Multilinear Restriction Theory

Ioan Bejenaru

Introduction

Given $f : \mathbb{R}^n \to \mathbb{C}$ its Fourier transform it is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

and call this the Fourier transform of f. The Fourier transform is well-defined for $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in C(\mathbb{R}^n)$.

The following inversion formula holds true

$$f(x) = c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)d\xi,$$

under reasonable assumptions on $\hat{f}$.
Motivated by many applications, it is important to study the case when $\hat{g}$ is supported on a hypersurface S :

$$\hat{g} = h\delta_S = hd\sigma.$$

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Basic Object.

Parametrizing the hypersurface, this is equivalent to studying the following object :

For $n \geq 1$, let $U \subset \mathbb{R}^n$ be an open, bounded neighborhood of the origin and let $\Sigma : U \to \mathbb{R}^{n+1}$ be a smooth parametrization of the n-dimensional submanifold $S = \Sigma(U)$ of $\mathbb{R}^{n+1}$ - a hypersurface. Define

$$\mathcal{E}f(x) = \int_U e^{ix\cdot\Sigma(\xi)} f(\xi) d\xi.$$

In this series of lectures we discuss various aspects of the multilinear estimate :

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^p} \lesssim \Pi_{i=1}^k \|f_i\|_{L^q}.$$

We continue with motivating the study of such objects.

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Harmonic Analysis

The linear restriction estimate : given a smooth compact hypersurface $S \subset \mathbb{R}^{n+1}, n \geq 1$ the linear restriction estimate $R_S(p \to q)$ holds true if

$$\|\hat{f}\|_{L^q(S,d\sigma)} \leq C(p,S)\|f\|_{L^p(\mathbb{R}^{n+1})}. \tag{1}$$

This justifies why, given $f \in L^p(\mathbb{R}^{n+1})$, one can meaningfully consider the restriction of $\hat{f}$ to S as an element $L^q(S,d\sigma)$. A priori this is true only if $p = 1$; indeed if S is a subset of a hyperplane estimate fails for $p > 1$. However if S has some non-vanishing principal curvatures, then improvements are available beyond the trivial case $p = 1$.
Using duality, the linear restriction estimate $R_S(p \to q)$ is equivalent to the adjoint linear restriction estimate $R_S^*(q' \to p')$:

$$\|\mathcal{E}f\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C(p,\Sigma)\|f\|_{L^{q'}(U)}, \tag{2}$$

where p', q' are the dual exponents to p, q used in (1). Establishing (1) or (2) for the full conjectured range of pairs (p,q), respectively (p',q') is a major open problem in Harmonic Analysis.

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The ball multiplier problem :

$$\|B_1 f\|_{L^p} \lesssim \|f\|_{L^p},$$

where $\widehat{B_1 f} = \chi_{B_1}(\xi)\hat{f}(\xi)$ and B_1 is the unit ball. It came as a surprise when Fefferman proved that, if $d \geq 2$, B_1 is not bounded unless $p = 2$.

The Bochner-Riesz means :

$$B_R^\delta f = (1 - \frac{|\xi|^2}{R^2})^\delta \chi_{B_1}(\xi)\hat{f}(\xi),$$

and the question is $\lim_{R\to\infty} B_R^\delta f = f$ in L^p; this is closely related to finding the relation between δ and the ranges of p for which B_R^δ is bounded on L^p.

$\partial B_1 = \mathbb{S}^{n-1}$ plays a key role in this problem, because this is where the cut-off switches to 0. This relates this problem to the restriction problem.

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Kakeya Conjecture

Kakeya asked the following question : what is the minimum volume that a set in $\mathbb{R}^n$ containing a unit segment in all directions can have ?
Besicovitch provided a surprising example of sets of arbitrarily small Lebesgue measure with this property ($n = 2$).

Relevance in Harmonic Analysis : Fefferman's example of B_1 not being bounded on $L^p, p \neq 2$ relies on such sets.

Restate the question in terms of geometric measure theory : what is the dimension (Hausdorff/Minkowski) of a Kakeya set ?

Conjecture : Full dimension n.

This is closely related to the Restriction Conjecture and Bochner-Riesz conjecture.

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PDE

Consider the transport equations :

$$u_t - b \cdot \nabla u = 0$$

Taking the space-time Fourier transform gives :

$$i(-\tau + b \cdot \xi)\hat{u} = 0.$$

$\hat{u}$ has to be 0 away from the hyperplane $\tau = b \cdot \xi$, and, thus, supported on the hyperplane $\tau = b \cdot \xi$; formally

$$\hat{u} = f(\xi)\delta_{\tau = b \cdot \xi}$$

where one should make sense of the object $\delta_{\tau = b \cdot \xi}$ (as a distribution) and $f(\xi)$ (last one is computed from initial data). We say that the hyperplane is the characteristic surface for the transport equation.

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Consider the free (homogenous) wave equation :

$$u_{tt} - \Delta u = 0 \Rightarrow (-\tau^2 + \xi^2)\hat{u} = 0.$$

Therefore $\hat{u}$ is supported on the cone $\tau = \pm|\xi|$ and formally

$$\hat{u} = f(\xi)\delta_{\tau=\pm|\xi|}$$

The cone is the characteristic surface for the wave equation.

Similarly, consider the Schrödinger equation :

$$iu_t - \Delta u = 0 \Rightarrow (\tau + \xi^2)\hat{u} = 0 \Rightarrow \hat{u} = f(\xi)\delta_{\tau=-|\xi|^2}.$$

The paraboloid is the characteristic surface for the Schrödinger equation.

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Relations with other areas of Mathematics.

The problem originated in Number Theory : counting lattice points inside a large domain $R\Omega$. Depending on the choice of Ω this is related to problems such as : average number of representation of a natural number as sum of squares, average number of integer divisors, etc.

Combinatorics, Incidence Geometry ; we'll highlight this later.

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The role of Curvature.

Probably the best understood problem is the following problem :

$$\|\mathcal{E}f\|_{L^p} \lesssim \|f\|_{L^2(U)}.$$

We use the (x,t) coordinates on the physical side and (ξ,τ) on the Fourier side. If $S = \Sigma(U)$ is flat, say $\tau = 0$ (transport equation), then $\Sigma(\xi) = \xi$ and

$$\mathcal{E}f(x,t) = \int_U e^{ix\cdot\xi} f(\xi) d\xi.$$

and $\mathcal{E}f(x,t)$ is constant in the t direction ; thus no integrability properties can be expected beyond

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Wave packets.

Setup : $\tau = \varphi(\xi), \xi \in U, \Sigma(\xi) = (\xi, \varphi(\xi))$ and t, x are dual variables :

$$\mathcal{E}f(x,t) = \int_U e^{i(x\cdot\xi + t\varphi(\xi))} f(\xi) d\xi.$$

Fix a time scale $T = R^2$. Discretize the Fourier space at scale $\Delta\xi = R^{-1}$ and the physical space at scale $\Delta x = R$. Given ξ_T, x_T define the tube T :

$$T := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_T + \nabla\varphi(\xi_T)t| \leq R\}.$$

Its center line goes into the direction of the normal $N(\Sigma(\xi_T))$ at S. If f is concentrated near ξ_T and $\check{f}$ at scale R^{-1} and it is supported near x_T at scale R, the $\mathcal{E}f$ is concentrated in T.
A wave packet decomposition

$$\mathcal{E}f = \sum_T \phi_T.$$

reveals the dispersion along tubes, thus allowing for improved L^p estimates.

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Tomas-Stein ; Strichartz estimates.

The classical result is due to Tomas-Stein : if S has n-nonvansihing principal curvatures (non-zero Gaussian curvature), then

$$\|\mathcal{E}f\|_{L^p} \lesssim \|f\|_{L^2}$$

for $p \geq \frac{2(n+2)}{n}$; $d = n+1$, n is the dimension of the ambient space for ξ.
More generally, if S has l n-nonvansihing principal curvatures then the above holds for $p \geq \frac{2(l+2)}{l}$.

These estimates, known as Strichartz estimates, played a crucial role in the field of dispersive PDE's : Schödinger equation, Wave equation, Klein-Gordon etc.

The wave packet is the known example to saturate the above estimate.

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Assume S_1 and S_2 are compact transversal hypersurfaces. Then the following L^2 bilinear estimate holds true :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^2} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2},$$

These estimates (and variations of) are widely used in the context of dispersive equation in conjunction with the $X^{s,b}$ estimates. Free solutions/waves to dispersive equation have the Fourier transform supported on characteristic hypersurfaces. The solutions to nonlinear dispersive equations do not enjoy the same property, but are expected to be "concentrated" on the same hypersurfaces; the $X^{s,b}$ space is a tool that quantifies the decay of such solutions away from corresponding characteristic hypersurface.

Combinations of the bilinear theory and the Strichartz theory ($L^p_t L^q_x$) led to major inroads in the theory of dispersive equations.

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Multilinear restriction estimates.

Assuming that $S_1, .., S_k$ are transversal hypersurfaces in $\mathbb{R}^{n+1}$, the following is conjectured to be true

$$\|\Pi_{i=1}^k \mathcal{E}_i f_i\|_{L^{\frac{2}{k-1}}(\mathbb{R}^{n+1})} \leq C \Pi_{i=1}^k \|f_i\|_{L^2(U_i)}.$$

Transversality : There exists $\nu > 0$ such that for any $\zeta_i \in S_i, i \in \{1, .., k\}$, the following holds true

$$vol(N_1(\zeta_1), .., N_k(\zeta_k)) \geq \nu.$$

Here *vol* is the standard volume form of k vectors in $\mathbb{R}^k$. No curvature assumptions are needed ; in fact, curvature complicates things.
There are two easy cases. If S_i are transversal hyperplanes then this is the classical Loomis-Whitney inequality ; for instance

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$$E(|\sum_{n=1}^{N} \omega_n a_n|^p) \approx \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{p}{2}},$$

for any $0 < p < \infty$, leads to the ....

Multilinear Kakeya.

We are given $n+1$ families of tubes $\mathcal{T}_1, .., \mathcal{T}_{n+1}$ such that each tube $T \in \mathcal{T}_i$ has the property that its core makes an angle $\ll 1$ with the vector e_i. We allow tubes in the same family to be parallel. The multilinear Kakeya conjecture is the following :

$$\|\Pi_{i=1}^{n+1}(\sum_{T_i \in \mathcal{T}_i} \chi_{T_i})\|_{L^{\frac{1}{n}}(\mathbb{R}^{n+1})} \leq C\Pi_{i=1}^{n+1} \# T_i.$$

where $\# T_i$ is the cardinality of the family of tubes T_i.

This is a version of the "joints problem" in incidence geometry, as the left hand-side counts the number of joints with multiplicity.

The nonlinear Kakeya is weaker than the original multilinear estimate, but it allows to obtain the near-optimal version :

$$\|\Pi_{i=1}^{k}\mathcal{E}_i f_i\|_{L^{\frac{2}{k-1}}(B(x;R))} \leq CR^{\epsilon}\Pi_{i=1}^{k}\|f_i\|_{L^2(U_i)}.$$

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Bennett, Carbery and Tao proved the near-optimal Kakeya and, as a consequence, the near-optimal multilinear restriction estimate. Guth provides a simplified argument for the near-optimal Kakeya, using ideas from capacity theory and highlights the underlying idea of an induction on scales type argument.

Recently, B. provides a direct argument for the near-optimal multilinear restriction estimate. This has the advantage it that provides good estimates when one of the factors has good localization properties - this plays a crucial role in improved estimates when curvature comes into play.

The most striking result is the sharp end-point case for the multilinear Kakeya. Here Guth introduces the polynomial partition method and uses ideas from algebraic topology. Carbery and Valdirmasson provide an alternative argument, using only basic theorems of Borsuk-Ulam type ; this is more familiar to an analyst, given that Brower's fixed point theorem relies on such results.

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Applications.

Bourgain and Guth improved the known ranges for the linear restriction conjecture in higher dimensions ; the bilinear restriction estimate is useful in lower dimensions, but not in in higher.

Guth takes further the polynomial partition ideas and improves the restriction theory ; recall that if $n = 2$ the conjecture is

$$\|\mathcal{E}f\|_{L^3} \lesssim \|f\|_{L^\infty}.$$

Guth proves this for $p > 3.25$; the bilinear theory does it with $p = \frac{10}{3} = 3.333..$, while Bourgain-Guth improved for $p > 3.29..$

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If $k = n + 1$ the exponent $p = \frac{2}{k-1} = \frac{2}{n}$ is sharp.

If $k < n + 1$, then the exponent $\frac{2}{k-1}$ is sharp in the generic case ; if one assumes curvature hypothesis, then it should improve.

Under transversality and appropriate curvature conditions the conjecture is that

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for any $p(k) \leq p \leq \infty$ were $p(k) = \frac{2(n+1+k)}{k(n+k-1)} < \frac{2}{k-1}$.

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This conjecture is mostly open. It was speculated by Bennett, Carbery and Tao : "simple heuristics suggest that the optimal k-linear restriction theory requires at least $n + 1 - k$ non-vanishing principal curvatures, but that further curvature assumptions have no further effect".

It was formulated by Bennett ; however some progress has been made only recently.

B. establishes the conjecture for $k - 1$ conical hypersurfaces - this is k-linear equivalent of Wolff's bilinear result for cones. This result will be discussed in the third lecture.

Guth formulates a weaker version of the above conjecture and proves it for subsets of the paraboloid using the polynomial partition method. He then uses it to further improve the linear restriction theory in higher dimensions.

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