Multilinear Restriction Theory

Ioan Bejenaru

The L^2 bilinear estimate

For $n \geq 1$, let $U \subset \mathbb{R}^n$ be an open, bounded neighborhood of the origin and let $\Sigma : U \to \mathbb{R}^{n+1}$ be a smooth parametrization of the n-dimensional submanifold $S = \Sigma(U)$ of $\mathbb{R}^{n+1}$ - a hypersurface. Define

$$\mathcal{E}f(x) = \int_U e^{ix\cdot\Sigma(\xi)} f(\xi) d\xi.$$

The L^2 bilinear estimate :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^2} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2},$$

where S_1, S_2 are assumed to be transversal :

$$|N_1(\zeta_1) \wedge N_2(\zeta_2)| \gtrsim 1, \quad \forall \zeta_i \in S_i.$$

Hyperplane case

If S_1 and S_2 are subsets of hyperplanes, say $\xi_1 = 0$ and $\xi_2 = 0$ respectively, then, with $g_i = \mathcal{E}_i f_i$, the estimate becomes

$$\|g_1(x_2, x_3, .., x_n) g_2(x_1, x_3, .., x_n)\|_{L^2} \lesssim \|g_1\|_{L^2} \|g_2\|_{L^2}.$$

Since S_1, S_2 are compact, it follows that

$$\|g_2(x_1, x_3, .., x_n)\|_{L^2_{x_1} L^\infty_{x_3,..,x_n}} \lesssim \|g_2\|_{L^2},$$

thus the estimate follows. Note that if the support of f_2 (or f_1) is small in one of the directions $\xi_3, .., \xi_n$ the L^∞ estimate improves and the bilinear estimate picks that up : if f_2 is supported in $|\xi_3| \leq \nu$ then

$$\|g_1(x_2, x_3, .., x_n) g_2(x_1, x_3, .., x_n)\|_{L^2} \lesssim \nu^{\frac{1}{2}} \|f_1\|_{L^2} \|f_2\|_{L^2}.$$

If f_2 is supported in $|\xi_3|, .., |\xi_n| \leq \nu$ then the fator improves to $\nu^{\frac{n-2}{2}}$.

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The nonlinear case

Assume S_1 and S_2 are general transversal hypersurfaces (not flat). Then the following L^2 bilinear estimate holds true :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^2} \lesssim \|f_1\|_{L^2}\|f_2\|_{L^2},$$

Advantage : using Plancherel one transfers the estimate to the Fourier side, where it becomes convolution :

$$\|\tilde{f}_1 d\sigma_1 * \tilde{f}_2 d\sigma_2\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|\tilde{f}_1\|_{L^2(S_1)}\|\tilde{f}_2\|_{L^2(S_2)} \approx \|f_1\|_{L^2(\mathbb{R}^n)}\|f_2\|_{L^2(\mathbb{R}^n)}.$$

where $\tilde{f}_1$ is the lift of f_1 to S_1, etc.
This is a well studied problem and refinements are known to hold

$$\|\tilde{f}_1 d\sigma_1 * \tilde{f}_2 d\sigma_2\|_{L^2(S_3)} \lesssim \|\tilde{f}_1\|_{L^2(S_1)}\|\tilde{f}_2\|_{L^2(S_2)}.$$

where S_3 is a hypersurface such that S_1, S_2, S_3 are transversal.

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Applications

This is equivalent to

$$\|\tilde{f}_1 d\sigma_1 * \tilde{f}_2 d\sigma_2 * \tilde{f}_3 d\sigma_3\|_{L^\infty} \lesssim \|\tilde{f}_1\|_{L^2(S_1)} \|\tilde{f}_2\|_{L^2(S_2)} \|\tilde{f}_3\|_{L^2(S_3)}.$$

which is a weaker version of the open problem :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2 \cdot \mathcal{E}_3 f_3\|_{L^1} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}.$$

Estimates of this type were first proven by Bennet, Carbery, Wright.

The following estimate is widely used in dispersive PDE's when $X^{s,b}$ type spaces occur :

$$\|\tilde{f}_1 d\sigma_1 * \tilde{f}_2 d\sigma_2\|_{L^2(S_3(\epsilon))} \lesssim C(\epsilon, \theta, \nu) \|\tilde{f}_1\|_{L^2(S_1)} \|\tilde{f}_2\|_{L^2(S_2)}.$$

where $S_3(\epsilon)$ is the neighborhood of size ϵ surface S_3.

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Curvature improves the bilinear estimates

Klainerman-Machedon conjectured that the best exponent for the bilinear estimate :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^p} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2},$$

should be $p = \frac{n+3}{n+1} < 2$; same result both for cone and paraboloid.

Results due to : Tao-Vargas , Wolff, Tao, Lee, Lee-Vargas, etc.

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Curvature - background.

Given a hypersurface $S \subset \mathbb{R}^{n+1}$, we define the Gauss map $g : S \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ ($\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$) by $g(\zeta) = N(\zeta)$, the normal to S at ζ. We identify $T_\zeta(S)$ and $T_{g(\zeta)}\mathbb{S}^n$, and define

$$dg_\zeta : T_\zeta S \to T_\zeta S, \quad dg_\zeta v = \frac{d}{dt}(N(\gamma(t)))|_{t=0}$$

where $\gamma \subset S$ is a curve with $\gamma(0) = \zeta, \gamma'(0) = v$. The shape operator $S_{N(\zeta)} : T_\zeta S \to T_\zeta S$ is defined by

$$S_{N(\zeta)} = -dg_\zeta,$$

where we keep the subscript $N(\zeta)$ to indicate that the shape operator depends on the choice of the normal vector field at S.

$S_{N(\zeta)}$ is symmetric, so there is an orthonormal basis of eigenvectors $\{e_i\}_{i=1,n}$ of $T_\zeta S$ with real eigenvalues $\{\lambda_i\}_{i=1,n}$. Then e_i are the principal directions and $\lambda_i = k_i$ are the principal curvatures of S. The Gaussian curvature is defined by $detS_N = \Pi_{i=1}^n \lambda_i$.

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Role of curvature in the bilinear restriction estimates.

Recall the bilinear restriction estimates :

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^p} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2}.$$

The role of transversality in the improved bilinear theory was clear :

$$|N_1(\zeta_1) \wedge N_2(\zeta_2)| \gtrsim 1, \quad \forall \zeta_i \in S_i. \tag{1}$$

The role of curvature is more intricate :

$$|S_{N_i(\zeta_i)} v \wedge n| \gtrsim |v||n|, \quad \forall \zeta_i \in C_i, v \in T_{\zeta_i} C_i, n \in (T_{\zeta_i} C_i)^\perp. \tag{2}$$

Here $C_i \subset S_i$ are submanifolds of S_i obtained by intersecting translates of S_1 and S_2. Note that C_i have dimension $n-1$, thus the shape operator ignores what happens in one direction on S_i ; this explains why the cone and paraboloid give the same result in the bilinear estimate.

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Theorem

(B.) If S_1 and S_2 satisfy (1) *and* (2) *then*

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^p} \lesssim \|f_1\|_{L^2}\|f_2\|_{L^2}.$$

holds true for all $p > \frac{n+3}{n+1}$.

The novelty of this result is its generality, the concise way of phrasing the role of the shape operator in this problem and the purely analytical argument (as opposed to the combinatorial approach).

Lee-Vargas have a result on bilinear estimates for k-conical hypersurfaces, where they uncover a similar condition to our (2) ; their argument seems to extend to a more general setup.

Our approach here follows the one introduced by Tao for the end-point problem for the cone.

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Free waves

Assume S (be it S_1 or S_2) has a parametrization $\tau = \varphi(\xi)$; then

$$\phi = \mathcal{E}f = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(\xi))}\hat{f}(\xi)d\xi$$

It follows that $\hat{\phi}(\xi, t) = e^{it\varphi(\xi)}\hat{f}(\xi)$ therefore ϕ_1 satisfies an ODE on the Fourier side, $\partial_t\hat{\phi}(\xi, t) = i\varphi(\xi)\hat{\phi}(\xi, t)$, and a linear PDE on the physical side, $\partial_t\phi = i\varphi(D)\phi$ with initial data $\phi(x, 0) = \check{f}(x)$. This justifies the wording : $\phi = \mathcal{E}f$ is a free wave.

We define the mass of a free wave by $M(\phi(t)) := \|\phi(t)\|_{L^2}^2$ and note that it is time independent :

$$M(\phi(t)) := \|\phi(t)\|_{L^2}^2 = \|\hat{\phi}(t)\|_{L^2}^2 = \|f\|_{L^2}^2 = M(\phi(0)).$$

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Wave Packets Theory

For free waves as above we introduce the following construction.
Let $\mathcal{L} = r^{-1}\mathbb{Z}^n \cap D$ and $L = r\mathbb{Z}^n$. With $x_T \in L, \xi_T \in \mathcal{L}$, define the tube $T := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_T + \nabla\varphi(\xi_T)t| \leq r\}$; denote by $\mathcal{T}$ the set of such tubes. For $T \in \mathcal{T}$, define the cut-off $\tilde{\chi}_T$ on $\mathbb{R}^{n+1}$ by

$$\tilde{\chi}_T(x,t) = \tilde{\chi}_{D(x_T - \nabla\varphi(\xi_T)t, t; r)}(x).$$

Let Q be a cube of radius $R \gg 1$ and ϕ be a free wave. For each $T \in \mathcal{T}$ there is a free wave ϕ_T, with $\hat{\phi}_T$ supported in a cube of size less than $CR^{-\frac{1}{2}}$. The map $\phi \to \phi_T$ is linear and

$$\phi = \sum_{T \in \mathcal{T}} \phi_T,$$

and the following estimates hold true

$$\sum_T \sup_{q \in Q_J(Q)} \tilde{\chi}_T(x_q, t_q)^{-N} \|\phi_T\|^2_{L^2(q)} \lesssim rM(\phi)$$

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$$\left(\sum_{q_0} M(\sum_T m_{q_0,T}\phi_T)\right)^{\frac{1}{2}} \lesssim M(\phi),$$

provided that the coefficients $m_{q_0,T} \geq 0$ satisfy

$$\sum_{q_0} m_{q_0,T} = 1, \qquad \forall T \in \mathcal{T}. \tag{3}$$

There are few more tweaks to this constructions :
- margin concept ;
- a small parameter c that helps keeping track of constants in the induction argument.

This wave packet construction is "blind" to potential presence of vanishing principal curvatures ; for instance, it can be simplified in the case of the cone.

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This wave packet construction is "blind" to potential presence of vanishing principal curvatures ; for instance, it can be simplified in the case of the cone.

Geometry

For $h > 0$, let $C_1(h) = S_1 \cap (h - S_2)$ and

$$\mathcal{CN}(C_1(h)) = \{\alpha N_1(\zeta), \zeta \in C_1(h), \alpha \in \mathbb{R}\}$$

be the cone generated by the normals to S_1 taken at points from $C_1(h)$ and passing through the origin. Note that $\mathcal{CN}(C_1(h)) \setminus \{0\}$ has maximal codimension 1. This hypersurface has a key property : For any $\zeta_2 \in S_2$, $N_2(\zeta_2)$ is transversal to each $N_1(\zeta_1)$, for any $\zeta_1 \in S_1$ (consequence of (1)). However, this does not imply that $N_2(\zeta_2)$ is transversal to the surface $\mathcal{CN}(C_1(h))$! Such a claim is the object of the following result.

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We prove that $N_2(\zeta_2)$ is transversal to $T_{\zeta_1}(\mathcal{CN}(C_1(h)))$, the tangent plane to $\mathcal{CN}(C_1(h))$ at ζ_1. Since $\mathcal{CN}(C_1(h))$ is a conic surface, its tangent space at the point $\alpha N_1(\zeta_1), \alpha \in \mathbb{R} \setminus \{0\}, \zeta_1 \in C_1(h)$, is spanned by $N_1(\zeta_1)$ and the linear subspace

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Energy estimates

For a given surface $S \subset \mathbb{R}^{n+1}$ we denote the neighborhood of size r of S by $S(r)$. For fixed t we define the time "slice" in $S(r)$ by $S_t(r) = \{x : (x,t) \in S(r)\}$.

Lemma

Let ψ be a free wave with $\hat{\psi}$ supported on S_2. Let $S = \mathcal{CN}(C_1(h))$. We assume that for any $\zeta \in S_2$, the vector $N_2(\zeta)$ is transversal to S in a uniform fashion. If $r \gtrsim 1$, the following holds true :

$$\|\psi\|_{L^2(S(r))} \lesssim r^{\frac{1}{2}} M(\psi)^{\frac{1}{2}}. \tag{4}$$

Note that is S were a planar surface, then the above estimate would follow from the standard energy estimates for ψ in various coordinate systems, a topic well studied in PDE's.

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$$\|\chi_{S(r)} e^{it\varphi_2(D)}\psi_0\|_{L^2(\mathbb{R}^{n+1})} \lesssim r^{\frac{1}{2}}\|\psi_0\|_{L^2(\mathbb{R}^n)}$$

which can be rewritten as follows

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where F inherits the Fourier localization properties of ψ. Using a TT^* argument, the two are equivalent to proving the following estimate

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where the kernel K is given by

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with η chosen so as to reflect the support properties of F, that are derived from those of ψ. The gradient of the phase function $\alpha(\xi) = x\cdot\xi + t\varphi_2(\xi)$ is $\nabla\alpha = x + t\nabla\varphi_2(\xi)$ and it can be easily seen that $|\nabla\alpha(\xi)| \gtrsim 1$ for $(x,t) \notin \mathcal{CN}(S_2) = \{\lambda N_2(\zeta) : \zeta \in S_2, \lambda \in \mathbb{R}\}$. In that case we have the improved estimate

$$|K(x,t)| \lesssim_N (1 + |x| + |t|)^{-N}.$$

Given two points $(x,t), (y,s) \in S(r)$ with $|t-s| \gg r$, by using the transversality property of $N_2(\zeta)$ to S, for any $\zeta \in S_2$, it follows that $(x-y, t-s) \notin \mathcal{CN}(S_2)$; from the bound above we conclude

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Proof of the main result

Main strategy - induction on scales. Define $A(R)$ to be the best constant in the estimate

$$\|\mathcal{E}_1 f_1 \cdot \mathcal{E}_2 f_2\|_{L^p(B(x,R))} \leq A(R)\|f_1\|_{L^2}\|f_2\|_{L^2}.$$

Then one seeks to quantify the growth of $A(R)$

$$A(CR) \leq (1 + c(R))A(R), \quad A(R^{1+}) \leq CA(R).$$

We go for the first choice and this is why we need the small parameter c - which is actually ignored in these notes.

One more notation : Given a cube Q of size R, we let $\mathcal{Q}_j(Q)$ be set of cubes of size $2^{-j}R$ filling Q. There will be 3 scales involved in this argument : large R, a bit smaller $2^{-C_0}R$ and small scale $2^{-j}R \approx R^{\frac{1}{2}}$.

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Proposition

Let Q be a cube of size $R \gg 1$ and let $\phi = e^{it\varphi_1(D)}\phi_0, \psi = e^{it\varphi_2(D)}\psi_0$ be free waves. Then there is a table $\Phi = \Phi_c(\phi, \psi, Q)$ with depth C_0 such that the following properties hold true :

$$\phi = \sum_{q \in \mathcal{Q}_{C_0}(Q)} \Phi^{(q)}, \tag{6}$$

$$M(\Phi) \lesssim M(\phi), \tag{7}$$

and for any $q', q'' \in \mathcal{Q}_{C_0}(Q), q' \neq q''$

$$\|\Phi^{(q')}\psi\|_{L^2((1-c)q'')} \lesssim R^{-\frac{n-1}{4}} M^{\frac{1}{2}}(\phi) M^{\frac{1}{2}}(\psi). \tag{8}$$

This "morally" suffices ; the following estimate is trivial

$$\begin{aligned}\|\Phi^{(q')}\psi\|_{L^1(Q)} &\lesssim R\|\Phi^{(q')}\psi\|_{L^\infty_t L^1_x(Q)}\\ &\lesssim R\|\Phi^{(q')}\|_{L^\infty_t L^2_x(Q)}\|\psi\|_{L^\infty_t L^2_x(Q)}\\ &\lesssim RM(\Phi^{(q')})^{\frac{1}{2}}M(\psi)^{\frac{1}{2}}.\end{aligned}$$

Interpolating between the above L^1 estimate and the improved L^2 estimate so as to cancel the power of R reveals

$$\|\Phi^{(q')}\psi\|_{L^p(q'')} \lesssim M(\Phi^{(q')})^{\frac{1}{2}}M(\psi)^{\frac{1}{2}}.$$

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We construct the wave packet decomposition for ϕ at time scale R. For any $q_0 \in \mathcal{Q}_{C_0}(Q)$ and $T \in \mathcal{T}$, we define

$$m_{q_0,T} := \sum_{\xi_2 \in \mathcal{L}} \|\tilde{\chi}_T \psi_{\xi_2}\|^2_{L^2(q_0)}$$

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$$\Phi^{(q_0)} := \sum_T \frac{m_{q_0,T}}{m_T} \phi_T.$$

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The mass estimate is a direct consequence of the theory of Wave packets.

All that is left to prove is (8), which is equivalent to

$$\sum_{q \in \mathcal{Q}_j(Q): d(q,q_0) \gtrsim cR} \|\Phi^{(q_0)}\psi\|^2_{L^2(q)} \lesssim c^{-C} r^{-(n-1)} M(\phi) M(\psi). \tag{9}$$

Note that the cubes q are selected at the finer scale dictated the size of cubes in $\mathcal{Q}_j(Q) : 2^{-j}R \approx R^{\frac{1}{2}}$.
It suffices to focus on the tubes which intersect q, thus it suffices to prove

$$\sum_{q \in \mathcal{Q}_j(Q): d(q,q_0) \gtrsim cR} \| \sum_{T \cap q \neq \emptyset} \frac{m_{q_0,T}}{m_T} \phi_T \psi \|^2_{L^2(q)}.$$

We further expand the above term as follows

$$= \sum_{q \in \mathcal{Q}_j(Q): d(q,q_0) \gtrsim cR} \| \sum_{\xi_2 \in \mathcal{L}} \sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0,T_1}}{m_{T_1}} \phi_{T_1} \psi_{\xi_2} \|^2_{L^2(q)},$$

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where $\tilde{\chi}_q$ is a smooth approximation of the characteristic function of q :
- $\tilde{\chi}_q \equiv 1$ on q, $\tilde{\chi}_q \equiv 0$ on $\mathbb{R}^{n+1} \setminus 2q$;
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On the Fourier side $\mathcal{F}_{x,t}(\tilde{\chi}_q)$ is highly concentrated in the region $|(\xi,\tau)| \leq r^{-1}$: $|\mathcal{F}(\tilde{\chi}_q)(\xi,\tau)| \lesssim_N \langle r(\xi,\tau)\rangle^{-N}$ for all $N \in \mathbb{N}$.

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Here by $(\xi_1,\xi_2)\in A(\xi,\tau)$ we mean that $(\xi,\tau)\in r^{-1}\mathbb{Z}^{n+1}$ and $|\xi_1+\xi_2-\xi|,|\varphi_1(\xi_1)+\varphi_2(\xi_2)-\tau|\lesssim r^{-1}$.

Alternative : ξ_2 is almost uniquely determined by $\xi_1\in A_1(\xi,\tau)$ via $\xi_2=\xi-\xi_1+\tilde{\xi},\tilde{\xi}\in\mathcal{L},|\tilde{\xi}|\lesssim r^{-1}$, where $A_1(\xi,\tau)$ is the set of ξ_1 for which there exists a ξ_2 such that $(\xi_1,\xi_2)\in A(\xi,\tau)$.

Key observations about the set $A_1(\xi,\tau)$. The set of solutions of the equation $(\xi_1,\varphi_1(\xi_1))+(\xi_2,\varphi_2(\xi_2))=\beta$ is $S_1\cap(\beta-S_2)=C_1(\beta)$. Let $S:=\mathcal{CN}(C_1(\beta))=\{\alpha N_1(\zeta):\zeta\in C_1(\beta),\alpha\in\mathbb{R}\}$ where $\beta\in r^{-1}\mathbb{Z}^{n+1}$ is such that $|\beta-(\xi_1+\xi_2,\varphi_1(\xi_1)+\varphi_2(\xi_2))|\lesssim r^{-1}$.

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The "thickened" surface

$$\tilde{S} := \{ T_1 : \xi_{T_1} \in A_1(\xi,\tau), T_1 \cap q \neq \emptyset, T_1 \cap q_0 \neq \emptyset \}$$

has the property that $\tilde{S} \cap q_0$ is a subset of the intersection of $q_0 \cap ((x_q, t_q) + S(r))$ where we recall that $S(r)$ is the neighborhood of size r to S.

For fixed (ξ, τ) we write

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Using the above and the obvious inequality $\frac{m_{q_0,T_1}}{m_{T_1}} \leq \frac{m_{q_0,T_1}^{\frac{1}{2}}}{m_{T_1}^{\frac{1}{2}}}$, we obtain :

$$\| \sum_{\xi_1 \in A_1(\xi,\tau)} \sum_{\substack{T_1 \cap q \neq \emptyset: \\ \xi_{T_1} = \xi_1}} \frac{m_{q_0,T_1}}{m_{T_1}} \phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q \|_{L^2}$$

$$\lesssim \left(\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \frac{\|\phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \right)^{\frac{1}{2}} \left(\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} m_{q_0,T_1} \tilde{\chi}_{T_1}(x_q, t_q) \right)^{\frac{1}{2}}$$

We claim the following estimate

$$\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} m_{q_0,T_1} \tilde{\chi}_{T_1}(x_q, t_q) \lesssim \sum_{\xi_2 \in \mathcal{L}} \|\chi \psi_{\xi_2}\|_{L^2}^2$$

$$\lesssim r \sum_{\xi_2 \in \mathcal{L}} M(\psi_{\xi_2}) \lesssim r M(\psi).$$

Using the above and the obvious inequality $\frac{m_{q_0,T_1}}{m_{T_1}} \leq \frac{m_{q_0,T_1}^{\frac{1}{2}}}{m_{T_1}^{\frac{1}{2}}}$, we obtain :

$$\| \sum_{\xi_1 \in A_1(\xi,\tau)} \sum_{\substack{T_1 \cap q \neq \emptyset: \\ \xi_{T_1} = \xi_1}} \frac{m_{q_0,T_1}}{m_{T_1}} \phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q \|_{L^2}$$

$$\lesssim \left(\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \frac{\| \phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q \|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \right)^{\frac{1}{2}} \left(\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} m_{q_0,T_1} \tilde{\chi}_{T_1}(x_q, t_q) \right)^{\frac{1}{2}}$$

We claim the following estimate

$$\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} m_{q_0,T_1} \tilde{\chi}_{T_1}(x_q, t_q) \lesssim \sum_{\xi_2 \in \mathcal{L}} \| \chi \psi_{\xi_2} \|_{L^2}^2$$

$$\lesssim r \sum_{\xi_2 \in \mathcal{L}} M(\psi_{\xi_2}) \lesssim r M(\psi).$$

Using the definition of m_{q_0,T_1} we identify the function

$$\chi = (\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \tilde{\chi}(x_q, t_q) \tilde{\chi}_{T_1}) \chi_{q_0}$$

which makes the first inequality true. χ has the following decay property :

$$\chi(x,t) \lesssim \left(1 + \frac{d((x,t),S)}{r}\right)^{-N}.$$

This is a consequence of the fact that the tubes T_1 passing thorough q separate inside q_0 as a consequence of (2) and the separation between q and q_0, that is $d(q,q_0) \gtrsim R$.
Based on the decay estimate for χ, we can use the energy estimate for each ψ_{ξ_2} to justify the second inequality above. The last inequality is obvious.

Using the definition of m_{q_0,T_1} we identify the function

$$\chi = \Big(\sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \tilde{\chi}(x_q, t_q) \tilde{\chi}_{T_1} \Big) \chi_{q_0}$$

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Based on the decay estimate for χ, we can use the energy estimate for each ψ_{ξ_2} to justify the second inequality above. The last inequality is obvious.

Next we claim the following estimate :

$$\sum_q \sum_{\xi \in \mathcal{L}} \sum_{\tau \in \mathcal{L}_1} \sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \frac{\|\phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim r^{-n} M(\phi). \tag{10}$$

This concludes the proofs of all claims of the Proposition.
We establish (10). Taking into account the frequency localization of $\phi_{T_1} \psi_{\xi_2}$ and the fast decay properties of $\mathcal{F}_{x,t}(\tilde{\chi}_q)$, we obtain

$$\|\phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q\|_{L^2}^2 \lesssim r^{-(n+1)} \|\phi_{T_1} \psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q\|_{L^1}^2$$

Therefore it suffices to show that

$$\sum_q \sum_{\xi} \sum_{\tau} \sum_{T_1 \in \mathcal{T}(A_1(\xi,\tau))} \frac{\|\phi_{T_1} \tilde{\chi}_q\|_{L^2}^2 \|\psi_{\xi_2(T_1,\xi,\tau)} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim r M(\phi).$$

The summation with respect to (ξ, τ) brings back all possible frequency interactions, hence the above is equivalent to proving

$$\sum_q \sum_{T_1 \cap q \neq \emptyset} \sum_{\xi_2 \in \mathcal{L}} \frac{\|\phi_{T_1} \tilde{\chi}_q\|_{L^2}^2 \|\psi_{\xi_2} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim r M(\phi).$$

Note that in the above estimate the frequency of T_1, $\xi_{T_1} = \xi_1$ is decoupled from ξ_2. By rearranging the sum, it suffices to show

$$\sum_{T_1} \sum_{q \cap T_1 \neq \emptyset} \sum_{\xi_2 \in \mathcal{L}} \frac{\|\phi_{T_1} \tilde{\chi}_q\|_{L^2}^2 \|\psi_{\xi_2} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim rM(\phi).$$

The inner sum is estimated as follows

$$\sum_{q \cap T_1 \neq \emptyset} \sum_{\xi_2 \in \mathcal{L}} \frac{\|\psi_{\xi_2} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim \sum_{\xi_2 \in \mathcal{L}} \frac{\|\psi_{\xi_2} \tilde{\chi}_{T_1}\|_{L^2}^2}{m_{T_1}} \lesssim 1.$$

We conclude the argument with the following claim

$$\sum_{T_1} \sup_q \|\phi_{T_1} \tilde{\chi}_q\|_{L^2}^2 \lesssim r \sum_{T_1} M(\phi_{T_1}) \lesssim rM(\phi),$$

which is obvious given the size of q in the temporal direction.