Linear Algebra I & II
Test

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Exercise 1
Which of the following tuples are:

a) linearly independent ?
b) a generating system of $\mathbb{R}^3$?
c) a basis of $\mathbb{R}^3$?

$$\mathcal{B}_1 := \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right) \quad \mathcal{B}_4 := \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right)$$

$$\mathcal{B}_2 := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad \mathcal{B}_5 := \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right)$$

$$\mathcal{B}_3 := \left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) \quad \mathcal{B}_6 := \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right)$$

Remarks :

- You do not need to justify your answers.
- Points are deducted for incorrect answers.
- Mark the properties satisfied by each system with an X in the table below.

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$
a)						
b)						
c)						

Exercise 2

Let K be a field and V, W two finite-dimensional K-vector spaces. Let $f : V \to W$ be a linear map and set $n := \dim(V)$ and $m := \dim(W)$. Mark each assertion true or false in the table below.

(A1) If $\mathrm{rk}(f) = 0$ then $\mathrm{Ker}(f) = V$.

(A2) If f is surjective, then $\mathrm{Ker}(f) = 0$.

(A3) If $\mathrm{rk}(f) = n$, then f is injective.

(A4) If $n = m$, then f is an isomorphism.

(A5) f is an isomorphism if and only if $n = m$ and f is injective.

Assertion	(A1)	(A2)	(A3)	(A4)	(A5)	(A6)
True						
False						

Remarks :

- You do not need to justify your answers.
- Points are deducted for incorrect answers.

Exercise 3 (8 points)

Let $a, b \in \mathbb{C}$. Determine the solutions $(x, y, z) \in \mathbb{C}^3$ of the following linear system:

$$\begin{cases} x + ay + z &= 1 \\ x - 2y + az &= -1 \\ -x + y - z &= b \end{cases}$$

Exercise 4

Let K be a field, $n \in \mathbb{N}$, $n \geq 2$, and let V be a K-vector space. Let $v_1, \ldots, v_n$ be linearly dependent vectors in V such that any $n-1$ of them are linearly independant.

(1) Show that there exists $\alpha_1, \ldots, \alpha_n \in K^\times$ such that $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$.

(2) Let $\alpha_1, \ldots, ,\alpha_n \in K^\times$ as in (1). Show that for all $\beta_1, \ldots, \beta_n \in K$ such that $\beta_1 v_1 + \ldots + \beta_n v_n = 0$, there exists $\lambda \in K$ such that $\beta_i = \lambda \alpha_i$ for all $i = 1, \ldots, n$.

Exercise 5

Let K be a field, $n \in \mathbb{N}$, and let V be a K-vector space of dimension n. Let $U, W \subseteq V$ be subspaces of dimension $n-1$. Show that $U + W = V$ if and only if $U \neq W$.

Exercise 6

Let K be a field and V a K-vector space of dimension 3. Let $\mathcal{B} = (v_1, v_2, v_3)$ be a basis of V, and let $\mathcal{B}^\vee := (\varphi_1, \varphi_2, \varphi_3)$ denote the dual basis of $\mathcal{B}$. Show that for all $v \in V$, one has

$$v \in \mathrm{Span}(v_1, v_2) \iff \varphi_3(v) = 0.$$

Exercise 7
Consider the matrix:

$$A := \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}.$$

(1) Show that $\mathrm{rk}(A) = 2$, and that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

(2) Determine $\det(A)$, $\mathrm{Tr}(A)$, χ_A, μ_A, the Jordan normal form of A, as well as the rational normal form of A.

Exercise 8
Determine the signature of the following symmetric matrices:

$$A := \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 9
Let K be a field of characteristic $\neq 2$. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are congruent. Are they similar?

Exercise 10
Let $A \in M_n(\mathbb{C})$ be a matrix. If A is similar to $2A$, show that A is nilpotent. Is the converse true?