mathrm{Hasse},w}$.

The case $j = 2$ and $m_i > 2$. In this case too, we may take w_1 and w_2 to be i-lower neighbours of w. Put:

$$w_1 := [m_1, \ldots, m_{i-1}, 2, m_i - 2, m_{i+1}, \ldots, m_k]$$
$$w_2 := [m_1, \ldots, m_{i-1}, 1, m_i - 1, m_{i+1}, \ldots, m_k]$$

In other words, w_1, w_2 are given respectively by partitioning m_i into $[2, m_i - 2]$ and $[1, m_i - 1]$. Consider the roots:

$$\alpha_1 := \alpha = (M_{i-1} + 1 \ \ M_{i-1} + 2)$$
$$\alpha_2 := (M_{i-1} + 2 \ \ M_{i-1} + 3).$$

It suffices to show $C^{\alpha_1}_{\text{Hasse},w_1} \cap C^{\alpha_2}_{\text{Hasse},w_2} \subset C^{\alpha}_{\text{Hasse},w}$. The equations satisfied by $C^{\alpha_1}_{\text{Hasse},w_1}$ and $C^{\alpha_2}_{\text{Hasse},w_2}$ are respectively:

$$(E_1): \quad \sum_{d=0}^{i-2} q^{k-d} x_{M_{i-d-1}+1} + q^{k-i+1} x_1 + \sum_{d=i}^{k-1} q^{d-i} x_{M_d+1} + x_{M_{i-1}+3} \qquad \leq_{M_{i-1}+2} 0$$
$$(E_2): \quad q^k x_{M_{i-1}+2} + \sum_{d=1}^{i-1} q^{k-d} x_{M_{i-d}+1} + q^{k-i} x_1 + \sum_{d=i}^{k-1} q^{d-i} x_{M_d+1} \qquad \leq_{M_{i-1}+3} 0.$$

Equation (E_1) is very similar to the one defining $C^{\alpha}_{\text{Hasse},w}$ (multiplied by q), except for the presence of the last term $x_{M_{i-1}+3}$ in (E_1). We can remove this term by using a linear combination with the second equation. Specifically, put $\delta := \frac{1}{\sum_{j=0}^{k} q^j} = \frac{(q-1)}{q^{k+1}-1}$. Since δ is positive, we have the inequality $(E_1) + \delta(E_2)$. Dividing throughout by $1 + \delta q$, we obtain precisely the inequality for $C^{\alpha}_{\text{Hasse},w}$.

The case $j = 2$ and $m_i = 2$. In this case, w admits only one i-lower neighbour, namely

$$w_2 := [m_1, \dots, m_{i-1}, 1, 1, m_{i+1}, \dots, m_k]$$

(which corresponds to the partition $2 = 1 + 1$). Therefore, we need to choose w_1 in a different block. Since we assume $\ell(w) \geq 2$, at least one other m_j is ≥ 2. We take

$$w_1 := [m_1, \dots, m_{j-1}, 1, m_j - 1, m_{j+1}, \dots, m_k]$$

(the j-lower neighbour corresponding to the partition of m_j into $[1, m_j - 1]$). Set:

$$\alpha_1 := \alpha = (M_{i-1} + 1 \quad M_{i-1} + 2)$$
$$\alpha_2 := (M_{j-1} + 1 \quad M_{j-1} + 2).$$

It suffices to show $C^{\alpha_1}_{\text{Hasse},w_1} \cap C^{\alpha_2}_{\text{Hasse},w_2} \subset C^{\alpha}_{\text{Hasse},w}$. Assume first that we can choose $j > i$. The equations satisfied by $C^{\alpha_1}_{\text{Hasse},w_1}$ and $C^{\alpha_2}_{\text{Hasse},w_2}$ are respectively:

$$(E_1): \quad \sum_{d=0}^{i-2} q^{k-d} x_{M_{i-d-1}+1} + q^{k-i+1} x_1 + \sum_{d=j}^{k-1} q^{d-i+1} x_{M_d+1} + q^{j-i} x_{M_{j-1}+2} + \sum_{d=i}^{j-1} q^{d-i} x_{M_d+1} \quad \leq_{M_{i-1}+2} 0.$$
$$(E_2): \quad \sum_{d=0}^{j-i-1} q^{k-d} x_{M_{d+i}+1} + q^{k-j+i} x_{M_{i-1}+2} + \sum_{d=1}^{i-1} q^{k-j+d} x_{M_d+1} + q^{k-j} x_1 + \sum_{d=j}^{k-1} q^{d-j} x_{M_d+1} \quad \leq_{M_{j-1}+2} 0.$$

Equation (E_1) is similar to the one defining $C^{\alpha}_{\text{Hasse},w}$. Specifically, the last terms x_{M_d+1} for $i \leq d \leq j-1$ are the same in both equations. The terms x_{M_d+1} for all other d and for x_1 are multiplied by an extra power of q in equation (E_1). Finally, the term $q^{j-i} x_{M_{j-1}+2}$ in (E_1) does not appear in the equation of $C^{\alpha}_{\text{Hasse},w}$. Using a similar strategy as before, we remove this term by using a linear combination with the second equation (E_2). Put $\delta := \frac{q^{j-i}}{\sum_{d=0}^{k} q^d} = \frac{q^{j-i}(q-1)}{q^{k+1}-1}$. Since δ is positive, we have the inequality $(E_1) +$

$\delta(E_2)$. In this equation, the variable $x_{j-1}+2$ has disappeared. We write the terms in decreasing order of the power of q as they appear in the equation of $C^{\alpha}_{\mathrm{Hasse},w}$, namely $x_{M_{i-1}+1}, x_{M_{i-2}+1}, \dots, x_1, x_{M_{k-1}}, \dots, x_{M_j+1}, x_{M_{j-1}+1}, \dots, x_{M_i+1}$. One sees immediately that the coefficents which appear in front of these terms in $(E_1)+\delta(E_2)$ are divided by q at each step between the terms $x_{M_{i-1}+1}$ and x_{M_j+1}, and between $x_{M_{j-1}+1}$ and x_{M_i+1}. It remains to show that the same happens between the terms x_{M_j+1} and $x_{M_{j-1}+1}$. The coefficient of x_{M_j+1} is $q^{j-i+1}+\delta$, and the coefficient of $x_{M_{j-1}+1}$ is $q^{j-i-1}+\delta q^k$. Since $\delta = \frac{q^{j-i}(q-1)}{q^{k+1}-1}$, one has indeed $q^{j-i+1}+\delta = q(q^{j-i-1}+\delta q^k)$. This shows that the equation $(E_1)+\delta(E_2)$ is a positive multiple of the equation for $C^{\alpha}_{\mathrm{Hasse},w}$.

It remains to treat the case when there is no $j > i$ such that $m_j \geq 2$. We choose $j < i$ with $m_j \geq 2$, and define w_1, w_2, α_1, α_2 as before. The equations satisfied by $C^{\alpha_1}_{\mathrm{Hasse},w_1}$ and $C^{\alpha_2}_{\mathrm{Hasse},w_2}$ are respectively:

$$(E_1): \quad \sum_{d=j}^{i-1} q^{k-i+d+1} x_{M_d+1} + q^{k-i+j} x_{M_{j-1}+2} + \sum_{d=1}^{j-1} q^{k-i+d} x_{M_d+1} + q^{k-i} x_1 + \sum_{d=i}^{k-1} q^{d-i} x_{M_d+1} \leq_{M_{i-1}+2} 0.$$

$$(E_2): \quad \sum_{d=1}^{j-1} q^{k-j+d+1} x_{M_d+1} + q^{k-j+1} x_1 + \sum_{d=i}^{k-1} q^{d-j+1} x_{M_d+1} + q^{i-j} x_{M_{i-1}+2} + \sum_{d=j}^{i-1} q^{d-j} x_{M_d+1} \leq_{M_{j-1}+2} 0.$$

As before, we remove the term $x_{M_{j-1}+2}$ in (E_1) using (E_2). Put $\delta := \frac{q^{k-i+j}}{\sum_{d=0}^{k} q^d} = \frac{q^{k-i+j}(q-1)}{q^{k+1}-1}$ and consider $(E_1)+\delta(E_2)$. Again, the coefficients of $x_{M_{i-1}+1}, x_{M_{i-2}+1}, \dots, x_1, x_{M_{k-1}}, \dots, x_{M_j+1}, x_{M_{j-1}+1}, \dots, x_{M_i+1}$ (in this order) are divided by q at each step, except perhaps for the coefficients of x_{M_j+1} and $x_{M_{j-1}+1}$. The former is $q^{k-i+1+j}+\delta$ and the latter is $q^{k-i+j-1}+q^k\delta$. Again, we have $q^{k-i+1+j}+\delta = q(q^{k-i+j-1}+q^k\delta)$ by definition of δ. This shows the result.

4.4.3 Main result

Our first main result is the strong version of the Hasse-regularity conjecture (see Conjecture 4.3.2) for unitary Shimura varieties of good reduction at a split prime. More generally, we take (X, ζ) to be an arbitrary pair satisfying Assumption 2.1.1.

Theorem 4.4.2. *Assume $G = \mathrm{GL}_{n,\mathbb{F}_q}$ and $(r,s) = (n-1,1)$. For any z-small element $w \in S_n$, the flag stratum Y_w is Hasse-regular.*

Proof. Since all z-small strata admit a system of Hasse invariants (Definition 4.1.2), we may construct a separating system $\mathbb{E} = (\mathbb{E}_w)_{w\in W}$ as follows. For z-small elements $w \in W$, we set $\mathbb{E}_w = E_w$ and we let $\{\chi\}_{\alpha\in E_w}$ be any system of characters satisfying Conditions (a) and (b) of Definition 2.5.1. For w not z-small, we set $\mathbb{E}_w = \emptyset$. We show by induction on $\ell(w)$ that for all z-small element w, the intersection cone $C^{+,\mathbb{E}}_w$ satisfies $C^{+,\mathbb{E}}_w \subset \langle C_{\mathrm{Hasse},w}\rangle$. For $\ell(w) = 1$ the result holds by Lemma 2.4.2. Suppose the result holds for all z-small strata of length $\leq d$ and let w be a z-small element of length $\ell(w) = d+1$. By Proposition 4.4.1, we obtain

$$\bigcap_{\alpha\in E_w} C^{+,\mathbb{E}}_{ws_\alpha} \subset \bigcap_{\alpha\in E_w} \langle C_{\mathrm{Hasse},ws_\alpha}\rangle \subset \langle C_{\mathrm{Hasse},w}\rangle$$

Since we clearly have $C^{\mathbb{E}}_{\mathrm{Hasse},w} \subset C_{\mathrm{Hasse},w}$, we deduce $C^{+,\mathbb{E}}_w \subset \langle C_{\mathrm{Hasse},w}\rangle$, which proves the result. By Theorem 2.5.3, we deduce that for any z-small element, $\langle C_{Y,w}\rangle = \langle C^{+,\mathbb{E}}_w\rangle = \langle C_{\mathrm{Hasse},w}\rangle$. This terminates the proof. □

In particular, for the element $w = z$, we deduce the following:

Corollary 4.4.3. *We have* $\langle C_{Y,z}\rangle = \{(k_1,\dots,k_n) \in \mathbb{Z}^n \mid k_i - k_n \leq 0 \text{ for all } i = 1,\dots,n\}$.

We also deduce from Theorem 4.4.2 the following approximation of the cone $\langle C_K(\overline{\mathbb{F}}_p)\rangle$:

Theorem 4.4.4. *We have $C_K(\overline{\mathbb{F}}_p) \subset C_{L-\mathrm{Min}}$. In other words, the weight $(k_1,\dots,k_n)$ of any nonzero mod p automorphic form satisfies:*

$$\sum_{i=1}^{j}(k_i - k_n) + \frac{1}{p}\sum_{i=j+1}^{n-1}(k_i - k_n) \leq 0 \quad \text{for all } j = 1,\dots,n-1.$$

Proof. We consider the sequence $(w_i^{(d)})_{i,d}$ for $1 \leq i \leq d+1$ and $1 \leq d < n-1$ which defines a path (in the terminology of section 4.2) from $\Lambda_1 = w_0$ to $\Lambda_{n-1} = z$. By Corollary 4.4.3, we have $C_{Y,z} \subset C_{L-\mathrm{Min}}$. Furthermore, by Corollary 4.2.3, the weight of the partial Hasse invariant $\mathrm{Ha}_i^{(d)}$ which cuts out the stratum $Y_{w^{(d)}_{i+1}}$ (for $1 \leq i \leq d$) in the closure of $Y_{w^{(d)}_i}$ lies in $C_{L-\mathrm{Min}}$. We deduce that $C_{Y,w} \subset C_{L-\mathrm{Min}}$ for each w in the chain. In particular, the result holds for w_0, which terminates the proof. □

Theorem 4.4.4 illustrates again the connection between group theory and geometry of Shimura varieties: The cone $C_{L-\mathrm{Min}}$ originates from a unipotent-invariance condition for automorphic forms on $G\text{-}\mathtt{Zip}^{\mu}$. Theorem 4.4.4 and its proof show that this condition also appears geometrically as a relationship between the flag strata of a Shimura variety.

Finally, we note that Theorem 4.4.4 provides a second, more precise proof of the containment $C_K(\mathbb{C}) \subset C_{\mathrm{GS}}$. Indeed, let $\mathscr{S}_K$ be an integral Shimura variety of Hodge-type of unitary type and signature $(n-1,1)$. At each split prime p of good reduction, we have $C_K(\overline{\mathbb{F}}_p) \subset C_{L-\mathrm{Min},p}$, where $C_{L-\mathrm{Min},p}$ denotes the L-minimal cone of the induced zip datum at p. We obtain

$$C_K(\mathbb{C}) \subset \bigcap_{\text{split } p} C_K(\overline{\mathbb{F}}_p) \subset \bigcap_{\text{split } p} C^{+,I}_{L-\mathrm{Min},p} = C_{\mathrm{GS}}.$$

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