

A vanishing theorem for vector-valued Siegel automorphic forms in characteristic p

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Abstract

We show that the space of vector-valued Siegel automorphic forms in characteristic p is zero when the weight is outside of an explicit locus. This result is a special case of a general conjecture about Hodge-type Shimura varieties formulated in previous work with W. Goldring.

Introduction

To study automorphic forms attached to a reductive group \mathbf{G} over \mathbb{Q} , it is often useful to view them as global section of certain vector bundles on a Shimura variety. Of course, this is only possible when the group \mathbf{G} admits such a variety. In this paper, we investigate automorphic forms attached to the group $\mathbf{G} = \mathrm{GSp}_{2n, \mathbb{Q}}$, called Siegel automorphic forms. We are particularly interested in the properties of automorphic forms with coefficients in $\overline{\mathbb{F}}_p$, for a prime number p . We will see that the geometric properties of the attached Siegel-type Shimura variety can be translated on the level of automorphic forms. This geometric approach was used in several previous papers joint with Wushi Goldring ([GK19a], [GK18], [GK22a], [GK22b]).

Let (\mathbf{G}, \mathbf{X}) be a Shimura datum of Hodge-type, in the sense of [Del79]. In particular, \mathbf{G} is a connected, reductive group over \mathbb{Q} . If $K \subset \mathbf{G}(\mathbb{A}_f)$ is a compact open subgroup, we denote by $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ the attached Shimura variety of level K . It is an algebraic variety defined over a number field \mathbf{E} , called the reflex field of the Shimura datum. For example, the Siegel-type Shimura variety $\mathcal{A}_{n, K}$ is a fundamental example of a Shimura variety, it parametrizes principally polarized abelian varieties of rank n endowed with a K -level structure. Let p be a prime of good reduction for (\mathbf{G}, \mathbf{X}) . By this we mean that the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified at p , and that K is of the form $K = K_p K^p$ with $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ hyperspecial and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ open compact. In particular, we can write $K_p = \mathbf{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ for a connected reductive \mathbb{Z}_p -model $\mathbf{G}_{\mathbb{Z}_p}$ of $\mathbf{G}_{\mathbb{Q}_p}$. By work of Kisin ([Kis10]) and Vasiu ([Vas99]), $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ admits a smooth canonical model \mathcal{S}_K over $\mathcal{O}_{\mathbf{E}_p}$ for any place $\mathfrak{p}|p$ of \mathbf{E} . We denote by $S_K := \mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} \overline{\mathbb{F}}_p$ its special fiber.

Let $\mu: \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$ be the cocharacter deduced from the Shimura datum (see section 2.3). It defines a parabolic subgroup $\mathbf{P} \subset \mathbf{G}_{\mathbb{C}}$ such that the centralizer of μ is a Levi subgroup $\mathbf{L} \subset \mathbf{P}$. Choose a Borel subgroup $\mathbf{B} \subset \mathbf{P}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$. Let Φ_+ denote the positive \mathbf{T} -roots with respect to \mathbf{B} , and $\Delta \subset \Phi_+$ the simple roots. Write $I := \Delta_{\mathbf{L}} \subset \Delta$ for the simple roots of \mathbf{L} and $\Phi_{\mathbf{L}, +}$ for the positive roots of \mathbf{L} . For any character $\lambda \in X^*(\mathbf{T})$, there is an automorphic vector bundle $\mathcal{V}_I(\lambda)$ on \mathcal{S}_K , modeled on the induced representation $\mathbf{V}_I(\lambda) := \mathrm{Ind}_{\mathbf{B}}^{\mathbf{P}}(\lambda)$ (see section 3.1 for details). For any $\mathcal{O}_{\mathbf{E}_p}$ -algebra R , we call elements of $H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} R, \mathcal{V}_I(\lambda))$ automorphic forms of level K and weight λ with coefficients in R .

In the papers [GK18, GK22a], Goldring and the author studied the set of $\lambda \in X^*(\mathbf{T})$ such that the space $H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} R, \mathcal{V}_I(\lambda))$ is nonzero. We denote this set by $C_K(R)$. It is an additive submonoid (i.e. a "cone") inside $X^*(\mathbf{T})$. This set depends on the level K , but its saturation $\langle C_K(R) \rangle$ does not. By the saturation $\langle C \rangle$ of a cone $C \subset X^*(\mathbf{T})$, we mean the set of $\lambda \in X^*(\mathbf{T})$ such that some positive multiple of λ lies in C . When $R = \mathbb{C}$ we proved in [GK22b] that $C_K(\mathbb{C})$ is contained in the Griffiths–Schmid cone, defined by

$$C_{\text{GS}} = \left\{ \lambda \in X^*(\mathbf{T}) \mid \begin{array}{l} \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for } \alpha \in I, \\ \langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for } \alpha \in \Phi_+ \setminus \Phi_{\mathbf{L},+} \end{array} \right\}.$$

It is expected that $\langle C_K(\mathbb{C}) \rangle = C_{\text{GS}}$ (this equality seems to be well-known to experts).

In this paper, we are interested in the case when $R = \overline{\mathbb{F}}_p$. In this case, there is no simple, general description of $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ in terms of the root datum. We conjectured that the cone $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ is entirely determined by the stack $G\text{-Zip}^\mu$ (the stack of G -zips), introduced by Moonen–Wedhorn ([MW04]) and Pink–Wedhorn–Ziegler ([PWZ11], [PWZ15]). Here G denotes the \mathbb{F}_p -reductive group $G := \mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. The stack $G\text{-Zip}^\mu$ is a smooth, finite stack over $\overline{\mathbb{F}}_p$. By a result of Zhang ([Zha18]), there is a natural smooth map

$$\zeta: S_K \rightarrow G\text{-Zip}^\mu.$$

The vector bundles $\mathcal{V}_I(\lambda)$ also exist on the stack $G\text{-Zip}^\mu$. Thus, we may consider the set C_{zip} of $\lambda \in X^*(\mathbf{T})$ such that $\mathcal{V}_I(\lambda)$ admits nonzero global sections on $G\text{-Zip}^\mu$. Pullback via ζ yields an inclusion $C_{\text{zip}} \subset C_K(\overline{\mathbb{F}}_p)$. We conjectured ([GK18, Conjecture C]):

Conjecture 1. *One has $\langle C_K(\overline{\mathbb{F}}_p) \rangle = \langle C_{\text{zip}} \rangle$.*

Goldring and the author proved this conjecture in *loc. cit.* for Hilbert–Blumenthal Shimura varieties, Picard modular surfaces at split primes and Siegel threefolds. We treated the case of unitary Shimura varieties attached to $\text{GU}(r, s)$ for $r + s \leq 4$ as well as GSp_6 in [GK22a] (with the exception of $\text{GU}(2, 2)$ at an inert prime). Since the cone C_{zip} is in general difficult to determine, Conjecture 1 does not give an explicit expression for $\langle C_K(\overline{\mathbb{F}}_p) \rangle$. However, in the paper [GK22b], we showed that Conjecture 1 can be used to determine a very explicit conjectural approximation (from above) for these cones. Assume for simplicity that G is split over \mathbb{F}_{p^2} and that p is split in \mathbf{E} . Let $W_{\mathbf{L}} = W(\mathbf{L}, \mathbf{T})$ be the Weyl group of \mathbf{L} . Note that $W_{\mathbf{L}} \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ acts naturally on the set $\Phi_+ \setminus \Phi_{\mathbf{L},+}$. Conjecture 1 implies the following, much more explicit conjecture.

Conjecture 2. *Let S_K be the special fiber of a Hodge-type Shimura variety at a prime p of good reduction which splits in the reflex field \mathbf{E} . Assume that the attached reductive \mathbb{F}_p -group G is split over \mathbb{F}_{p^2} . Then, if $f \in H^0(S_K, \mathcal{V}_I(\lambda))$ is a nonzero automorphic form of weight $\lambda \in X^*(\mathbf{T})$, we have*

$$\sum_{\alpha \in \mathcal{O} \setminus S} \langle \lambda, \alpha^\vee \rangle + \frac{1}{p} \sum_{\alpha \in S} \langle \lambda, \alpha^\vee \rangle \leq 0$$

for all $W_{\mathbf{L}} \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ -orbits $\mathcal{O} \subset \Phi_+ \setminus \Phi_{\mathbf{L},+}$ and all subsets $S \subset \mathcal{O}$.

The above inequalities are rather sharp, and define a good approximation (from above) of the set $\langle C_{\text{zip}} \rangle$, hence also conjecturally of $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ (see [GK22a, Figure 1] for a visual illustration of these cones in the case $G = \text{GSp}_6$). The assumption that p is split in \mathbf{E} implies that the ordinary locus of S_K is non-empty. We proved that Conjecture 2 holds for unitary Shimura varieties of signature $(n - 1, 1)$ at split primes in [GK22b]. We prove in

this paper that Conjecture 2 holds for Siegel-type Shimura varieties. However, we cannot prove the stronger Conjecture 1 for these varieties (except for $n \leq 3$). In the case of GSp_{2n} , characters are parametrized by an $n+1$ -tuple $\lambda = (a_1, \dots, a_n, b) \in \mathbb{Z}^{n+1}$ satisfying $\sum_{i=1}^n a_i \equiv b \pmod{2}$ (where the natural multiplier of GSp_{2n} corresponds to $(0, \dots, 0, 2)$). Concretely, we show the following:

Theorem 1. *Conjecture 2 holds for good reduction Siegel-type Shimura varieties $\mathcal{A}_{n,K}$. Specifically, if $f \in H^0(\mathcal{A}_{n,K} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \mathcal{V}_I(\lambda))$ is a nonzero mod p automorphic of weight $\lambda = (a_1, \dots, a_n, b)$, we have:*

$$\sum_{i=1}^j a_i + \frac{1}{p} \sum_{i=j+1}^n a_i \leq 0 \quad \text{for all } j = 1, \dots, n-1. \quad (1)$$

Equivalently, if for any $j \in \{1, \dots, n-1\}$ the inequality (1) is not satisfied, then the space $H^0(\mathcal{A}_{n,K} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \mathcal{V}_I(\lambda))$ is zero. Therefore, we may view the above result as a vanishing theorem for the cohomology of Shimura varieties in degree 0. The proof of Theorem 1 uses the flag space $\mathrm{Flag}(\mathcal{A}_{n,K})$ of $\mathcal{A}_{n,K}$. It is a moduli space introduced by Ekedahl–van der Geer in [EvdG09], which parametrizes pairs (x, \mathcal{F}_\bullet) , where x is a point of $\mathcal{A}_{n,K}$ and \mathcal{F}_\bullet is a full symplectic flag in the de Rham cohomology of the abelian variety at x , which refines the Hodge filtration. There is a natural projection map

$$\pi: \mathrm{Flag}(\mathcal{A}_{n,K}) \rightarrow \mathcal{A}_{n,K}, \quad ((x, \mathcal{F}_\bullet) \mapsto x.$$

For any character $\lambda \in X^*(\mathbf{T})$, there is a line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $\mathrm{Flag}(\mathcal{A}_{n,K})$ such that $\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}(\lambda)$. In particular, automorphic forms of weight λ coincide with global sections of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on the flag space. The special fiber $\mathrm{Flag}_K := \mathrm{Flag}(\mathcal{A}_{n,K}) \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$ admits a stratification $(\mathrm{Flag}_{K,w})_{w \in W}$ where w varies in the Weil group W of $G = \mathrm{GSp}_{2n}$. Write $w_{\max} = w_{0,I}w_0$, where w_0 and $w_{0,I}$ are the longest elements of W and $W_{\mathbf{I}}$ respectively. The following theorem is the first step in the proof of Theorem 1.

Theorem 2. *Let $f \in H^0(\overline{\mathrm{Flag}}_{K,w_{\max}}, \mathcal{V}_{\mathrm{flag}}(\lambda))$ be a nonzero section over the Zariski closure of the flag stratum $\mathrm{Flag}_{K,w_{\max}}$. If we write $\lambda = (a_1, \dots, a_n, b)$, we have $a_i \leq 0$ for all $i = 1, \dots, n$.*

The above result is proved by embedding Hilbert–Blumenthal Shimura varieties (attached to a totally real field \mathbf{F} where p splits) into $\mathcal{A}_{n,K}$. This embedding lifts naturally to an embedding into the flag space of $\mathcal{A}_{n,K}$, and on the special fiber the image is contained in the Zariski closure $\overline{\mathrm{Flag}}_{K,w_{\max}}$. Since Conjecture 1 holds for Hilbert–Blumenthal varieties by previous results, we deduce that any section $f \in H^0(\overline{\mathrm{Flag}}_{K,w_{\max}}, \mathcal{V}_{\mathrm{flag}}(\lambda))$ vanishes on the image of the embedding. The final step to prove Theorem 2 is to use Hecke operators to show that f vanishes everywhere. For this, we consider the stable base locus $\mathbb{B}_{K^p}(\lambda)$ of $\mathcal{V}_{\mathrm{flag}}(\lambda)$ (i.e. the intersection of the base loci of all positive powers of $\mathcal{V}_{\mathrm{flag}}(\lambda)$) and we show:

Theorem 3. *The set $\mathbb{B}_{K^p}(\lambda) \subset S_K$ is stable by prime-to- p Hecke operators.*

Finally, we use a well-known Theorem of Chai on the density of prime-to- p Hecke orbits of ordinary points to terminate the proof of Theorem 2. Lastly, to prove Theorem 1, we use the flag stratum $\mathrm{Flag}_{K,w_{\max}}$ as a starting point and propagate the inequalities afforded by Theorem 2 using an appropriately chosen increasing sequence of elements in W , until we reach the maximal element w_0 .

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1 Shimura varieties

1.1 Siegel-type Shimura varieties

Let $n \geq 1$ be an integer and let $\Lambda = \mathbb{Z}^{2n}$, endowed with the symplectic pairing $\Psi: \Lambda \times \Lambda \rightarrow \mathbb{Z}$ defined by the $2n \times 2n$ symplectic matrix

$$\begin{pmatrix} & J_0 \\ -J_0 & \end{pmatrix}, \quad \text{where} \quad J_0 = \begin{pmatrix} & & 1 \\ & \ddots & \\ & & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}).$$

Let GSp_{2n} be the reductive group over \mathbb{Z} such that for any \mathbb{Z} -algebra R , we have

$$\mathrm{GSp}_{2n}(R) = \{g \in \mathrm{GL}_R(\Lambda \otimes_{\mathbb{Z}} R) \mid \exists c(g) \in R^\times, \Psi_R(gx, gy) = c(g)\Psi_R(x, y)\}.$$

The map $\mathrm{GSp}_{2n} \rightarrow \mathbb{G}_m$, $g \mapsto c(g)$ is the multiplier character. The subgroup defined by the condition $c(g) = 1$ is denoted by Sp_{2n} . Let \mathcal{H}_n^\pm be the set of symmetric complex matrices of size $n \times n$ whose imaginary part is positive definite or negative definite. The pair $(\mathrm{GSp}_{2n, \mathbb{Q}}, \mathcal{H}_n^\pm)$ is called a Shimura datum of Siegel-type.

We fix a prime number p and set $K_p := \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. It is a hyperspecial subgroup of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$. For any open compact subgroup $K^p \subset \mathrm{GSp}_{2n}(\mathbb{A}_f^p)$, write $K = K_p K^p \subset \mathrm{GSp}_{2n}(\mathbb{A}_f)$ and let $\mathcal{A}_{n, K}$ be stack over $\mathbb{Z}_{(p)}$ such that for any $\mathbb{Z}_{(p)}$ -scheme S , the S -valued points of $\mathcal{A}_{n, K}$ parametrize the triples $(A, \lambda, \bar{\eta})$ satisfying the following conditions:

- A is an abelian scheme of relative dimension n over S .
- $\lambda: A \rightarrow A^\vee$ is a $\mathbb{Z}_{(p)}$ -multiple of a polarization whose degree is prime to p .
- $\eta: \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{A}_f^p \rightarrow H^1(A, \mathbb{A}_f^p)$ is an isomorphism of sheaves of \mathbb{A}_f^p -modules on S . We impose that η is compatible with the symplectic pairings induced by Ψ and λ . We write $\bar{\eta} = \eta K^p$ for the K^p -orbit of η .

For K^p small enough, $\mathcal{A}_{n, K}$ is a smooth, quasi-projective $\mathbb{Z}_{(p)}$ -scheme of relative dimension $\frac{n(n+1)}{2}$. We will always make this assumption on K^p . We are interested in the geometry of the special fiber $\mathcal{A}_{n, K} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$. We fix an algebraic closure $k = \overline{\mathbb{F}}_p$ and define:

$$\overline{\mathcal{A}}_{n, K} := \mathcal{A}_{n, K} \otimes_{\mathbb{Z}_{(p)}} k.$$

1.2 Hodge-type Shimura varieties

More generally, let (\mathbf{G}, \mathbf{X}) be a Shimura datum of Hodge-type [Del79, 2.1.1]. Recall that this means that there is an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \rightarrow (\mathrm{GSp}_{2n, \mathbb{Q}}, \mathcal{H}_n^\pm)$ for some $n \geq 1$. Here \mathbf{G} is a connected, reductive group over \mathbb{Q} . The symmetric domain \mathbf{X} gives rise to a well-defined $\mathbf{G}(\overline{\mathbb{Q}})$ -conjugacy class of cocharacters $[\mu]$ of $\mathbf{G}_{\overline{\mathbb{Q}}}$. Write $\mathbf{E} = \mathbf{E}(\mathbf{G}, \mathbf{X})$ for the reflex field of (\mathbf{G}, \mathbf{X}) (i.e. the field of definition of $[\mu]$). For any open compact subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ be Deligne's canonical model ([Del79]) at level K defined over \mathbf{E} . When K is small enough, $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ is a smooth, quasi-projective scheme over \mathbf{E} , and its \mathbb{C} -valued points are given by

$$\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash (\mathbf{G}(\mathbb{A}_f) / K \times \mathbf{X}).$$

Let p be a prime of good reduction. By this, we mean that K can be written in the form $K = K_p K^p$ where $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ is hyperspecial and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is open compact. We will only consider open compact subgroups of this form. In particular, the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified and there exists a connected reductive \mathbb{Z}_p -model $\mathbf{G}_{\mathbb{Z}_p}$ such that $K_p = \mathbf{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$. For any

place \mathfrak{p} above p in \mathbf{E} , Kisin ([Kis10]) and Vasiu ([Vas99]) constructed a smooth canonical model \mathcal{S}_K of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ over $\mathcal{O}_{\mathbf{E}_{\mathfrak{p}}}$. The embedding $(\mathbf{G}, \mathbf{X}) \rightarrow (\mathrm{GSp}_{2n, \mathbb{Q}}, \mathcal{H}_n^{\pm})$ induces an embedding $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \rightarrow \mathcal{A}_{n, K_0} \otimes_{\mathbb{Z}_{(p)}} \mathbf{E}_{\mathfrak{p}}$ for a compatible open compact subgroups $K_0 \subset \mathrm{GSp}_{2n}(\mathbb{A}_f)$. By [Kis10, Theorem 2.4.8], \mathcal{S}_K is the normalization of the scheme-theoretical image of $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ inside \mathcal{A}_{n, K_0} . Put

$$S_K := \mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_{\mathfrak{p}}}} k$$

Let $K'^p \subset K^p$ be two compact open subgroups of $\mathbf{G}(\mathbb{A}_f^p)$. There is a natural projection morphism

$$\pi_{K', K}: \mathcal{S}_{K'} \rightarrow \mathcal{S}_K \quad (1.2.1)$$

which is finite etale. We call $\pi_{K', K}$ the change-of-level map.

1.3 Hecke operators

We review the definition of Hecke operators on \mathcal{S}_K . For any $g \in \mathbf{G}(\mathbb{A}_f^p)$, consider the compact open subgroup $K' = K \cap gKg^{-1}$. There are two natural maps $\mathcal{S}_{K \cap gKg^{-1}} \rightarrow \mathcal{S}_K$. The first one is the natural change-of-level map $\pi_{K', K}$ defined in (1.2.1). The second one is induced on \mathbb{C} -points by the map

$$\mathbf{G}(\mathbb{A}_f)/K' \rightarrow \mathbf{G}(\mathbb{A}_f)/K, \quad xK' \mapsto gxK.$$

We denote the induced map simply by $g: \mathcal{S}_{K \cap gKg^{-1}} \rightarrow \mathcal{S}_K$. This construction yields a correspondance with finite etale maps

$$\begin{array}{ccc} & \mathcal{S}_{K \cap gKg^{-1}} & \\ \pi \swarrow & & \searrow g \\ \mathcal{S}_K & & \mathcal{S}_K \end{array} \quad (1.3.1)$$

The Hecke algebra $\mathbb{T}_{K^p}(\mathbf{G})$ (with coefficients in \mathbb{Z}) attached to K^p is the ring of K^p -bi-invariant, locally constant functions $\mathbf{G}(\mathbb{A}_f^p) \rightarrow \mathbb{Z}$ with compact support. Any such function can be written as a sum of characteristic functions of double cosets $K^p g K^p$ ($g \in \mathbf{G}(\mathbb{A}_f^p)$). Multiplication is defined by convolution with respect to the left-Haar measure ν normalized by $\nu(K^p) = 1$. The algebra $\mathbb{T}_{K^p}(\mathbf{G})$ acts on several natural objects related to \mathcal{S}_K via the correspondance (1.3.1). We will only consider the action on 0-cycles on the special fiber of \mathcal{S}_K . Concretely, if $x \in S_K(k)$ and $g \in \mathbf{G}(\mathbb{A}_f^p)$, we define

$$T_{K^p, g}(x) = g_* \pi^*(x). \quad (1.3.2)$$

We view $T_{K^p, g}(x)$ as a formal sum of points of $S_K(k)$. We denote by $\mathcal{H}^p(x)$ the set of all points of $S_K(k)$ appearing in the support of the 0-cycle $T_{K^p, g}(x)$ for some $g \in \mathbf{G}(\mathbb{A}_f^p)$. We call $\mathcal{H}^p(x)$ the prime-to- p Hecke orbit of x . Denote by S_K^{ord} the ordinary locus of S_K , i.e. the set of points $x \in S_K$ whose image in $\mathcal{A}_{n, K_0} \otimes_{\mathbb{Z}_{(p)}} k$ corresponds to an ordinary abelian variety. S_K^{ord} is nonempty if and only if $\mathbf{E}_{\mathfrak{p}} = \mathbb{Q}_p$. By [vH, Theorem I], one has the following theorem:

Theorem 1.3.1. *Assume that $\mathbf{E}_{\mathfrak{p}} = \mathbb{Q}_p$. For any point $x \in S_K^{\mathrm{ord}}$, the set $\mathcal{H}^p(x)$ is Zariski dense in S_K .*

In the case of Siegel-type Shimura varieties, Theorem 1.3.1 was first proved by Chai in [Cha95].

1.4 Hilbert–Blumenthal Shimura varieties

In this section, we recall the definition of Hilbert–Blumenthal Shimura varieties (also known as Hilbert modular varieties). Let \mathbf{F}/\mathbb{Q} be a totally real extension of degree n . We denote by \mathbf{H} the reductive \mathbb{Q} -group whose R -points (for any \mathbb{Q} -algebra R) are defined by

$$\mathbf{H}(R) = \{g \in \mathrm{GL}_2(\mathbf{F} \otimes_{\mathbb{Q}} R) \mid \det(g) \in R^\times\}.$$

It is a subgroup of the Weil restriction $\mathrm{Res}_{\mathbf{F}/\mathbb{Q}}(\mathrm{GL}_{2,\mathbf{F}})$. We assume that p is unramified in \mathbf{F} . In this case, the group \mathbf{H} is unramified at p and the lattice $\mathcal{O}_{\mathbf{F}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset \mathbf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ yields a reductive \mathbb{Z}_p -model $\mathbf{H}_{\mathbb{Z}_p}$ of $\mathbf{H}_{\mathbb{Q}_p}$. We set $K'_p := \mathbf{H}_{\mathbb{Z}_p}(\mathbb{Z}_p)$. It is a hyperspecial subgroup of $\mathbf{H}(\mathbb{Q}_p)$. For any open compact subgroup $K'^p \subset \mathbf{H}(\mathbb{A}_f^p)$, we write $K' := K'_p K'^p$. The Hilbert–Blumenthal Shimura variety $\mathcal{H}_{\mathbf{F},K'}$ of level K' is the $\mathbb{Z}_{(p)}$ -stack whose S -valued points (for any $\mathbb{Z}_{(p)}$ -scheme S) parametrize quadruplets $(A, \lambda, \iota, \bar{\eta})$ satisfying the following conditions:

- A is an abelian scheme of relative dimension n over S .
- $\lambda: A \rightarrow A^\vee$ is a $\mathbb{Z}_{(p)}$ -multiple of a polarization of degree prime to p .
- $\iota: \mathcal{O}_{\mathbf{F}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \rightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a ring homomorphism.
- $\eta: (\mathbf{F} \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^2 \rightarrow H^1(A, \mathbb{A}_f^p)$ is an \mathbf{F} -linear isomorphism of sheaves of \mathbb{A}_f^p -modules on S , and $\bar{\eta} = \eta K'^p$ is the K'^p -orbit of η .

The homomorphism ι yields an action of $\mathcal{O}_{\mathbf{F}}$ on the dual abelian variety A^\vee . We impose that the polarization λ be $\mathcal{O}_{\mathbf{F}}$ -linear for this action.

Next, we construct a morphism $\mathcal{H}_{\mathbf{F},K'} \rightarrow \mathcal{A}_{n,K}$ for an appropriate choice of level structures $K'^p \subset \mathbf{H}(\mathbb{A}_f^p)$ and $K^p \subset \mathrm{GSp}_{2n,\mathbb{Q}}$. Write $\mathbf{G} := \mathrm{GSp}_{2n,\mathbb{Q}}$ and $\mathbf{G}_{\mathbb{Z}_p} = \mathrm{GSp}_{2n,\mathbb{Z}_p}$. First, we construct an embedding $\mathbf{H} \rightarrow \mathbf{G}$ of reductive \mathbb{Q} -groups. Consider the symplectic form

$$\psi_0: \mathbf{F}^2 \times \mathbf{F}^2 \rightarrow \mathbb{Q}, \quad ((x_1, y_1), (x_2, y_2)) \mapsto \mathrm{Tr}_{\mathbf{F}/\mathbb{Q}}(x_1 y_2 - x_2 y_1).$$

For any matrix $A \in \mathrm{GL}_2(\mathbf{F})$ such that $\det(A) \in \mathbb{Q}^\times$ and $x, y \in \mathbf{F}^2$, one has

$$\Psi_0(Ax, Ay) = \det(A) \Psi_0(x, y).$$

Hence, we have an inclusion $\mathbf{H} \subset \mathrm{GSp}(\Psi_0)$. Fix an isomorphism $\gamma: (\mathbf{F}^2, \Psi_0) \rightarrow (\mathbb{Q}^{2n}, \Psi)$ of symplectic spaces over \mathbb{Q} . We obtain an embedding of reductive \mathbb{Q} -groups

$$u: \mathbf{H} \rightarrow \mathbf{G}, \quad f \mapsto \gamma \circ f \circ \gamma^{-1}. \quad (1.4.1)$$

Since p is unramified in \mathbf{F} , we may further assume that γ induces an isomorphism $\mathcal{O}_{\mathbf{F}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then, $u_{\mathbb{Q}_p}$ extends to an embedding $u_{\mathbb{Z}_p}: \mathbf{H}_{\mathbb{Z}_p} \rightarrow \mathbf{G}_{\mathbb{Z}_p}$ of reductive \mathbb{Z}_p -groups. For any compact open subgroups $K'^p \subset \mathbf{H}(\mathbb{A}_f^p)$ and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ such that $u(K'^p) \subset K^p$, there is a natural morphism of $\mathbb{Z}_{(p)}$ -schemes

$$\tilde{u}_{K',K}: \mathcal{H}_{\mathbf{F},K'} \rightarrow \mathcal{A}_{n,K} \quad (1.4.2)$$

defined as follows. Let $x = (A, \lambda, \iota, \bar{\eta})$ be a point of $\mathcal{H}_{\mathbf{F},K'}$, where $\bar{\eta} = \eta K'^p$ and $\eta: (\mathbf{F} \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^2 \rightarrow H^1(A, \mathbb{A}_f^p)$ is an \mathbf{F} -linear isomorphism. Then $\tilde{u}_{K',K}$ sends x to the point $(A, \lambda, (\eta \circ \gamma^{-1}) K^p)$ of $\mathcal{A}_{n,K}$ (the isomorphism $\eta \circ \gamma^{-1}: \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{A}_f^p \rightarrow H^1(A, \mathbb{A}_f^p)$ is compatible with the symplectic forms, and thus gives rise to a level structure). It is possible to choose K', K so that $\tilde{u}_{K',K}$ is a closed embedding, but we will not need this.

To conclude this section, we consider the case when p splits in \mathbf{F} . If we let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the prime ideals of \mathbf{F} dividing p , we get an identification

$$\mathcal{O}_{\mathbf{F}} \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \prod_{i=1}^n \mathcal{O}_{\mathbf{F}}/\mathfrak{p}_i \simeq \mathbb{F}_p^n.$$

Write $H := \mathbf{H}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $G := \mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ for the special fibers. We obtain an isomorphism

$$H \simeq \{(A_1, \dots, A_n) \in \mathrm{GL}_{2, \mathbb{F}_p} \mid \det(A_1) = \dots = \det(A_n)\}.$$

Furthermore, the embedding $u: H \rightarrow G$ identifies with the following map.

$$u: \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & & & & & & & b_1 \\ & a_2 & & & & & & b_2 \\ & & \ddots & & & & & \\ & & & a_n & b_n & & & \\ & & & c_n & d_n & & & \\ & & & & & \ddots & & \\ & & c_2 & & & & d_2 & \\ c_1 & & & & & & & d_1 \end{pmatrix}.$$

2 The stack of G -zips

The stack of G -zips was introduced by Moonen–Wedhorn ([MW04]) and Pink–Wedhorn–Ziegler ([PWZ11, PWZ15]). We start by recalling its definition.

2.1 Definition

Fix an algebraic closure $k = \overline{\mathbb{F}_p}$ of \mathbb{F}_p . For a k -scheme X , we denote by $X^{(p)}$ its p -th power Frobenius twist and by $\varphi: X \rightarrow X^{(p)}$ its relative Frobenius. Let $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$ be the automorphism $x \mapsto x^p$. Let G be a connected, reductive group over \mathbb{F}_p endowed with a cocharacter $\mu: \mathbb{G}_{m, k} \rightarrow G_k$. We call the pair (G, μ) a cocharacter datum. The cocharacter μ induces a decomposition $\mathrm{Lie}(G) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Lie}(G)_n$, where $\mathrm{Lie}(G)_n$ is the subspace where $\mathbb{G}_{m, k}$ acts by the character $x \mapsto x^n$ via μ . We obtain a pair of opposite parabolic subgroups $P_{\pm} \subset G_k$, defined by the conditions

$$\mathrm{Lie}(P_-) = \bigoplus_{n \leq 0} \mathrm{Lie}(G)_n \quad \text{and} \quad \mathrm{Lie}(P_+) = \bigoplus_{n \geq 0} \mathrm{Lie}(G)_n. \quad (2.1.1)$$

We set $P := P_-$ and $Q := P_+^{(q)}$. Let $L := \mathrm{Cent}(\mu)$ be the centralizer of μ , it is a Levi subgroup of P . Put $M := L^{(q)}$, which is a Levi subgroup of Q . We have a Frobenius map $\varphi: L \rightarrow M$. We call the tuple

$$\mathcal{Z}_{\mu} := (G, P, Q, L, M)$$

the zip datum attached to (G, μ) (this terminology slightly differs from [PWZ15, Definition 3.6]). If L is defined over \mathbb{F}_p , we have $M = L^{(p)} = L$. This will be the case for all zip data considered in this paper. Let $\theta_L^P: P \rightarrow L$ be the projection onto the Levi subgroup L modulo the unipotent radical $R_u(P)$ and define $\theta_M^Q: Q \rightarrow M$ similarly. The zip group is defined by

$$E := \{(x, y) \in P \times Q \mid \varphi(\theta_L^P(x)) = \theta_M^Q(y)\}. \quad (2.1.2)$$

Let E act on G by the rule $(x, y) \cdot g := xgy^{-1}$. The stack of G -zips of type μ is the quotient stack

$$G\text{-Zip}^{\mu} := [E \backslash G_k].$$

It can also be defined as a moduli stack of torsors ([PWZ15, Definition 1.4]). The association $(G, \mu) \mapsto G\text{-Zip}^{\mu}$ is functorial in the following sense. Let (H, μ_H) and (G, μ_G) be two

cocharacter data and let $f: H \rightarrow G$ be a homomorphism defined over \mathbb{F}_p satisfying $\mu_G = f_k \circ \mu_H$. Then by [GK19b, §2.1, §2.2], f induces a natural morphism of stacks $f_{\text{zip}}: H\text{-Zip}^{\mu_H} \rightarrow G\text{-Zip}^{\mu_G}$ which makes the diagram below commute (the vertical maps are the natural projections).

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ H\text{-Zip}^{\mu_H} & \xrightarrow{f_{\text{zip}}} & G\text{-Zip}^{\mu_G}. \end{array} \quad (2.1.3)$$

2.2 Parametrization of strata

Let (G, μ) be a cocharacter datum, with attached zip datum $\mathcal{Z}_\mu = (G, P, Q, L, M)$. The stack $G\text{-Zip}^\mu$ is a smooth stack over k whose underlying topological space is finite. We review below the parametrization of the points of $G\text{-Zip}^\mu$, following [PWZ11]. We first fix some group-theoretical data.

- For simplicity, we assume that there is an Borel pair (B, T) defined over \mathbb{F}_p satisfying $B \subset P$ and such that μ factors through T (this condition can always be achieved after possibly changing μ to a conjugate cocharacter). In particular, we have an action of $\text{Gal}(k/\mathbb{F}_p)$ on $X^*(T)$.
- Let $W = W(G_k, T)$ be the Weyl group of G_k . The group $\text{Gal}(k/\mathbb{F}_p)$ acts on W and the actions of $\text{Gal}(k/\mathbb{F}_p)$ and W on $X_*(T)$ are compatible in a natural sense.
- Write Φ for the set of T -roots, Φ_+ for the positive roots with respect to B (in our convention, a root α is positive if the corresponding α -root group U_α is contained in the opposite Borel B^+). Let Δ denote the set of simple roots.
- For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. The system $(W, \{s_\alpha \mid \alpha \in \Delta\})$ is a Coxeter system.
- Write $\ell: W \rightarrow \mathbb{N}$ for the length function, and \leq for the Bruhat order on W . Let w_0 denote the longest element of W .
- Write $I \subset \Delta$ for the set of simple roots contained in L .
- For a subset $K \subset \Delta$, let W_K denote the subgroup of W generated by $\{s_\alpha \mid \alpha \in K\}$. Write $w_{0,K}$ for the longest element in W_K .
- Let ${}^K W$ denote the subset of elements $w \in W$ which have minimal length in the coset $W_K w$. Then ${}^K W$ is a set of representatives of $W_K \backslash W$. The longest element in the set ${}^K W$ is $w_{0,K} w_0$.
- We set

$$z = \sigma(w_{0,I})w_0. \quad (2.2.1)$$

The triple (B, T, z) is a W -frame, in the terminology of [GK19b, Definition 2.3.1] (we simply call such a triple a frame).

- For $w \in W$, fix a representative $\dot{w} \in N_G(T)$, such that $(w_1 w_2)^\cdot = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, [ABD⁺66, XXIII, §6]). If no confusion occurs, we simply write w instead of \dot{w} .
- For $w, w' \in {}^I W$, write $w' \preceq w$ if there exists $w_1 \in W_I$ such that $w' \leq w_1 \sigma(w_1)^{-1}$. This defines a partial order on ${}^I W$ ([PWZ11, Corollary 6.3]).
- For $w \in {}^I W$, define G_w as the E -orbit of $\dot{w} z^{-1}$.

We now explain the parametrization of E -orbits in G_k .

Theorem 2.2.1 ([PWZ11, Theorem 7.5]).

- (1) *Each E -orbit is smooth and locally closed in G_k .*

- (2) The map $w \mapsto G_w$ is a bijection from ${}^I W$ to the set of E -orbits in G_k .
- (3) For $w \in {}^I W$, one has $\dim(G_w) = \ell(w) + \dim(P)$.
- (4) The Zariski closure of G_w is

$$\overline{G}_w = \bigsqcup_{w' \in {}^I W, w' \preceq w} G_{w'}. \quad (2.2.2)$$

For each $w \in {}^I W$, we define

$$\mathcal{X}_w := [E \backslash G_w].$$

It is a locally closed substack of $G\text{-Zip}^\mu = [E \backslash G_k]$. We call the \mathcal{X}_w the zip strata of $G\text{-Zip}^\mu$. We obtain a stratification $G\text{-Zip}^\mu = \bigsqcup_{w \in {}^I W} \mathcal{X}_w$ and the closure relations between strata are given by (2.2.2). In particular, there is a unique open E -orbit $U_\mu \subset G$ corresponding to the longest element

$$w_{\max} := w_{0,I} w_0$$

in ${}^I W$. We always have $1 \in U_\mu$. The open substack $\mathcal{U}_\mu := \mathcal{X}_{w_{\max}}$ is called the μ -ordinary locus of $G\text{-Zip}^\mu$, following usual terminology pertaining to Shimura varieties.

2.3 G -zips and Shimura varieties of Hodge-type

We retain the notations of section 1.2. Let $S_K = \mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} k$ be the special fiber of a Hodge-type Shimura variety at a place of good reduction. Define $G := \mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. We can find a cocharacter μ in the conjugacy class $[\mu]$ such that μ extends to a cocharacter of $\mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{E}_p}$ ([Kim18, Corollary 3.3.11]). We denote again by μ the cocharacter of G_k obtained by reducing modulo p . By §2.1, there is a stack of G -zips of type μ attached to the cocharacter datum (G, μ) . Zhang ([Zha18, 4.1]) constructed a smooth morphism

$$\zeta_K: S_K \rightarrow G\text{-Zip}^\mu, \quad (2.3.1)$$

whose fibers are called the Ekedahl–Oort strata of S_K . This map is also surjective by [SYZ19, Corollary 3.5.3(1)]. The map ζ_K commutes with change-of-level maps, in the sense that for any compact open subgroups $K'^p \subset K^p \subset \mathbf{G}(\mathbb{A}_f^p)$, there is a commutative diagram

$$\begin{array}{ccc} S_{K'} & & \\ \downarrow \pi_{K',K} & \searrow \zeta_{K'} & \\ & G\text{-Zip}^\mu & \\ & \nearrow \zeta_K & \\ S_K & & \end{array}$$

For this reason, we often omit the subscript K and denote these maps by ζ . For $w \in {}^I W$, write

$$S_{K,w} := \zeta^{-1}(\mathcal{X}_w)$$

for the corresponding Ekedahl–Oort stratum. The Ekedahl–Oort stratum $S_{K,w_{\max}}$ corresponding to the maximal element $w_{\max} = w_{0,I} w_0 \in {}^I W$ is called the μ -ordinary locus. When $\mathbf{E}_p = \mathbb{Q}_p$, this is simply the classical ordinary locus of S_K , and is often denoted by S_K^{ord} .

The Siegel case

We consider the case $G = \mathrm{GSp}_{2n, \mathbb{F}_p}$ (as defined in section 1.1). We endow G with the cocharacter

$$\mu_G: \mathbb{G}_{m, k} \rightarrow G_k, \quad t \mapsto \begin{pmatrix} tI_n & \\ & I_n \end{pmatrix}.$$

Let (u_1, \dots, u_{2n}) denote the canonical basis of $V = k^{2n}$, and set

$$V_P := \mathrm{Span}_k(u_{n+1}, \dots, u_{2n}).$$

Let P be the parabolic subgroup which stabilizes the filtration $0 \subset V_P \subset V$. Similarly, let Q be the parabolic subgroup opposite to P stabilizing the subspace $V_Q := \mathrm{Span}_k(u_1, \dots, u_n)$. The intersection $L := P \cap Q$ is a common Levi subgroup to P and Q . The zip datum attached to (G, μ) is $\mathcal{Z}_\mu = (G, P, Q, L, L)$. Let $E \subset P \times Q$ be the group defined in (2.1.2) and let $G\text{-Zip}^{\mu_G} := [E \backslash G_k]$ be the attached stack of G -zips. We fix $B \subset G$ to be the Borel subgroup of lower-triangular matrices in G , and we let $T \subset B$ be the maximal torus consisting of diagonal matrices in G . The Weyl group $W = W(G, T)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ and can be identified with the group of permutations $w \in \mathfrak{S}_{2n}$ satisfying

$$w(i) + w(2n + 1 - i) = 2n + 1. \quad (2.3.2)$$

The Weyl group $W_L = W(L, T)$ is isomorphic to \mathfrak{S}_n and identifies with the subgroup of permutations in \mathfrak{S}_{2n} satisfying (2.3.2) and stabilizing the subset $\{1, \dots, n\}$. As explained in §2.3, there is a smooth surjective morphism

$$\zeta_G: \overline{\mathcal{A}}_{n, K} \rightarrow G\text{-Zip}^{\mu_G}$$

whose fibers are the Ekedahl–Oort strata of $\overline{\mathcal{A}}_{n, K}$.

The Hilbert–Blumenthal case

Next, we consider the Hilbert–Blumenthal case. We simply write H for the special fiber of the reductive group $\mathbf{H}_{\mathbb{Z}_p}$ defined in section 1.4. The embedding $u_{\mathbb{Z}_p}: \mathbf{H}_{\mathbb{Z}_p} \rightarrow \mathbf{G}_{\mathbb{Z}_p}$ induces an embedding

$$u: H \rightarrow G$$

(to simplify the notation, we continue to denote it by u). If we let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the prime ideals of $\mathcal{O}_{\mathbf{F}}$ above p and write $\kappa_i := \mathcal{O}_{\mathbf{F}}/\mathfrak{p}_i$, we may view H as a subgroup

$$H \subset \prod_{i=1}^r \mathrm{Res}_{\kappa_i/\mathbb{F}_p} \mathrm{GL}_{2, \kappa_i}.$$

The group H_k identifies with

$$\{(A_1, \dots, A_n) \in \mathrm{GL}_{2, k}^n \mid \det(A_1) = \dots = \det(A_n)\}.$$

We consider the stack of H -zips attached to the cocharacter

$$\mu_H: \mathbb{G}_{m, k} \rightarrow H_k, \quad t \mapsto \left(\begin{pmatrix} t & \\ & 1 \end{pmatrix}, \dots, \begin{pmatrix} t & \\ & 1 \end{pmatrix} \right).$$

Write B_H (resp. B_H^+) for the subgroup consisting of tuples $(A_1, \dots, A_n) \in H_k$ of lower-triangular (resp. upper-triangular) matrices. It is clear that B_H, B_H^+ are Borel subgroups

of H defined over \mathbb{F}_p . Write $T_H = B_H \cap B_H^-$ for the maximal torus consisting of tuples of diagonal matrices in H_k . The zip datum \mathcal{Z}_{μ_H} attached to μ_H is $\mathcal{Z}_{\mu_H} = (H, B_H, B_H^+, T_H, T_H)$. Let $H\text{-Zip}^{\mu_H}$ be the stack of H -zips attached to (H, μ_H) . Note that the embedding $u: H \rightarrow G$ is compatible with the cocharacters μ_H and μ_G . Therefore, u induces by functoriality (see diagram (2.1.3)) a morphism of stacks

$$u_{\text{zip}}: H\text{-Zip}^{\mu_H} \rightarrow G\text{-Zip}^{\mu_G}.$$

For simplicity, we omit the cocharacters in the notation and write respectively $H\text{-Zip}$ and $G\text{-Zip}$ for the above stacks.

Write $\zeta_H: \overline{\mathcal{H}}_{\mathbf{F}, K'} \rightarrow H\text{-Zip}$ for the map given by (2.3.1). For a fixed choice of compatible level structures $K' \subset \mathbf{H}(\mathbb{A}_f^p)$ and $K \subset \mathbf{G}(\mathbb{A}_f^p)$, we sometimes write $X_H := \overline{\mathcal{H}}_{\mathbf{F}, K'}$ and $X_G := \overline{\mathcal{A}}_{n, K}$ in order to uniformize the notation. Write simply \tilde{u} for the map $\tilde{u}_{K', K}: \overline{\mathcal{H}}_{\mathbf{F}, K'} \rightarrow \overline{\mathcal{A}}_{n, K}$ given by (1.4.2). By the construction of the maps ζ_H and ζ_G , there is a commutative diagram:

$$\begin{array}{ccc} X_H & \xrightarrow{\tilde{u}} & X_G \\ \zeta_H \downarrow & & \downarrow \zeta_G \\ H\text{-Zip} & \xrightarrow{u_{\text{zip}}} & G\text{-Zip} \end{array} \quad (2.3.3)$$

2.4 The flag space

2.4.1 Flag strata

We briefly consider the general setting before specializing to the groups H, G of section 2.3. Let (G, μ) be a cocharacter datum and let $\mathcal{Z}_\mu = (G, P, Q, L, M)$ be the attached zip datum. Fix a Borel subgroup $B \subset P$. We defined in [GK19a, Part 1, §2] the stack of G -zip flags $G\text{-ZipFlag}^\mu$ (attached to B). It can be defined as the quotient stack

$$G\text{-ZipFlag}^\mu = [E \backslash (G_k \times P/B)].$$

Here, E acts on $G_k \times (P/B)$ by $(x, y) \cdot (g, hB) = (xgy^{-1}, xhB)$ for all $(x, y) \in E$, $g \in G$, $h \in P$. The first projection induces a morphism of stacks

$$\pi: G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu$$

whose fibers are isomorphic to P/B . Now, let X be a k -scheme endowed with a morphism $\zeta: X \rightarrow G\text{-Zip}^\mu$. We define the flag space $\text{Flag}(X)$ of X as the fiber product

$$\begin{array}{ccc} \text{Flag}(X) & \xrightarrow{\zeta_{\text{flag}}} & G\text{-ZipFlag}^\mu \\ \pi_X \downarrow & & \downarrow \pi \\ X & \xrightarrow{\zeta} & G\text{-Zip}^\mu. \end{array} \quad (2.4.1)$$

We briefly recall the flag stratification of $G\text{-ZipFlag}^\mu$, defined in [GK19a, §2.3]. Define $\text{Sbt} := [B \backslash G_k / B]$ (we call this stack the Schubert stack of G). There is a natural map $\psi: G\text{-ZipFlag}^\mu \rightarrow \text{Sbt}$, defined as follows. The natural inclusion $G_k \rightarrow G_k \times P/B$ identifies $G\text{-ZipFlag}^\mu$ with the quotient $[E' \backslash G_k]$, where $E' = (B \times G_k) \cap E$. One sees immediately that $E' \subset B \times {}^z B$ (where $z = \sigma(w_{0, I})w_0$ as defined in (2.2.1)), so that there is a canonical

projection $[E' \backslash G_k] \rightarrow [B \backslash G_k / {}^z B]$. Finally, the map $G_k \rightarrow G_k$, $g \mapsto g\dot{z}$ yields an isomorphism $[B \backslash G_k / {}^z B] \rightarrow [B \backslash G_k / B] = \text{Sbt}$. By composing these maps, we obtain a smooth, surjective morphism

$$\psi: G\text{-ZipFlag}^\mu \rightarrow \text{Sbt}. \quad (2.4.2)$$

By the Bruhat stratification of G , the points of the underlying topological space of Sbt correspond bijectively to the elements of the Weyl group W . Specifically, the element $w \in W$ corresponds to the locally closed point

$$\text{Sbt}_w := [B \backslash BwB / B].$$

For $w \in W$, we denote by $\mathcal{F}_{G,w}$ or simply \mathcal{F}_w the preimage $\psi^{-1}(\text{Sbt}_w)$. It is locally closed in $G\text{-ZipFlag}^\mu$. We call the locally closed subsets $(\mathcal{F}_w)_{w \in W}$ the flag strata of $G\text{-ZipFlag}^\mu$. Each flag stratum \mathcal{F}_w is smooth, and the Zariski closure $\overline{\mathcal{F}_w}$ is normal. The image of a flag stratum under the projection $\pi: G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu$ is a union of zip strata. It is difficult to determine in general which zip strata appear in $\pi(\mathcal{F}_w)$. However, in the case $w \in {}^I W$, we have $\pi(\mathcal{F}_w) = \mathcal{X}_w$. Moreover, the map $\pi: \mathcal{F}_w \rightarrow \mathcal{X}_w$ is finite etale ([Kos18, Proposition 2.2.1]). In particular, for $w_{\max} = w_{0,I}w_0$ (the longest element of ${}^I W$), we have $\pi(\mathcal{F}_{w_{\max}}) = \mathcal{U}_\mu$ and the map $\pi: \mathcal{F}_{w_{\max}} \rightarrow \mathcal{U}_\mu$ is finite etale. For $w \in W$, we write

$$\text{Flag}(X)_w := \zeta_{\text{flag}}^{-1}(\mathcal{F}_w) \quad (2.4.3)$$

for the corresponding flag stratum in $\text{Flag}(X)$, where ζ_{flag} is the map defined in diagram (2.4.1). If ζ is smooth, then each flag stratum $\text{Flag}(X)_w$ is smooth and its Zariski closure $\overline{\text{Flag}(X)}_w$ is normal.

2.4.2 Functoriality

The flag space satisfies functoriality properties, as explained in [GK19b, §5.3]. Let (H, μ_H) and (G, μ_G) be two cocharacter data and let $f: H \rightarrow G$ be a homomorphism over \mathbb{F}_p compatible with μ_H and μ_G . Write $\mathcal{Z}_H = (H, P_H, Q_H, L_H, M_H)$ and $\mathcal{Z}_G = (G, P_G, Q_G, L_G, M_G)$ for the zip data associated with (H, μ_H) and (G, μ_G) respectively. Furthermore, choose Borel subgroups $B_H \subset P_H$ and $B_G \subset P_G$ such that $f(B_H) \subset B_G$ (this is always possible). There is a natural morphism of k -stacks $f_{\text{flag}}: H\text{-ZipFlag}^{\mu_H} \rightarrow G\text{-ZipFlag}^{\mu_G}$ which makes the diagram below commute

$$\begin{array}{ccc} H\text{-ZipFlag}^{\mu_H} & \xrightarrow{f_{\text{flag}}} & G\text{-ZipFlag}^{\mu_G} \\ \pi_H \downarrow & & \downarrow \pi_G \\ H\text{-Zip}^{\mu_H} & \xrightarrow{f_{\text{zip}}} & G\text{-Zip}^{\mu_G}. \end{array} \quad (2.4.4)$$

Furthermore, the map f_{flag} sends a flag stratum of $H\text{-ZipFlag}^{\mu_H}$ to a flag stratum of $G\text{-ZipFlag}^\mu$ (indeed, the map ψ defined in (2.4.2) is also functorial). Now, assume further that we have k -schemes X_H, X_G endowed with morphisms $\zeta_H: X_H \rightarrow H\text{-Zip}^{\mu_H}$ and $\zeta_G: X_G \rightarrow G\text{-Zip}^{\mu_G}$, together with a morphism $\tilde{f}: X_H \rightarrow X_G$ making the diagram below commute.

$$\begin{array}{ccc} X_H & \xrightarrow{\tilde{f}} & X_G \\ \zeta_H \downarrow & & \downarrow \zeta_G \\ H\text{-Zip}^{\mu_H} & \xrightarrow{f_{\text{zip}}} & G\text{-Zip}^{\mu_G}. \end{array}$$

Then, by taking fiber products, we obtain a natural morphism $\tilde{f}_{\text{flag}}: \text{Flag}(X_H) \rightarrow \text{Flag}(X_G)$ such that the diagram below commutes.

$$\begin{array}{ccccc}
\text{Flag}(X_H) & \xrightarrow{\zeta_{H,\text{flag}}} & H\text{-ZipFlag}^{\mu_H} & & \\
\downarrow \pi_{X,H} & \searrow \tilde{f}_{\text{flag}} & \downarrow \pi_H & \searrow f_{\text{flag}} & \\
& \text{Flag}(X_G) & \xrightarrow{\zeta_{G,\text{flag}}} & G\text{-ZipFlag}^{\mu_G} & \\
& \downarrow \pi_{X,G} & \downarrow \pi_H & \downarrow \pi_G & \\
X_H & \xrightarrow{\zeta_H} & H\text{-Zip}^{\mu_H} & & \\
& \searrow \tilde{f} & \downarrow \pi_H & \searrow f_{\text{zip}} & \\
& X_G & \xrightarrow{\zeta_G} & G\text{-Zip}^{\mu_G} &
\end{array}$$

We denoted by $\pi_H, \pi_G, \pi_{X,H}, \pi_{X,G}, \zeta_{H,\text{flag}}, \zeta_{G,\text{flag}}$ the obvious maps for each group.

We now return to the setting of section 2.3. In particular, we write $X_H = \overline{\mathcal{H}}_{\mathbf{F},K'}$, $X_G = \overline{\mathcal{A}}_{n,K}$ for a compatible choice of level structures K', K , and we have the commutative diagram (2.3.3). Note that the parabolic subgroup attached to μ_H is the Borel subgroup B_H of H . Therefore, the stacks $H\text{-Zip}$ and $H\text{-ZipFlag}$ coincide, and similarly we have $X_H = \text{Flag}(X_H)$. Hence, the above diagram collapses in this case to the following, slightly simpler commutative diagram. To simplify the notation further, we write $\text{Flag}_G := \text{Flag}(X_G)$.

$$\begin{array}{ccc}
X_H & \xrightarrow{\zeta_H} & H\text{-Zip} \\
\downarrow \tilde{u}_{\text{flag}} & & \downarrow u_{\text{flag}} \\
\text{Flag}_G & \xrightarrow{\zeta_{G,\text{flag}}} & G\text{-ZipFlag} \\
\downarrow & & \downarrow \\
X_G & \xrightarrow{\zeta_G} & G\text{-Zip}
\end{array}
\quad (2.4.5)$$

\tilde{u} (curved arrow from X_H to X_G) and u_{zip} (curved arrow from $H\text{-Zip}$ to $G\text{-Zip}$)

Let $w_{\max} = w_{0,I}w_0$ be the maximal element of ${}^I W$. The flag stratum $\mathcal{F}_{G,w_{\max}}$ of $G\text{-ZipFlag}$ will play an important role for us. By definition, we have $\mathcal{F}_{G,w_{\max}} = [E' \setminus Bw_{\max}Bz^{-1}]$. Concretely, the set $Bw_{\max}Bz^{-1}$ is the subset of G_k of matrices of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad A_i \in M_n(k), \text{ with } A_1 \in \text{GL}_n(k) \text{ and lower-triangular.}$$

The Zariski closure of $Bw_{\max}Bz^{-1}$ has a similar description, but the condition $A_1 \in \text{GL}_n(k)$ is removed.

Proposition 2.4.1. *The map*

$$u_{\text{flag}}: H\text{-Zip} \rightarrow G\text{-ZipFlag}$$

maps the μ_H -ordinary locus (i.e. the unique open zip stratum of $H\text{-Zip}$) into the flag stratum $\mathcal{F}_{G,w_{\max}}$ of $G\text{-ZipFlag}$. In particular, the image of u_{flag} is contained in $\overline{\mathcal{F}}_{G,w_{\max}}$.

Proof. The identity element $1 \in H$ lies in the μ_H -ordinary locus, and its image by u is $1 \in G$. By the above description of $\mathcal{F}_{w_{\max}}$, this point lies in $Bw_{\max}Bz^{-1}$. Since u_{flag} sends a flag stratum of H into a flag stratum of G , we deduce that any point in the μ_H -ordinary locus of H is sent to a point of $\mathcal{F}_{G,w_{\max}}$ by u_{flag} . \square

3 Automorphic forms in characteristic p

3.1 Automorphic vector bundles

We first review automorphic vector bundles on the generic fiber of Hodge-type Shimura varieties. We retain the notation of section 1.2. Let \mathbf{P}_{\pm} be the pair of opposite parabolic subgroups of $\mathbf{G}_{\mathbb{C}}$ defined by our chosen cocharacter $\mu: \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$, as in (2.1.1), and write $\mathbf{P} := \mathbf{P}_{-}$. The Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \otimes_{\mathbf{E}} \mathbb{C}$ carries a natural \mathbf{P} -torsor. Therefore, any algebraic representation $\rho: \mathbf{P} \rightarrow \mathrm{GL}(W)$ (where W is a finite-dimensional \mathbb{C} -vector space) gives rise to a vector bundle $\mathcal{V}(\rho)$ on $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \otimes_{\mathbf{E}} \mathbb{C}$. Let $\mathbf{B} \subset \mathbf{P}$ be a Borel subgroup. For a character $\lambda \in X^*(\mathbf{T})$, we may view λ as a one-dimensional representation of \mathbf{B} and consider the induced representation

$$\mathbf{V}_I(\lambda) := \mathrm{Ind}_{\mathbf{B}}^{\mathbf{P}}(\lambda).$$

We denote by $\mathcal{V}_I(\lambda)$ the vector bundle attached to $\mathbf{V}_I(\lambda)$ and call it the automorphic vector bundle attached to λ . Note that $\mathbf{V}_I(\lambda) = 0$ when λ is not I -dominant (we say that a character λ is I -dominant if $\langle \lambda, \alpha^{\vee} \rangle \geq 0$ for all $\alpha \in I$).

The vector bundle $\mathbf{V}_I(\lambda)$ admits an extension to the integral model \mathcal{S}_K . Indeed, by our choice of μ (see section 2.3), we have a cocharacter $\mu: \mathbb{G}_{m, \mathcal{O}_{\mathbf{E}_p}} \rightarrow \mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{E}_p}$. It induces therefore a parabolic subgroup $\mathcal{P} \subset \mathbf{G}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbf{E}_p}$. By construction of the integral model \mathcal{S}_K , there is a natural \mathcal{P} -torsor on \mathcal{S}_K . For each $\lambda \in X^*(\mathbf{T})$, consider the \mathcal{P} -representation

$$\mathbf{V}_I(\lambda)_0 := H^0(\mathcal{P}/\mathcal{B}, \mathcal{L}_{\lambda}),$$

where \mathcal{L}_{λ} is the line bundle on \mathcal{P}/\mathcal{B} naturally attached to λ (strictly speaking, the line bundle \mathcal{L}_{λ} may not be defined over $\mathcal{O}_{\mathbf{E}_p}$, so it may be necessary to pass to some ring extension contained in $\overline{\mathbb{Z}_p}$). By applying this representation to the \mathcal{P} -torsor of \mathcal{S}_K , we obtain a vector bundle $\mathcal{V}_I(\lambda)$ on $\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} \overline{\mathbb{Z}_p}$ which extends the one on the generic fiber. We denote again its special fiber by $\mathcal{V}_I(\lambda)$. It is a vector bundle on S_K modeled on the k -representation $V_I(\lambda) = \mathrm{Ind}_B^P(\lambda)$, where B and P denote the special fibers of \mathcal{B} and \mathcal{P} respectively.

Let Ω be a field endowed with a ring homomorphism $\mathcal{O}_{\mathbf{E}_p} \rightarrow \Omega$. An element of the space

$$H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} \Omega, \mathcal{V}_I(\lambda))$$

will be called an automorphic form of weight λ and level K with coefficients in Ω . In this paper, we are mainly interested in the case $\Omega = k$.

3.2 Vector bundles on $G\text{-Zip}^{\mu}$

Let (G, μ) be a cocharacter datum with attached zip datum $\mathcal{Z}_{\mu} = (G, P, Q, L, M)$ and zip group E (see (2.1.2)). Let $\rho: E \rightarrow \mathrm{GL}(W)$ be an algebraic representation of E on a finite-dimensional k -vector space W . By [IK21a, §2.4.2], we can attach a vector bundle $\mathcal{V}(\rho)$ on $G\text{-Zip}^{\mu}$ using the associated sheaf construction of [Jan03, I.5.8]. If $\rho: P \rightarrow \mathrm{GL}(W)$ is an algebraic representation of P , we view it as an E -representation via the first projection $\mathrm{pr}_1: E \rightarrow P$, and we denote again the associated vector bundle by $\mathcal{V}(\rho)$. For $\lambda \in X^*(T)$, view λ as a one-dimensional representation of B and consider the induced P -representation

$$V_I(\lambda) := \mathrm{Ind}_B^P(\lambda).$$

The vector bundle on $G\text{-Zip}^\mu$ attached to $V_I(\lambda)$ is denoted by $\mathcal{V}_I(\lambda)$. The vector bundles $\mathcal{V}_I(\lambda)$ on $G\text{-Zip}^\mu$ and S_K are compatible via the map $\zeta: S_K \rightarrow G\text{-Zip}^\mu$, in the sense that $\zeta^*(\mathcal{V}_I(\lambda))$ coincides with the vector bundle $\mathcal{V}_I(\lambda)$ defined in section 3.1.

By a similar construction, each algebraic B -representation $\rho: B \rightarrow \text{GL}(W)$ gives rise to a vector bundle $\mathcal{V}_{\text{flag}}(\rho)$ on the flag space $G\text{-ZipFlag}^\mu$. In particular, we have a family of line bundles $\mathcal{V}_{\text{flag}}(\lambda)$ for $\lambda \in X^*(T)$. We denote again by $\mathcal{V}_{\text{flag}}(\lambda)$ the line bundle $\zeta_{\text{flag}}^*(\mathcal{V}_{\text{flag}}(\lambda))$ on the flag space $\text{Flag}(S_K)$. For a B -representation ρ , the push-forward of $\mathcal{V}_{\text{flag}}(\rho)$ via the map $\pi: G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu$ is given by

$$\pi_*(\mathcal{V}_{\text{flag}}(\rho)) = \mathcal{V}(\text{Ind}_B^P(\rho))$$

by [IK21b], Proposition 3.2.1. In particular, for any character $\lambda \in X^*(T)$ we have the identification

$$\pi_*(\mathcal{V}_{\text{flag}}(\lambda)) = \mathcal{V}_I(\lambda). \quad (3.2.1)$$

Hence, we have an identification $H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) = H^0(G\text{-ZipFlag}^\mu, \mathcal{V}_{\text{flag}}(\lambda))$. For a general pair (G, μ) , this space can be described as the part of the Brylinski–Kostant filtration of $V_I(\lambda)$ stable under the action of a certain finite group (see [IK21a, Theorem 3.4.1]). When P is defined over \mathbb{F}_p , it has a simpler description: For each $\chi \in X^*(T)$, write $V_I(\lambda)_\chi$ for the χ -weight space of $V_I(\lambda)$ and define

$$V_I(\lambda)_{\leq 0} = \bigoplus_{\substack{\langle \chi, \alpha^\vee \rangle \leq 0 \\ \text{for all } \alpha \in \Delta \setminus I}} V_I(\lambda)_\chi. \quad (3.2.2)$$

Then we have an identification ([Kos19, Theorem 3.7.2])

$$H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) = V_I(\lambda)_{\leq 0} \cap V_I(\lambda)^{L(\mathbb{F}_p)}.$$

Assume now that S_K is the good reduction special fiber of a Hodge-type Shimura variety as in section 2.3, and $G\text{-Zip}^\mu$ is the attached stack of G -zips. Denote by π_K the projection $\pi_K: \text{Flag}(S_K) \rightarrow S_K$ (see diagram (2.4.1)). Then formula (3.2.1) holds also on the level of S_K , i.e. one has $\pi_{K,*}(\mathcal{V}_{\text{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. Therefore, the space of automorphic forms of weight λ can be identified with

$$H^0(S_K, \mathcal{V}_I(\lambda)) = H^0(\text{Flag}(S_K), \mathcal{V}_{\text{flag}}(\lambda)). \quad (3.2.3)$$

We are thus reduced to studying the line bundle $\mathcal{V}_{\text{flag}}(\lambda)$ on the flag space of S_K .

3.3 Extension to the toroidal compactification

Retain the notation and assumptions of section 2.3. By [MS11, Theorem 1], we can find a sufficiently fine cone decomposition Σ and a smooth toroidal compactification \mathcal{S}_K^Σ of \mathcal{S}_K over $\mathcal{O}_{\mathbf{E}_p}$. The family $(\mathcal{V}_I(\lambda))_{\lambda \in X^*(\mathbf{T})}$ admits a canonical extension $(\mathcal{V}_I^\Sigma(\lambda))_{\lambda \in X^*(\mathbf{T})}$ to \mathcal{S}_K^Σ . The following result is known as the Koecher principle.

Theorem 3.3.1 ([LS18, Theorem 2.5.11]). *Let Ω be a field which is an $\mathcal{O}_{\mathbf{E}_p}$ -algebra, and assume that $\mathcal{V}_I(\lambda)$ is defined over Ω . The natural map*

$$H^0(\mathcal{S}_K^\Sigma \otimes_{\mathcal{O}_{\mathbf{E}_p}} \Omega, \mathcal{V}_I^\Sigma(\lambda)) \rightarrow H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_p}} \Omega, \mathcal{V}_I(\lambda))$$

is a bijection, except when $\dim(\text{Sh}_K(\mathbf{G}, \mathbf{X})) = 1$ and $\mathcal{S}_K^\Sigma \setminus \mathcal{S}_K \neq \emptyset$.

The Koecher principle holds for the Shimura varieties considered in this article. Consider the special fiber $S_K^\Sigma := \mathcal{S}_K^\Sigma \otimes_{\mathcal{O}_{\mathbb{E}_p}} k$. By [GK19a, Theorem 6.2.1], the map $\zeta: S_K \rightarrow G\text{-Zip}^\mu$ extends naturally to a map of stacks

$$\zeta^\Sigma: S_K^\Sigma \rightarrow G\text{-Zip}^\mu$$

and by [And, Theorem 1.2], the extended map ζ^Σ is again smooth. Since ζ is surjective, ζ^Σ is also surjective. By construction, the pull-back by ζ^Σ of $\mathcal{V}_I(\lambda)$ coincides with the canonical extension $\mathcal{V}_I^\Sigma(\lambda)$.

3.4 Finite etale maps

Let $f: Y \rightarrow X$ be a finite etale map of degree $n \geq 1$ between noetherian schemes. If \mathcal{L} is a line bundle on X , it is known that one has a natural trace map

$$\text{Tr}: H^0(Y, f^*(\mathcal{L})) \rightarrow H^0(X, \mathcal{L}).$$

We generalize this construction to other symmetric functions. We first consider the case when $\mathcal{L} = \mathcal{O}_X$ and define maps

$$c_i: H^0(Y, \mathcal{O}_Y) \rightarrow H^0(X, \mathcal{O}_X)$$

for each $0 \leq i \leq n-1$ as follows. We first examine the case when $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and B is a free A -algebra. In this case, for each $b \in B$, we may consider the characteristic polynomial $\chi_b(X)$ of the A -linear map $B \rightarrow B$, $x \mapsto bx$. Then, define $c_i: B \rightarrow A$ as the i -th coefficient of $\chi_b(X)$, for each $0 \leq i \leq n-1$. Next, assume that B is a finite etale A -algebra. In this case B is a locally free A -algebra, so by taking an appropriate open covering of A , we may extend the definition of $\chi_b(X)$ and $c_i(b)$ by glueing. This defines a map of sets $c_i: B \rightarrow A$. Similarly, for an arbitrary finite etale map $f: Y \rightarrow X$ of degree n , we choose an open affine covering of X and glue the maps c_i defined in the affine case. We obtain a map $c_i: H^0(Y, \mathcal{O}_Y) \rightarrow H^0(X, \mathcal{O}_X)$. Next, we consider an arbitrary line bundle \mathcal{L} on X . First assume that $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ where B is a free A -algebra, and M is a free A -module of rank one. Choose any element $m \in M$ such that $M = Am$ and define a map (of sets) by

$$c_{i,M}: B \otimes_A M \rightarrow M^{\otimes i}, \quad b \otimes (am) \mapsto c_i(b)(am) \otimes \cdots \otimes (am)$$

This map is well-defined since $c_i(ab) = a^i c_i(b)$ for any $a \in A$. By glueing, we may define $c_{i,M}$ when M is a locally free A -module, and eventually we obtain a map

$$c_{i,\mathcal{L}}: H^0(Y, f^*(\mathcal{L})) \rightarrow H^0(X, \mathcal{L}^i)$$

in the general setting.

3.5 Stable base locus

Let \mathcal{L} be a line bundle on a scheme X . Recall that the base locus of \mathcal{L} is the set of points $x \in X$ such that any section $s \in H^0(X, \mathcal{L})$ vanishes at x . Denote by $B(\mathcal{L})$ the base locus of \mathcal{L} and set:

$$\mathbb{B}(\mathcal{L}) := \bigcap_{d \geq 1} B(\mathcal{L}^d).$$

The set $\mathbb{B}(\mathcal{L})$ is called the stable base locus of \mathcal{L} . Clearly, it is a Zariski closed subset of X . If $f: Y \rightarrow X$ is a morphism of schemes and \mathcal{L} is a line bundle on X , then it is clear that $f(B(f^*(\mathcal{L}))) \subset B(\mathcal{L})$. Similarly, we have an inclusion $f(\mathbb{B}(f^*(\mathcal{L}))) \subset \mathbb{B}(\mathcal{L})$, in other words

$$\mathbb{B}(f^*(\mathcal{L})) \subset f^{-1}(\mathbb{B}(\mathcal{L})).$$

Proposition 3.5.1. *Let K be an algebraically closed field and let X, Y be K -varieties. Assume that $f: Y \rightarrow X$ is a finite etale map. For any line bundle \mathcal{L} on X , we have*

$$\mathbb{B}(f^*(\mathcal{L})) = f^{-1}(\mathbb{B}(\mathcal{L})).$$

Proof. Let $y \in Y$ and $x := f(y) \in X$ such that $x \in \mathbb{B}(\mathcal{L})$. We need to show that $y \in \mathbb{B}(f^*(\mathcal{L}))$. Let $s \in H^0(Y, f^*(\mathcal{L})^d)$ be a global section. We will show that s vanishes at each point of $f^{-1}(x)$. By assumption, for any $0 \leq i \leq n-1$ (where $n = \deg(f)$), the section $c_i(s) \in H^0(X, \mathcal{L}^{id})$ vanishes at x . Let $U = \text{Spec}(A)$ be an open affine neighborhood of x and write $f^{-1}(U) = \text{Spec}(B)$. We may assume that B is a free A -module and that \mathcal{L} is trivial on U . Since K is algebraically closed, it suffices to consider closed points, hence we may assume that x corresponds to a maximal ideal $\mathfrak{m} \subset A$. The fiber $f^{-1}(x)$ consists of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ of B . Taking tensor product with A/\mathfrak{m} , we obtain a Cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{c_i} & A \\ \downarrow & & \downarrow \\ B/\mathfrak{m}B & \xrightarrow{c_i} & A/\mathfrak{m} \end{array}$$

Since K is algebraically closed, we have $A/\mathfrak{m} = K$. The inclusion $\mathfrak{m}B \subset \bigcap_{i=1}^n \mathfrak{m}_i = \prod_{i=1}^n \mathfrak{m}_i$, yields a natural projection

$$B/\mathfrak{m}B \rightarrow B/\prod_{i=1}^n \mathfrak{m}_i = \prod_{i=1}^n B/\mathfrak{m}_i = K^n.$$

Since B is a free A -module of rank n , $B/\mathfrak{m}B$ has dimension n over K , so we deduce that this projection map is an isomorphism. In particular, the lower horizontal map in the above Cartesian diagram identifies with the map

$$c_i: K^n \rightarrow K, \quad (x_1, \dots, x_n) \mapsto c_i(x_1, \dots, x_n)$$

where $c_i(x_1, \dots, x_n)$ is the coefficient of degree i in the polynomial

$$(X - x_1)(X - x_2) \dots (X - x_n)$$

Write $z_i := s + \mathfrak{m}_i \in K$ for the value of s at the point $\mathfrak{m}_i \in \text{Spec}(B)$. By assumption, we have $c_i(z_1, \dots, z_n) = 0$ for all $0 \leq i \leq n-1$. It follows that $(X - z_1)(X - z_2) \dots (X - z_n) = X^n$, and therefore $z_i = 0$ for all i . This terminates the proof. \square

3.6 Hecke-equivariance and automorphic vector bundles

We apply Proposition 3.5.1 in the context of Shimura varieties. First, we need to define Hecke operators on the level of the flag space $\text{Flag}(S_K)$. For each compact open subgroup $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and each $g \in \mathbf{G}(\mathbb{A}_f^p)$, recall that we have the Hecke correspondance given by

diagram (1.3.1). Passing to the special fiber and taking the fiber product with $G\text{-ZipFlag}^\mu$ over $G\text{-Zip}^\mu$, we obtain a diagram

$$\begin{array}{ccc} & \text{Flag}(S_{K \cap gK^p g^{-1}}) & \\ \pi_{\text{flag}} \swarrow & & \searrow g_{\text{flag}} \\ \text{Flag}(S_K) & & \text{Flag}(S_K) \end{array}$$

with finite etale maps. We deduce from this that for any $\lambda \in X^*(\mathbf{T})$, there is a natural action of the Hecke algebra $\mathbb{T}_{K^p} := \mathbb{T}_{K^p}(\mathbf{G})$ on the space $H^0(\text{Flag}(S_K), \mathcal{V}_{\text{flag}}(\lambda))$. Similarly, the above diagram is obviously compatible with the flag stratifications, so we obtain actions of \mathbb{T}_{K^p} on the spaces

$$H^0(\text{Flag}(S_K)_w, \mathcal{V}_{\text{flag}}(\lambda)) \quad \text{and} \quad H^0(\overline{\text{Flag}}(S_K)_w, \mathcal{V}_{\text{flag}}(\lambda))$$

for any $w \in W$. Furthermore, \mathbb{T}_{K^p} acts by correspondences on the 0-cycles of $\text{Flag}(S_K)_w$ and $\overline{\text{Flag}}(S_K)_w$. For $x \in \text{Flag}(S_K)$ and $g \in \mathbf{G}(\mathbb{A}_f^p)$, we define a 0-cycle

$$T_{K^p, g}(x) = g_{\text{flag}, *}\pi_{\text{flag}}^*(x) \quad (3.6.1)$$

similarly to the case of S_K (see (1.3.2)). We view equation (3.6.1) as a formal sum of points of $\text{Flag}(S_K)$.

Definition 3.6.1. *Let A be a subset of S_K (resp. $\text{Flag}(S_K)$). We say that A is stable by Hecke operators if for any $x \in A$ and any $g \in \mathbf{G}(\mathbb{A}_f^p)$, all points in the formal sum $T_{K^p, g}(x)$ lie in A .*

Fix $w \in W$ and $\lambda \in X^*(T)$. Write

$$\mathbb{B}_{K^p, w}(\lambda)$$

for the stable base locus of the line bundle $\mathcal{V}_{\text{flag}}(\lambda)$ restricted to $\overline{\text{Flag}}(S_K)_w$ (the Zariski closure of $\text{Flag}(S_K)_w$). By Proposition 3.5.1, for any inclusion $K'^p \subset K^p$ of compact open subgroups of $\mathbf{G}(\mathbb{A}_f^p)$, we have

$$\pi_{\text{flag}}^{-1}(\mathbb{B}_{K^p, w}(\lambda)) = \mathbb{B}_{K'^p, w}(\lambda) \quad (3.6.2)$$

where $\pi = \pi_{K', K}$ denotes the change of level map with respect to $K'^p \subset K^p$. We deduce the following result:

Theorem 3.6.2. *The set $\mathbb{B}_{K^p, w}(\lambda)$ is stable by Hecke operators.*

Proof. Let $x \in \mathbb{B}_{K^p, w}(\lambda)$ and $g \in \mathbf{G}(\mathbb{A}_f^p)$. By (3.6.2), any point in $\pi_{\text{flag}}^{-1}(x)$ lies in the set $\mathbb{B}_{K \cap gK g^{-1}, w}(\lambda)$. Therefore, $g_{\text{flag}}(\pi_{\text{flag}}^{-1}(x))$ is contained in $\mathbb{B}_{K^p, w}(\lambda)$. \square

In the case $w = w_{\max}$, this theorem has the following consequence:

Corollary 3.6.3. *Assume that $\mathbf{E}_p = \mathbb{Q}_p$. Let $\lambda \in X^*(T)$ and assume that*

$$\mathbb{B}_{K^p, w_{\max}}(\lambda) \neq \emptyset.$$

Then $\mathbb{B}_{K^p, w_{\max}}(\lambda) = \overline{\text{Flag}}(S_K)_{w_{\max}}$. In other words, for any $r \geq 1$, one has

$$H^0(\overline{\text{Flag}}(S_K)_{w_{\max}}, \mathcal{V}_{\text{flag}}(\lambda)^r) = 0.$$

Proof. Since $\mathbf{E}_p = \mathbb{Q}_p$, we may use Theorem 1.3.1. Choose $x \in \mathbb{B}_{K^p, w_{\max}}(\lambda)$ and consider the prime-to- p Hecke orbit $\mathcal{H}^p(x) \subset \text{Flag}(S_K)_{w_{\max}}$. By Theorem 3.6.2, $\mathcal{H}^p(x)$ is contained in the stable base locus $\mathbb{B}_{K^p, w_{\max}}(\lambda)$. If we write $x' := \pi(x) \in S_K^{\text{ord}}$, the Hecke orbit $\mathcal{H}^p(x')$ is Zariski dense in S_K by Theorem 1.3.1. By Lemma 3.6.4 below, we deduce that $\overline{\mathcal{H}^p(x)}$ is dense in $\text{Flag}(S_K)_{w_{\max}}$. Since $\mathbb{B}_{K^p, w_{\max}}(\lambda)$ is closed, it follows that $\mathbb{B}_{K^p, w_{\max}}(\lambda) = \overline{\text{Flag}(S_K)_{w_{\max}}}$. \square

Lemma 3.6.4. *Let $A \subset \text{Flag}(S_K)_{w_{\max}}$ be a subset such that $\pi(A)$ is Zariski dense in S_K . Then A is Zariski dense in $\text{Flag}(S_K)_{w_{\max}}$.*

Proof. We have a Cartesian diagram

$$\begin{array}{ccc} \text{Flag}(S_K)_{w_{\max}} & \longrightarrow & \mathcal{F}_{w_{\max}} \\ \downarrow & & \downarrow \\ S_{K, w_{\max}} & \xrightarrow{\zeta} & \mathcal{X}_{w_{\max}} \end{array}$$

Here, $\mathcal{X}_{w_{\max}} = \mathcal{U}_\mu$ denotes the unique open zip stratum of $G\text{-Zip}^\mu$ (see section 2.2) and $S_{K, w_{\max}} = S_K^{\text{ord}}$ is the ordinary locus (see section 2.3). Furthermore, the vertical maps are surjective, finite etale. The stacks $\mathcal{F}_{w_{\max}}$ and $\mathcal{X}_{w_{\max}}$ are connected, so the above diagram induces a bijection between connected components of $\text{Flag}(S_K)_{w_{\max}}$ and connected components of S_K . By the above, the Zariski closure \overline{A} of A intersected with any connected component of $\text{Flag}(S_K)_{w_{\max}}$ has dimension $\dim(S_K) = \dim(\text{Flag}(S_K)_{w_{\max}})$. Since $\text{Flag}(S_K)_{w_{\max}}$ is smooth, each connected component is irreducible, so we deduce that A is dense in $\text{Flag}(S_K)_{w_{\max}}$. \square

4 The cone conjecture and consequences

4.1 Statement

The cone conjecture was first formulated in [GK18]. We can state it in the broader context of a general k -scheme X endowed with a morphism $\zeta: X \rightarrow G\text{-Zip}^\mu$. For such a pair (X, ζ) , define $\text{Flag}(X)$ as in §2.4.1. For any $w \in {}^I W$, write $X_w := \zeta^{-1}(\mathcal{X}_w)$ for the corresponding locally closed subset of X . Similarly, for $w \in W$ define $\text{Flag}(X)_w$ as in equation (2.4.3). We make the following assumptions on (X, ζ) :

Assumption 4.1.1.

- (1) ζ is smooth.
- (2) The restriction of ζ to any connected component of X is surjective.
- (3) For any $w \in W$ of length 1, the Zariski closure of $\text{Flag}(X)_w$ is pseudo-complete.

In assumption (3) above, a k -scheme Y is called pseudo-complete if $\mathcal{O}_Y(Y) = k$. For example, (3) is satisfied if X is a proper k -scheme. Since $G\text{-Zip}^\mu$ is smooth over k , (1) implies that X is a smooth k -scheme. The pair $(S_K^\Sigma, \zeta^\Sigma)$ defined in section 3.3 satisfies Assumption 4.1.1, by [GK22b, §1.1.4]. For $\lambda \in X^*(T)$, the pullback $\zeta^*(\mathcal{V}_I(\lambda))$ is a vector bundle on X that we denote again by $\mathcal{V}_I(\lambda)$. Define the cone of X as follows:

$$C_X := \{\lambda \in X^*(T) \mid H^0(X, \mathcal{V}_I(\lambda)) \neq 0\}.$$

When X is connected, one can check immediately that C_X is a cone in $X^*(T)$ (i.e. an additive monoid). Write $X_{+, I}^*(T)$ for the set of I -dominant characters. Since $\mathcal{V}_I(\lambda) = 0$ when λ is not I -dominant, we obviously have $C_X \subset X_{+, I}^*(T)$. We define similarly

$$C_{\text{zip}} := \{\lambda \in X^*(T) \mid H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0\}$$

and call C_{zip} the zip cone of (G, μ) . Using the description of $H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda))$ given by (3.2.2), the cone C_{zip} relates to the representation theory of reductive groups. Since ζ is surjective, we obtain by pull-back an injective map

$$\zeta^*: H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda)) \rightarrow H^0(X, \mathcal{V}_I(\lambda)).$$

This shows that we have an inclusion $C_{\text{zip}} \subset C_X$. For a subset $C \subset X^*(T)$, we define the saturation of C by:

$$\langle C \rangle := \{\lambda \in X^*(T) \mid \exists N \geq 1, N\lambda \in C\}.$$

Conjecture 4.1.2. *If (X, ζ) satisfies Assumption 4.1.1, we have $\langle C_X \rangle = \langle C_{\text{zip}} \rangle$.*

We retain the notation from §2.3 and consider the Shimura variety \mathcal{S}_K . For a field Ω endowed with a map $\mathcal{O}_{\mathbb{E}_p} \rightarrow \Omega$, put

$$C_K(\Omega) := \{\lambda \in X^*(T) \mid H^0(\mathcal{S}_K \otimes_{\mathcal{O}_{\mathbb{E}_p}} \Omega, \mathcal{V}_I(\lambda)) \neq 0\}.$$

This set highly depends on the level K , but the saturation $\langle C_K(\Omega) \rangle$ is independent of K by [Kos19, Corollary 1.5.3]. We are interested in the case when Ω has characteristic p . By the Koecher principle (Theorem 3.3.1), the global sections of $\mathcal{V}_I(\lambda)$ over S_K and those of $\mathcal{V}_I^\Sigma(\lambda)$ over S_K^Σ coincide. Since the pair $(S_K^\Sigma, \zeta^\Sigma)$ satisfies Assumption 4.1.1, Conjecture 4.1.2 applied to the pair $(S_K^\Sigma, \zeta^\Sigma)$ predicts that $\langle C_K(k) \rangle = \langle C_{\text{zip}} \rangle$, where $k = \overline{\mathbb{F}}_p$ is an algebraic closure of \mathbb{F}_p .

The above conjecture is partly inspired by a result of Diamond–Kassaei ([DK17, Corollary 8.2]) stating that the weight of any nonzero Hilbert modular form in characteristic p is spanned by the weight of partial Hasse invariants (this was proved independently in [GK18]). This result shows that the Ekedahl–Oort stratification (and hence the stack $G\text{-Zip}^\mu$) encodes information about the weights of all automorphic forms. We have verified Conjecture 4.1.2 in [GK18], [GK22a] for several pairs (G, μ) :

Theorem 4.1.3 ([GK18], [GK22a]). *Conjecture 4.1.2 holds in the following cases. In each case, assume that μ is a minuscule cocharacter of G .*

- (a) G is an \mathbb{F}_p -form of GL_2^n .
- (b) $G = \text{GL}_{3, \mathbb{F}_p}$ or $G = \text{GL}_{4, \mathbb{F}_p}$,
- (c) $G = \text{GSp}_{4, \mathbb{F}_p}$,
- (d) $G = \text{GSp}_{6, \mathbb{F}_p}$ and $p \geq 5$,
- (e) $G = \text{GU}(3)_{\mathbb{F}_p}$,
- (f) $G = \text{GU}(4)_{\mathbb{F}_p}$ and the type of μ is not $(2, 2)$.

In particular, the above result applies to Hilbert modular varieties, several unitary Shimura varieties at split or inert primes of good reduction, and Siegel-type Shimura varieties in rank ≤ 3 .

4.2 Approximations of C_{zip}

In general, it is a difficult problem to determine the zip cone C_{zip} . In [IK22], many subcones of C_{zip} are given. For example, define the Griffiths–Schmid cone C_{GS} by

$$C_{\text{GS}} = \left\{ \lambda \in X^*(\mathbf{T}) \mid \begin{array}{l} \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for } \alpha \in I, \\ \langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for } \alpha \in \Phi_+ \setminus \Phi_{\mathbf{L}, +} \end{array} \right\}.$$

It was shown in [IK22, Theorem 6.4.2] that C_{GS} is always contained in $\langle C_{\text{zip}} \rangle$. This inclusion is consistent with Conjecture 4.1.2, in the following sense. The cone C_{GS} is expected to

coincide with $\langle C_K(\mathbb{C}) \rangle$ (this seems to be known to experts, even though no explicit proof can be found in the literature; the inclusion $\langle C_K(\mathbb{C}) \rangle \subset C_{\text{GS}}$ is proved explicitly in [GK22b]). Furthermore, by a mod p reduction argument ([Kos19, Proposition 1.8.3]), we have an inclusion $C_K(\mathbb{C}) \subset C_K(\overline{\mathbb{F}}_p)$. Therefore, it was predicted from Conjecture 4.1.2 that C_{GS} is contained in $\langle C_{\text{zip}} \rangle$, which we indeed verified. Other natural subcones of C_{zip} are defined in [IK22]: The Hasse cone C_{Hasse} , the highest weight cone C_{hw} , the lowest weight cone C_{lw} , etc. We omit the definitions of these cones here.

The above subcones give lower bounds for the sets C_{zip} and $C_K(\overline{\mathbb{F}}_p)$. However, they do not provide an upper bound, i.e. an explicit condition on $\lambda \in X^*(T)$ for the space $H^0(S_K, \mathcal{V}_I(\lambda))$ to vanish. In the paper [GK22b], we determined an upper bound for $\langle C_{\text{zip}} \rangle$. We showed that the cone $\langle C_{\text{zip}} \rangle$ is contained in a (rather) explicit cone, called the unipotent-invariance cone. We omit the general definition, which is somewhat technical. The situation simplifies greatly if we make the following assumptions:

- (1) The parabolic P is defined over \mathbb{F}_p .
- (2) The group G is split over \mathbb{F}_{p^2} .

In this case, one can give a quite sharp and explicit approximation from above for the cone $\langle C_{\text{zip}} \rangle$. Let $W_L = W(L, T)$ be the Weyl group of L . Note that $W_L \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ acts naturally on the set $\Phi_+ \setminus \Phi_{L,+}$. We proved ([GK22b, Theorem 3.2.5] and §3.3):

Proposition 4.2.1. *Any $\lambda \in \langle C_{\text{zip}} \rangle$ satisfies*

$$\sum_{\alpha \in \mathcal{O} \setminus S} \langle \lambda, \alpha^\vee \rangle + \frac{1}{p} \sum_{\alpha \in S} \langle \lambda, \alpha^\vee \rangle \leq 0 \quad (4.2.1)$$

for all $W_L \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ -orbits $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$ and all subsets $S \subset \mathcal{O}$.

Following the notation and terminology of [GK22b], we denote by $C_{L-\text{Min}}$ the set of $\lambda \in X^*(T)$ satisfying the above inequalities (4.2.1), and call $C_{L-\text{Min}}$ the L -minimal cone. We also define

$$C_{L-\text{Min}}^{+,I} := C_{L-\text{Min}} \cap X_{+,I}^*(T). \quad (4.2.2)$$

Since $\langle C_{\text{zip}} \rangle$ and $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ are always contained in the I -dominant cone $X_{+,I}^*(T)$, the cone $C_{L-\text{Min}}^{+,I}$ is actually more relevant to us. The number of inequalities necessary to define $C_{L-\text{Min}}^{+,I}$ is usually significantly smaller than for $C_{L-\text{Min}}$, but it is difficult to determine a set of minimal inequalities for it. In particular, Conjecture 4.1.2 implies the following:

Conjecture 4.2.2. *Let S_K be the special fiber of a Hodge-type Shimura variety at a place p of good reduction. Assume that p splits in the reflex field \mathbf{E} and that the attached reductive \mathbb{F}_p -group G is split over \mathbb{F}_{p^2} . Then the space $H^0(S_K, \mathcal{V}_I(\lambda))$ is zero outside of the locus defined by the inequalities (4.2.1).*

4.3 Partial Hasse invariants

For $w \in W$, let E_w denote the set of positive roots α such that $ws_\alpha < w$ and $\ell(ws_\alpha) = \ell(w) - 1$. In other words, the elements ws_α for $\alpha \in E_w$ are precisely the lower neighbors of w in W with respect to the Bruhat order. For a pair $(\chi, \eta) \in X^*(T)^2$, let $\mathcal{V}_{\text{Sbt}}(\chi, \eta)$ be the attached line bundle on the Schubert stack Sbt . We review Chevalley's formula for the stratum Sbt_w .

Theorem 4.3.1 ([GK19a, Theorem 2.2.1]). *Let $w \in W$. One has the following:*

- (1) $H^0(\text{Sbt}_w, \mathcal{V}_{\text{Sbt}}(\chi, \eta)) \neq 0 \iff \eta = -w^{-1}\chi$.

- (2) $\dim_k H^0(\text{Sbt}_w, \mathcal{V}_{\text{Sbt}}(\chi, -w^{-1}\chi)) = 1$.
(3) For any nonzero $f \in H^0(\text{Sbt}_w, \mathcal{V}_{\text{Sbt}}(\chi, -w^{-1}\chi))$, one has

$$\text{div}(f) = - \sum_{\alpha \in E_w} \langle \chi, w\alpha^\vee \rangle \overline{\text{Sbt}}_{ws_\alpha}.$$

For each $\chi \in X^*(T)$ and $w \in W$, we denote by $f_{w,\chi}$ a nonzero element in the space $H^0(\text{Sbt}_w, \mathcal{V}_{\text{Sbt}}(\chi, -w^{-1}\chi))$. By Theorem 4.3.1, the section $f_{w,\chi}$ extends to the Zariski closure $\overline{\text{Sbt}}_w$ if and only if $\langle \chi, w\alpha^\vee \rangle \leq 0$ for all $\alpha \in E_w$. By [GK19a, Lemma 3.1.1 (b)], the pullback of $\mathcal{V}_{\text{Sbt}}(\chi, \eta)$ via the morphism $\psi: G\text{-ZipFlag}^\mu \rightarrow \text{Sbt}$ defined in (2.4.2) is:

$$\psi^*(\mathcal{V}_{\text{Sbt}}(\chi, \nu)) = \mathcal{V}_{\text{flag}}(\chi + pw_{0,I}w_0\sigma^{-1}(\nu))$$

(note that *loc. cit.* contains a typo; it should be σ^{-1} instead of σ). For $w \in W$, define

$$h_w: X^*(T) \rightarrow X^*(T), \quad \chi \mapsto -w\chi + pw_{0,I}w_0\sigma^{-1}(\chi). \quad (4.3.1)$$

For all $\chi \in X^*(T)$ and all $w \in W$, define

$$\text{Ha}_{w,\chi} := \psi^*(f_{w,-w\chi}).$$

By construction, we have

$$\text{Ha}_{w,\chi} \in H^0(\mathcal{F}_w, \mathcal{V}_{\text{flag}}(h_w(\chi))).$$

Since ψ is smooth, the multiplicity of $\text{Ha}_{w,\chi}$ along \mathcal{F}_{ws_α} is the same as the multiplicity of $f_{w,-w\chi}$ along Sbt_{ws_α} , namely $\langle \chi, \alpha^\vee \rangle$. We are particularly interested in sections which vanish exactly on the Zariski closure of a single stratum $\overline{\mathcal{F}}_{ws_\alpha} \subset \overline{\mathcal{F}}_w$ (where $\alpha \in E_w$). This kind of section is sometimes called a partial Hasse invariant.

Definition 4.3.2. We say that $w \in W$ admits a separating system of partial Hasse invariants if the elements $\{\alpha^\vee \mid \alpha \in E_w\}$ are linearly independent in the \mathbb{Q} -vector space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If w admits a separating system of partial Hasse invariants, then each flag stratum of codimension 1 in $\overline{\mathcal{F}}_w$ can be cut out by a partial Hasse invariant. Indeed, in this case, for each $\beta \in E_w$, we can find a character $\chi_\beta \in X^*(T)$ satisfying

$$\begin{cases} \langle \chi_\beta, \beta^\vee \rangle > 0 & \text{and} \\ \langle \chi_\beta, \alpha^\vee \rangle = 0 & \text{for all } \alpha \in E_w \setminus \{\beta\}. \end{cases} \quad (4.3.2)$$

For this character χ_β , the corresponding section Ha_{w,χ_β} is a partial Hasse invariant which cuts out exactly \mathcal{F}_{ws_β} inside $\overline{\mathcal{F}}_w$. Note that the weight of Ha_{w,χ_β} is

$$\text{ha}_{w,\chi_\beta} := h_w(\chi_\beta) = -w\chi_\beta + pw_{0,I}w_0\sigma^{-1}(\chi_\beta). \quad (4.3.3)$$

4.4 Regularity of strata

Although we are mainly interested in proving results pertaining to automorphic forms in $H^0(S_K, \mathcal{V}_I(\lambda))$, it is useful to consider sections on smaller flag strata or zip strata, or the Zariski closures thereof. This strategy was carried out in [GK19a], where exact sequences between cohomology groups of various strata play a key role to move Hecke eigen-systems from higher degree cohomology to degree 0. For this reason, we introduce the following, somewhat cumbersome but useful notation and terminology. For each $w \in W$, denote by

$X_{+,w}^*(T)$ the set of characters $\chi \in X^*(T)$ such that $\langle \chi, \alpha^\vee \rangle \geq 0$ for all $\alpha \in E_w$. We may call such characters " w -dominant". Define the Hasse cone of w by:

$$C_{\text{Hasse},w} := h_w(X_{+,w}^*(T))$$

where h_w is the map (4.3.1). By Chevalley's formula (Theorem 4.3.1), the section $\text{Ha}_{w,\chi}$ extends to $\overline{\mathcal{F}}_w$ if and only if $\chi \in X_{+,w}^*(T)$. Therefore, the set $C_{\text{Hasse},w}$ is the set of all $\lambda \in X^*(T)$ such that $\mathcal{V}_{\text{flag}}(\lambda)$ admits a nonzero section over $\overline{\mathcal{F}}_w$ which arises by pullback from a section over $\overline{\text{Sbt}}_w$. In general, there are many sections on $\overline{\mathcal{F}}_w$ which do not arise by pullback in this fashion. To account for them, we define the following cone:

$$C_{\text{flag},w} := \{\lambda \in X^*(T) \mid H^0(\overline{\mathcal{F}}_w, \mathcal{V}_{\text{flag}}(\lambda)) \neq 0\}.$$

By the above discussion, it is clear that $C_{\text{Hasse},w} \subset C_{\text{flag},w}$.

Consider now a k -scheme X endowed with a surjective map $\zeta: X \rightarrow G\text{-Zip}^\mu$ (for example $X = S_K$ or $X = S_K^\Sigma$). Let $\text{Flag}(X)$ be the flag space of X (diagram (2.4.1)). Let $(\text{Flag}(X)_w)_{w \in W}$ be the flag stratification on $\text{Flag}(X)$ and write $\overline{\text{Flag}}(X)_w$ for the Zariski closure of $\text{Flag}(X)_w$. Define similarly:

$$C_{X,w} := \{\lambda \in X^*(T) \mid H^0(\overline{\text{Flag}}(X)_w, \mathcal{V}_{\text{flag}}(\lambda)) \neq 0\}. \quad (4.4.1)$$

Since ζ is surjective, the pullback map via ζ is injective, so we get inclusions

$$C_{\text{Hasse},w} \subset C_{\text{flag},w} \subset C_{X,w}. \quad (4.4.2)$$

Note that when $w = w_0$, the identification (3.2.3) shows that $C_{\text{flag},w_0} = C_{\text{zip}}$ and similarly $C_{X,w_0} = C_X$. For $w = w_0$, we simply write $C_{\text{Hasse},w_0} = C_{\text{Hasse}}$. Concretely, C_{Hasse} is the set of non-negative linear combinations of the weights of the global partial Hasse invariants, similarly to those studied in [DK17] and [IK21b].

Definition 4.4.1. *Let $w \in W$.*

- (a) *We say that the flag stratum $\text{Flag}(X)_w$ is Hasse-regular if $\langle C_{X,w} \rangle = \langle C_{\text{Hasse},w} \rangle$.*
- (b) *We say that the flag stratum $\text{Flag}(X)_w$ is flag-regular if $\langle C_{X,w} \rangle = \langle C_{\text{flag},w} \rangle$.*

In view of the inclusions (4.4.2), it is clear that any Hasse-regular stratum is flag-regular. Conjecture 4.1.2 asserts that under Assumption 4.1.1, the maximal stratum $\text{Flag}(X)_{w_0}$ is flag-regular. When X is a Hilbert–Blumenthal modular variety, this stratum is even Hasse-regular by Diamond–Kassaei's result [DK17, Corollary 8.2], proved independently in [GK18, Corollary 4.2.4] (note that in this case $X = \text{Flag}(X)$, and flag strata coincide with usual Ekedahl–Oort strata). Concretely, this means that the weight of any nonzero Hilbert modular form over $\overline{\mathbb{F}}_p$ is spanned by the weights of the partial Hasse invariants. The same holds for Siegel threefolds and Picard modular surfaces at a split prime ([GK18, Corollary 5.1.2]). However, in general the stratum $\text{Flag}(X)_{w_0}$ is not Hasse-regular because the inclusion $\langle C_{\text{Hasse}} \rangle \subset \langle C_{\text{zip}} \rangle$ can be strict. For a complete characterization of the cases when the equality $\langle C_{\text{Hasse}} \rangle = \langle C_{\text{zip}} \rangle$ holds, see [IK22, Theorem 4.3.1].

Returning to the case of Hilbert–Blumenthal modular varieties (attached to a totally real field \mathbf{F}), the paper [GK18] also covers the case of smaller strata. There is an explicit and straight-forward criterion (see *loc. cit.* Theorem 4.2.3) on the element $w \in W$ for the Hasse-regularity of $S_{K,w}$. When p is split in \mathbf{F} , all strata are Hasse-regular. For general Hodge-type Shimura varieties with good reduction at p , we conjectured the following ([GK22b]):

Conjecture 4.4.2. *Assume that P is defined over \mathbb{F}_p . Under Assumption 4.1.1, the flag stratum $\text{Flag}(X)_{w_{\max}}$ is Hasse-regular.*

When $X = S_K$, the condition that P is defined over \mathbb{F}_p is equivalent to the non-emptiness of the ordinary locus of S_K . In the paper [GK22b], we prove Conjecture 4.4.2 in the case of PEL unitary Shimura varieties of signature $(n-1, 1)$ at a split prime. The flag stratum parametrized by w_{\max} plays a central role in our strategy. When P is defined over \mathbb{F}_p , the cone $C_{\text{Hasse}, w_{\max}}$ has a very simple form:

$$C_{\text{Hasse}, w_{\max}} = \{\lambda \in X^*(T) \mid \langle \lambda, \alpha^\vee \rangle \leq 0, \text{ for all } \alpha \in \Phi_+ \setminus \Phi_{L,+}\}.$$

Hence, Conjecture 4.4.2 means that if $H^0(\overline{\text{Flag}}(X)_{w_{\max}}, \mathcal{V}_{\text{flag}}(\lambda))$ is nonzero, then λ has a non-positive scalar product with all roots in $\Phi_+ \setminus \Phi_{L,+}$.

In the case of PEL unitary Shimura varieties of signature $(n-1, 1)$ at a split prime, Conjecture 4.4.2 was used as one of the main ingredients in the proof of Conjecture 4.2.2 in the article [GK22b]. In the present paper concerned with the Siegel case, we will see in the next section that a similar approach can be carried out.

5 Vanishing result for Siegel automorphic forms

In this section, we prove Conjectures 4.2.2 and 4.4.2 in the case of Siegel-type Shimura varieties. We will adopt a similar strategy as in the unitary case mentioned above, treated in [GK22b]. However, the extremely tedious brute force computations to prove Conjecture 4.4.2 will be replaced by a much more elegant argument.

5.1 Main result

Recall that $C_K(\overline{\mathbb{F}_p})$ is always contained in the I -dominant cone $X_{+,I}^*(T)$, so we consider the intersection $C_{L-\text{Min}} \cap X_{+,I}^*(T)$. It is often the case that many of the $W_L \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ -orbits $\mathcal{O} \subset \Phi_+ \setminus \Phi_{L,+}$ and subsets $S \subset \mathcal{O}$ appearing in (4.2.1) will contribute trivially to this intersection, and can be removed without changing the result. Let us examine the case of $G = \text{GSp}_{2n, \mathbb{F}_p}$. We retain the notation of §1.1. Write T for the diagonal torus of G . It is generated by the subgroup $Z \subset T$ of nonzero scalar matrices and the subtorus $T_0 := T \cap \text{Sp}_{2n}$. Explicitly, T_0 is given by

$$T_0 = \{\text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \in \mathbb{G}_m\}.$$

Identify $X^*(T_0) = \mathbb{Z}^n$ such that (a_1, \dots, a_n) corresponds to the character

$$\text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mapsto \prod_{i=1}^n t_i^{a_i}.$$

We also identify $X^*(Z) = \mathbb{Z}$, such that $b \in \mathbb{Z}$ corresponds to the character $\text{diag}(t, \dots, t) \mapsto t^b$. The natural map $X^*(T) \rightarrow X^*(T_0) \times X^*(Z) = \mathbb{Z}^n \times \mathbb{Z}$, $\chi \mapsto (\chi|_{T_0}, \chi|_Z)$ is injective, and identifies $X^*(T)$ with the set

$$X^*(T) = \left\{ (a_1, \dots, a_n, b) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^n a_i \equiv b \pmod{2} \right\}. \quad (5.1.1)$$

Write e_1, \dots, e_n for the canonical basis of \mathbb{Z}^n , and write again e_i for the element $(e_i, 0) \in \mathbb{Z}^{n+1}$. The positive roots of G are

$$\Phi_+ := \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \sqcup \{2e_i \mid 1 \leq i \leq n\}.$$

Hence, the set $\Phi_+ \setminus \Phi_{L,+}$ consists of exactly two W_L -orbits, namely

$$\mathcal{O}_1 = \{2e_i \mid 1 \leq i \leq n\} \quad \text{and} \quad \mathcal{O}_2 = \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

One can show that the inequalities (4.2.1) corresponding to subsets $S \subset \mathcal{O}_2$ become redundant when intersecting with $X_{+,I}^*(T)$, so we may restrict to subsets of \mathcal{O}_1 . Similarly, only subsets of the form $S = \{2e_1, 2e_2, \dots, 2e_j\}$ (for $j = 1, \dots, n-1$) are relevant. Therefore, the minimal set of inequalities necessary to define $C_{L-\text{Min}}^{+,I}$ (see (4.2.2)) inside $X_{+,I}^*(T)$ is given as follows:

$$C_{L-\text{Min}}^{+,I} = \left\{ (a_1, \dots, a_n, b) \in X_{+,I}^*(T) \mid \sum_{i=1}^j a_i + \frac{1}{p} \sum_{i=j+1}^n a_i \leq 0 \text{ for } j = 1, \dots, n-1 \right\}. \quad (5.1.2)$$

The following theorem is our main result on the topic of Siegel automorphic forms. We explain the proof in section 5.5.

Theorem 5.1.1. *If $f \in H^0(\overline{\mathcal{A}}_{n,K}, \mathcal{V}_I(\lambda))$ is a nonzero automorphic form over $k = \overline{\mathbb{F}}_p$, the weight $\lambda = (a_1, \dots, a_n, b)$ satisfies the inequalities (5.1.2). In other words, if λ fails to satisfy at least one of the above inequalities, one has*

$$H^0(\overline{\mathcal{A}}_{n,K}, \mathcal{V}_I(\lambda)) = 0.$$

In the case $n = 3$, it was shown for $p \geq 5$ in [GK22a] that the cone $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ coincides with $\langle C_{\text{zip}} \rangle$ and is given as follows:

$$\langle C_K(\overline{\mathbb{F}}_p) \rangle = \left\{ (a_1, a_2, a_3, b) \in X_{+,I}^*(T) \mid \begin{array}{l} p^2 a_1 + a_2 + p a_3 \leq 0 \\ p a_1 + p^2 a_2 + a_3 \leq 0 \end{array} \right\}.$$

This cone is only slightly smaller than the cone $C_{L-\text{Min}}^{+,I}$, which is given by

$$C_{L-\text{Min}}^{+,I} = \left\{ (a_1, a_2, a_3, b) \in X_{+,I}^*(T) \mid \begin{array}{l} p a_1 + a_2 + a_3 \leq 0 \\ p a_1 + p a_2 + a_3 \leq 0 \end{array} \right\}$$

(see [GK22a, Figure 1] for a visual plot of these cones). This shows that the upper bound for $\langle C_K(\overline{\mathbb{F}}_p) \rangle$ provided by Theorem 5.1.1 is a good approximation.

5.2 Bruhat order of W

Recall that the Weyl group W of GSp_{2n} can be identified with the set of permutations $w \in \mathfrak{S}_{2n}$ satisfying $w(i) + w(2n+1-i) = 2n+1$ for all $1 \leq i \leq 2n$. For $w \in W$, put

$$\begin{aligned} \mathcal{M}(w) &:= \{(i, j) \mid 1 \leq i < j \leq n \text{ and } w(i) > w(j)\} \\ \mathcal{N}(w) &:= \{(i, j) \mid 1 \leq i \leq j \leq n \text{ and } w(i) + w(j) > 2n+1\}. \end{aligned}$$

Write $M(w)$, $N(w)$ for the number of elements of $\mathcal{M}(w)$ and $\mathcal{N}(w)$ respectively. The length $\ell(w)$ of $w \in W$ is given by

$$\ell(w) = M(w) + N(w).$$

For $w \in W$ and $1 \leq i, j \leq 2n$, define

$$r_w(i, j) := |\{1 \leq k \leq i \mid w(k) \leq j\}|.$$

For two elements $w_1, w_2 \in W$, one has an equivalence

$$w_1 \leq w_2 \iff r_{w_1}(i, j) \geq r_{w_2}(i, j) \quad \text{for all } 1 \leq i, j \leq 2n. \quad (5.2.1)$$

For any permutation $\tau \in \mathfrak{S}_{2n}$, write $M_\tau \in \text{GL}_{2n}$ for the permutation matrix of τ . We say that a pair (i, j) is admissible for τ if it satisfies the following conditions:

- $1 \leq i < j \leq 2n$
- $\tau(i) > \tau(j)$
- There is no $i < k < j$ such that $\tau(j) < \tau(k) < \tau(i)$.

In other words, (i, j) is admissible for τ if and only if the submatrix of M_τ with corners $(i, \tau(i))$ and $(j, \tau(j))$ has only zero coefficients, except for these two corners. Now, we let $w \in W$ and define three sets $\mathcal{E}_w^1, \mathcal{E}_w^2, \mathcal{E}_w^3$ as follows:

$$\begin{aligned}\mathcal{E}_w^1 &:= \{(i, j) \mid \text{admissible for } w, \text{ and } 1 \leq i < j \leq n\} \\ \mathcal{E}_w^2 &:= \{(i, j) \mid \text{admissible for } w, i \leq n < j \text{ and } w(i), w(j) \leq n\} \\ \mathcal{E}_w^3 &:= \{(i, j) \mid \text{admissible for } w, i \leq n \text{ and } j = 2n + 1 - i\}.\end{aligned}$$

Finally, we set

$$\mathcal{E}_w = \mathcal{E}_w^1 \sqcup \mathcal{E}_w^2 \sqcup \mathcal{E}_w^3.$$

For $(i, j) \in \mathcal{E}_w$, define a positive root $\gamma(i, j) \in \Phi_+$ as follows:

$$\gamma(i, j) := \begin{cases} e_i - e_j & \text{if } (i, j) \in \mathcal{E}_w^1 \\ e_i + e_{2n+1-j} & \text{if } (i, j) \in \mathcal{E}_w^2 \\ 2e_i & \text{if } (i, j) \in \mathcal{E}_w^3. \end{cases}$$

One has the following lemma:

Lemma 5.2.1. *The map γ is a bijection of \mathcal{E}_w onto the set E_w .*

Proof. We first show that when $(i_0, j_0) \in \mathcal{E}_w$, the positive root $\alpha = \gamma(i_0, j_0)$ lies in E_w . By the characterization of the Bruhat order (see (5.2.1)), we clearly have $ws_\alpha < w$. Next, we need to show that $\ell(ws_\alpha) = \ell(w) - 1$. We consider the following cases:

- The case $(i_0, j_0) \in \mathcal{E}_w^1$: We construct a bijection $\epsilon: \mathcal{M}(w) \setminus \{(i_0, j_0)\} \rightarrow \mathcal{M}(ws_\alpha)$. Let $(i, j) \in \mathcal{M}(w) \setminus \{(i_0, j_0)\}$.

$$\begin{cases} \text{if } i < i_0 \text{ and } w(i_0) > w(i) > w(j_0), & \text{we set } \epsilon(i, j_0) = (i, i_0) \\ \text{if } j_0 < j \text{ and } w(i_0) > w(j) > w(j_0), & \text{we set } \epsilon(i_0, j) = (j_0, j). \end{cases}$$

In all other cases, set $\epsilon(i, j) = (i, j)$. Then one checks easily that ϵ is a bijection $\epsilon: \mathcal{M}(w) \setminus \{(i_0, j_0)\} \rightarrow \mathcal{M}(ws_\alpha)$. Therefore, we have $M(ws_\alpha) = M(w) - 1$. On the other hand, if two permutations τ, τ' satisfy $\{\tau(1), \dots, \tau(n)\} = \{\tau'(1), \dots, \tau'(n)\}$, then one clearly has $N(\tau) = N(\tau')$. Hence we deduce $N(ws_\alpha) = N(w)$ and thus $\ell(ws_\alpha) = \ell(w) - 1$.

- The case $(i_0, j_0) \in \mathcal{E}_w^2$: In this case, one has $(w(j_0), w(i_0)) \in \mathcal{E}_{w^{-1}}^1$. We have $ws_\alpha = s_{w \cdot \alpha} w$. By the first case above, we deduce

$$\ell(ws_\alpha) = \ell(s_{w \cdot \alpha} w) = \ell(w^{-1} s_{w \cdot \alpha}) = \ell(w^{-1}) - 1 = \ell(w) - 1.$$

- The case $(i_0, j_0) \in \mathcal{E}_w^3$: Denote by A the set of pairs (i, i_0) satisfying $1 \leq i < i_0$ and $w(j_0) < w(i) < w(i_0)$. One shows easily the following:

$$\begin{aligned}\mathcal{M}(ws_\alpha) &= \mathcal{M}(w) \sqcup A \\ \mathcal{N}(ws_\alpha) &= \mathcal{N}(w) \setminus (A \cup \{(i_0, i_0)\})\end{aligned}$$

It follows that $\ell(ws_\alpha) = \ell(w) - 1$. This terminates the proof of $\gamma(\mathcal{E}_w) \subset E_w$.

Finally, we show that any $\alpha \in E_w$ is of the form $\gamma(i_0, j_0)$ for $(i_0, j_0) \in \mathcal{E}_w$. We consider the case $\alpha = e_{i_0} - e_{j_0}$ for $1 \leq i_0 < j_0 \leq n$. Since $ws_\alpha < w$, it is clear that $w(i_0) > w(j_0)$. We check that (i_0, j_0) is admissible for w . If $i_0 < k < j_0$ satisfies $w(i_0) > w(k) > w(j_0)$, then define $\beta := e_{i_0} - e_k$. Using (5.2.1), one sees immediately that $w > ws_\beta > ws_\alpha$. This contradicts the assumption $\ell(ws_\alpha) = \ell(w) - 1$. Hence (i_0, j_0) is admissible for w . In the case $\alpha = e_{i_0} + e_{j_0}$ (resp. $\alpha = 2e_{i_0}$), a similar argument applies to show that (i_0, j_0) (resp. $(i_0, 2n + 1 - i_0)$) is admissible. We deduce the result. \square

5.3 Auxilliary sequence

Using the same strategy as in [GK22b], we define a descending path in W from the longest element w_0 of W to $w_{\max} = w_{0,I}w_0$ (the longest element of IW) satisfying certain requirements. Specifically, we construct a sequence of elements $\tau_1, \tau_2, \dots, \tau_N$ satisfying the following conditions:

- (a) $\tau_1 = w_0$ and $\tau_N = w_{\max}$.
- (b) $\tau_1 > \dots > \tau_N$ and $\ell(\tau_{i+1}) = \ell(\tau_i) - 1$ for each $i = 1, \dots, N - 1$.
- (c) Each τ_i admits a separating system of partial Hasse invariants (Definition 4.3.2).

For $1 \leq d \leq n$, let $\Lambda_d \in W$ be the element corresponding to the following permutation:

$$\Lambda_d := \left(\begin{array}{c|c|c} & & 1 \\ & & \ddots \\ & & 1 \\ \hline & & 1 \\ & & \ddots \\ & 1 & \\ \hline 1 & & \\ & \ddots & \\ & 1 & \end{array} \right)$$

where the upper right block and the lower left block have size $d \times d$ and the middle block has size $(2n - 2d) \times (2n - 2d)$. In particular, $\Lambda_1 = w_0$ is the longest element of W and $\Lambda_n = w_{0,I}w_0 = w_{\max}$ is the longest element of IW . For each $1 \leq d \leq n - 1$, we construct a path from Λ_d to Λ_{d+1} as follows. Define

$$\begin{aligned} \tau_d^{(0)} &= \Lambda_d \\ \tau_d^{(1)} &= \tau_d^{(0)} s_{\alpha_1} \quad \text{with } \alpha_1 = e_1 - e_{d+1} \\ \tau_d^{(2)} &= \tau_d^{(1)} s_{\alpha_2} \quad \text{with } \alpha_2 = e_2 - e_{d+1} \\ &\vdots \\ \tau_d^{(d-1)} &= \tau_d^{(d-2)} s_{\alpha_{d-1}} \quad \text{with } \alpha_{d-1} = e_{d-1} - e_{d+1} \\ \tau_d^{(d)} &= \tau_d^{(d-1)} s_{\alpha_d} \quad \text{with } \alpha_d = e_d - e_{d+1} \end{aligned}$$

At each step, the coefficient 1 in the $d + 1$ -th row moves down by one. Therefore, after d steps, we have $\tau_d^{(d)} = \Lambda_{d+1}$. It is easy to check that $\ell(\tau_d^{(i+1)}) = \ell(\tau_d^{(i)}) - 1$. By this construction, we obtain a path in W from Λ_d to Λ_{d+1} . By concatenating these paths, we obtain a path from $\Lambda_1 = w_0$ to $\Lambda_n = w_{\max}$.

Proposition 5.3.1. *For all $1 \leq d \leq n - 1$ and all $0 \leq i \leq d - 1$, the element $\tau_d^{(i)}$ admits a separating system of partial Hasse invariants.*

Proof. We use Lemma 5.2.1. For each $1 \leq d \leq n - 2$ and each $0 \leq i \leq d - 1$, the element $\tau_d^{(i)}$ has exactly n lower neighbors. By the shape of the permutation matrix of $\tau_d^{(i)}$, we see that the set $E_{\tau_d^{(i)}}$ is made of five different parts, namely:

$$\begin{aligned} E_{\tau_d^{(i)}} &= \{e_k - e_{d+2} \mid 1 \leq k \leq i\} \cup \{e_{d+1} - e_{d+2}\} \cup \{e_k - e_{d+1} \mid i + 1 \leq k \leq d\} \\ &\quad \cup \{e_k - e_{k+1} \mid d + 2 \leq k \leq n - 1\} \cup \{2e_n\}. \end{aligned}$$

It is straight-forward to check that the above elements are linearly independent. Finally, in the case $d = n - 1$, the number of lower neighbors of $\tau_{n-1}^{(i)}$ is $n - i - 1$ for all $0 \leq i \leq n - 2$. Specifically, we have

$$E_{\tau_{n-1}^{(i)}} = \{e_k - e_n \mid i + 1 \leq k \leq n - 1\}.$$

Again, the above elements are linearly independent. This terminates the proof. \square

In particular, there exists a partial Hasse invariant on the flag stratum parametrized by $\tau_d^{(i)}$ whose vanishing locus is exactly the Zariski closure of the stratum parametrized by $\tau_d^{(i+1)}$, for each $1 \leq d \leq n - 1$ and each $0 \leq i \leq d - 1$. We first examine the case $1 \leq d \leq n - 2$. In this case, the elements of $E_{\tau_d^{(i)}}$ form a basis of \mathbb{Z}^n , so there is a unique such partial Hasse invariant (up to multiple). To determine the weight of this partial Hasse invariant, recall that $\tau_d^{(i+1)} = \tau_d^{(i)} s_\alpha$ with $\alpha = e_{i+1} - e_{d+1}$. We consider the character

$$\chi_d^{(i)} := e_{i+1}.$$

By the proof of Proposition 5.3.1, $\chi_d^{(i)}$ is orthogonal to all elements of $E_{\tau_d^{(i)}}$ different from $\alpha = e_{i+1} - e_{d+1}$, and satisfies $\langle \chi_d^{(i)}, \alpha^\vee \rangle = 1$. By (4.3.3) the corresponding partial Hasse invariant $\text{Ha}_{\tau_d^{(i)}, \chi_d^{(i)}}$ has weight

$$\begin{aligned} \text{ha}_{\tau_d^{(i)}, \chi_d^{(i)}} &= -\tau_d^{(i)} \chi_d^{(i)} + pw_{0,I} w_0 \chi_d^{(i)} \\ &= e_{d-i+1} - pe_{n-i}. \end{aligned} \tag{5.3.1}$$

We denote the above character simply by $\text{ha}_d^{(i)}$.

Lemma 5.3.2. *For each $1 \leq d \leq n - 1$ and each $0 \leq i \leq d - 1$, the weight $\text{ha}_d^{(i)}$ lies in the L -minimal cone (5.1.2).*

Proof. This follows immediately from (5.3.1), noting that $d - i + 1 \leq n - i$. \square

5.4 Hasse-regularity

In this section, we prove conjecture 4.4.2 for $\overline{\mathcal{A}}_{n,K}$. For $w \in W$, write $C_{K,w}$ for the cone attached to the flag stratum $\text{Flag}(\overline{\mathcal{A}}_{n,K})_w$, as defined in (4.4.1). Recall that we parametrize characters in $X^*(T)$ by $n + 1$ -tuples (a_1, \dots, a_n, b) satisfying $\sum_{i=1}^n a_i \equiv b \pmod{2}$ (see (5.1.1)).

Theorem 5.4.1. *The flag stratum $\text{Flag}(\overline{\mathcal{A}}_{n,K})_{w_{\max}}$ is Hasse-regular. Explicitly, one has*

$$\langle C_{K,w_{\max}} \rangle = \langle C_{\text{Hasse},w_{\max}} \rangle = \{(a_1, \dots, a_n, b) \in X^*(T) \mid a_i \leq 0, i = 1, \dots, n\}.$$

The above theorem has also consequences for automorphic forms over $\overline{\mathbb{F}}_p$. Indeed, let $f \in H^0(\overline{\mathcal{A}}_{n,K}, \mathcal{V}_I(\lambda))$ be an automorphic form and assume that $\lambda = (a_1, \dots, a_n, b)$ satisfies $a_j > 0$ for some $j = 1, \dots, n$. Then, if we view f as a section of $\mathcal{V}_{\text{flag}}(\lambda)$ on the flag space of $\overline{\mathcal{A}}_{n,K}$, f must vanish identically on the flag stratum $\text{Flag}(\overline{\mathcal{A}}_{n,K})_{w_{\max}}$. Equivalently, the flag stratum $\text{Flag}(\overline{\mathcal{A}}_{n,K})_{w_{\max}}$ is contained in the base-locus of $\mathcal{V}_{\text{flag}}(\lambda)$ whenever λ has at least one positive coordinate.

To prove Theorem 5.4.1, we use an embedding from a Hilbert–Blumenthal Shimura variety into $\overline{\mathcal{A}}_{n,K}$. Note that the cones $\langle C_{K,w} \rangle$ are independent of the choice of the level K (by the same formal argument as [Kos19, Corollary 1.5.3]). Therefore, we may change K

freely if necessary. Choose a totally real field \mathbf{F} such that p splits in \mathbf{F} , and consider the associated Hilbert–Blumenthal Shimura variety as in section 1.4. Choose open compact subgroups $K'^p \subset \mathbf{H}(\mathbb{A}_f^p)$ and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ such that $u(K'^p) \subset K^p$, where $u: \mathbf{H} \rightarrow \mathbf{G}$ is the embedding (1.4.1). We adopt the same notation as in section 2.3 and diagram (2.4.5), namely we write $X_H := \overline{\mathcal{H}}_{\mathbf{F}, K'}$ and $X_G := \overline{\mathcal{A}}_{n, K}$. Write also Flag_G for the flag space of X_G .

Proof of Theorem (5.4.1). We first note that the formation of the line bundles $\mathcal{V}_{\text{flag}}(\lambda)$ is functorial. To explain this, we briefly consider the more general setting of diagram (2.4.4), where H, G are arbitrary connected reductive \mathbb{F}_p -groups endowed with cocharacters μ_H, μ_G respectively and $f: H \rightarrow G$ is a compatible embedding. Write B_H, B_G for the Borel subgroups as in §2.4.2 and choose maximal tori $T_H \subset B_H$ and $T_G \subset B_G$ such that $f(T_H) \subset T_G$. It is clear by construction that for any $\lambda \in X^*(T_G)$, one has

$$f_{\text{flag}}^*(\mathcal{V}_{\text{flag}}(\lambda)) = \mathcal{V}_{\text{flag}}(f^*(\lambda)) \quad (5.4.1)$$

where $f^*(\lambda)$ is the character $\lambda \circ f: T_H \rightarrow \mathbb{G}_m$. We now return to the present setting. The embedding $u: H \rightarrow G$ induces an isomorphism $u: T_H \rightarrow T$ of tori. For $\lambda = (a_1, \dots, a_n, b) \in X^*(T)$, the character $u^*(\lambda)$ is given by

$$\left(\begin{pmatrix} xt_1 & \\ & xt_1^{-1} \end{pmatrix}, \dots, \begin{pmatrix} xt_n & \\ & xt_n^{-1} \end{pmatrix} \right) \mapsto x^b \prod_{i=1}^n t_i^{a_i}.$$

Let $\lambda = (a_1, \dots, a_n, b) \in X^*(T)$ and assume that $a_j > 0$ for some $j \in \{1, \dots, n\}$. We want to show that $H^0(\overline{\text{Flag}}_{G, w_{\max}}, \mathcal{V}_I(\lambda)) = 0$, where $\overline{\text{Flag}}_{G, w_{\max}}$ is the Zariski closure of $\text{Flag}_{G, w_{\max}}$. Let $s \in H^0(\overline{\text{Flag}}_{G, w_{\max}}, \mathcal{V}_I(\lambda))$ be a section. By Proposition 2.4.1, the μ_H -ordinary locus (since p splits in \mathbf{F} , this is simply the usual ordinary locus) of X_H is mapped by \tilde{u}_{flag} (see diagram (2.4.5)) into the flag stratum $\text{Flag}_{G, w_{\max}}$. Thus, we obtain a map

$$\tilde{u}_{\text{flag}}: X_H \rightarrow \overline{\text{Flag}}_{G, w_{\max}}. \quad (5.4.2)$$

Consider the pullback $\tilde{u}_{\text{flag}}^*(s)$ via the map (5.4.2). By the above functoriality property (5.4.1), this is a section of $\mathcal{V}_{\text{flag}}(u^*(\lambda))$ over X_H . By [GK18, Corollary 4.2.4] or [DK17, Corollary 8.2], we know that Conjecture 4.1.2 holds for Hilbert–Blumenthal Shimura varieties. Since we are assuming that p splits in \mathbf{F} , the cone of X_H is simply given by the condition $a_i \leq 0$ for all $i = 1, \dots, n$. Hence, since we assumed $a_j > 0$ for some $j \in \{1, \dots, n\}$, we deduce:

$$H^0(X_H, \mathcal{V}_{\text{flag}}(u^*(\lambda))) = 0.$$

In particular, $\tilde{u}_{\text{flag}}^*(s) = 0$. This implies that any section of $\mathcal{V}_{\text{flag}}(\lambda)$ over $\overline{\text{Flag}}_{G, w_{\max}}$ must vanish at each point in the image of \tilde{u}_{flag} . Furthermore, for any $r \geq 1$ we have $\mathcal{V}_{\text{flag}}(\lambda)^r = \mathcal{V}_{\text{flag}}(r\lambda)$, so the same argument shows that any section of $\mathcal{V}_{\text{flag}}(\lambda)^r$ over $\overline{\text{Flag}}_{G, w_{\max}}$ is zero on the image of \tilde{u}_{flag} . In particular, the stable base locus of λ on the stratum $\overline{\text{Flag}}_{G, w_{\max}}$ satisfies

$$\mathbb{B}_{K^p, w_{\max}}(\lambda) \neq \emptyset.$$

By Corollary 3.6.3, we obtain

$$H^0(\overline{\text{Flag}}_{G, w_{\max}}, \mathcal{V}_{\text{flag}}(\lambda)^r) = 0, \quad r \geq 1.$$

This terminates the proof of Theorem 5.4.1. \square

5.5 Proof of the main result

In this final section, we prove Theorem 5.1.1. This part of the argument is similar to the one used in [GK22b]. Assume that $w \in W$ admits a separating system of partial Hasse invariants (Definition 4.3.2). Let $\beta \in E_w$ and let $\chi_\beta \in X^*(T)$ be a character satisfying

$$\begin{cases} \langle \chi_\beta, \beta^\vee \rangle > 0 & \text{and} \\ \langle \chi_\beta, \alpha^\vee \rangle = 0 & \text{for all } \alpha \in E_w \setminus \{\beta\}. \end{cases}$$

as in (4.3.2). Write $m = \langle \chi_\beta, \beta^\vee \rangle$. Then, the corresponding section $\text{Ha}_{w, \chi_\beta}$ vanishes exactly on $\overline{\mathcal{F}}_{ws_\beta}$, with multiplicity m . Write $\text{ha}_{w, \chi_\beta}$ for the weight of $\text{Ha}_{w, \chi_\beta}$ (see (4.3.3)). Let $s \in H^0(\overline{\text{Flag}}_{G, w}, \mathcal{V}_{\text{flag}}(\lambda))$ and assume that s vanishes with multiplicity $d \geq 0$ along $\overline{\text{Flag}}_{G, ws_\beta}$ (since Zariski closures of flag strata are normal, it makes sense to talk about multiplicities). If $d = 0$, then the restriction of s to $\overline{\text{Flag}}_{G, ws_\beta}$ is nonzero. If $d > 0$, then s^m can be written as $s^m = (\text{Ha}_{w, \chi_\beta})^d s'$ where s' is not identically zero on $\overline{\text{Flag}}_{G, ws_\beta}$. We deduce from the above discussion:

$$\langle C_{K, w} \rangle \subset \langle C_{K, ws_\beta} \rangle + \mathbb{N} \text{ha}_{w, \chi_\beta} \quad (5.5.1)$$

where $C_{K, w}$ is the cone of the flag stratum $\text{Flag}_{G, w}$, as in section 5.4. Therefore, we can propagate information about a particular stratum to the strata above it, provided that there exists a separating system of partial Hasse invariants.

Proof of Theorem 5.1.1. By Theorem 5.4.1, the cone $\langle C_{K, w_{\max}} \rangle$ is given by the condition $a_i \leq 0$ for all $i = 1, \dots, n$. Therefore, it is clear that $\langle C_{K, w_{\max}} \rangle$ is contained in $C_{L-\text{Min}}$. Consider the sequence

$$(\tau_d^{(i)})_{\substack{1 \leq d \leq n-1 \\ 0 \leq i \leq d-1}}$$

constructed in section 5.3. By Proposition 5.3.1, each element in this sequence admits a separating system of partial Hasse invariants. Furthermore, Lemma 5.3.2 shows that for each element of the sequence, the weight of the partial Hasse invariant $\text{ha}_d^{(i)}$ (for $1 \leq d \leq n-1$ and $0 \leq i \leq d-1$) lies in $C_{L-\text{Min}}$. By using the inclusion (5.5.1) recursively, we deduce $\langle C_{K, w_0} \rangle \subset C_{L-\text{Min}}$. Since $C_{K, w_0} = C_K(\overline{\mathbb{F}}_p)$, we deduce that $\langle C_K(\overline{\mathbb{F}}_p) \rangle \subset C_{L-\text{Min}}$. The result follows. \square

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