The cone conjecture for Hilbert–Blumenthal Shimura varieties via intersection cones

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Abstract

We give a simpler and more general proof that the weight of a mod p Hilbert modular form is spanned by the weights of partial Hasse invariants. We also cover the case of Ekedahl–Oort strata of smaller dimension. For the global stratum, this result was also proved by Diamond–Kassaei by other methods in [DK23].

1 Introduction

The main purpose of this short paper is to provide a more straight-forward proof of the cone conjecture for Hilbert–Blumenthal Shimura varieties. This conjecture, which can be formulated for arbitrary Hodge-type Shimura varieties, asserts in this case that the weight of any nonzero Hilbert modular form in characteristic p is spanned (over $\mathbb{Q}_{\geq 0}$) by the weights of partial Hasse invariants. This result was initially proved in [DK17] (extended in [DK23]) by Diamond–Kassaei and around the same time in [GK18] using a completely different approach. We discuss below the differences between these papers in terms of results and methods.

The main results of this paper concern the special fiber of Hilbert–Blumenthal Shimura varieties at a prime p of good reduction. Fix a totally real extension $F/\mathbb{Q}$ of degree $n := [F : \mathbb{Q}] > 1$. Recall that Hilbert–Blumenthal Shimura varieties are moduli spaces of polarized abelian varieties endowed with an action of $\mathcal{O}_F$. From the point of view of Deligne ([Del79], they are attached to the connected, reductive $\mathbb{Q}$-group $\mathbf{G}$ defined by

$$\mathbf{G}(R) = \{g \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} R) \mid \det(g) \in R^\times\}$$

for any $\mathbb{Q}$-algebra R. Fix a prime number p which is unramified in F. Write Σ for the set of embeddings $F \to \overline{\mathbb{Q}}_p$ and choose an ordering $\Sigma = \{\tau_1, \dots, \tau_n\}$. The base change of $\mathbf{G}$ to F then identifies with the subgroup of tuples $(g_1, \dots, g_n) \in \mathrm{GL}_2 \times \cdots \times \mathrm{GL}_2$ satisfying the condition $\det(g_1) = \cdots = \det(g_n)$. Since p is unramified, the group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on Σ, and hence on $\{1, \dots, n\}$. We denote the action of the Frobenius homomorphism $x \mapsto x^p$ by $i \mapsto \sigma(i)$ for $i \in \{1, \dots, n\}$.

Fix an open compact subgroup $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$. Since p is assumed unramified, the lattice $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ yields a reductive $\mathbb{Z}_p$-model $\mathcal{G}$ of $\mathbf{G}_{\mathbb{Q}_p}$. Set $K_p := \mathcal{G}(\mathbb{Z}_p)$. By works of Deligne–Pappas, the Shimura variety at level $K = K^p K_p$ admits a smooth integral model over $\mathbb{Z}_p$. We are interested in its special fiber over $\mathbb{F}_p$, that we simply denote by X. It is a smooth $\mathbb{F}_p$-scheme of dimension n. For each subset $S \subset \{1, \dots, n\}$, there is a corresponding Ekedahl–Oort stratum $X_S \subset X$, which is a locally closed subscheme of X of dimension $|S|$. For example, the maximal Ekedahl–Oort stratum corresponds to $S = \{1, \dots, n\}$ and coincides with the locus where the underlying abelian variety is ordinary. A stratum X_S is contained in the Zariski closure of X_T if and only if $S \subset T$.

The scheme X comes equipped with line bundles $\omega_1, \dots, \omega_n$ whose tensor product $\omega := \bigotimes_{i=1}^n \omega_i$ is the usual Hodge line bundle. For each $\lambda = (k_1 \dots, k_n) \in \mathbb{Z}^n$, define a line bundle

$$\omega(\lambda) := \bigotimes_{i=1}^{n} \omega_i^{k_i}. \tag{1.0.1}$$

For each subset $S \subset \{1, \dots, n\}$, there exists a system of generalized partial Hasse invariants on the Zariski closure $\overline{X}_S$. By this we mean that there exist sections $\mathsf{Ha}_{S,i}$ (for each $i \in S$) such that $\mathsf{Ha}_{S,i}$ vanishes exactly on the set $\overline{X}_{S\setminus\{i\}}$. Furthermore, for $i \notin S$, there are natural sections $\mathsf{Ha}_{S,i}$ that are entirely non-vanishing on $\overline{X}_S$ (see section 4.1 for details). Specifically, the section $\mathsf{Ha}_{S,i}$ lies in $H^0(\overline{X}_S, \omega(\mathsf{ha}_{S,i}))$ where

$$\mathsf{ha}_{S,i} := \begin{cases} \mathbf{e}_i - q\mathbf{e}_{\sigma(i)} & \text{if } i \in S, \\ \mathbf{e}_i + q\mathbf{e}_{\sigma(i)} & \text{if } i \notin S \end{cases}$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the standard basis of $\mathbb{Z}^n$. When $S = \{1, \dots, n\}$, the sections $\mathsf{Ha}_{S,i}$ are the usual partial Hasse invariants constructed by Andreatta–Goren in [AG05]. We denote them simply by Ha_i for $i = 1, \dots, n$ and we write $\mathsf{ha}_i := \mathbf{e}_i - q\mathbf{e}_{\sigma(i)}$ for their weights.

Define a cone $C_{\mathsf{pHa},S} \subset \mathbb{Z}^n$ as the cone spanned over $\mathbb{Z}_{\geq 0}$ by the weights $\mathsf{ha}_{S,i}$ for $i \in \{1, \dots, n\}$. On the other hand, we define a second cone $C_{X,S} \subset \mathbb{Z}^n$ as the set of $\lambda \in \mathbb{Z}^n$ such that the space $H^0(\overline{X}_S, \omega(\lambda))$ is nonzero. Obviously, one has an inclusion $C_{\mathsf{pHa},S} \subset C_{X,S}$. For a cone $C \subset \mathbb{Z}^n$, define the saturation $\mathcal{C}$ of C as the set of $\lambda \in \mathbb{Z}^n$ such that some positive multiple of λ lies in C. We always denote the saturation with the calligraphic letter $\mathcal{C}$. For example, $\mathcal{C}_{\mathsf{pHa},S}$ and $\mathcal{C}_{X,S}$ denote the saturations of $C_{\mathsf{pHa},S}$ and $C_{X,S}$ respectively. For each subset S, we have an inclusion $\mathcal{C}_{\mathsf{pHa},S} \subset \mathcal{C}_{X,S}$.

In Definition 4.3.1, we define the notion of admissible subsets of $\{1, \dots, n\}$ by an explicit condition on S. When p splits in F, any subset is admissible. The whole set $S = \{1, \dots, n\}$ is always admissible. If p is inert and S is a set of the form $S = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$ and $i_{j+1} - i_j$ odd (for $j = 1, \dots, s-1$) and $i_1 + n - i_s$ odd, then S is admissible. The main result of [GK18] is

Theorem 1. *If S is an admissible subset of $\{1, \dots, n\}$, then $\mathcal{C}_{\mathsf{pHa},S} = \mathcal{C}_{X,S}$.*

This result gives a vanishing result for the space of Hilbert modular forms defined on the stratum $\overline{X}_S$. Namely, for admissible S one has $H^0(\overline{X}_S, \omega(\lambda)) = 0$ whenever λ lies in the complement of $\mathcal{C}_{\mathsf{pHa},S}$. In particular, we deduce for $S = \{1, \dots, n\}$ the following corollary:

Corollary 2. *Let $f \in H^0(X, \omega(\lambda))$ be a nonzero Hilbert modular form of weight λ. Then λ is spanned over $\mathbb{Q}_{\geq 0}$ by the weights of partial Hasse invariants $\mathsf{ha}_1, \dots, \mathsf{ha}_n$.*

In our original paper [GK18], we proved this result by an extremely tedious computation. In this short paper, we propose to use the notion of intersection-sum cones introduced in [GK22] to give a more elementary and systematic proof. The advantage of this method is that it can adapt to any reductive group G and any Shimura variety of Hodge-type, as demonstrated in *loc. cit.*. Moreover, we believe that the proof presented in this paper is more enlightening than the original proof.

Finally, we briefly discuss the results of Diamond–Kassaei in [DK23] and the differences with our paper. The results of *loc. cit.* show Corollary 2 as a consequence of a result on divisibility of Hilbert modular forms by partial Hasse invariants ([DK23, Theorem 7.1]). They only consider global sections over X and do not treat the case of a general stratum X_S for $S \subset \{1, \dots, n\}$, as opposed to our Theorem 1. However, the results of *loc. cit.* also cover the case when p is ramified in F, whereas we assume p to be unramified. On the other hand, we work in the general setting of an arbitrary scheme endowed with a smooth, surjective morphism $X \to G\text{-}\mathtt{Zip}^{\mu}$, where $G\text{-}\mathtt{Zip}^{\mu}$ is the stack of G-zips of Pink–Wedhorn–Ziegler ([PWZ15]). Therefore our results reach beyond the case of Hilbert–Blumenthal Shimura varieties.

We now give an overview of each section. In section 2, we review the basic definitions pertaining to the stack of G-zips in the general setting. In section 3, we recall the Cone Conjecture for schemes endowed with a smooth, surjective map $X \to G\text{-}\mathtt{Zip}^{\mu}$ (which include Shimura varieties). We also explain the strategy to prove it using the notion of intersection-sum cones of strata. Finally, section 4 is dedicated to the proof of Theorem 1 using the approach of intersection-sum cones in the case when G is the reductive group defined above in this introduction.

2 Review of G-zips

2.1 Stack of G-zips

Fix an algebraic closure k of $\mathbb{F}_q$ (in applications to Shimura varieties, we always take $q = p$). Let G be a connected reductive $\mathbb{F}_q$-group. Denote by $\varphi\colon G \to G$ the Frobenius homomorphism. Fix a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$. We call (G, μ) a cocharacter datum. From μ, we obtain a zip datum $\mathcal{Z}_\mu$ as explained in [IK21a, §2.2.2]. We recall the construction. First, μ defines a pair of opposite parabolics $P_{\pm}(\mu)$, where $P_{+}(\mu)(k)$ (resp. $P_{-}(\mu)(k)$) consists of the elements $g \in G(k)$ such that the map

$$\mathbb{G}_{\mathrm{m},k} \to G_k;\ t \mapsto \mu(t)g\mu(t)^{-1} \quad (\text{resp. } t \mapsto \mu(t)^{-1}g\mu(t))$$

extends to a regular map $\mathbb{A}^1_k \to G_k$. The centralizer of μ is a Levi subgroup $L(\mu) = P_{+}(\mu) \cap P_{-}(\mu)$. Then, define $P := P_{-}(\mu)$, $Q := (P_{+}(\mu))^{(q)}$, $L := L(\mu)$ and $M := L^{(q)}$. Let $\varphi\colon L \to M$ be the Frobenius homomorphism. The tuple $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ is called the zip datum attached to (G, μ). Then, define the zip group E by:

$$E := \{(x, y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}. \tag{2.1.1}$$

Here, $\theta^P_L\colon P \to L$ denotes the map that sends $x \in P$ to its Levi component $\overline{x} \in L$ (and similarly for θ^Q_M). Pink–Wedhorn–Ziegler define the stack of G-zips of type μ, denoted by $G\text{-}\mathtt{Zip}^{\mu}$ in [PWZ15, Definition 1.4]. It can be defined as the quotient stack

$$G\text{-}\mathtt{Zip}^{\mu} = [E \backslash G_k]\,.$$

where E acts on G by $(x, y) \cdot g := xgy^{-1}$ for all $(x, y) \in E$ and all $g \in G$. It also has an interpretation as a moduli stack of certain torsors.

2.2 The flag space

For convenience, we assume that there exists a Borel pair (B, T) defined over $\mathbb{F}_q$ such that $B \subset P$ and such that μ factors through T. Let $\Phi^+ \subset X^*(T)$ (resp. Φ^+_L) be the set of positive T-roots in G (resp. L), where positivity is defined with respect to the opposite Borel of B. Write Δ (resp. $I := \Delta_L$) for the subsets of simple roots of G (resp. L), and let W (resp. W_I) be the Weyl group of Φ (resp. Φ_L). For $\alpha \in \Phi$, let s_α be the corresponding root reflection. Then $(W, \{s_\alpha | \alpha \in \Delta\})$ is a Coxeter system; denote by $\ell\colon W \to \mathbb{N}$ its length function and by $\leq$ the Bruhat-Chevalley order. Write w_0 (resp. $w_{0,I}$) for the longest element of W (resp. W_I). Let ${}^IW \subset W$ be the subset of elements $w \in W$ which are of minimal length in their right coset W_Iw. We write $X^*_+(T)$ for the set of dominant characters. Similarly, the I-dominant characters of T are denoted by $X^*_{+,I}(T)$. We set

$$z := \sigma(w_{0,I})w_0.$$

The stack of zip flags $G\text{-}\mathtt{ZipFlag}^\mu$ was defined in [GK19a, §2.1]. It can be defined as the quotient stack $[E'\backslash G]$ where $E' = E \cap (B \times G)$ (it also has a modular interpretation in terms of torsors). There is a natural projection map $\pi\colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ whose fibers are isomorphic to P/B. Let X be a k-scheme endowed with a morphism of stacks $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$. Form the fiber product

$$\begin{array}{ccc} \mathrm{Flag}(X) & \xrightarrow{\zeta_{\mathrm{flag}}} & G\text{-}\mathtt{ZipFlag}^\mu \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\zeta]{} & G\text{-}\mathtt{Zip}^\mu \end{array}$$

We call $\mathrm{Flag}(X)$ the flag space of X ([GK19a, §9.1]). By [GK19b, §4.1], there is a natural smooth, surjective morphism of stacks

$$\Psi\colon\ G\text{-}\mathtt{ZipFlag}^\mu \to \mathrm{Sbt} := [B\backslash G/B]. \tag{2.2.1}$$

The stack Sbt is called the Schubert stack, it is finite and its points are parametrized by the Weyl group W. It admits a natural stratification $(\mathrm{Sbt}_w)_{w\in W}$ by locally closed substacks, corresponding to the Bruhat stratification of G, i.e.

$$\mathrm{Sbt}_w := [B\backslash BwB/B]\,. \tag{2.2.2}$$

By pullback, the fibers of Ψ define a stratification of $G\text{-}\mathtt{ZipFlag}^\mu$ by locally closed substacks $(\mathcal{F}_w)_w$, with the same closure relations. Pulling back via ζ_{flag}, we obtain a locally closed stratification $(\mathrm{Flag}(X)_w)_{w\in W}$ on $\mathrm{Flag}(X)$.

2.3 The zip cone

As in [IK21a, §2.4], we can attach to any P-representation (V,ρ) a vector bundle $\mathcal{V}(\rho)$ on $G\text{-}\mathtt{Zip}^\mu$, similarly to the usual associated sheaf construction of [Jan03, §5.8]. For $\lambda \in X^*(T)$, denote by $V_I(\lambda)$ the P-representation $\mathrm{Ind}_B^P(\lambda)$ and by $\rho_{I,\lambda})$ the corresponding map $P \to \mathrm{GL}_k(V_I(\lambda))$. Note that $\rho_{I,\lambda}$ is trivial on the unipotent radical $R_{\mathrm{u}}(P)$, so we may view it as an L-representation (with highest weight λ). Denote by $\mathcal{V}_I(\lambda)$ the vector bundle on $G\text{-}\mathtt{Zip}^\mu$ attached to $V_I(\lambda)$, and call it the automorphic vector bundle attached to λ. Similarly, we can define a line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $G\text{-}\mathtt{ZipFlag}^\mu$ such that $\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$, as in [IK21b, §3.2]. In particular, we can identify the space of global sections $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ with the space $H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathrm{flag}}(\lambda))$. Similarly, if X is a k-scheme endowed with a smooth map $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$, the map $\pi\colon \mathrm{Flag}(X) \to X$ satifies $\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$, hence

$$H^0(X, \mathcal{V}_I(\lambda)) = H^0(\mathrm{Flag}(X), \mathcal{V}_{\mathrm{flag}}(\lambda)). \tag{2.3.1}$$

We define the zip cone of (G,μ) as in [Kos19, §1.2] and [IK22, §3] by

$$C_{\mathsf{zip}} := \{\lambda \in X^*(T) \mid H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0\}.$$

This can be seen as a group-theoretical version of the set of possible weights of nonzero automorphic forms in characteristic p. Since $V_I(\lambda) = 0$ when λ is not I-dominant, we clearly have $C_{\mathsf{zip}} \subset X^*_{+,I}(T)$. One can see that C_{zip} is a cone in $X^*(T)$ (i.e an additive submonoid containing 0 ([Kos19, Lemma 1.4.1]). For a cone $C \subset X^*(T)$, define its saturation as:

$$\mathcal{C} := \{\lambda \in X^*(T) \mid \exists N \geq 1, N\lambda \in C\}.$$

We always use the letter $\mathcal{C}$ to denote the saturation. For example, we write $\mathcal{C}_{\mathsf{zip}}$ for the saturation of C_{zip}. One can define various subcones of C_{zip}, which are more tractable (see [IK22, §3.7]). Here, we only recall the definition of the cone C_{pHa}, called the cone of partial Hasse invariants.

Definition 2.3.1 ([Kos19, Definition 1.7.1])**.** *Define C_{pHa} as the image of $X^*_+(T)$ by the map*

$$h\colon X^*(T) \to X^*(T), \quad \lambda \mapsto \lambda - qw_{0,I}(\sigma^{-1}\lambda).$$

Similarly, write $\mathcal{C}_{\mathsf{pHa}}$ for the saturation of C_{pHa}. The cone C_{pHa} is a subcone of C_{zip}. It can be interpreted as the set of weights of automorphic forms on $G\text{-}\mathtt{Zip}^\mu$ which arise by pullback from the stack Sbt via the map Ψ defined in (2.2.1). The vanishing locus of sections arising in this way is a union of codimension one strata $\overline{\mathcal{F}}_w$. In [IK21b], we termed such sections (flag) partial Hasse invariants.

3 The cone conjecture

3.1 Statement

Let X be a k-scheme endowed with a morphism $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$. We make the following assumption:

Assumption 3.1.1.
(a) *ζ is smooth.*
(b) *The restriction of ζ to every connected component of X is surjective.*
(c) *For all $w \in W$ such that $\ell(w) = 1$, $\overline{\mathrm{Flag}(X)}_w$ is proper.*

We define

$$C_X := \{\lambda \in X^*(T) \mid H^0(X, \mathcal{V}_I(\lambda)) \neq 0\}.$$

Elements of $H^0(X, \mathcal{V}_I(\lambda))$ may be called automorphic forms of weight λ on X, by analogy with the terminology of Shimura varieties. We always have inclusions

$$C_{\mathsf{pHa}} \subset C_{\mathsf{zip}} \subset C_X \subset X^*_{+,I}(T). \tag{3.1.1}$$

Conjecture 3.1.2. *Under Assumption 3.1.1, we have $\mathcal{C}_X = \mathcal{C}_{\mathsf{zip}}$.*

It was determined in [IK22, Theorem 4.3.1] when the equality $\mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$ holds:

Theorem 3.1.3. *The following are equivalent:*
(i) *One has $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$.*
(ii) *L is defined over $\mathbb{F}_q$ and σ acts on Δ_L by $-w_{0,L}$.*

If the equivalent condition of Theorem 3.1.3 are satisfied, we say that (G, μ) is of Hasse-type. In this note, we are interested in the case of Hilbert–Blumenthal Shimura varieties, in which case we have $I = \emptyset$ (also, for Shimura varieties the group G is always defined over $\mathbb{F}_p$, so we take $q = p$). In particular, the condition of Theorem 3.1.3 is obviously satisfied, therefore one has $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$. Combining Theorem 3.1.3 with Conjecture 3.1.2, we obtain the following:

Conjecture 3.1.4. *Suppose that (G,μ) is of Hasse-type and let (X,ζ) be a pair satisfying Assumption 3.1.1. Then we have $\mathcal{C}_X = \mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$.*

In this note, we explain a new proof of the following theorem.

Theorem 3.1.5. *Assume that G is an $\mathbb{F}_q$-form of the group $\mathrm{SL}_{2,k}^n$ and that $\mu\colon \mathbb{G}_{m,k} \to G_k$ is non-trivial on each factor of G_k. Then Conjecture 3.1.2 holds.*

One can also change $\mathrm{SL}_{2,k}$ to a group with the same adjoint groups. In particular, Theorem 3.1.5 applies to Hilbert–Blumenthal Shimura varieties. Actually, we will show a much stronger result, which also gives information regarding sections on other strata. Roughly speaking, we will prove that a version of the result $\mathcal{C}_X = \mathcal{C}_{\mathsf{pHa}}$ also holds for various strata of X, those which are called admissible (see Theorem 4.3.2 below).

3.2 Intersection-sum cones

We first explain the general strategy to prove Conjecture 3.1.2 for a general group G. We implement this strategy in §4 in the specific setting of Theorem 3.1.5. For each $w \in W$, we define the cone of partial Hasse invariants $C_{\mathsf{pHa},w}$ of w as follows. First, define a map $h_w\colon X^*(T) \to X^*(T)$ by

$$h_w\colon \chi \mapsto -w\chi + q w_{0,I} w_0 \sigma^{-1}(\chi).$$

Then, for $w \in W$, define E_w as the set of positive roots α such that $ws_\alpha < w$ (with respect to the Bruhat order) and $\ell(ws_\alpha) = \ell(w) - 1$. Define $X^*_{+,w}(T) \subset X^*(T)$ as the subset of $\chi \in X^*(T)$ such that $\langle \chi, \alpha^\vee \rangle \geq 0$ for all $\alpha \in E_w$. We set

$$C_{\mathsf{pHa},w} := h_w(X^*_{+,w}(T)).$$

This cone is the analogue of C_{pHa} for smaller strata. Its interpretation is the following: $C_{\mathsf{pHa},w}$ is the set of weights $\lambda \in \mathbb{Z}^n$ such that $\mathcal{V}_{\mathrm{flag}}(\lambda)$ admits a nonzero section on $\overline{\mathcal{F}}_w$ arising by pullback from a section over the stratum $\overline{\mathrm{Sbt}}_w$ in the Schubert stack (see (2.2.2)). In the case $w = w_0$, the cone C_{pHa,w_0} coincides with the cone C_{pHa} of Definition 2.3.1. To simplify, we will make the following assumption:

Assumption 3.2.1. *For all $w \in W$ and all $\alpha \in E_w$, there exists $\chi_\alpha \in X^*(T)$ such that*
(a) $\langle \chi_\alpha, \alpha^\vee \rangle > 0$
(b) $\langle \chi_\alpha, \beta^\vee \rangle = 0$ *for all $\beta \in E_w \setminus \{\alpha\}$.*

This assumption is not always satisfied. It is satisfied for the groups considered in Theorem 3.1.5 (for more general groups, one can still carry out the strategy using a subset of strata which satisfy this assumption). Under Assumption 3.2.1, each stratum $\mathcal{F}_w$ carries a section $\mathsf{Ha}_{w,\alpha}$ (for each $\alpha \in E_w$) defined over $\overline{\mathcal{F}}_w$, whose vanishing locus is exactly $\overline{\mathcal{F}}_{ws_\alpha}$. We now define the intersection-sum cones C_w^+ of w as follows.

Definition 3.2.2. *For $\ell(w) = 1$, set $C_w^+ := C_{\mathsf{pHa},w}$. For $\ell(w) \geq 2$, define inductively*

$$C_w^+ := C_{\mathsf{pHa},w} + \bigcap_{\alpha \in E_w} C_{ws_\alpha}^+.$$

Again, we write $\mathcal{C}_w^+$ for the saturation of C_w^+. We now explain the connection with Conjecture 3.1.2. Let (X,ζ) be a pair satisfying Assumption 3.1.1. Define the cone of X at $w \in W$ as follows:

$$C_{X,w} := \{\lambda \in X^*(T) \mid H^0(\overline{\mathrm{Flag}(X)}_w, \mathcal{V}_{\mathrm{flag}}(\lambda)) \neq 0\}. \tag{3.2.1}$$

Here $\overline{\mathrm{Flag}(X)_w}$ denotes the Zariski closure of $\mathrm{Flag}(X)_w$ endowed with the reduced closed subscheme structure. For $w = w_0$, we have $C_{X,w_0} = C_X$, because $H^0(\mathrm{Flag}(X), \mathcal{V}_{\mathsf{flag}}(\lambda))$ identifies with $H^0(X, \mathcal{V}_I(\lambda))$ (see (2.3.1)). Write $\mathcal{C}_{X,w}$ for the saturation of $C_{X,w}$. Under Assumption 3.2.1, we have:

Theorem 3.2.3 ([GK22, Theorem 2.3.8])**.** *For each $w \in W$, we have $\mathcal{C}_{X,w} \subset \mathcal{C}^+_w$.*

We deduce from (3.1.1) the following corollary.

Corollary 3.2.4.
(1) *Assume that $\mathcal{C}^+_{w_0} \subset \mathcal{C}_{\mathsf{zip}}$. Then we have $\mathcal{C}_{\mathsf{zip}} = \mathcal{C}_X = \mathcal{C}^+_{w_0}$.*
(2) *Assume that $\mathcal{C}^+_{w_0} \subset \mathcal{C}_{\mathsf{pHa}}$. Then we have $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}} = \mathcal{C}_X = \mathcal{C}^+_{w_0}$.*
In particular Conjecture 3.1.2 holds in those cases.

This result reduces the proof of Conjecture 3.1.2 to showing the inclusion $\mathcal{C}^+_{w_0} \subset \mathcal{C}_{\mathsf{zip}}$ or $\mathcal{C}^+_{w_0} \subset \mathcal{C}_{\mathsf{pHa}}$, which is a purely group-theoretical statement, and is independent of the scheme X. An other advantage of this method is that it behaves well under $\mathbb{F}_q$-products. Assume that $G = G_1 \times G_2$ where G_1, G_2 are $\mathbb{F}_q$-groups. Let (B_i, T_i) denote a Borel pair in G_i for $i = 1, 2$, and let $\mu_i \colon \mathbb{G}_{\mathrm{m},k} \to T_{i,k}$ be a cocharacter. Then it is clear from the definition that all cones C_{zip}, $C_{\mathsf{pHa},w}$, $C_{X,w}$, C^+_w decompose as $C_1 \times C_2$ where $C_i \subset X^*(T_i)$ is the corresponding cone of (G_i, μ_i). Hence, if the assumption of (1) or (2) is satified for G_1 and G_2, then it is also satisfied for G. In particular if we prove Conjecture 3.1.2 using the strategy of intersection-sum cones, then we automatically obtain the results for products (over $\mathbb{F}_q$) of such groups. However, it is unclear whether Conjecture 3.1.2 itself is stable by $\mathbb{F}_q$-products of groups.

4 Groups of type A_1

In this section, we prove Theorem 3.1.5 by showing that $\mathcal{C}^+_{w_0} \subset \mathcal{C}_{\mathsf{pHa}}$ is satisfied in that case. We let G be an $\mathbb{F}_q$-form of $\mathrm{SL}^n_{2,k}$. Then, G is isomorphic over $\mathbb{F}_q$ to a product of groups of the form $\mathrm{Res}_{\mathbb{F}_q^m/\mathbb{F}_q}(\mathrm{SL}_{2,\mathbb{F}_{q^m}})$. By the previous discussion regarding $\mathbb{F}_q$-products, we are reduced to the case of a Weil restriction. Note also that in this case, we have $P = B$, hence $G\text{-}\mathtt{ZipFlag}^\mu = G\text{-}\mathtt{Zip}^\mu$ and $\mathrm{Flag}(X) = X$. As previously mentioned, the arguments are not sensitive to changing the group to a group with the same adjoint group. Therefore, the following applies also to the group G appearing in the context of Hilbert–Blumenthal Shimura varieties.

4.1 Group theory

Let $n \geq 1$ be an integer and let G be an $\mathbb{F}_q$-form of $\mathrm{SL}^n_{2,k}$. We can write G as a product $G_1 \times \cdots \times G_r$ where

$$G_i := \mathrm{Res}_{\mathbb{F}_{q^{m_i}}/\mathbb{F}_q}(\mathrm{SL}_{2,\mathbb{F}_{q^{m_i}}})$$

for positive integers $m_1, \ldots, m_r$ satisfying $\sum_{i=1}^r m_i = n$. We fix an isomorphism $\iota \colon G_k \to \mathrm{SL}^n_{2,k}$, which yields a partition

$$\{1, \ldots, n\} = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$$

corresponding to the orbits of the Galois group on the factors. We write $i \mapsto \sigma(i)$ for the action of the Frobenius element $\sigma \in \mathrm{Gal}(k/\mathbb{F}_q)$ on $\{1, \ldots, n\}$. For example, in the case $r = 1$, we can choose ι so that the Frobenius element acts as a cycle

$$^{\sigma}(x_1, \ldots, x_n) := (\sigma(x_2), \ldots, \sigma(x_n), \sigma(x_1)). \tag{4.1.1}$$

on $G(k)$, and hence $\sigma(i) = i+1$ (modulo n). Let $T_0 \subset \mathrm{SL}_{2,k}$ be the diagonal torus. We identify $X^*(T_0) = \mathbb{Z}$ by sending $m \in \mathbb{Z}$ to the character $\mathrm{diag}(x, x^{-1}) \mapsto x^m$. Define $T := T_0^n \subset G_k$ and identify similarly $X^*(T) = \mathbb{Z}^n$. Let $B_0 \subset \mathrm{SL}_{2,k}$ be the Borel subgroup of lower-triangular matrices, and define $B := B_0^n \subset G$. Similarly, write B_- for the opposite Borel. The Weyl group $W := W(G,T)$ is $W = \{\pm 1\}^n$. An element $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in W$ acts on $X^*(T) = \mathbb{Z}^n$ by $\epsilon\dot{\lambda} = (\epsilon_1 a_1, \dots, \epsilon_n a_n)$ for all $\lambda = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Identify elements of W with subsets $S \subset \{1, \dots, n\}$ by the map

$$W \to \mathcal{P}(\{1, \dots, n\}), \quad \epsilon = (\epsilon_1, \dots, \epsilon_n) \mapsto \{i \in \{1, \dots, n\} \mid \epsilon_i = -1\}. \tag{4.1.2}$$

Let $\mu_0 \colon \mathbb{G}_{\mathrm{m},\mathbb{F}_q} \to \mathrm{SL}_{2,\mathbb{F}_q}$ be the cocharacter $t \mapsto \mathrm{diag}(t, t^{-1})$, and define $\mu \colon \mathbb{G}_{\mathrm{m},\mathbb{F}_q} \to G$ by $t \mapsto (\mu_0(t), \dots, \mu_0(t))$. Write $G\text{-}\mathtt{Zip}^\mu$ for the corresponding stack of G-zips. Note that since $I = \emptyset$, it is the same as the stack $G\text{-}\mathtt{ZipFlag}^\mu$ defined in §2.2. Recall that $G\text{-}\mathtt{Zip}^\mu = [E \backslash G]$ where $E \subset B \times B_-$ is the zip group defined in (2.1.1).

4.2 Partial Hasse invariant cones

Define a Zariski open subset $U \subset \mathrm{SL}_{2,k}$ as the non-vanishing locus of the function

$$h \colon \mathrm{SL}_{2,k} \to \mathbb{A}^1_k, \quad h \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a.$$

Denote by $Z \subset \mathrm{SL}_{2,k}$ the zero locus of h (note that Z is a reduced subscheme). For a subset $S \subset \{1, \dots, n\}$, define the set G_S by:

$$G_S := \prod_{i=1}^n G_{S,i} \quad \text{where} \quad G_{S,i} := \begin{cases} U & \text{if } i \in S \\ Z & \text{if } i \notin S. \end{cases}$$

For each subset S, the corresponding stratum $\mathcal{F}_S \subset G\text{-}\mathtt{Zip}^\mu$ is the locally closed substack

$$\mathcal{F}_S := [E \backslash G_S].$$

Write $C_{\mathsf{pHa},S}$ instead of $C_{\mathsf{pHa},w}$ where w is the element corresponding to S via the identification (4.1.2). Denote by $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ the natural basis of $\mathbb{Z}^n$. For a subset $S \subset \{1, \dots, n\}$ and $1 \le i \le n$, define:

$$\mathsf{ha}_{S,i} := \begin{cases} \mathbf{e}_i - q\mathbf{e}_{\sigma(i)} & \text{if } i \in S, \\ \mathbf{e}_i + q\mathbf{e}_{\sigma(i)} & \text{if } i \notin S. \end{cases} \tag{4.2.1}$$

Then one shows easily that $C_{\mathsf{pHa},S}$ is given by

$$C_{\mathsf{pHa},S} = \left\{ \sum_{i=1}^n a_i \mathsf{ha}_{S,i} \;\middle|\; a_i \in \mathbb{Z}_{\ge 0} \text{ for } i \in S, \;\; a_i \in \mathbb{Z} \text{ for } i \notin S \right\}.$$

4.3 Results

Let (X, ζ) be a pair satisfying Assumption 3.1.1. Recall that $\mathrm{Flag}(X) = X$ in our case. For a subset $S \subset \{1, \dots, n\}$, denote by $X_S \subset X$ the corresponding locally closed subset, endowed with the reduced structure. Concretely, X_S is the preimage by ζ of the locally closed substack $\mathcal{F}_S \subset G\text{-}\mathtt{Zip}^\mu$. Let $\overline{X}_S$ be the Zariski closure of X_S. Denote also by $C_{X,S}$ the cone of X at the stratum S, as defined in (3.2.1). We have an action of $\mathrm{Gal}(k/\mathbb{F}_q)$ on the set $\{1, \dots, n\}$. Hence, $\mathrm{Gal}(k/\mathbb{F}_q)$ also acts on the set of all subsets of $\{1, \dots, n\}$. It also acts on the set of pairs (S, j) where $S \subset \{1, \dots, n\}$ is a subset and $j \in S$. First, consider the case when $S \subset \Sigma_i$ for some $1 \leq i \leq r$, and $j \in S$ is an element. In this case, we have a bijection

$$\beta_j \colon \{1, \dots, m_i\} \to \Sigma_i, \quad d \mapsto \sigma^d(j).$$

We say that the pair (S, j) is admissible if $\beta_j^{-1}(S) = \{u_1, \dots, u_s\}$ with $u_1 < \cdots < u_s$ and all the integers $u_{i+1} - u_i$ for $i = 1, \dots, s-1$ are odd. For any pair (S, j) such that $j \in S$, we say that (S, j) is admissible if the pair $(S \cap \Sigma_i, j)$ is admissible, where $i \in \{1, \dots, r\}$ is the unique element such that $j \in \Sigma_i$. Finally, we define:

Definition 4.3.1. *Let $S \subset \{1, \dots, n\}$ be a subset. We say that S is admissible if (S, j) is an admissible pair for each $j \in S$.*

For example, $\{1, \dots, n\}$ is always admissible. When $r = 1$, we fix the Galois action given in (4.1.1). In this case, a subset $S = \{u_1, \dots, u_s\}$ (with $u_1 < \cdots < u_s$) is admissible if and only if $i_{j+1} - i_j$ (for $j = 1, \dots, s-1$) and $i_1 + n - i_s$ are all odd numbers. When $r = n$, all subsets are admissible. The bulk of the proof is to show the following result, which is [GK18, Theorem 4.2.3] :

Theorem 4.3.2. *Let $S \subset \{1, \dots, n\}$ be an admissible subset. Then $\mathcal{C}_{X,S} = \mathcal{C}_{\mathsf{pHa},S}$.*

By taking $S = \{1, \dots, n\}$, we obtain Theorem 3.1.5. We also have the following result, which shows that the space of Hilbert modular forms defined on a specifica stratum vanishes in some cases:

Corollary 4.3.3. *Let $S \subset \{1, \dots, n\}$ be an admissible subset. For any $\lambda \in \mathbb{Z}^n$ such that $\lambda \notin \mathcal{C}_{\mathsf{pHa},S}$, we have*

$$H^0(\overline{X}_S, \omega(\lambda)) = 0.$$

Here, $\omega(\lambda)$ is the line bundle defined in (1.0.1) in the introduction, which was denoted by $\mathcal{V}_{\mathrm{flag}}(\lambda)$ for a general group in section 2.2. In the remainder of this paper, we illustratate how the techniques of intersection-sum cones (Theorem 3.2.3 and Corollary 3.2.4) give a straightforward proof of Theorem 4.3.2.

4.4 Inclusions between cones

As we previously explained, we can reduce to the case of a Weil restriction (i.e. to the case when $r = 1$). Therefore, for the remainder of the paper, we will assume that $r = 1$ and we consider the Galois action given by (4.1.1), which corresponds to $\sigma(i) = i + 1$ (taken modulo n). Hence, the weights $\mathsf{ha}_{S,i}$ are given by

$$\mathsf{ha}_{S,i} := \begin{cases} \mathbf{e}_i - q\mathbf{e}_{i+1} & \text{if } i \in S, \\ \mathbf{e}_i + q\mathbf{e}_{i+1} & \text{if } i \notin S. \end{cases}$$

The tuple $\mathcal{B}_S = (\mathsf{ha}_{S,1}, \dots, \mathsf{ha}_{S,n})$ is a $\mathbb{Q}$-basis of $\mathbb{Q}^n$. As in (4.2.1), the cone $C_{\mathsf{pHa},S}$ is the set of integral linear combinations of the $\mathsf{ha}_{S,i}$ with nonnegative coefficents for $i \in S$. For a subset $S \subset \{1, \dots, n\}$ and $j \in S$, define the partial Hasse invariant cone of S with respect to $j \in S$ as follows:

$$C^{(j)}_{\mathsf{pHa},S} := \left\{ \sum_{i=1}^{n} a_i \mathsf{ha}_{S,i} \;\middle|\; a_i \in \mathbb{Z} \text{ for } i \neq j, \ \ a_j \in \mathbb{Z}_{\geq 0} \right\}.$$

Then, for each $j \in S$, the cone $C^{(j)}_{\mathsf{pHa},S}$ is a half-space of $\mathbb{Z}^n$ and $C_{\mathsf{pHa},S}$ can be written as $C_{\mathsf{pHa},S} = \bigcap_{j\in S} C^{(j)}_{\mathsf{pHa},S}$. For convenience, we will consider rational coefficients and study the $\mathbb{Q}_{\geq 0}$-subcones of $\mathbb{Q}^n$ generated by $C_{\mathsf{pHa},S}$ and $C^{(j)}_{\mathsf{pHa},S}$ respectively. We write $\mathbb{C}_{\mathsf{pHa},S}$ and $\mathbb{C}^{(j)}_{\mathsf{pHa},S}$ respectively for these subsets of $\mathbb{Q}^n$. We give an explicit equation for the cone $\mathbb{C}^{(j)}_{\mathsf{pHa},S}$ in the canonical basis $(\mathbf{e}_i)_{1\leq i\leq n}$ of $\mathbb{Q}^n$. For $1 \leq a, b \leq n$, define $S(a,b)$ as the set $\{a \leq i \leq b \mid i \in S\}$, and put

$$\gamma(S,a,b) := (-1)^{|S(a,b)|}.$$

In particular, for $a > b$ we have $S(a,b) = \emptyset$ and $\gamma(S,a,b) = 1$. We put $s := |S|$. For $j \in S$ and each $1 \leq i \leq n$, define

$$C_i(S,j) := \begin{cases} (-1)^{n+1+s+i+j} \cdot \gamma(S,i,j) \cdot q^{1-i} & \text{for } 1 \leq i \leq j \\ (-1)^{1+i+j} \cdot \gamma(S,j+1,i-1) \cdot q^{n+1-i} & \text{for } j+1 \leq i \leq n. \end{cases}$$

Furthermore, for $\lambda = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, define $E^{(j)}_S(\lambda) = \sum_{i=1}^n C_i(S,j)x_i$.

Lemma 4.4.1. *We have* $\mathbb{C}^{(j)}_{\mathsf{pHa},S} = \left\{\lambda \in \mathbb{Q}^n \;\middle|\; E^{(j)}_S(\lambda) \leq 0\right\}$.

Proof. It is easy to check that $E^{(j)}_S(\mathsf{ha}_{S,i}) = 0$ for all $i \neq j$. Finally, $E^{(j)}_S(\mathsf{ha}_{S,j})$ has the same sign as $-C_{j+1}(S,j) = -q^{n-j} < 0$. The result follows. □

For each $1 \leq r \leq n$ and $1 \leq i \leq r$, define $T_r := \{1, \ldots, r\}$ and $T^{(i)}_r := \{1, \ldots, i-1\} \cup \{i+1, \ldots, r\}$. In other words, $T^{(i)}_r$ is the set obtained by removing i from T_r. Note that $T^{(r)}_r = T_{r-1}$.

Lemma 4.4.2. *Let $1 \leq r \leq n$. We have* $\mathbb{C}^{(1)}_{\mathsf{pHa},T^{(r)}_r} \cap \mathbb{C}^{(r)}_{\mathsf{pHa},T^{(1)}_r} \subset \mathbb{C}^{(1)}_{\mathsf{pHa},T_r}$.

Proof. By Lemma 4.4.1, the cone $\mathbb{C}^{(1)}_{\mathsf{pHa},T_r}$ is the set of $\lambda = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $E_{T_r,1}(\lambda) \leq 0$, where

$$E^{(1)}_{T_r}(\lambda) := (-1)^{n+r}x_1 + \sum_{i=2}^{r+1} q^{n+1-i}x_i + \sum_{i=r+2}^{n} (-1)^{i+r+1}q^{n+1-i}x_i.$$

Similarly, the cones $\mathbb{C}^{(1)}_{\mathsf{pHa},T^{(r)}_r}$ and $\mathbb{C}^{(r)}_{\mathsf{pHa},T^{(1)}_r}$ are respectively given by the inequalities

$$(-1)^{n+r+1}x_1 + \sum_{i=2}^{r} q^{n+1-i}x_i + \sum_{i=r+1}^{n} (-1)^{i+r}q^{n+1-i}x_i \leq 0, \tag{4.4.1}$$

$$(-1)^{n+r}x_1 + \sum_{i=2}^{r}(-1)^{n+r-1}q^{1-i}x_i + \sum_{i=r+1}^{n} (-1)^{i+r+1}q^{n+1-i}x_i \leq 0. \tag{4.4.2}$$

Define $a, b \in \mathbb{Q}$ by $a := \frac{1+(-1)^{n+r}q^{-n}}{1+(-1)^{n+r+1}q^{-n}}$and $b := \frac{2}{1+(-1)^{n+r+1}q^{-n}}$. It is clear that a, b are positive numbers. If we multiply (4.4.1) by a, (4.4.2) by b, and add up these inequalities, we find $E^{(1)}_{T_r}(\lambda) \leq 0$. The result follows. □

Lemma 4.4.2 can be generalized a follows. For a subset $S \subset \{1, \ldots, n\}$ and $u \in S$, define $S^{(u)} := S \setminus \{u\}$. Suppose $S = \{u_1, \ldots, u_r\}$ with $u_1 < \cdots < u_r$ and $r \geq 2$. The proof of the following Lemma is completely similar to Lemma 4.4.2, modulo a change of variable.

Lemma 4.4.3. *We have an inclusion* $\mathbb{C}^{(u_1)}_{\mathsf{pHa},S^{(u_r)}} \cap \mathbb{C}^{(u_r)}_{\mathsf{pHa},S^{(u_1)}} \subset \mathbb{C}^{(u_1)}_{\mathsf{pHa},S}$.

We now explain a similar result regarding the various cones of partial Hasse invariants.

Lemma 4.4.4. *Let* $2 \le r \le n$. *Assume that* r *and* n *are of same parity. Then*

$$\left(\bigcap_{i=1}^{r-1} \mathbb{C}^{(i+1)}_{\mathsf{pHa},T_r^{(i)}}\right) \cap \mathbb{C}^{(1)}_{\mathsf{pHa},T_r^{(r)}} \subset \mathbb{C}^{(1)}_{\mathsf{pHa},T_r}.$$

Proof. For $1 \le i \le r-1$, the cone $\mathbb{C}^{(i+1)}_{\mathsf{pHa},T_r^{(i)}}$ is given by the set of $\lambda = (x_1, \dots, x_n) \in \mathbb{Z}^n$ satisfying the inequality $E^{(i+1)}_{T_r^{(i)}}(\lambda) \le 0$, where

$$E^{(i+1)}_{T_r^{(i)}}(\lambda) := \sum_{d=1}^{i} q^{1-d} x_d - q^{-1} x_{i+1} + \sum_{d=i+2}^{r} q^{n+1-d} x_d + \sum_{d=r+1}^{n} (-1)^{d+r+1} q^{n+1-d} x_d.$$

In the above equation, we used that r and n have the same parity, hence $(-1)^{r+n} = 1$. Similarly, $\mathbb{C}^{(1)}_{\mathsf{pHa},T_r^{(r)}}$ is defined by $E^{(1)}_{T_r^{(r)}}(\lambda) \le 0$, where

$$E^{(1)}_{T_r^{(r)}}(\lambda) = -x_1 + \sum_{d=2}^{r} q^{n+1-d} x_d + \sum_{d=r+1}^{n} (-1)^{d+r} q^{n+1-d} x_d.$$

Finally, $\mathbb{C}^{(1)}_{\mathsf{pHa},T_r}$ is defined by $E^{(1)}_{T_r}(\lambda) \le 0$, where

$$E^{(1)}_{T_r}(\lambda) = -x_1 + \sum_{d=2}^{r+1} q^{n+1-d} x_d + \sum_{d=r+2}^{n} (-1)^{d+r+1} q^{n+1-d} x_d.$$

One can check easily that

$$\sum_{i=1}^{r-1} 2^{r-i} q^n (q^n+1)^{i-1} E^{(i+1)}_{T_r^{(i)}}(\lambda) + (q^n+1)^{r-1} E^{(1)}_{T_r^{(r)}}(\lambda) = P_r(q^n) E^{(1)}_{T_r}(\lambda) \tag{4.4.3}$$

where $P_r(x)$ is the polynomial $P_r(x) = x \sum_{i=1}^{r-1} 2^{r-i}(x+1)^{i-1} - (x+1)^{r-1}$. The coefficient of x^d in $P_r(x)$ is positive for all $d > 0$, and it is -1 for $d = 0$. In particular, we have $P_r(q^n) > 0$. Hence (4.4.3) shows that if $E^{(i+1)}_{T_r^{(i)}}(\lambda) \le 0$ for each $1 \le i \le r-1$ and $E^{(1)}_{T_r^{(r)}}(\lambda) \le 0$, then also $E^{(1)}_{T_r}(\lambda) \le 0$. This proves the result. □

4.5 Generalization to admissible subsets

We can generalize Lemma 4.4.4 as follows. For any subset $S \subset \{1, \dots, n\}$ and any $i \in S$, define i^+ as the "next element" in S. More precisely, if $S = \{u_1, \dots, u_r\}$ with $u_1 < \cdots < u_r$, define $u_j^+ := u_{j+1}$ for $1 \le j < r$ and $u_r^+ := u_1$. For any subset S such that $|S| > 1$, we define

$$\widehat{C}_{\mathsf{pHa},S} := \bigcap_{u \in S} C^{(u^+)}_{\mathsf{pHa},S^{(u)}}$$

Similarly, we define $\widehat{\mathbb{C}}_{\mathsf{pHa},S}$ as the $\mathbb{Q}_{\ge 0}$-subcone generated in $\mathbb{Q}^n$ by $\widehat{C}_{\mathsf{pHa},S}$.

Lemma 4.5.1. *Suppose that* $r := |S| > 1$ *and that* (S, j) *is an admissible pair. Assume furthermore that* r *and* n *are of same parity. Then*

$$\widehat{\mathbb{C}}_{\mathsf{pHa},S} \subset \mathbb{C}^{(j)}_{\mathsf{pHa},S}$$

Proof. The proof is completely similar to Lemma 4.4.4. Write $S = \{u_1, \dots, u_r\}$ with $u_1 < \cdots < u_r$. Using the Galois action, we may assume that $j = u_1$. Then we find again the relation

$$\sum_{i=1}^{r-1} 2^{r-i} q^n (q^n+1)^{i-1} E^{(u_{i+1})}_{S^{(u_i)}}(\lambda) + (q^n+1)^{r-1} E^{(u_1)}_{S^{(u_r)}}(\lambda) = P_r(q^n) E^{(u_1)}_S(\lambda)$$

where $P_r(x)$ is the polynomial $P_r(x) = x \sum_{i=1}^{r-1} 2^{r-i}(x+1)^{i-1} - (x+1)^{r-1}$. The result follows as in Lemma 4.4.4. □

Finally, we state a useful lemma used in the next section. Assume that $S = \{u_1, \dots, u_r\}$ with $u_1 < \cdots < u_r$ and $r \geq 2$. Furthermore, assume that (S, u_1) is an admissible pair. Then we have

Lemma 4.5.2.
(1) *The pair* $(S^{(u_r)}, u_1)$ *is admissible.*
(2) *Assume that the parity of* r *and* n *are different. Then* $(S^{(u_1)}, u_r)$ *is admissible.*

Proof. The first assertion is immediate. To show (2), it suffices to show that $n - u_r + u_2$ is odd. Since (S, u_1) is admissible, $u_r - u_2$ has the same parity as r. The result follows. □

4.6 Intersection-sum cones

Denote by C^+_S the intersection-sum cone corresponding to the stratum S. We recall its definition. For S such that $|S| = 1$, we set $C^+_S := C_{\mathsf{pHa},S}$. For $|S| > 1$, we define inductively

$$C^+_S := C_{\mathsf{pHa},S} + \bigcap_{u \in S} C^+_{S^{(u)}}.$$

Proposition 4.6.1. *If* (S, j) *is an admissible pair, then* $\mathcal{C}^+_S \subset \mathcal{C}^{(j)}_{\mathsf{pHa},S}$.

Proof. It suffices to show $C^+_S \subset \mathbb{C}^{(j)}_{\mathsf{pHa},S}$. Since $C_{\mathsf{pHa},S}$ is obviously contained in $\mathbb{C}^{(j)}_{\mathsf{pHa},S}$, it is enough to show $\bigcap_{u\in S} C^+_{S^{(u)}} \subset \mathbb{C}^{(j)}_{\mathsf{pHa},S}$. We prove the result by induction on $|S|$. Write $r := |S|$. We may assume that $S = \{u_1, \dots, u_r\}$ with $u_1 < \cdots < u_r$ and $j = u_1$. First, assume that r and n have different parity. In this case, the pairs $(S^{(u_r)}, u_1)$ and $(S^{(u_1)}, u_r)$ are admissible by Lemma 4.5.2. By induction, we have $C^+_{S^{(u_r)}} \subset \mathbb{C}^{(u_1)}_{\mathsf{pHa},S^{(u_r)}}$ and $C^+_{S^{(u_1)}} \subset \mathbb{C}^{(u_r)}_{\mathsf{pHa},S^{(u_1)}}$. We deduce by Lemma 4.4.3 that

$$\bigcap_{u\in S} C^+_{S^{(u)}} \subset C^+_{S^{(u_1)}} \cap C^+_{S^{(u_r)}} \subset \mathbb{C}_{\mathsf{pHa},u_1}.$$

We now assume that r and n have the same parity. In this case, we consider the pairs $(S^{(u)}, u^+)$ for each $u \in S$. It is clear that they are all admissible. Hence, we have by induction $C^+_{S^{(u)}} \subset \mathbb{C}^{(u^+)}_{\mathsf{pHa},S^{(u)}}$. By Lemma 4.5.1, we deduce

$$\bigcap_{u\in S} C^+_{S^{(u)}} \subset \bigcap_{u\in S} \mathbb{C}^{(u^+)}_{\mathsf{pHa},S^{(u)}} = \widehat{\mathbb{C}}_{\mathsf{pHa},S} \subset \mathbb{C}^{(u_1)}_{\mathsf{pHa},S}.$$

This terminates the proof. □

Corollary 4.6.2. *If* S *is an admissible subset, then* $\mathcal{C}^+_S \subset \mathcal{C}_{\mathsf{pHa},S}$.

Proof. By definition, (S, j) is admissible for all $j \in S$. We deduce from Proposition 4.6.1 that $C^+_S \subset \bigcap_{j\in S} \mathcal{C}^{(j)}_{\mathsf{pHa},S} = \mathcal{C}_{\mathsf{pHa},S}$. The result follows. □

Finally, we complete the proof of Theorem 4.3.2. Since we always have $\mathcal{C}_{\mathsf{pHa},S} \subset \mathcal{C}_{X,S} \subset \mathcal{C}^+_S$ by Theorem 3.2.3, we deduce from Corollary 4.6.2 that these three cones must coincide.

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