Universität Bielefeld

SFB 701

Bilinear Restriction Estimates and Applications

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Interactions between Harmonic and Geometric Analysis

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Bilinear Extension Estimates

For $f \in L^2(\mathbb{R}^n)$ let

$$e^{it|\nabla|}f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{it|\xi|}e^{ix\cdot\xi}d\xi.$$

Note that $e^{it|\nabla|}$ is a homogeneous (or free) solution to the wave equation and is (essentially) the extension operator for the cone $\{\tau = |\xi|\}$.

Bilinear Extension

Suppose $\operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g} \subset \{|\xi| \approx 1\}$. *For which* p *do we have*

$$\big\|e^{it|\nabla|}fe^{it|\nabla|}g\big\|_{L^p_{t,x}(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}?$$

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- Immediate observation:

  $\widehat{f}$, $\widehat{g}$ have compact support $\Longrightarrow e^{it|\nabla|} f e^{it|\nabla|} g$ has compact Fourier support

  hence by Bernstein's inequality followed by Holder

  $$\left\| e^{it|\nabla|} f e^{it|\nabla|} g \right\|_{L^\infty_{t,x}} \lesssim \left\| e^{it|\nabla|} f e^{it|\nabla|} g \right\|_{L^\infty_t L^1_x} \lesssim \|f\|_{L^2} \|g\|_{L^2}$$

  and thus the case $p = \infty$ is always true.

Bilinear Extension Estimates

Can do better by exploiting the curvature of the cone. More precisely, the Strichartz estimate

$$\|e^{it|\nabla|}f\|_{L^{2\frac{n+1}{n-1}}_{t,x}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2_x}$$

implies that, after an application of Holder,

$$\begin{aligned}\big\|e^{it|\nabla|}f e^{it|\nabla|}g\big\|_{L^{\frac{n+1}{n-1}}_{t,x}} &\lesssim \|e^{it|\nabla|}f\|_{L^{2\frac{n+1}{n-1}}_{t,x}} \|e^{it|\nabla|}g\|_{L^{2\frac{n+1}{n-1}}_{t,x}} \\ &\lesssim \|f\|_{L^2_x}\|g\|_{L^2_x}.\end{aligned}$$

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Hence

curvature $\Longrightarrow$ bilinear extension estimate for $\dfrac{n+1}{n-1} \leqslant p \leqslant \infty.$

In general range is sharp (just take $f = g$ and use fact that linear Strichartz is sharp).

Bilinear Extension Estimates

Alternative approach is to exploit transversality. For example, we have

Theorem

Assume that $\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}$, $\operatorname{supp} \widehat{g} \subset \{|\xi + e_1| \ll 1\}$. *Then*

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- Proof follows by a change of variables together with Plancheral and Cauchy-Schwartz.
- Version is true for general phases $e^{it\Phi_1(\nabla)} f$, $e^{it\Phi_2(\nabla)} g$ under the transversality assumption

$$|\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| \gtrsim 1$$

for $\xi \in \operatorname{supp} \widehat{f}$, $\eta \in \operatorname{supp} \widehat{g}$.

Bilinear Extension Estimates

$$\big\|e^{it|\nabla|}f e^{it|\nabla|}g\big\|_{L^p_{t,x}(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

To summarise

$$\text{curvature} \implies \frac{n+1}{n-1} \leqslant p \leqslant \infty$$

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- The fully transverse case $p = 2$ is a bilinear estimate, and does not need any curvature. In particular is true for hyperplanes (i.e. solns to transport equation).

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- Can improve range of p by exploiting both Transversality and Curvature. First progress below $p = 2$ due to Bourgain '91.

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- The fully transverse case $p = 2$ is a bilinear estimate, and does not need any curvature. In particular is true for hyperplanes (i.e. solns to transport equation).
- Can improve range of p by exploiting both Transversality and Curvature. First progress below $p = 2$ due to Bourgain '91.
- Conjecture of Klainerman-Machedon: Under suitable transversality and curvature assumptions, the bilinear extension estimate holds for $p > \frac{n+3}{n+1}$.

Bilinear Extension Estimates

Theorem (Wolff '01)

Let $\frac{n+3}{n+1} < p \leqslant \infty$ *and assume*

$$\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp} \widehat{g} \subset \{|\xi + e_1| \ll 1\}.$$

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- Although the above was stated for the cone, it is also true for general surfaces under appropriate Curvature and Transversality assumptions. A precise statement will be given later.

Applications

- Linear Restriction.
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  Bilinear extension estimates originally devised to improve the range of linear restriction estimates (i.e. Bourgain, Tao-Vargas-Vega).

- Improved Strichartz Estimates.
  For instance we have the following estimate due to J. Ramos '12

$$\|e^{it|\nabla|}f\|_{L^{2\frac{n+1}{n-1}}_{t,x}} \lesssim \|f\|_{\dot{B}^{\frac{1}{2}}_{2,q}}$$

  with $q = 2\frac{n+1}{n-1}$ ($q = 2$ corresponds to standard Strichartz bound). Stronger versions of this estimate (also due to J. Ramos'12) play a key role in the profile decomposition for the linear wave equation.

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For instance, for waves $u = e^{it|\nabla|} f$, $v = e^{it|\nabla|} g$ we have

$$\| \partial_t u \partial_t v - \nabla u \cdot \nabla v \|_{L^q_{t,x}} \lesssim \| f \|_{\dot{H}^s} \| g \|_{\dot{H}^s}$$

for $q > \frac{n+3}{n+1}$ and $s = \frac{n+2}{2} - \frac{n+1}{q}$ (see Lee-Vargas '08).

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  for $q > \frac{n+3}{n+1}$ and $s = \frac{n+2}{2} - \frac{n+1}{q}$ (see Lee-Vargas '08).

  The null form $Q_0(u, v) = \partial_t u \partial_v - \nabla u \cdot \nabla v$ is a substitute for the lack of transversality, in particular, this estimate fails for a general bilinear form like $|\nabla u \cdot \nabla v|$.

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and
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- Implied constant independent of $\mathcal{I}$, $\mathcal{J}$, (so the number of intervals plays no role) and the families $(f_I)_{I\in\mathcal{I}}$, $(g_J)_{J\in\mathcal{J}}$.

Observations

$$\Big\| \Big(\sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{it|\nabla|} f_I \Big) \Big(\sum_J \mathbb{1}_J(t) e^{it|\nabla|} g_J \Big) \Big\|_{L^p_{t,x}} \lesssim \Big(\sum_I \|f_I\|^2_{L^2_x} \Big)^{\frac{1}{2}} \Big(\sum_J \|g_J\|^2_{L^2_x} \Big)^{\frac{1}{2}}$$

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- The function $u = \sum_I \mathbb{1}_I(t) e^{it|\nabla|} f_I$ is known as a (rescaled U^2) atom, or alternatively as an ℓ^2 family of free solutions. Thus we are asking:

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- Bilinear restriction for ℓ^2 families $\Longrightarrow$ bilinear restriction for homogeneous solutions
  (just take $\mathcal{I} = \{\mathbb{R}\}$, $\mathcal{J} = \{\mathbb{R}\}$ to be the trivial partitions).

Observations

- Bilinear restriction for free solutions $\Longrightarrow$ bilinear restriction for ℓ^p families, since

$$\begin{aligned}\Big\| \Big(\sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{it|\nabla|} f_I \Big) \Big(\sum_{J \in \mathcal{J}} \mathbb{1}_J(t) e^{it|\nabla|} g_J \Big) \Big\|_{L^p_{t,x}} \\ &= \Big(\sum_{I,J} \| e^{it|\nabla|} f_I e^{it|\nabla|} g_J \|^p_{L^p_{t,x}(I \cap J \times \mathbb{R}^n)} \Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{I,J} \| e^{it|\nabla|} f_I \|^p_{L^2_x} \| e^{it|\nabla|} g_J \|^p_{L^2_x} \Big)^{\frac{1}{p}} \\ &= \Big(\sum_{I \in \mathcal{I}} \| f_I \|^p_{L^2_x} \Big)^{\frac{1}{p}} \Big(\sum_{J \in \mathcal{J}} \| g_J \|^p_{L^2_x} \Big)^{\frac{1}{p}}\end{aligned}$$

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$$\lesssim \Big(\sum_{I,J} \| e^{it|\nabla|} f_I \|^p_{L^2_x} \| e^{it|\nabla|} g_J \|^p_{L^2_x} \Big)^{\frac{1}{p}}$$

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- In the interesting region $p < 2$, this is weaker than the estimate we want, since $\ell^p \subset \ell^2$!

Bilinear Restriction for ℓ^2 families

Theorem (C.-Herr'16, wave version)

Let $p > \frac{n+3}{n+1}$. *Let* $u = \sum_I \mathbb{1}_I(t) e^{it|\nabla|} f_I$, $v = \sum_J \mathbb{1}_J(t) e^{it|\nabla|} g_J$ *be* ℓ^2 *families with*

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- For $n = 2$ the range $p > \frac{13}{7}$ was obtained in Sterbenz-Tataru'10 via the homogeneous estimate and an interpolation argument.
- proof follows argument of Tao'01, Lee-Vargas'10 (does not follow from homogeneous case).

General Version: Assumptions

We assume we have phases $\Phi_j : \Lambda_j \to \mathbb{R}$ satisfying, for some constants $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, N

1. (Transversality) For all $\xi \in \Lambda_1$, $\eta \in \Lambda_2$ we have

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2. (Curvature) Let $\Sigma_j(a,h) = \{\xi \in \Lambda_j \cap (\Lambda_k + h) | \Phi_j(\xi) = \Phi_k(\xi - h) + a\}$. Then for all $(a,h) \in \mathbb{R}^{1+n}$, $\xi, \xi' \in \Sigma_j(a,h)$, and $\eta \in \Lambda_k$ we have

$$|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \wedge (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta))| \geqslant \mathbf{C}_2|\xi - \xi'|.$$

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3. (Regularity) $\Phi_j \in C^N(\Lambda_j)$ and
$$\sup_{|\kappa|\leqslant N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leqslant \mathbf{C}_3.$$

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$$|(\nabla\Phi_j(\xi) - \nabla\Phi_j(\xi')) \wedge (\nabla\Phi_j(\xi) - \nabla\Phi_k(\eta))| \geqslant \mathbf{C}_2|\xi - \xi'|.$$
3. (Regularity) $\Phi_j \in C^N(\Lambda_j)$ and
$$\sup_{|\kappa| \leqslant N} \|\partial^\kappa \Phi_j\|_{L^\infty(\Lambda_j)} \leqslant \mathbf{C}_3.$$

- Conditions on phases are based on assumptions used in Lee-Vargas'10, Bejenaru'16.

General Version

Theorem (C.-Herr'16)

Let $p > \frac{n+3}{n+1}$. Assume that the phases Φ_1 and Φ_2 satisfy the transversality, curvature, and regularity assumptions.
Let $u = \sum_I \mathbb{1}_I(t) e^{it\Phi_1(\nabla)} f_I$, $v = \sum_J \mathbb{1}_J(t) e^{it\Phi_2(\nabla)} g_J$ be ℓ^2 families with

$$\operatorname{supp} \widehat{u} \subset \Lambda_1, \qquad \operatorname{supp} \widehat{v} \subset \Lambda_2.$$

Then

$$\|uv\|_{L^p} \lesssim \Big(\sum_I \|f_I\|_{L^2_x}^2 \Big)^{\frac{1}{2}} \Big(\sum_J \|g_J\|_{L^2_x}^2 \Big)^{\frac{1}{2}}$$

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- Case of homogeneous solutions: Lee-Vargas'10, Bejenaru'16.
- $\Phi_j = (m_j^2 + |\xi|^2)^{\frac{1}{2}}$ and $\Lambda_1 = \{|\xi - e_1| \ll 1\}$, $\Lambda_2 = \{|\xi + e_1| \ll 1\}$ satisfies conditions with constant independent of the masses m_1, m_2.

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Transference Principle

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- Existence for dispersive PDE requires careful choice of Banach space $X \subset C(I, H^s)$. Should think of X as containing pertubations of free solutions (say for the wave equation $e^{it|\nabla|}f$).

- Need to prove estimates for functions in X. For instance, may want to prove

$$\|uv\|_{L^p_{t,x}} \lesssim \|u\|_X \|v\|_X.$$

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- The Transference Principle states that it is enough to prove estimates for homogeneous solutions. Thus

$$\begin{aligned} \|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^p_{t,x}} &\lesssim \|f\|_{L^2_x} \|g\|_{L^2_x} \\ &\implies \|uv\|_{L^p_{t,x}} \lesssim \|u\|_X \|v\|_X \end{aligned}$$

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- Typical example is $X = X^{s,b}$, point is that can write elements of $X^{s,b}$ as averages of free solutions, namely,

$$u(t,x) = \int_{\mathbb{R}} e^{it\tau}e^{it|\nabla|}f_\tau d\tau$$

with $\int_{\mathbb{R}} \|f_\tau\|_{L^2}d\tau \lesssim \|u\|_{X^{0,b}}$.

Weak Transference Principle

Important endpoint spaces do not satisfy transference principle. Instead we only have a weaker variant.

- X satisfies the weak transference principle if

$$\Big\| \Big(\sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{it|\nabla|} f_I \Big) \Big(\sum_J \mathbb{1}_J(t) e^{it|\nabla|} g_J \Big) \Big\|_{L^p_{t,x}} \lesssim \Big(\sum_I \|f_I\|^2_{L^2_x} \Big)^{\frac{1}{2}} \Big(\sum_J \|g_J\|^2_{L^2_x} \Big)^{\frac{1}{2}}$$

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- Typical examples of function spaces satisfying weak transference but not transference include U^2, V^2, and null frame type spaces.
- Main result can then be restated as:

  If X satisfies weak transference $\Longrightarrow$ Bilinear restriction estimates hold in X

Applications to DKG system

Dirac-Klein-Gordon system for a spinor $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ coupled with a scalar field $\phi : \mathbb{R}^{1+3} \to \mathbb{R}$ is given by

$$\left.\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi \\ \Box\phi + m^2\phi &= \overline{\psi}\psi \end{aligned}\right\} \qquad (DKG)$$

- We use summation convention, $\overline{\psi} = \psi^\dagger \gamma^0$, $\psi^\dagger$ is conjugate transpose, $\Box = \partial^2 - \Delta$ is the wave operator, and $M, m > 0$.

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- Classical model in relativistic quantum mechanics
- The Dirac matrices $\gamma^\mu \in \mathbb{C}^{4\times 4}$ are given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

with the Pauli matrices σ^j given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Scattering and GWP for DKG

We consider the Cauchy problem for (DKG) with data

$$\psi(0) = \psi_0 : \mathbb{R}^3 \to \mathbb{C}^4, \qquad \big(\phi(0), \partial_t \phi(0)\big) = (\phi_0, \phi_1) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}.$$

Let $\langle \Sigma \rangle^\sigma$ denote σ spherical derivatives ($\Sigma_{ji} = x_j \partial_i - x_j \partial_j$).

Theorem (C.-Herr'16)

Suppose that $2M \geqslant m > 0$ *and* $\sigma > 0$ *or* $m > 2M > 0$ *and* $\sigma > \frac{7}{30}$. *There exists* $\delta > 0$ *such that if*

$$\| \langle \Sigma \rangle^\sigma \psi_0 \|_{L^2_x} + \| \langle \Sigma \rangle^\sigma (\phi_0, \phi_1) \|_{H^{\frac{1}{2}} \times H^{-\frac{1}{2}}} \leqslant \delta$$

then Cauchy problem is globally well-posed, and moreover the solution scatters to free solutions as $t \to \pm\infty$.

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- First GWP and scattering result in the resonant case $m > 2M$.

Previous Results

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- GWP and scattering in nonresonant case $2M > m$: subcritical Bejenaru-Herr '14, endpoint Besov case with angular regularity Wang'15
- In the case $n = 1$ related results can be found in Machihara'07, Machihara-Nakanishi-Tsugawa'10, C.'13...

Sketch of Proof

- Diagonlisation of Dirac operator: $\psi = \psi_+ + \psi_-$ where

$$\Pi_\pm = \frac{1}{2}\Big(I + \frac{1}{\langle \xi \rangle_M}\big(\xi_j \gamma^0 \gamma^j + M\gamma^0\big)\Big), \qquad \psi_\pm = \Pi_\pm(\nabla)\psi.$$

(and $\langle \xi \rangle_M = (M^2 + |\xi|^2)^{\frac{1}{2}}$) DKG system then equivalent to

$$(-i\partial_t \pm \langle \nabla \rangle_M)\psi_\pm = \Pi_\pm(\nabla)\big(\Re(\phi_+)\gamma^0\psi\big)$$
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- Duhamel Formula, duality: problem reduces to proving estimates for

$$\int_{\mathbb{R}^{1+3}} \phi_+ \overline{\psi_\pm^{(1)}} \psi_\pm^{(2)} dt dx$$

  key role play by the resonance function

$$r_{m,M} = \big|\langle \xi - \eta \rangle_m \mp_1 \langle \xi \rangle_M \pm_2 \langle \eta \rangle_M\big|$$

  (measures how far ϕ, $\psi^{(1)}$ and $\psi^{(2)}$ are from free solutions).

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- If $2M > m$ then get lower bound on $r_{m,M} \Longrightarrow$ problem nonresonant.
- If $2M < m$ then no lower bound and no null structure $\Longrightarrow$ seems to cause difficulties in closing argument
- But in resonant case, have transversality when $r_{m,M} = 0$!!! Thus can apply bilinear restriction estimate + spherical Strichartz $\Longrightarrow$ gives result.

Summary

- Obtain (sharp) bilinear restriction estimates for families of ℓ^2 solutions

$$\Big\| \Big(\sum_{I \in \mathcal{I}} \mathbb{1}_I(t) e^{it|\nabla|} f_I \Big) \Big(\sum_J \mathbb{1}_J(t) e^{it|\nabla|} g_J \Big) \Big\|_{L^p_{t,x}} \lesssim \Big(\sum_I \|f_I\|_{L^2_x}^2 \Big)^{\frac{1}{2}} \Big(\sum_J \|g_J\|_{L^2_x}^2 \Big)^{\frac{1}{2}}.$$

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Thank you for listening!!