Linear Algebra II
1. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
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Summer term 2016
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Groupwork

Exercise G1 (Three equivalent conditions for isometries)
Let $(V, \langle , \rangle)$ be an inner product space. Show that the following statements are equivalent:

(i) f is an isometry.

(ii) $\|f(x)\| = \|x\|$ for all $x \in V$.

(iii) If $(v_1, ..., v_n)$ is an orthonormal basis of V, then so is $(f(v_1), ..., f(v_n))$.

Exercise G2 (Orthogonal group is a subgroup)
Show that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

Exercise G3 (A counter-example to surjectivity of isometries)
Let H be the real vector space of sequences $(x_n)_{n\in\mathbb{N}_0}$ with $\sum_{n=0}^{\infty} x_n^2 < +\infty$.

(a) Prove that $\langle x, y\rangle := \sum_{n=0}^{\infty} x_n y_n$ defines an inner product on H.

(b) Let $T : H \to H$ denote the endomorphism mapping $(x_n)_{n\in\mathbb{N}_0}$ to $(0, x_0, x_1, x_2, ...)$. Show that T is an isometry of H, that is not surjective.

Exercise G4 (Matrix of an endomorphism with respect to a base)
We endow $\mathbb{R}$ with respect to the usual inner product. Let $\mathcal{B} = (e_1, e_2)$ be the basis of $\mathbb{R}^2$ given by $e_1 = (1, 1)$ and $e_2 = (2, 1)$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, x - 2y)$.

(a) Compute the matrix of f with respect to $\mathcal{B}$.

(b) Compute the matrix of f^* with respect to $\mathcal{B}$.

Homework

Exercise H1 (Normal endomorphisms) (4 points)
Let $(V, \langle , \rangle)$ be an inner product space.

(a) Let $f : V \to V$ be an endomorphism. Show that $\mathrm{Ker}(f) = \mathrm{Im}(f^*)^{\perp}$

(b) If f is normal, then $\mathrm{Im}(f) = \mathrm{Im}(f^*)$.

(c) Let $f, g : V \to V$ be normal endomorphisms. Show that $f \circ g = 0$ if and only if $g \circ f = 0$.

Exercise H2 (Orthogonal projections)
Let $(V, \langle,\rangle)$ be an inner product space, and let $p : V \to V$ be an endomorphism. Recall that p is called a projection if there exists subspaces V_1, V_2 such that $V = V_1 \oplus V_2$, and $p(x_1 + x_2) = x_1$ for all $x_i \in V_i$ and $i = 1, 2$. Furthermore, p is an orthogonal projection if there exists such V_1, V_2 with $V = V_1 \perp V_2$.

(a) Show that p is a projection if and only if $p^2 = p$.

(b) Show that the following statements are equivalent:
   (i) p is an orthogonal projection.
   (ii) $p^* = p$ and $p^2 = p$.

Exercise H3 (Eigenvalues)
Let f be a self-adjoint endomorphism of a unitary space V. Show that the following statements are equivalent:

(i) f has real positive eigenvalues.

(ii) For all $x \in V \setminus \{0\}$, $\langle f(x), x\rangle > 0$.

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2. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
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Summer term 2016
21. April 2016

Groupwork

Exercise G1 (Determinant of an isometry)
Let f be an isometry of a $\mathbb{K}$-inner product space. Show that $|\det(f)| = 1$.

Exercise G2 (Normal endomorphisms and stable subspaces)
Let $(V, \langle\cdot,\cdot\rangle)$ be a $\mathbb{K}$-inner product space and let $f : V \to V$ be a normal endomorphism. If $P = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{K}[X]$ is a polynomial, set

$$P(f) := a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 \operatorname{Id}_V \in \operatorname{End}(V).$$

Furthermore, we denote by $\overline{P}$ the polynomial $\overline{a}_nX^n + \overline{a}_{n-1}X^{n-1} + \cdots + \overline{a}_1X + \overline{a}_0$.

(a) Show that there exists $P \in \mathbb{K}[X]$ such that $f^* = P(f)$.

Hint : Do first the unitary case. For this, show that there exists a polynomial P such that $P(\lambda_i) = \overline{\lambda_i}$ for all eigenvalues λ_i of f. For the euclidean case, use the unitary case and note that $\frac{P+\overline{P}}{2}$ has real coefficients for all $P \in \mathbb{C}[X]$.

(b) Let $U \subseteq V$ be a subspace such that $f(U) \subseteq U$. Show that $f^*(U) \subseteq U$. Furthermore, show that the restriction $f_U : U \to U$ of f to U is normal.

(c) Let $U \subseteq V$ be a subspace such that $f(U) \subseteq U$. Show that $f(U^\perp) \subseteq U^\perp$.

Exercise G3 (Set of normal endomorphisms)
Let V be a $\mathbb{K}$-inner-product space. Is the set of normal endomorphisms of V a subspace of $\operatorname{End}_{\mathbb{K}}(V)$?

Exercise G4 (G-invariant inner-product)
Let V be a $\mathbb{K}$-vector space and let G be a finite subgroup of $GL_{\mathbb{K}}(V)$. Show that there exists a $\mathbb{K}$-inner product $\langle\cdot,\cdot\rangle$ on V such that $\langle g(x), g(y)\rangle = \langle x, y\rangle$ for all $g \in G$ and for all $x, y \in V$.

Exercise G5 (Finite subgroups of $O_2(\mathbb{R})$)
Let G be a finite subgroup of $O_2(\mathbb{R})$ with n elements.

(a) Show that

$$C := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \ a, b \in \mathbb{R}, (a, b) \neq (0, 0) \right\}$$

is a subgroup of $GL_2(\mathbb{R})$ and that

$$C \to \mathbb{C}^\times, \qquad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto a + ib$$

is a bijective group homomorphism.

(b) Show that there exists a bijective group homomorphism $SO_2(\mathbb{R}) \to S^1 := \{z \in \mathbb{C}, |z| = 1\}$.

(c) Assume $G \subseteq SO_2(\mathbb{R})$. Show that there exists $A \in G$ such that $G = \{I_2, A, A^2, \ldots, A^{n-1}\}$.

(d) Show that any element $A \in O_2(\mathbb{R})$ with $\det(A) = -1$ is an orthogonal reflection (see Exercise H1) and that $A^2 = I_2$.

(e) If G is not contained in $SO_2(\mathbb{R})$, show that there exists $m \in \mathbb{N}$ such that $n = 2m$, and that $G \cap SO_2(\mathbb{R})$ has m elements.

Homework

Exercise H1 (Reflections) (4+4+4 points)

Let V be a euclidean space, and let $H \subset V$ be a subspace.

(a) Show that there exists a unique endomorphism $r_H \in \mathrm{End}(V)$ satisfying $r_H(x) = x$ for all $x \in H$ and $r_H(x) = -x$ for all $x \in H^\perp$.

(b) Show that r_H is an isometry. When H is a hyperplane (i.e., a subspace of V with $\dim(H) = \dim(V) - 1$), then r_H is called the *orthogonal reflection with respect to* H.

(c) For all $f \in O(V)$, show that $f \circ r_H \circ f^{-1} = r_{f(H)}$.

Exercise H2 (Commuting normal endomorphisms) (3+9+12* points)

Let $(V, \langle , \rangle)$ be a $\mathbb{K}$-inner product space and let $f, g : V \to V$ be normal endomorphisms such that $f \circ g = g \circ f$.

(a) Let $\lambda \in \mathbb{K}$ be an eigenvalue of f. Show that the eigenspace $E_\lambda(f)$ is g-invariant.

(b) Assume that V is a unitary space. Show that there exists an orthonormal basis of V for which both f and g have a diagonal matrix.

*(c) Assume now that V is a euclidean space. We want to show that there exists an orthonormal basis for which both f and g are in normal form (i.e have matrices of the form given by Theorem 1.17). We choose an orthonormal basis $\mathcal{B}$ of V, and denote respectively by A and B the matrices of f and g with respect to $\mathcal{B}$. We may view A and B as endomorphisms of $\mathbb{R}^n$ or $\mathbb{C}^n$ where $n = \dim_{\mathbb{R}}(V)$.

   (i) Let $x = (x_1, ..., x_n) \in \mathbb{C}^n$. Define $\overline{x} := (\overline{x}_1, ..., \overline{x}_n)$. Show that $\langle x, \overline{x} \rangle_{\mathbb{C}} \cap \mathbb{R}^n$ is a $\mathbb{R}$-subspace of $\mathbb{R}^n$ of dimension ≤ 2. When does it have dimension $0, 1, 2$?

   (ii) Prove that there exists a subspace of dimension 1 or 2 in $\mathbb{R}^n$ that is stable by both A and B.

   (iii) Deduce the result by induction. Hint: Use Exercise G2(c).

Exercise H3 (The orthogonal group is generated by reflections) (12 points)

Let V be a euclidean space. Show that for every $g \in O(V)$ there exists $m \in \mathbb{N}$ and orthogonal reflections $r_{H_1}, \ldots, r_{H_m}$ (see Exercise H1) such that $g = r_{H_1} \circ r_{H_2} \circ \cdots \circ r_{H_m}$.

Hint : First prove the result for $\dim(V) \leq 2$, then use induction.

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3. Exercise Sheet

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Summer term 2016
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Groupwork

Exercise G1 (Positive definite matrix)
Consider the following matrix:

$$A := \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) Show that A is positive definite.

(b) Determine the unique symmetric positive definite matrix B such that $A = B^2$.

Exercise G2 (Square root of A^*A and AA^*)
Let $A \in GL_n(\mathbb{C})$ be a matrix. For a positive definite hermitian matrix B, we denote by $\sqrt{B}$ the unique positive definite hermitian matrix such that $(\sqrt{B})^2 = B$.

(a) Show that A^*A and AA^* are positive definite hermitian matrices.

(b) For $x \in \mathbb{C}^n$, show that $\|Ax\| = \|\sqrt{A^*A}x\|$ and $\|A^*x\| = \|\sqrt{AA^*}x\|$.

(c) Recall that there is a unique unitary matrix U_A such that $A = U_A\sqrt{A^*A}$. Show that there is a unique unitary matrix V_A such that $A = \sqrt{AA^*}V_A$. Show that $V_{A^*} = U_A^*$.

(d) Show that $U_A = V_A$. Deduce that $U_{A^*} = U_A^*$.

(e) Assume $A = SU = U'S$, where $U, U' \in U(n)$ and S is hermitian positive definite. Show that A is normal and that $U = U'$.

Exercise G3 (Function on Spectrum)
For a matrix $A \in M_n(\mathbb{C})$, we denote by $\sigma(A)$ the set of eigenvalues of A, called the spectrum of A. Let A be a normal matrix, and let $f : \sigma(A) \to \mathbb{C}$ be a function, let $A \in M_n(\mathbb{C})$ be a normal matrix. There exists $U \in U(n)$ such that $A = U^*DU$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal coefficients $\lambda_1, \dots, \lambda_n$. We define:

$$f(A) := U^* \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n))U.$$

(a) Show that $f(A)$ is independant of the choice of the the decomposition $A = U^*DU$. Hint : Choose a polynomial $P \in \mathbb{C}[X]$ such that $P(\lambda_i) = f(\lambda_i)$ for all $i = 1, \dots, n$, and prove that $f(A) = P(A)$.

(b) Let $f, g : \sigma(A) \to \mathbb{C}$ be two functions. Prove that:

$$\begin{aligned}(f+g)(A) &= f(A) + g(A)\\ (fg)(A) &= f(A)g(A)\\ \overline{f}(A) &= (f(A))^*\end{aligned}$$

(c) Show that $f(A)$ is normal and $\sigma(f(A)) = f(\sigma(A))$.

(d) Let $f : \sigma(A) \to \mathbb{C}$, $\lambda \mapsto \overline{\lambda}$. Show that $f(A) = A^*$.

(e) Assume A is invertible, and let $f : \sigma(A) \to \mathbb{C}$, $\lambda \mapsto \frac{1}{\lambda}$. Show that $f(A) = A^{-1}$.

(f) Let $f : \sigma(A) \to \mathbb{C}$ and $g : \sigma(f(A)) \to \mathbb{C}$. Show that $(g \circ f)(A) = g(f(A))$.

(g) Prove the equivalences:

$$\begin{aligned}f(A) \text{ is unitary} &\iff f \text{ has values in } S^1 = \{z \in \mathbb{C}, |z| = 1\}\\ f(A) \text{ is self-adjoint} &\iff f \text{ has values in } \mathbb{R}\\ f(A) \text{ is positive definite} &\iff f \text{ has values in } \{x \in \mathbb{R}, x > 0\}\\ f(A) \text{ is an orthogonal projection} &\iff f \text{ has values in } \{0, 1\}.\end{aligned}$$

Exercise G4 (Polynomial division)
Compute the division with remainder of P by Q in the following cases:

(a) $P = X^7 + 2X^3 - X^2 + X + 1$ and $Q = X^5 + X^3 - X^2 + X - 3$.

(b) $P = X^{12} - 1$ and $Q = X^4 - 1$.

Homework

Exercise H1 (Iwasawa decomposition) (12 points)
Determine the Iwasawa decomposition of the matrix

$$A := \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Exercise H2 (Polar decomposition, Cartan decomposition) (12 points)
Determine the polar decomposition and a Cartan decomposition of the matrix

$$\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$$

Exercise H3 (Euclidean ring) (2+4+6+6* points)

(a) Show that the following sets form rings with respect to the addition and multiplication induced by $\mathbb{C}$:

$$\mathbb{Z}[i] := \{a + bi,\ a, b \in \mathbb{Z}\}$$
$$\mathbb{Q}[i] := \{a + bi,\ a, b \in \mathbb{Q}\}.$$

(b) Show that every element of $\mathbb{Q}[i]$ is of the form z/w for $z, w \in \mathbb{Z}[i]$ with $w \neq 0$ (this means that $\mathbb{Q}[i]$ is the quotient field of $\mathbb{Z}[i]$ in the language introduced in §4 of the lecture).

(c) Show that $\mathbb{Z}[i]$ is a euclidean ring for the euclidean norm function $\mathbb{Z}[i] \setminus \{0\} \to \mathbb{N}$, $z \mapsto |z|^2$.

*(d) Show that $\mathbb{Z}[i]^\times := \{z \in \mathbb{Z}[i];\ \exists\, w \in \mathbb{Z}[i] \colon zw = 1\} = \{1, -1, i, -i\}$.

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4. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

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Summer term 2016
5. Mai 2016

Groupwork

Exercise G1 (Equivalence relation)
Let X be a finite set with n elements, and let $f : X \to X$ be a mapping such that $f \circ f = \mathrm{Id}_X$.

(a) Show that the relation

$$x \sim y \Leftrightarrow (y = x \text{ or } y = f(x))$$

defines an equivalence relation on X.

(b) Assume that n is odd. Show that f has a fixed point, i.e there exists $x \in X$ such that $f(x) = x$.

(c) Generalize this result for a mapping f satisfying $f^{(p)} := f \circ \ldots \circ f = \mathrm{Id}_X$, where p is a prime number. Show that if n is not divisible by p, then f has a fixed point. Hint : Use the equivalence relation: $y \sim x \Leftrightarrow \exists k \in \mathbb{Z}, y = f^{(k)}(x)$, where by definition $f^{(0)} := \mathrm{Id}_X$ and $f^{(k)} := (f^{-1})^{(-k)}$ for $k < 0$.

Exercise G2 (Simple roots)

(a) For a polynomial $P = \sum_{i=0}^{n} a_i X^i \in K[X]$, we define its derived polynomial as:

$$P'(X) := \sum_{i=1}^{n} i a_i X^{i-1}.$$

Show that $(P+Q)' = P' + Q'$ and $(PQ)' = P'Q + PQ'$ for all polynomials $P, Q \in K[X]$.

(b) An element $\alpha \in K$ is a simple root of P if P is divisible by $X - \alpha$ but not by $(X-\alpha)^2$. Prove that α is a simple root if and only if $P(\alpha) = 0$ and $P'(\alpha) \neq 0$.

(c) Let P be an irreducible polynomial in $\mathbb{Q}[X]$. Show that P has only simple roots in $\mathbb{C}$.

Exercise G3 (System of congruences)

(a) Let $a, b \in \mathbb{Z}$ be coprime integers. Show that there exists $x, y \in \mathbb{Z}$ such that

$$\begin{cases} x \equiv 1 \pmod{a} \\ x \equiv 0 \pmod{b} \end{cases} \qquad \begin{cases} y \equiv 0 \pmod{a} \\ y \equiv 1 \pmod{b} \end{cases}$$

(b) Let $a_0, b_0 \in \mathbb{Z}$ and consider the following system of congruences:

$$\begin{cases} z \equiv a_0 \pmod{a} \\ z \equiv b_0 \pmod{b} \end{cases}$$

Show that the set of elements $z \in \mathbb{Z}$ satisfying the above congruences is

$$\{a_0 x + b_0 y + kab,\ k \in \mathbb{Z}\}.$$

(c) Determine the integers $z \in \mathbb{Z}$ satisfying the following congruences:

$$\begin{cases} z \equiv 3 \pmod{5} \\ z \equiv 7 \pmod{3} \end{cases}$$

(d) Using a similar method, determine the integers $z \in \mathbb{Z}$ satisfying the following congruences:

$$\begin{cases} z \equiv 1 \pmod{3} \\ z \equiv 2 \pmod{7} \\ z \equiv 3 \pmod{11} \end{cases}$$

Exercise G4 (Greatest common divisor)
Let K be a field. For two polynomials $P, Q \in K[X]$, we denote by $\gcd_K(P, Q)$ the unic monic polynomial that is a greatest common divisor of P and Q in the ring $K[X]$.

(a) Let $P, Q \in \mathbb{Q}[X]$. Show that $\gcd_{\mathbb{Q}}(P, Q) = \gcd_{\mathbb{C}}(P, Q)$.

(b) Let $P, Q \in \mathbb{Q}[X]$. Show that P, Q are coprime in $\mathbb{Q}[X]$ if and only if they have no common root in $\mathbb{C}$.

Homework

Exercise H1 (Euclidean algorithm) (4+4+4 points)
Using the Euclidean algorithm, determine the GCD of the following elements a, b in the euclidean ring R:

(a) $a = 91091$, $b = 1729$ in $R = \mathbb{Z}$.

(b) $a = X^7 + 2X^6 - 3X^5 + X^2 - X + 1$, $b = X^5 + X^2 - 2X + 1$ in $R = \mathbb{R}[X]$.

(c) $a = 10$, $b = i - 7$ in $R = \mathbb{Z}[i]$ (see Ex. H3 on Ex. sheet 3).

Exercise H2 (Irreducible decomposition) (2+2+2+2+2+2 points)
Determine the decomposition of the following polynomials into irreducible factors:

$$X^2+1 \in \mathbb{R}[X]$$
$$X^2+1 \in \mathbb{C}[X]$$
$$X^2+1 \in (\mathbb{Z}/2\mathbb{Z})[X]$$
$$X^2+1 \in (\mathbb{Z}/3\mathbb{Z})[X]$$
$$7X^2-8X+5 \in \mathbb{R}[X]$$
$$aX^2+bX+c \in \mathbb{R}[X] \text{ with } ac<0$$

Exercise H3 (Ideals and quotient ring) (2+3+4+3 points)
Let R be a commutative ring and let $X \subseteq R$ be a subset. Define a relation on R by $x \sim y$ if $x-y \in X$.

(a) Show that $\sim$ is an equivalence relation if and only if X is a subgroup of $(R,+)$.

(b) Let X be a subgroup of $(R,+)$. Show that the map

$$+\colon (R/\sim)\times(R/\sim) \longrightarrow R/\sim,\ ([x],[y]) \mapsto [x+y]$$

is well defined.

(c) Show that the map

$$\cdot\colon (R/\sim)\times(R/\sim) \longrightarrow R/\sim,\ ([x],[y]) \mapsto [xy]$$

is well defined if and only if for all $a \in R$ and $x \in X$ one has $ax \in X$. A subgroup X of $(R,+)$ satisfying this condition is called ideal of R. Show that in this case $R/\sim$ is a commutative ring with respect to the addition and multiplication defined above.

(d) Determine the ideals of the ring $\mathbb{Z}$. If R is a field, what are its ideals?

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5. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

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Summer term 2016
12. Mai 2016

Groupwork

Exercise G1 (Long exact sequences, dimensions)

(a) Consider a short exact sequence of finite-dimensional vector spaces:

$$0 \to V' \to V \to V'' \to 0.$$

Show that $\dim(V) = \dim(V') + \dim(V'')$.

(b) Let

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} V_{n-1} \xrightarrow{f_{n-1}} V_n \to 0$$

be a long exact sequence of vector spaces. Show that one obtains short exact sequences:

$$0 \to \mathrm{Im}(f_i) \to V_{i+1} \to \mathrm{Im}(f_{i+1}) \to 0$$

for all $i = 1, \dots, n-2$.

(c) Assume further that the vector spaces $V_1, \dots, V_n$ in (b) are finite-dimensional. Prove that:

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

Exercise G2 (Quotient of polynomial ring)

Let K be a field and $P \in K[X]$ be a nonzero polynomial. Give a basis of the K-vector space $K[X]/(P)$, where (P) is the principal ideal of $K[X]$ generated by P, and show $\dim(K[X]/(P)) = \deg(P)$.

Exercise G3 (Semi-inner-product)

Let V be an $\mathbb{R}$-vector space and let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form such that $B(v, v) \geq 0$ for all $v \in V$ (then B is called a *semi-inner product*). Define

$$W := V^{\perp} := \{x \in V \,;\, \forall\, y \in V, B(x, y) = 0\}.$$

(a) Show that the following map is well-defined and is an inner-product on V/W :

$$(V/W) \times (V/W) \to K, \quad ([x]_W, [y]_W) \mapsto B(x, y).$$

(b) Let V be the space of Riemann-integrable $\mathbb{R}$-valued functions on the interval $[0,1]$. For functions $f, g \in V$, define:

$$B(f,g) := \int_0^1 f(x)g(x)dx.$$

Show that B is a semi-inner-product. Give an example of a nonzero function in $V^{\perp}$.

*(c) Show that B induces an inner-product on the subspace $V' \subset V$ of continuous $\mathbb{R}$-valued functions on the interval $[0,1]$.

Exercise G4 (Operations on principal ideals)
Let R be a euclidean ring, and let $a, b \in R$ be two elements.

(a) Show that the set

$$I := \{ax + by\ ;\ x, y \in R\}$$

is the ideal generated by a greatest common divisor of a and b.

(b) Show that the intersection $J := (a) \cap (b)$ is an ideal which is generated by a lowest common multiple of a and b.

(c) Let d be a gcd of a and b and let m be an lcm of a and b. Show that $(ab) = (dm)$.

Homework

Exercise H1 (Matrices and polynomials) (3+4+5 points)
Let K be a field, and let $A \in M_n(K)$. Consider the map

$$\varphi : K[X] \to M_n(K),\ P \mapsto P(A)$$

(a) Show that φ is a K-linear map.

(b) Show that $\mathrm{Ker}(\varphi)$ is an ideal $\neq \{0\}$ of $K[X]$.

(c) Let $n = 2$, $K = \mathbb{R}$ and let A be the matrix

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Determine the unique monic generator of the ideal $\mathrm{Ker}(\varphi)$[1]. Give a K-basis for $\mathrm{Im}(\varphi)$ and determine its dimension.

Exercise H2 (A dimension formula) (6+3+3 points)
Let V be a K-vector space, and let $V_1, V_2 \subseteq V$ be subspaces.

(a) Show that one has a short exact sequence

$$0 \to V_1 \cap V_2 \to V_1 \to (V_1 + V_2)/V_2 \to 0.$$

(b) Deduce that one has an isomorphism

$$V_1/(V_1 \cap V_2) \simeq (V_1 + V_2)/V_2.$$

(c) Assume that V_1 and V_2 are finite-dimensional. Deduce from (b) that

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim(V_1) + \dim(V_2).$$

[1] Such a generator is then the minimal polynomial of A.

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6. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

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Summer term 2016
19. Mai 2016

Groupwork

Exercise G1 (Compute a high power of a matrix)
Consider the matrix:

$$A := \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \in M_2(\mathbb{Q})$$

(a) Determine the minimal polynomial and the characteristic polynomial of A. Is A diagonalizable?

(b) Compute A^{10}. Hint: Determine the remainder of the polynom division of X^{10} by μ_A.

Exercise G2 (Square roots of a matrix)
Consider the symmetric matrix

$$A := \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

(a) Show that $\chi_A = \mu_A$ and that A is positive definite.

(b) Let $p \in \mathbb{R}[X]$ and $M := p(A)$. Show that $M^2 = A$ if and only if μ_A divides $p^2 - X$. Determine the set of polynomials $p \in \mathbb{R}[X]$ satisfying this relation.

(c) Compute the unique symmetric positive definite matrix M such that $M^2 = A$.

Exercise G3 (Minimal polynomials)
Let V be a finite-dimensional K-vector space, $W \subset V$ a subspace, and $f : V \to V$ an endomorphism satisfying $f(W) \subset W$. We denote respectively by μ, μ_W and $\mu_{V/W}$ the minimal polynomials of f, the restriction of f to W, and the induced endomorphism $f_{V/W}$ of V/W.

(a) Show that $\operatorname{lcm}(\mu_W, \mu_{V/W}) \mid \mu \mid \mu_W \mu_{V/W}$.

(b) Give an example where $\mu \neq \operatorname{lcm}(\mu_W, \mu_{V/W})$.

Exercise G4 (Eigenvalues)
Let $A \in M_n(\mathbb{C})$ be a matrix.

(a) Let $P \in \mathbb{C}[X]$ such that $P(A) = 0$. Show that $P(\lambda) = 0$ for all eigenvalues $\lambda \in K$ of A.

(b) Show that the following are equivalent:

(i) A is nilpotent (i.e., there exists $k \in \mathbb{N}$ such that $A^k = 0$.)

(ii) One has $\mu_A = X^r$ for some $r \geq 1$.

(iii) All eigenvalues of A are 0.

Homework

Exercise H1 (Minimal polynomial) (12 points)

Let K be a field and let $\lambda \in K$. For all $n \geq 1$ compute the minimal polynomial of the following $n \times n$ matrix:

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in M_n(K).$$

Exercise H2 (A one-line proof of Cayley-Hamilton's theorem?) (12 points)

Explain why the following proof is wrong: Let K be a field and let $n \in \mathbb{N}$. Let $A \in M_n(K)$ be a matrix. One has $\chi_A = \det(XI_n - A)$. Hence by substituting A for the indeterminate X, we get

$$\chi_A(A) = \det(A.I_n - A) = \det(0) = 0.$$

Exercise H3 (Nilpotency degree) (8+4 points)

Let V be an n-dimensional vector space and $f : V \to V$ a nilpotent endomorphism (i.e., there exists $k \in \mathbb{N}$ such that $f^k = 0$). Let $m \geq 1$ be the smallest integer such that $f^m = 0$.

(a) Let $x \in V$ with $f^{m-1}(x) \neq 0$. Show that the system $(x, f(x), ..., f^{m-1}(x))$ is linearly independant and deduce that $m \leq n$.

(b) Give another proof of $m \leq n$ by using the theorem of Cayley-Hamilton.

Linear Algebra II
7. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Jean-Stefan Koskivirta, Florian Sokoli

Summer term 2016
26. Mai 2016

Groupwork

Exercise G1 (Matrix equation)
Let $k \in \mathbb{N}$ be an odd number. Show that there exists no matrix $A \in M_k(\mathbb{R})$ such that $A^2 = 3A - 7I_k$. Is there such a matrix in $M_k(\mathbb{C})$? Is there such a matrix in $M_k(\mathbb{F}_2)$?

Exercise G2 (Rational normal form)
In each of the following cases, determine the rational normal form of the matrix A:

(a) $A \in M_n(K)$, $\mu_A = (X-\lambda)^{n-1}$ and $\chi_A = (X-\lambda)^n$ for $\lambda \in K$, $n \geq 2$.

(b) $A \in M_4(\mathbb{R})$, $\mu_A = (X^2+X+1)(X-1)$.

(c) $A \in M_3(\mathbb{R})$, $A \neq 0$ and $A^3 = -A$

Exercise G3 (A proof of Cayley-Hamilton's theorem)
In this exercise, we give another proof of the theorem of Cayley-Hamilton for matrices in $M_n(\mathbb{C})$ with $n \geq 1$.

(a) Show the relation $\chi_A(A) = 0$ when $A \in M_n(\mathbb{C})$ is diagonalizable.

(b) Show that for every matrix $A \in M_n(\mathbb{C})$ there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of diagonalizable matrices $A_k \in M_n(\mathbb{C})$ that converges to A (with respect to some norm on $M_n(\mathbb{C})$; recall that all norms are equivalent, hence it does not matter which norm one takes).

Hint: First let A be upper-triangular. Show that there exists a sequence $(A_k)_k$ converging to A such that A_k has pairwise distinct eigenvalues. Then use the fact that every matrix in $M_n(\mathbb{C})$ is triangularizable.

(c) Deduce the theorem of Cayley-Hamilton.

Exercise G4 (An identity for characteristic polynomials)
In this exercise, we prove that $\chi_{AB} = \chi_{BA}$ for $A, B \in M_n(K)$.

(a) Show this relation assuming $A \in GL_n(K)$.

(b) For $A, B \in M_n(K)$ arbitrary, consider the function

$$\phi : K \longrightarrow K[X], \quad \lambda \mapsto \chi_{(A-\lambda I_n)B} - \chi_{B(A-\lambda I_n)}.$$

Assume that K is an infinite field. Show that there exists infinitely many $\lambda \in K$ such that $\phi(\lambda) = 0$.

(c) Deduce that $\chi_{AB} = \chi_{BA}$ when K is an infinite field. Hint : Show that $\phi(\lambda) = 0$ for all $\lambda \in K$, in particular for $\lambda = 0$. Note that for all $k \in \mathbb{N}_0$, the function $\lambda \mapsto c_k(\phi(\lambda))$ is polynomial, where $c_k(P)$ denotes the k-th coefficient of the polynomial $P \in K[X]$.

(d) Prove that $\chi_{AB} = \chi_{BA}$ for an arbitrary field K.
Hint: If K is finite, then observe that the field of fractions of $K[X]$ is an infinite field.

Homework

Exercise H1 (Conjugacy classes) (2+4+3+3 points)

(a) Show that the relation

$$A \sim B \Leftrightarrow A \text{ is similar to } B$$

is an equivalence relation on the set $M_n(K)$. An equivalence class for this relation is called a *conjugacy class*.

(b) Let $p = (X^2+1)^2(X-2)^3 \in \mathbb{R}[X]$. Determine the number of conjugacy classes of matrices in $M_7(\mathbb{R})$ with characteristic polynomial equal to p.

(c) Determine the number of conjugacy classes of matrices in $M_7(\mathbb{C})$ with characteristic polynomial equal to p.

(d) Given two monic polynomials $\chi, \mu \in K[X]$ such that μ divides χ and such that μ and χ have the same irreducible divisors in $K[X]$, show that there exists a matrix $A \in M_n(K)$ with $n = \deg(\chi)$ such that $\mu_A = \mu$ and $\chi_A = \chi$.

Exercise H2 (A matrix endomorphism) (4+4+4 points)

Let $n \geq 2$ be an integer. Consider the endomorphism

$$\psi : M_n(K) \longrightarrow M_n(K),\ A \mapsto A^t$$

(a) Show that $\psi^2 = Id_{M_n(K)}$. If $\mathrm{char}(K) \neq 2$, show that ψ is diagonalizable. If $\mathrm{char}(K) = 2$, show that ψ is not diagonalizable.

(b) Determine the minimal and characteristic polynomial of ψ.

(c) Determine the rational normal form of ψ.

Exercise H3 (Rational normal form) (12 points)

Determine characteristic polynomial, minimal polynomial, and the rational normal form of the matrix:

$$A := \begin{pmatrix} 1 & -1 & 2 & a \\ 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathbb{C})$$

where $a \in \mathbb{C}$.

Linear Algebra II
8. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Jean-Stefan Koskivirta, Florian Sokoli

Summer term 2016
2. Juni 2016

Groupwork

Exercise G1 (Diagonalizability and invariant subspaces)
Let $f \in \mathrm{End}(V)$ be an endomorphism.

(a) Assume that f is diagonalizable. Let $W \subseteq V$ be an f-invariant subspace. Show that the restriction $f_W : W \to W$ is diagonalizable.

(b) Assume $V = V_1 \oplus V_2$, where W_1, W_2 are f-invariant subspaces. Show that f is diagonalizable if and only if f_{V_1} and f_{V_2} are diagonalizable.

Exercise G2 (Jordan normal form)
In each of the following cases, determine the Jordan normal form.

(a) $A \in M_n(K)$ and $\mu_A = X - \lambda$ for $\lambda \in K$.

(b) $B \in M_4(K)$, $B^2 = B$ and $\mathrm{rk}(B) = 3$.

(c) $C \in M_n(K)$ with $\mu_C = (X-2)^2$ and $\chi_C = (X-2)^3$.

(d) $D \in M_5(K)$ with $\mu_D = X^2$ and $\mathrm{rk}(D) = 3$.

Exercise G3 (Diagonalizability and powers of an endomorphism)

(a) Let $A \in GL_n(\mathbb{C})$ such that A^r is diagonalizable for some $r \geq 1$. Show that A is diagonalizable over $\mathbb{C}$.

(b) Does (a) remain true if one replaces $\mathbb{C}$ by $\mathbb{R}$?

(c) Does (a) remain true if one removes the assumption that A is invertible ?

(d) Does (a) remain true if one replaces $\mathbb{C}$ by a field of characteristic p ?

Exercise G4 (Endomorphism of space of polynomials)
Let n be an integer, and denote by $K_{n-1}[X]$ the K-vector space of polynomials of degree $\leq n-1$. Consider the endomorphism:

$$f : K_{n-1}[X] \to K_{n-1}[X], \quad P \mapsto P'.$$

Determine the Jordan normal form of f.

Homework

Exercise H1 (Similar matrices) (12 points)

Show that the following matrices are pairwise non-similar:

$$A_1 := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, A_2 := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, A_3 := \begin{pmatrix} 1 & 2 & 7 & 6 \\ 0 & 2 & 5 & 8 \\ 7 & 1 & 0 & 7 \\ 3 & 1 & 2 & 2 \end{pmatrix}$$

$$A_4 := \begin{pmatrix} 1 & 4 & 1 & 4 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 \\ 4 & 1 & 4 & 1 \end{pmatrix}, A_5 := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 9 & 8 & 2 & 0 \\ 7 & 6 & 0 & 2 \end{pmatrix}.$$

Exercise H2 (Fitting decomposition) (4+4+4 points)

Let $f : V \to V$ be an endomorphism of a finite-dimensional K-vector space.

(a) Show that there exists a pair $(V_{\text{nilp}}, V_{\text{isom}})$ of f-stable subspaces, such that $V = V_{\text{nilp}} \oplus V_{\text{isom}}$ and such that the restriction $f : V_{\text{nilp}} \to V_{\text{nilp}}$ is nilpotent and $f : V_{\text{isom}} \to V_{\text{isom}}$ is an isomorphism.

(b) Show that the pair $(V_{\text{nilp}}, V_{\text{isom}})$ is unique with these properties.

(c) Let $g \in \text{End}(V)$ such that $f \circ g = g \circ f$. Show that $g(V_{\text{nilp}}) \subset V_{\text{nilp}}$ and $g(V_{\text{isom}}) \subset V_{\text{isom}}$

Exercise H3 (Normal forms) (3+3+3+3 points)

In each of the following cases, determine the rational normal form, the minimal polynomial, the characteristic polynomial. If the matrix is triangularizable, determine the Jordan normal form.

(a) $A \in M_n(K)$, $n \geq 2$ with $\text{tr}(A) = b \in K$ and $\text{rk}(A) = 1$.

(b) $B \in M_3(K)$ nilpotent with $\text{rk}(B) = 2$

(c) $C \in M_4(\mathbb{R})$ such that $C^4 = I_4$, $\text{tr}(C) \neq 0$, $\dim(\text{Ker}(C - I_4)) = 2$.

(d) $D \in M_4(\mathbb{R})$, with only eigenvalues $1, 2$ in $\mathbb{C}$, and $\text{tr}(D) = 5$ and $\text{rk}(D - I_4) = 3$

Linear Algebra II
9. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Jean-Stefan Koskivirta, Florian Sokoli

Summer term 2016
9. Juni 2016

Groupwork

Exercise G1 (Equivalent conditions)
Let $A \in M_n(K)$ be a matrix and $\lambda \in K$. Show that the following properties are equivalent:

(i) A is similar to the Jordan block $J_n(\lambda)$.

(ii) The matrix $A - \lambda I_n$ is nilpotent of rank $n-1$.

(iii) A is similar to a matrix of the form

$$J := \begin{pmatrix} \lambda & x_{1,2} & \cdots & x_{1,n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{n-1,n} \\ & & & \lambda \end{pmatrix}$$

where $x_{i,i+1} \neq 0$ for all $i = 1, \dots, n-1$.

(iv) A is similar to a matrix of the form ${}^tJ$ with J as in (iii).

(v) $\mu_A = (X - \lambda)^n$.

Exercise G2 (Generalized Eigenspaces)
Let f be an endomorphism of a finite dimensional K-vector space V. For a polynomial p in $K[X]$ and for $k \in \mathbb{N}$, define:

$$V_k(p) := \operatorname{Ker}(p^k(f)), \quad k \geq 1,$$

and $m_k(p) := \dim_K(V_k(p))$. Note that for $\lambda \in K$ the eigenspace of f for the eigenvalue λ is $V_1(X - \lambda)$.

(a) Show that $V_k(p) \subseteq V_{k+1}(p)$ and $m_{k+1}(p) \geq m_k(p)$ for all $k \geq 1$. Conclude that there exists N such that $V_k(p) = V_N(p)$ for all $k \geq N$. We denote by $k_\infty(p)$ the smallest integer N satisfying this condition.

(b) Show that if $m_k(p) = m_{k+1}(p)$ for some $k \geq 1$, then $V_r(p) = V_k(p)$ for all $r \geq k$.

(c) Show that the sequence $a_k := m_{k+1}(p) - m_k(p)$ is decreasing (i.e., $a_{k+1} \leq a_k$ for all k). Hint: Show that $p(f)$ induces an injective K-linear map

$$V_{k+2}(p)/V_{k+1}(p) \longrightarrow V_{k+1}(p)/V_k(p).$$

(d) Assume that p is irreducible and monic. Show that the following are equivalent:
   (i) There exists a $k \in \mathbb{N}$ such that $V_k(p) \neq 0$.
   (ii) $V_1(p) \neq 0$.
   (iii) p divides μ_f.

   Show in this case that $k_\infty(p)$ is the exponent of p in the irreducible decomposition of μ_f in $K[X]$.

(e) Show that one has a decomposition into generalized eigenspaces:

$$V = \bigoplus_{p|\mu_f} V_{k_\infty(p)}(p)$$

   where the sum is over all irreducible monic factors p of μ_f in $K[X]$.

Exercise G3 (Similar matrices in low dimensions)
Let K be a field, $n \in \mathbb{N}$ and $A, B \in M_n(K)$ be matrices.

(a) Let $n = 2$. Show that A and B are similar if and only if $\mu_A = \mu_B$.

(b) Let $n = 3$. Show that A and B are similar if and only if $\mu_A = \mu_B$ and $\chi_A = \chi_B$.

(c) Does (b) remain true for $n = 4$?

Exercise G4 (Diagonalizability and field extensions)
Let $n \in \mathbb{N}$ and $K \subseteq L$ be a field extension, and let $A \in M_n(K)$ be a matrix. If A is triangularizable over K and diagonalizable over L, show that A is already diagonalizable over K.

Homework

Exercise H1 (Transpose) (12 points)
Let K be a field and $A \in M_n(K)$ a matrix. Show that A is similar to its transpose ${}^tA$.
Hint: Use that K is a subfield of an algebraically closed field.

Exercise H2 (Invariant subspaces) (5+2+5 points)
Let f be an endomorphism of a finite-dimensional K-vector space V. We say that an f-invariant subspace $W \subset V$ is nontrivial if $W \neq 0, V$.

(a) Show that V has no nontrivial f-invariant subspaces if and only if $\mu_f = \chi_f$ and μ_f is irreducible in $K[X]$.

(b) For K algebraically closed, determine the endomorphisms with no nontrivial invariant subspaces.

(c) Show that V cannot be written as a direct sum of nontrivial f-invariant subspaces if and only if $\mu_f = \chi_f = p^k$ for some irreducible polynomial $p \in K[X]$.

Exercise H3 (Powers of matrices) (9+3 points)
Let $n \in \mathbb{N}$.

(a) For all $k \geq 1$, show that the map $GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$, $A \mapsto A^k$ is surjective.

(b) Is the map $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$, $A \mapsto A^k$ also surjective ?

Linear Algebra II
10. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Jean-Stefan Koskivirta, Florian Sokoli

Summer term 2016
16. Juni 2016

Groupwork

Exercise G1 (Exact sequences and dual)

(a) Consider an exact sequence of finite-dimensional K-vector spaces

$$\cdots \to V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} V_{i+2} \to \ldots .$$

Show that the dual sequence

$$\cdots \to V_{i+2}^\vee \xrightarrow{f_{i+1}^\vee} V_{i+1}^\vee \xrightarrow{f_i^\vee} V_1^\vee \to \ldots$$

is exact[1].

(b) Let $f : V \to W$ be a K-linear map. We define the cokernel of f as the quotient $\mathrm{Coker}(f) := W/\mathrm{Im}(f)$. Show that there is an exact sequence

$$0 \to \mathrm{Ker}(f) \to V \xrightarrow{f} W \to \mathrm{Coker}(f) \to 0$$

and deduce that there is a natural isomorphism:

$$\mathrm{Ker}(f^\vee) \simeq \mathrm{Coker}(f)^\vee.$$

Exercise G2 (Linear forms on K^n)

For an n-tuple $\underline{\lambda} := (\lambda_1, \ldots, \lambda_n) \in K^n$, denote by $\varphi_{\underline{\lambda}}$ the map

$$\varphi_{\underline{\lambda}} : K^n \to K,\ (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \lambda_i x_i.$$

(a) Show that $K^n \to (K^n)^\vee$, $\underline{\lambda} \mapsto \varphi_{\underline{\lambda}}$ is an isomorphism.

(b) For $(\underline{\lambda}_1, \ldots, \underline{\lambda}_n) \in (K^n)^n$, show that $(\varphi_{\underline{\lambda}_1}, \ldots, \varphi_{\underline{\lambda}_n})$ is a basis of $(K^n)^\vee$ if and only if the $n \times n$-matrix whose i-th column is $\underline{\lambda}_i$ is invertible.

[1] The result holds also for arbitrary not necessarily finite-dimensional vector spaces.

Exercise G3 (Tridual)
Let V be a K-vector space. Let $\iota_V : V \to V^{\vee\vee}$ denote the canonical biduality homomorphism for V. Similarly, let $\iota_{V^\vee} : V^\vee \to V^{\vee\vee\vee}$ be the canonical biduality homomorphism of $V^\vee$. Show the identity:

$$(\iota_V)^\vee \circ \iota_{V^\vee} = \mathrm{Id}_{V^\vee}.$$

Exercise G4 (Evaluation morphism)
Let V be a K-vector space and let

$$\iota : V \to V^{\vee\vee}$$

be the canonical homomorphism. Recall that ι is injective. Show that ι is surjective if and only if V is finite-dimensional. Hint: For V infinite-dimensional, let $\mathcal{B} = (e_i)_{i\in I}$ be a basis of V. First show that the subspace $W \subset V^\vee$ generated by $e_i^\vee \in V^\vee$ is strictly contained in $V^\vee$. Then consider a nonzero linear form on $V^\vee$ that is zero on W.

Homework

Exercise H1 (Dual basis) (12 points)
Let $\mathcal{B} = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$. Find all vectors $x \in \mathbb{R}^3$ such that $\mathcal{B}_x := (e_1, e_2, x)$ is a basis of $\mathbb{R}^3$ and determine the dual basis $(\mathcal{B}_x)^\vee$.

Exercise H2 (Annihilator) (3+3+3+3 points)
Let V be a vector space and $(U_i \subseteq V)_{i\in I}$ be a family of subspaces.

(a) Assume V is finite-dimensional, and let $U \subset V$ be a subspace. Show the identity:

$$U^{oo} = U$$

where we identified V with $V^{\vee\vee}$ via the biduality isomorphism $\iota : V \to V^{\vee\vee}$.

(b) Show that

$$\left(\bigcup_{i\in I} U_i\right)^o = \left(\sum_{i\in I} U_i\right)^o = \bigcap_{i\in I} U_i^o.$$

(c) Show that

$$\sum_{i\in I} U_i^o \subseteq \left(\bigcap_{i\in I} U_i\right)^o$$

and for V finite-dimensional, show that this inclusion is an equality.

(d) Give an example where the inclusion in (b) is not an equality.

Exercise H3 (Projections) (6+6 points)

Let V be a K-vector space of finite dimension n and let $U \subseteq V$ be a subspace of dimension $r \leq n$.

(a) Show that an endomorphism $f \in \mathrm{End}(V)$ is a projection (i.e., $f^2 = f$) with image U if and only if there exists a basis $(e_1, \ldots, e_n)$ of V with $e_1, \ldots, e_r \in U$ and

$$f(x) = \sum_{i=1}^{r} e_i^\vee(x) e_i$$

for all $x \in V$.

(b) Assume that $(V, \langle , \rangle)$ is an inner-product space. Show that an endomorphism $f \in \mathrm{End}(V)$ is an orthogonal projection with image U if and only if there exists an orthonormal basis $(e_1, \ldots, e_n)$ of V with $e_1, \ldots, e_r \in U$ and

$$f(x) = \sum_{i=1}^{r} \langle e_i, x \rangle e_i$$

for all $x \in V$.

Linear Algebra II
11. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Jean-Stefan Koskivirta, Florian Sokoli

Summer term 2016
23. Juni 2016

Groupwork

Exercise G1 (Trace as a bilinear form)
Let $n \geq 1$ be an integer. Consider the map: $\psi : M_n(K) \times M_n(K) \to K$, $(A,B) \mapsto \mathrm{Tr}({}^tAB)$. For $1 \leq p,q \leq n$, denote by $E_{p,q}$ the $n \times n$-matrix whose (i,j)-coefficient is 1 if $(i,j) = (p,q)$ and 0 otherwise. Let $\mathcal{B} = (E_{p,q})_{1 \leq p,q \leq n}$ be the n^2-tuple consisting of these matrices, in lexicographic order, i.e $\mathcal{B} = (E_{1,1}, E_{1,2}, \ldots, E_{1,n}, E_{2,1}, \ldots, E_{n,n})$.

(a) Show that ψ is a symmetric bilinear form on $M_n(K)$.

(b) Determine the matrix $M_{\mathcal{B}}(\beta)$. Show that ψ is nondegenerate.

(c) Let $W \subset M_n(K)$ denote the subspace of upper-triangular matrices. Determine the orthogonal $W^\perp$.

(d) Determine $S_n(K)^\perp$, where $S_n(K)$ denotes the set of symmetric matrices.

Exercise G2 (Biorthogonal)
Let β be a symmetric or skew-symmetric bilinear form on a K-vector space V, and let $U \subset V$ be a subspace.

(a) Show that $U \subseteq U^{\perp\perp}$.

(b) Assume now that V is finite-dimensional. Show that $U + V^\perp = U^{\perp\perp}$.

(c) Deduce that $U = U^{\perp\perp}$ if and only if $V^\perp \subseteq U$.

Exercise G3 (Quadratic forms)
A quadratic form on a finite-dimensional K-vector space V is a map $q : V \to K$ satisfying the following properties:

(i) For all $\lambda \in K$, and all $x \in V$, one has $q(\lambda x) = \lambda^2 q(x)$.

(ii) The map $\beta_q : V \times V \to K$, $(x,y) \mapsto q(x+y) - q(x) - q(y)$ is a bilinear form on V.

We denote by $\mathcal{Q}(V)$ the set of quadratic forms on V.

(a) Show that $\mathcal{Q}(V)$ is a subspace of the K-vector space of functions $V \to K$ (endowed with natural addition and scalar multiplication).

(b) Show that the product of two linear forms on V is a quadratic form.

(c) If β is a bilinear form on V, show that $q_\beta(x) := \beta(x,x)$ defines a quadratic form on V. Show that this defines a K-linear map

$$\begin{array}{ccc} \mathcal{L}^2(V) & \xrightarrow{\phi} & \mathcal{Q}(V) \\ \beta & \mapsto & q_\beta \end{array}$$

(d) Show that $\mathrm{Ker}(\phi) = \mathrm{Alt}(V)$, the space of alternating forms on V. Deduce that $\dim_K(\mathcal{Q}(V)) \geq \frac{n(n+1)}{2}$. Show that for all $\lambda_1, \lambda_2 \in V^\vee$, one has $\lambda_1\lambda_2 \in \mathrm{Im}(\phi)$.

(e) For a quadratic form q, show that β_q is symmetric. If we denote by $\mathrm{Sym}(V)$ the K-vector space of symmetric bilinear forms on V, show that this defines a K-linear map

$$\begin{array}{ccc} \mathcal{Q}(V) & \xrightarrow{\psi} & \mathrm{Sym}(V) \\ q & \mapsto & \beta_q \end{array}$$

(f) Show that $\phi \circ \psi = 2\,\mathrm{Id}_{\mathcal{Q}(V)}$. Show that for all $\beta \in \mathcal{L}^2(V)$, the symmetric bilinear form $\psi(\phi(\beta))$ is given by $(x,y) \mapsto \beta(x,y) + \beta(y,x)$.

(g) When $\mathrm{char}(K) \neq 2$, show that ψ is an isomorphism. Show that the restriction of ϕ to $\mathrm{Sym}(V)$ is an isomorphism (in particular, ϕ is surjective). Deduce the dimension of $\mathcal{Q}(V)$ as a K-vector space.

(h) Now assume $\mathrm{char}(K) = 2$. Show that $\mathrm{Im}(\psi)$ is the space $\mathrm{Alt}(V) \subset \mathrm{Sym}(V)$ of alternating forms. Deduce that there is an exact sequence

$$0 \to W \to \mathcal{Q}(V) \to \mathrm{Alt}(V) \to 0$$

where W is the space of functions $q : V \to K$ satisfying

$$q(x+y) = q(x) + q(y)$$
$$q(\lambda x) = \lambda^2 q(x)$$

for all $x, y \in V$ and for all $\lambda \in K$. Deduce from (d) that $\dim(W) \geq n$.

(i) Show that $\dim_K(W) = n$, deduce the dimension of $\mathcal{Q}(V)$ over K. Deduce that ϕ is surjective. Hint: For a basis $(e_1, \ldots, e_n)$ of V, show that $W \to K^n$, $q \mapsto (q(e_1), \ldots, q(e_n))$ is an isomorphism.

(j) When $\mathrm{char}(K) \neq 2$, show that any function $q : V \to K$ satisfying the following conditions is a quadratic form:
   (i') For all $x \in V$, one has $q(2x) = 4q(x)$.
   (ii) The map $\beta_q : V \times V \to K$, $(x,y) \mapsto q(x+y) - q(x) - q(y)$ is a bilinear form on V.

Exercise G4 (Bilinear form and dual)
Let V be a K-vector space. For $v, w \in V$, define a map:

$$\beta_{v,w} : V^\vee \times V^\vee \to K, \quad (\phi, \chi) \mapsto \phi(v)\chi(w).$$

(a) Show that $\beta_{v,w}$ is a bilinear form on $V^\vee$.

(b) Let T be the subspace of $\mathcal{L}^2(V^\vee)$ generated by the elements $\beta_{v,w}$ for $v, w \in V$. For a linear form $\lambda \in T^\vee$, define a function $\beta_\lambda : V \times V \to K$ by $\beta_\lambda(v,w) = \lambda(\beta_{v,w})$. Show that this defines an isomorphism of K-vector spaces

$$T^\vee \longrightarrow \mathcal{L}^2(V), \quad \lambda \mapsto \beta_\lambda.$$

Homework

Exercise H1 (Totally isotropic subspaces) (12 points)
Let β be a symmetric or skew-symmetric bilinear form on a finite-dimensional K-vector space V, and let $U \subset V$ be a totally isotropic space. Show the formula:

$$\dim_K(U) \leq \dim_K(V) - \frac{\mathrm{rk}(\beta)}{2}.$$

Exercise H2 (A bilinear form on a polynomial ring) (3+4+2+3 points)
Let $V = K_{n-1}[X]$ be the K-vector space of polynomials of degree $\leq n-1$. For an integer $k \in \mathbb{N}$, let $a_k(p)$ denote the k-th coefficient of a polynomial $p \in K[X]$.

(a) Show that the map $\beta_k : V \times V \to K$, $(p,q) \mapsto a_k(pq)$ is a symmetric bilinear form.

(b) Let $\mathcal{B} = (1, X, ..., X^{n-1})$ denote the canonical basis of V. Compute the matrix of β_k in $\mathcal{B}$.

(c) For what values of k is β_k nondegenerate?

(d) Find a totally isotropic subspace $W \subset V$ of maximal dimension.

Exercise H3 (Existence of adjoints) (8+4 points)
Let β be a non-degenerate bilinear form on a finite-dimensional K-vector space V, and let $f \in \mathrm{End}(V)$ be an endomorphism.

(a) Show that there exists a unique endomorphism $f^* \in \mathrm{End}(V)$ such that for all $x, y \in V$, one has

$$\beta(f(x), y)) = \beta(x, f^*(y)).$$

Show that there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f^*} & V \\ {\scriptstyle f_\beta}\downarrow & & \downarrow{\scriptstyle f_\beta} \\ V^\vee & \xrightarrow{f^\vee} & V^\vee \end{array}$$

where $f_\beta : V \to V^\vee$ is the map $w \mapsto \beta(\cdot, w)$.

(b) Let $\mathcal{B}$ be a basis of V and let $\mathcal{B}^\vee$ be the dual basis. Prove the formula:

$$M_{\mathcal{B}}(f^*) = B^{-1} M_{\mathcal{B}}(f^\vee) B$$

where $B = M_{\mathcal{B}}(\beta)$.

Linear Algebra II
12. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

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Groupwork

Exercise G1 (Signature)
Consider the functions:

$$\psi_1 : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}, \quad (A,B) \mapsto \mathrm{Tr}({}^tAB)$$
$$\psi_2 : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}, \quad (A,B) \mapsto \mathrm{Tr}(AB)$$

(a) Show that ψ_1, ψ_2 are symmetric bilinear forms.

(b) Determine the signatures of ψ_1 and ψ_2. Hint: Show that $M_n(\mathbb{R}) = S_n(\mathbb{R}) \perp A_n(\mathbb{R})$ where $S_n(\mathbb{R})$ denotes the set of symmetric matrices in $M_n(\mathbb{R})$, and $A_n(\mathbb{R})$ the set of skew-symmetric matrices in $M_n(\mathbb{R})$.

Exercise G2 (Charactersitic polynomials and bilinear forms)

(a) Show that if $\chi_A = \chi_B$ for two diagonalizable matrices $A, B \in M_n(K)$, then $\mathrm{rk}(A) = \mathrm{rk}(B)$.

(b) Give a counter-example to (a) when we do not assume that A and B are diagonalizable.

(c) Let $A, B \in GL_n(\mathbb{R})$ be symmetric matrices such that $\chi_A = \chi_B$. Show that A and B have the same signature.

(d) Let $A, B \in M_n(\mathbb{R})$ be symmetric matrices such that $\chi_A = \chi_B$. Show that A and B are congruent.

Exercise G3 (Gram-Schmidt algorithm for anisotropic symmetric forms)
Let K be a field of characteristic $\neq 2$, and let $f \in K[X]$ be an irreducible polynomial of degree ≥ 2. We denote by $K(X)$ the field of fractions of $K[X]$.

(a) Show that the bilinear form on $K(X)^2$ given by the matrix

$$A := \begin{pmatrix} X & 1 \\ 1 & f \end{pmatrix}$$

is anisotropic.

(b) Let $K = \mathbb{R}$ and $f = X^2 + 1$. Use the Gram-Schmidt algorithm to determine an orthogonal basis of $K(X)^2$. Find a matrix $P \in GL_2(\mathbb{R}(X))$ such that ${}^tPAP$ is diagonal.

Exercise G4 (Symplectic group)
A non-degenerate, alternating bilinear form is called a *symplectic form*. Let V be a K-vector space of even dimension $2n$, and let ψ be a symplectic form on V. We define the symplectic group of (V,ψ) by:

$$Sp(V,\psi) := \{f \in GL_K(V),\ \psi(f(x), f(y)) = \psi(x,y),\ \forall x, y \in V\}.$$

A maximal totally isotropic subspace $U \subset V$ is called a *Lagrangian*.

(a) Show that U is a Lagrangian if and only if U is totally isotropic and $\dim_K(U) = n$. Show that the image of a Lagrangian by an element of $Sp(V,\psi)$ is again a Lagrangian.

(b) Let U be a Lagrangian and let $(e_1, \ldots, e_n)$ be a basis of U. Show that there exists a tuple $(f_1, \ldots, f_n) \in V^n$ such that $\mathcal{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ is a basis of V, and such that:

$$M_{\mathcal{B}}(\beta) = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}.$$

Hint:To construct f_1, choose first a vector in the complement of U in $\mathrm{span}(e_2, \ldots, e_n)^{\perp}$, then rescale. Afterwards proceed by induction.

(c) Show that for all Lagrangian subspaces U_1, U_2 in V, there exists $f \in Sp(V,\psi)$ such that $f(U_1) = U_2$.

(d) Let U_1, U_2 be isotropic subspaces of V of the same dimension. Show that there exists $f \in Sp(V,\psi)$ such that $f(U_1) = U_2$.

Exercise G5 (Determinant of isometries)
Let β be a non-degenerate bilinear form on a finite-dimensional vector space V. Show that any isometry of V has determinant ± 1.

Homework

Exercise H1 (Signature)
Consider the symmetric matrix:

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{R}).$$

(a) Determine the signature of A.

(b) Find a matrix $P \in GL_3(\mathbb{R})$ such that ${}^tPAP$ is diagonal.

Exercise H2 (Isometric bilinear forms)

In each case, determine if the matrices A and B are congruent in $M_m(K)$:

(a) $K = \mathbb{C}$, $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 3 \\ 1 & 3 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 8 \end{pmatrix}$

(b) $K = \mathbb{R}$, A and B as in (a).

(c) $K = \mathbb{Q}$, $A = \begin{pmatrix} 7 & -2 \\ -2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 19 & 3 \\ 3 & 11 \end{pmatrix}$

(d) $K = \mathbb{R}$, A and B as in (c).

(e) $K = \mathbb{Z}/5\mathbb{Z}$, A and B as in (c).

Exercise H3 (An isometric transformation)

Let K be a field, and $\alpha, \beta \in K^\times$ such that $\alpha + \beta \in K^\times$. Then show :

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \text{ is congruent to } \begin{pmatrix} \alpha + \beta & 0 \\ 0 & (\alpha + \beta)\alpha\beta \end{pmatrix}.$$

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13. Exercise Sheet

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Groupwork

Exercise G1 (Theorem of Witt does not hold for degenerate spaces)
Give an example of a degenerate quadratic space (V,β) over a field K of characteristic $\neq 2$, a subspace $U \subseteq V$, and an injective homomorphism of quadratic spaces $s\colon (U,\beta_U) \to (V,\beta)$ than cannot be extended to an isometry $(V,\beta) \to (V,\beta)$.

Exercise G2 (Generalized Witt decomposition)
Let (V,β) be a quadratic space over a field of characteristic $\neq 2$. Show that there exists a decomposition

$$(V,\beta) = V^{\perp} \perp H \perp V_a,$$

where (H,β_H) is a hyperbolic space, (V_a,β_{V_a}) is anisotropic, and where both are determined uniquely up to isometry.

Exercise G3 (Minkowski space of spacetime)
Let $V = \mathbb{R}^4$ endowed with a Lorentz form, i.e., a symmetric bilinear form g of signature $(1,3)$. An $x \in \mathbb{R}^4$ is called *time-like* (resp. *space-like*, resp. *light-like*) if $g(x,x) < 0$ (resp. $g(x,x) > 0$, resp. $g(x,x) = 0$).

(a) Show that the set of time-like vectors is the disjoint union of two open cones in $\mathbb{R}^4$ ("open" with respect to some norm on $\mathbb{R}^4$). Here a subset $C \subseteq \mathbb{R}^n$ is called a *cone* if

$$x,y \in C \implies x+y \in C, \qquad \lambda \in \mathbb{R}_{>0}, x \in C \implies \lambda x \in C.$$

We call one of the cones the *future cone* and one the *past cone* (it does not matter which we cone we choose as future cone).

(b) Let $x,y \in V$ be time-like vectors. If both are in the future cone with $g(x,x) = g(y,y) = 1$, then show that $g(x,y) \geq 1$. If x is in the future cone and $g(x,y) > 0$, then y is in the future cone.

Exercise G4 (Generalization of Witt's cancellation to degenerate spaces)
Let K be a field of characteristic $\neq 2$. Let (V,β) and (V',β') be quadratic spaces with orthogonal decompositions $V = U_1 \perp U_2$ and $V' = U_1' \perp U_2'$. Suppose that (V,β) and (V',β') are isometric and that (U_1,β_{U_1}) and $(U_1',\beta_{U_1'})$ are isometric. Show that (U_2,β_{U_2}) and $(U_2',\beta_{U_2'})$ are isometric.
Hint: The following steps might be useful:

(a) Let $1 \leq r \leq n$, $A \in M_r(K)$ be symmetric matrices with A invertible and assume that the matrices $n \times n$-block matrices

$$\begin{pmatrix} 0 & 0 & 0 & A \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 0 & A' \end{pmatrix}$$

are congruent. Show that A and A' are congruent. Deduce Witt's cancelation if U_1 is totally isotropic and U_2 is non-degenerate.

(b) Next show that cancelation holds if U_1 is totally isotropic.

(c) Show the general case: Use induction to assume that U_1 is one-dimensional and use the second step to assume that (U_1, β_{U_1}) is anisotropic. Then proceed as in the first step of the proof of Witt's theorem.

Homework

Exercise H1 (Signature)
Let $n \in \mathbb{N}$ and let $A \in M_n(\mathbb{R})$ be a symmetric matrix. For $k = 1, \ldots, n$ let A_k be the matrix consisting of the first k rows and k columns of A. Suppose that $\det(A_k) \neq 0$ for all $k = 1, \ldots, n$. Show that the signature of A is $(n-s, s)$, where s is the number of sign changes in the sequence $1, \det(A_1), \det(A_2), \ldots, \det(A_n)$.

Exercise H2 (Product of reflections)
Let V be an $\mathbb{R}$-vector space of dimension $n \geq 2$ with a non-degenerate symmetric bilinear form β. Find n reflections $s_1, \ldots, s_n \in O(V, \beta)$ such that $\mathrm{Id}_V = s_1 \circ s_2 \circ \cdots \circ s_n$.

Exercise H3 (Determinant as quadratic form)
Let K be a field of characteristic $\neq 2$.

(a) Show that $\det\colon M_2(K) \to K$ is a quadratic form (Exercise Sheet 11, G3) and let

$$\beta : M_2(K) \times M_2(K) \to K, \qquad (A, B) \mapsto \det(A+B) - \det(A) - \det(B)$$

be the corresponding symmetric bilinear form.

(b) Let $K = \mathbb{R}$. What is the signature of β? What is the signature of the restriction of β to the subspace $\{A \in M_2(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}$.