Average decay estimates for Fourier transforms of measures

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Introduction

- $\mathfrak{M}(A)$: a family of nonnegative Borel regular measures with compact support in A.
- Frostman's Lemma: For a Borel set $A \subset \mathbb{R}^d$,
  $\dim A = \sup\{\alpha \in [0,d] : \exists \mu \in \mathfrak{M}(A)$ such that
  $$\mu(B(x,r)) \le r^\alpha, \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0\}. \tag{1}$$
  (If $A \subset \mathbb{S}^{d-1}$, then (1) holds for the spherical measure μ with $\alpha = d-1$.)
- For $0 < \alpha < d$, the α-dimensional energy of μ is defined by
  $$I_\alpha(\mu) = \iint |x-y|^{-\alpha} d\mu(y) d\mu(x). \tag{2}$$
- Relation between (1) and (2):
  - $\int |x-y|^{-\alpha} d\mu(y) = \alpha \int_0^\infty \mu(B(x,r)) r^{-\alpha-1} dr$,
  - (1) implies $I_\beta(\mu) < \infty$ for $0 < \beta < \alpha$.
  - If $I_\alpha(\mu)$ is finite, there exists ν satisfying (1) such that $\nu(X) \le 2\mu(X)$ for any Borel set X.
- By this,
  $$\dim A = \sup\{\alpha \in [0,d] : \exists \mu \in \mathfrak{M}(A) \text{ such that } I_\alpha(\mu) < \infty\}.$$

For $0 < \alpha < d$, we have the identity

$$I_\alpha(\mu) = \iint |x-y|^{-\alpha} d\mu(x) d\mu(y) = C_{\alpha,d} \int |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi.$$

L^2 average over balls

- If $I_\alpha(\mu) < \infty$ and $\lambda \geq 1$, then

$$\int_{B(0,1)} |\widehat{\mu}(\lambda\xi)|^2 d\xi \lesssim \lambda^{-\alpha} I_\alpha(\mu).$$

- If $B(0,1)$ is replaced by a smooth submanifold of lower dimension (e.g. sphere, cone, curve, etc.), it is expected that the decay rate gets worse.
- **The curvature property** of the underlying submanifolds becomes important.

Theorem (Erdoğan, 2004)

Let $I_\alpha(\mu) < \infty$. Let Σ be a smooth compact submanifold with measure $d\nu$ such that

$$|\widehat{\nu}(\xi)| \lesssim |\xi|^{-a}, \quad \nu(B(x,\rho)) \lesssim \rho^b.$$

Then there exists a constant $C > 0$ such that

$$\int_\Sigma |\widehat{\mu}(\lambda\xi)|^2 d\nu(\xi) \leq C\lambda^{-\zeta(\alpha)} I_\alpha(\mu)$$

for $\zeta(\alpha) = \max(\min(\alpha, a), \alpha - d + b)$.

L^2 spherical average

If $I_\alpha(\mu) < \infty$, then what is the optimal decay rate ζ such that

$$\int_{\mathbb{S}^{d-1}} |\widehat{\mu}(\lambda\xi)|^2 d\nu(\xi) \leq C\lambda^{-\zeta} I_\alpha(\mu)?$$

- $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-(d-1)/2}$, $\nu(B(x,\rho)) \lesssim \rho^{d-1}$.
  By the above theorem, $\zeta = \max(\min(\alpha, (d-1)/2), \alpha - 1)$.
- This estimate has been studied extensively after P. Mattila's contribution (1987) to Falconer distance set problem.
  (See P. Sjölin, J. Bourgain, T. Wolff, B. Erdoğan, R. Lucà–K. Rogers, B. Shayya...)
- Known results: The L^2 spherical average holds with decay rate

$$\zeta \geq \begin{cases} \min(\alpha, \frac{d-1}{2}), & \text{if } 0 < \alpha \leq \frac{d}{2}, \text{ (Mattila)} \\ \alpha - 1 + \frac{d+2-2\alpha}{4}, & \text{if } \frac{d}{2} < \alpha \leq \frac{d+2}{2}, \text{ (Erdoğan)} \\ \alpha - 1, & \text{if } \frac{d+2}{2} \leq \alpha \leq d. \text{ (Sjölin)} \end{cases}$$

For $d \geq 3$,

$$\zeta \geq \alpha - 1 + \frac{(d-\alpha)^2}{(d-1)(2d-\alpha-1)}. \text{ (Lucà–Rogers)}$$

Remarks

- Their improvements were based on sophisticated method, such as bilinear and multilinear estimate with induction on scale, polynomial partitioning, which were developed in the study of the Fourier restriction problem (and Bochner-Riesz conjecture).
- $\mathbb{S}^{d-1}$ can be replaced by any smooth submanifold with nonvanishing curvature. (Sjölin, 1997)
- For $d = 2$, the sharp average estimate is known.

L^2 circular average

For $0 < \alpha < 2$,

$$\int_{-\pi}^{\pi} |\widehat{\mu}(\lambda e^{i\theta})|^2 d\theta \leq C\lambda^{-\zeta} I_\alpha(\mu). \tag{3}$$

- necessary condition : $\zeta \leq \max(\min(\alpha, 1/2), \alpha/2)$.
- Mattila (1987), Sjölin (1997) : $\zeta \leq \max(\min(\alpha, 1/2), \alpha - 1)$.
- Wolff (1999) : $\zeta < \alpha/2$.
- Erdoğan–Oberlin (2013) : the same result as Wolff's, for a certain class of general curves in $\mathbb{R}^2$.

Two observations by Wolff (1999) to obtain (3):

1. *we may assume that* $\mu(B(x,r)) \le C_\mu\, r^\alpha$ *for all* $x \in \mathbb{R}^d$ *and* $r > 0$ *instead of* $I_\alpha(\mu) < \infty$*;*
   If $I_\alpha(\mu) < \infty$ for a positive Borel measure supported in $B(0,1)$, then $\mu = \sum_{1 \le j \le O(\log R)} \mu_j$ for some $R > 1$ and each μ_j satisfies

$$\mu_j(\mathbb{R}^d) \sup_{(x,r) \in \mathbb{R}^d \times [R^{-1}, \infty)} r^{-\alpha} \mu_j(B(x,r)) \lesssim I_\alpha(\mu).$$

2. *assuming* $I_\alpha(\mu) = 1$, *it suffices to show*

$$\left| \int \widehat{g}(x) d\mu(x) \right| \leq C\lambda^{\kappa} \|g\|_{L^2},$$

where $\operatorname{supp} g \subset \lambda\mathbb{S}^1 + O(1)$. Here $\kappa = (1-\zeta)/2$.

$$\boxed{\left|\int \widehat{g}(x) d\mu(x)\right| \leq C\lambda^{\kappa}\|g\|_{L^2}}$$

$$\Downarrow$$

$$\boxed{\int |\widehat{\mu}(\lambda e^{it})|^2 dt \lesssim \lambda^{-(1-2\kappa)}}$$

- Wolff obtained the sharp estimates using a refinement of the two dimensional Kakeya maximal theorem.

[Sketch of proof of 2]

By duality, we have

$$\int_{\lambda\mathbb{S}^1+O(1)} |\widehat{\mu}(\xi)|^2 d\xi = \sup_{\|g\|_{L^2}\le 1} \left| \int_{\lambda\mathbb{S}^1+O(1)} \widehat{\mu}(\xi) g(\xi) d\xi \right| \lesssim \lambda^{2\kappa}.$$

Let ψ be a Schwartz function which is equal to 1 on the support of μ. Then by rapid decay of $\widehat{\psi}$,

$$\begin{aligned}
\int |\widehat{\mu}(\lambda e^{it})|^2 dt &= \int |\widehat{\psi} * \widehat{\mu}(\lambda e^{it})|^2 dt \\
&\lesssim \int_{\mathbb{R}^d} \int |\widehat{\psi}(\lambda e^{it} - \xi)| dt |\widehat{\mu}(\xi)|^2 d\xi \\
&\lesssim \frac{1}{\lambda} \int \frac{|\widehat{\mu}(\xi)|^2}{(1+\mathrm{dist}(\lambda\mathbb{S}^1, \xi))^N} d\xi \\
&\lesssim \frac{1}{\lambda} \left(\int_{\lambda\mathbb{S}^1+O(1)} + \sum_{j=1}^{\infty} 2^{-Nj} \int_{\lambda\mathbb{S}^1+O(2^j)} |\widehat{\mu}(\xi)|^2 d\xi \right) \\
&\lesssim \lambda^{2\kappa-1}.
\end{aligned}$$

- Since μ is a finite measure,

$$\left|\int \widehat{g}(x)d\mu(x)\right| \leq \|\mu\|^{\frac{1}{q'}} \|\widehat{g}\|_{L^q(d\mu)}$$

for any $q \geq 1$.
- It suffices to obtain $\kappa = \kappa(q)$ for which

$$\|\widehat{g}\|_{L^q(d\mu)} \leq C\lambda^{\kappa}\|g\|_{L^2}$$

holds for some $C > 0$.

$$\boxed{\|\widehat{g}\|_{L^q(d\mu)} \leq C\lambda^{\kappa}\|g\|_{L^2}}$$

$$\Downarrow$$

$$\boxed{\int |\widehat{\mu}(\lambda e^{it})|^2 dt \lesssim \lambda^{-(1-2\kappa)}}$$

Question : Let μ be an α-dimensional measure. For $\lambda \geq 1$, find optimal rate κ such that

$$\boxed{\|\widehat{g}\|_{L^q(d\mu)} \lesssim \lambda^{\kappa}\|g\|_{L^2},}$$

where g is supported in a $O(1)$-neighborhood of a hypersurface.
For example,

$$\begin{cases} \lambda\mathbb{S}^1 \\ \Gamma_\lambda = \{(x,t) : |x| = t, \lambda \leq t \leq 2\lambda\} \subset \mathbb{R}^{2+1} \end{cases}$$

Sharp estimates for those cases were proved by Erdoğan (2004).

In this talk, we will consider a generalization of them:

$$\boxed{\lambda\mathbb{S}^1} \dashrightarrow \boxed{\lambda\gamma(t) \text{ in } \mathbb{R}^d}$$

$$\boxed{\Gamma_\lambda \text{ in } \mathbb{R}^{2+1}} \dashrightarrow \boxed{\Gamma_\lambda \text{ in } \mathbb{R}^{d+1}}$$

Theorem (Erdoğan, 2004)

- *Let* $\alpha \in (1,2)$*, and* $\lambda \geq 1$*.*
- $g \in L^2(\mathbb{R}^2)$ *is supported in*

$$\lambda\,\mathbb{S}^1 + O(1) = \{x \in \mathbb{R}^2 : \lambda - 1 < |x| < \lambda + 1\}.$$

- $\kappa(\alpha, q) = \max\left(\frac{1}{2} - \frac{\alpha}{4},\, \frac{1}{4} + \frac{1-\alpha}{2q},\, \frac{1}{2} - \frac{\alpha}{q}\right).$

For each $q \geq 1$ *and for any* $\epsilon > 0$*, there exists a constant* $C > 0$ *such that*

$$\|\widehat{g}\|_{L^q(d\mu)} \leq C\, C_\mu^{\frac{1}{q}}\, \lambda^{\kappa(\alpha,q)+\epsilon} \|g\|_{L^2(\mathbb{R}^2)}.$$

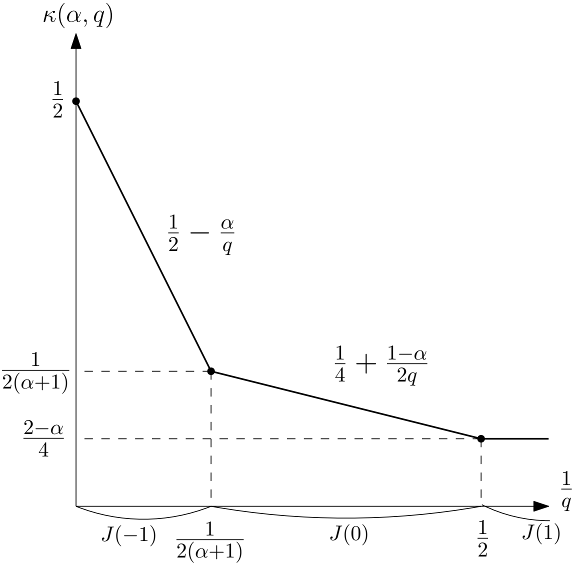


Figure: $d = 2$, $1 < \alpha < 2$.

Main results: space curves

Now we consider

$$\gamma(t) = \left(t, \frac{t^2}{2!}, \frac{t^3}{3!}\right), t \in I = [0,1].$$

(Then $\det(\gamma'(t), \gamma''(t), \gamma'''(t)) = 1$ for all $t \in I$.)

Suppose that μ is supported in $B(0,1)$ and satisfies the growth condition

$$\mu(B(x,r)) \le C_\mu r^\alpha \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0.$$

If $\operatorname{supp} g \subset \lambda\gamma(t) + O(1)$, what is the optimal rate κ such that

$$\|\widehat{g}\|_{L^q(d\mu)} \le C_\mu \lambda^\kappa \|g\|_{L^2} \ ?$$

Let us define some quantities. For $K = 1, 2, \dots, d$ and $0 < \alpha \leq K$, we set

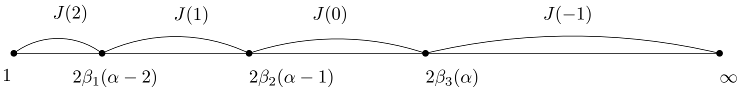
$$\beta_K(\alpha) = ([K-\alpha]+1)\alpha + \frac{(K-1-[K-\alpha])(K-[K-\alpha])}{2}.$$

Here $[x]$ denotes the integer part of x.
For example,

$$\begin{cases} \beta_3(3) = 1+2+3, & \text{if } \alpha = 3, \\ \beta_3(\alpha) = 1+2+\alpha, & \text{if } 2 < \alpha \leq 3, \\ \beta_3(\alpha) = 1+\alpha+\alpha, & \text{if } 1 < \alpha \leq 2, \\ \beta_3(\alpha) = \alpha+\alpha+\alpha, & \text{if } 0 < \alpha \leq 1. \end{cases}$$

Note that $\beta_K(\alpha)$ generalizes the number $\beta_d(d) = d(d+1)/2$ which appears in the studies on Fourier restriction estimates for space curves.

Using this, we define intervals:
if $2 < \alpha \leq 3$, then



Here, $\beta_1(\alpha-2) = \alpha - 2$, $\beta_2(\alpha-1) = 1 + \alpha - 1 = \alpha$, and $\beta_3(\alpha) = 1 + 2 + \alpha = 3 + \alpha$.

If $1 < \alpha \leq 2$, then there are 3 intervals: $J(\ell)$, $\ell = -1, 0, 1$.

Necessary conditions for $2 < \alpha \leq 3$:

$$\kappa \geq \begin{cases} \frac{1}{2} - \frac{\alpha}{q}, & \text{if } \; q \in J(-1), \\ \frac{1}{2} - \frac{\alpha}{q} + \frac{1}{3}\left(\frac{\beta_3(\alpha)}{q} - \frac{1}{2}\right), & \text{if } \; q \in J(0), \\ \frac{1}{2} - \frac{\alpha-1}{q} + \frac{1}{2}\left(\frac{\beta_2(\alpha-1)}{q} - \frac{1}{2}\right), & \text{if } \; q \in J(1), \\ \frac{1}{2} - \frac{\alpha-2}{q} + \left(\frac{\beta_1(\alpha-2)}{q} - \frac{1}{2}\right), & \text{if } \; q \in J(2). \end{cases}$$

$$\kappa \geq (3-\alpha)/4, \quad \text{for any } \; q \geq 1.$$

Theorem (Choi-H-Lee)

Let $2 < \alpha \leq 3$. The necessary conditions above are also sufficient conditions for

$$\|\widehat{g}\|_{L^q(d\mu)} \leq C_\mu \lambda^\kappa \|g\|_{L^2}.$$

(Except for the endpoint.)

In general, if $d - 1 < \alpha \leq d$ i.e. $[d - \alpha] = 0$, then we obtain the sharp estimate.

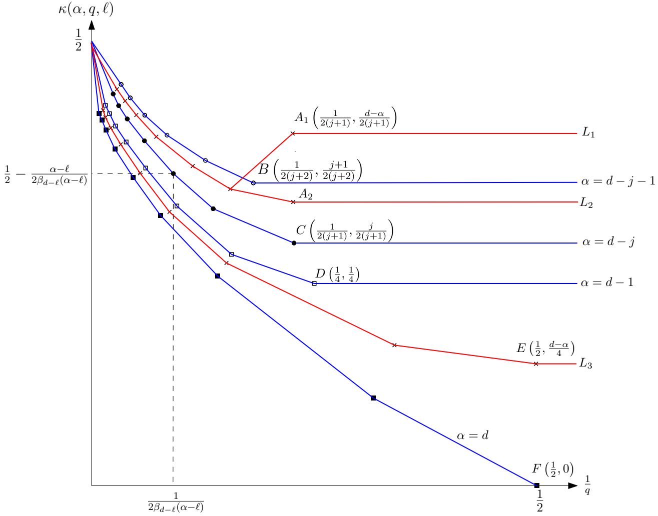
However, as α decreases, the estimate is sharp only for large q. (For small $q(\geq 2)$, the esitmate gets worse.)

For each $\ell = -1, 0, 1, \ldots, d-1-[d-\alpha]$ and $q \in J(\ell)$, set

$$\kappa(\alpha, q, \ell) = \begin{cases} \frac{1}{2} - \frac{\alpha}{q}, & \text{if } \ \ell = -1, \\ \frac{1}{2} - \frac{\alpha-\ell}{q} + \frac{1}{d-\ell}\left(\frac{\beta_{d-\ell}(\alpha-\ell)}{q} - \frac{1}{2}\right), & \text{if } \ 0 \le \ell \le d-3-[d-\alpha], \\ \frac{1}{2} - \frac{\alpha-\ell}{q} + \frac{1}{\mathfrak{J}_\ell}\left(\frac{\beta_{d-\ell}(\alpha-\ell)}{q} - \frac{1}{2}\right), & \text{if } \ \ell = d-2-[d-\alpha], \\ \min\left(\frac{d-\alpha}{4}, \frac{d-\alpha}{2([d-\alpha]+1)}\right), & \text{if } \ \ell = d-1-[d-\alpha], \end{cases}$$

where $\mathfrak{J}_\ell = \begin{cases} d-\ell = 2, & \text{if } [d-\alpha] = 0, \\ |J(d-2-[d-\alpha])|/2 & \text{if } [d-\alpha] \ge 1. \end{cases}$

Here $|J(\ell)|$ denotes the length of $J(\ell)$.



For integer α, $\kappa(\alpha,q,\ell)$ decreases.

If $d-j-1<\alpha<d-j$ and $j=[d-\alpha]\geq 1$, then $\kappa(\alpha,q,\ell)$ may increase.

Main results (Choi-H-Lee)

Let $\gamma : I = [0,1] \to \mathbb{R}^d$ be of a C^{d+1} curve satisfying

$$\det(\gamma'(t), \gamma''(t), \cdots, \gamma^{(d)}(t)) \neq 0 \text{ for } t \in I.$$

Theorem

For any $\epsilon > 0$ and for $q \in J(\ell), \ell = -1, 0, \ldots, d-1-[d-\alpha]$,

$$\|\widehat{g}\|_{L^q(d\mu)} \leq C_\mu \lambda^{\kappa(\alpha,q,\ell)+\epsilon} \|g\|_{L^2}.$$

Theorem

Suppose $0 < \alpha < d$ and $I_\alpha(\mu) = 1$, then for $\lambda > 1$ there exists a constant $C > 0$ such that

$$\int_0^1 |\widehat{\mu}(\lambda\gamma(t))|^2 dt \leq C \lambda^{-(1-2\kappa(\alpha,q,\ell))+2\epsilon}.$$

Sketch of proof

Using the bilinear argument due to Erdoğan (or Wolff), we obtain

Theorem

Suppose that $d \geq 2$ and $d-1 \leq \alpha \leq d$. Let γ, μ, and g be given as above. Then, for $\lambda > 1$, $q \geq 2$ and $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\|\widehat{g}\|_{L^q(d\mu)} \leq C_\mu \, \lambda^{\kappa+\epsilon} \|g\|_{L^2}$$

for $\kappa = \max(\frac{1}{4} + \frac{d-\alpha-1}{2q}, \frac{1}{2} + \frac{d-\alpha-2}{q})$.

This gives sharp result $\frac{1}{4} + \frac{d-\alpha-1}{2q}$ for $q \in [2, 2(\alpha - d + 3)] = J(d-2)$, while $\frac{1}{2} + \frac{d-\alpha-2}{q}$ for $q \geq 2(\alpha - d + 3)$ is not sharp.

To obtain a better result for large q, we make use of **induction on scale argument with multilinear estimates**.

Oscillatory integral operators

For $\lambda \geq 1$ let us consider an oscillatory integral operator defined by

$$\mathcal{E}_\lambda^\gamma f(x) = a(x) \int_I e^{i\lambda x \cdot \gamma(t)} f(t)dt,$$

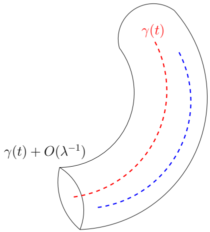
where a is a bounded function supported in $B(0,1)$ and $\|a\|_\infty \leq 1$.

- This is an adjoint form of the Fourier restriction to the curve $\lambda\gamma$.
- $\lambda\gamma(I) + O(1)$ can be foliated into a set of $O(1)$-translations of the curve $\lambda\gamma$.

$$\boxed{\|\mathcal{E}_\lambda^\gamma f\|_{L^q(d\mu)} \lesssim \lambda^{-\vartheta} \|f\|_{L^2(I)}}$$

$$\Updownarrow$$

$$\boxed{\|\widehat{g}\|_{L^q(d\mu)} \leq C\lambda^{\frac{1}{2}-\vartheta} \|g\|_{L^2(\mathbb{R}^d)}}$$



Theorem (Lee–H, 2014)

Let $0 < \alpha \le d$ and $\lambda \ge 1$. Let $\gamma \in C^{d+1}$ and μ be given as above. Then

$$\|\mathcal{E}_\lambda^\gamma f\|_{L^q(d\mu)} \lesssim \lambda^{-\frac{\alpha}{q}} \|f\|_{L^p(I)}$$

holds for $1 \le p, q \le \infty$ satisfying $d/q \le 1 - 1/p$, $q \ge 2d$ and

$$\frac{\beta_d(\alpha)}{q} + \frac{1}{p} < 1, \quad q > \beta_d(\alpha) + 1.$$

Immediately it follows that

$$\|\widehat{g}\|_{L^q(d\mu)} \le C\lambda^{\frac{1}{2} - \frac{\alpha}{q}} \|g\|_{L^2(\mathbb{R}^d)}$$

for $q > \max(2\beta_d(\alpha), 2d)$.

Lemma

Let $\gamma : I \to \mathbb{R}^d$ *be a smooth curve. Let* $A_0, A_1, \ldots, A_{d-1}$*, and* $\{\mathcal{I}^i\}$*,* $i = 1, \ldots, d-1$ *be defined as in the above. Then, for any* $x \in \mathbb{R}^d$*, there is a constant* C*, independent of* γ, x*,* $A_0, A_1, \ldots, A_{d-1}$*, such that*

$$\begin{aligned} |\mathcal{E}_\lambda^\gamma f(x)| \le C \sum_{i=1}^{d-1} A_{i-1}^{-2(i-1)} \max_{\mathcal{I}^i} |\mathcal{E}_\lambda^\gamma f_{\mathcal{I}^i}(x)| \\ + C A_{d-1}^{-2(d-1)} \max_{\substack{\mathcal{I}_1^{d-1}, \mathcal{I}_2^{d-1}, \ldots, \mathcal{I}_d^{d-1}; \\ \Delta(\mathcal{I}_1^{d-1}, \mathcal{I}_2^{d-1}, \ldots, \mathcal{I}_d^{d-1}) \ge A_{d-1}}} \left| \prod_{i=1}^{d} \mathcal{E}_\lambda^\gamma f_{\mathcal{I}_i^{d-1}}(x) \right|^{\frac{1}{d}}. \end{aligned} \tag{4}$$

Here $\mathcal{I}_i^j$ *denotes the element in* $\{\mathcal{I}^i\}$ *and* $\Delta(\mathcal{I}_1^{d-1}, \ldots, \mathcal{I}_d^{d-1}) = \min_{1 \le j < m \le d} \mathrm{dist}(\mathcal{I}_j^{d-1}, \mathcal{I}_m^{d-1})$.

The induction quantity

For $\lambda \geq 1$, $1 \leq p, q \leq \infty$, and $\epsilon > 0$, we define $Q_\lambda = Q_\lambda(p, q, \epsilon)$ by setting

$$Q_\lambda = \sup\{ \|\mathcal{E}_\lambda^\gamma f\|_{L^q(d\mu)} : \mu \in \mathfrak{M}(\alpha, 1),\ \gamma \in \Gamma(\epsilon),\ \|f\|_{L^p(I)} \leq 1,\ a \in \mathfrak{A}\},$$

where $\mathfrak{A}$ is a set of measurable functions supported in $B(0,1)$ and $\|a\|_\infty \leq 1$. It is clear that Q_λ is finite for any $\lambda > 0$.

We want: $Q_\lambda \lesssim \lambda^{-\alpha/q}$.

We have the following two estimates

- $\left\| \max_{\mathcal{I}^i} |\mathcal{E}_\lambda^\gamma f_{\mathcal{I}^i}| \right\|_{L^q(d\mu)} \leq A_i^{1-\frac{1}{p}-\frac{\beta_d(\alpha)}{q}} Q_\lambda \|f\|_p$,
- For $\mathcal{I}_{d-1}^i$, $1 \leq i \leq d$ such that $\Delta(\mathcal{I}_{d-1}^i, \mathcal{I}_{d-1}^j) \geq A_{d-1}$,

$$\left\| \left| \prod_{i=1}^d \mathcal{E}_\lambda^\gamma f_{\mathcal{I}_i^{d-1}}(x) \right|^{\frac{1}{d}} \right\|_{L^q(d\mu)} \leq C A_{d-1}^{-C} \lambda^{-\frac{\alpha}{q}} \|f\|_p.$$

Hence, we get

$$Q_\lambda \leq C \sum_{i=1}^{d-1} A_{i-1}^{-C} A_i{}^{1-\frac{1}{p}-\frac{\beta_d(\alpha)}{q}} Q_\lambda + C A_{d-1}^{-C} \lambda^{-\frac{\alpha}{q}}.$$

As long as $1 - \frac{1}{p} - \frac{\beta_d(\alpha)}{q} > 0$, we can choose $A_1, \ldots, A_{d-1}$, successively, so that $C A_{i-1}^{-C} \, A_i{}^{1-\frac{1}{p}-\frac{\beta_d(\alpha)}{q}} < \frac{1}{2d}$ for $i = 1, \ldots, d-1$. Therefore, we obtain $Q_\lambda \leq C\lambda^{-\frac{\alpha}{q}}$.

However this is not enough in order to obtain the estimate for $q \leq 2\beta_d(\alpha)$.

The nondegenerate curves in $\mathbb{R}^d$ are also nondegenerate in $\mathbb{R}^k$ when they are projected into $\mathbb{R}^k \times \{0\}$. For example, considering $\gamma(t) = (t, \dots, t^d)$, we see that $\gamma_k(t) = (t, \dots, t^k, 0, \dots, 0)$ is nondegenerate in $\mathbb{R}^k \times \{0\}$.

Theorem

For each integer $\ell = 0, 1, \dots, d - 1 - [d - \alpha]$, there exists a constant C_ℓ such that

$$\|\mathcal{E}_\lambda^\gamma f\|_{L^q(d\mu)} \le C_{\mu,\ell}\, \lambda^{-\frac{\alpha-\ell}{q}} \|f\|_{L^p(I)}$$

holds for $f \in L^p(I)$ and $\lambda \ge 1$ whenever $(d-\ell)/q + 1/p \le 1$, $q \ge 2(d-\ell)$ and

$$\frac{\beta_{d-\ell}(\alpha-\ell)}{q} + \frac{1}{p} < 1, \quad q > \beta_{d-\ell}(\alpha-\ell) + 1.$$

Recall that $\ell = 0$ case was already obtained.
The induction on scale argument due to Bourgain–Guth works well in each k-linear step.

Main results: cone in $\mathbb{R}^{d+1}$

Let $g \in L^2(\mathbb{R}^3)$ be supported in

$$\Gamma_\lambda + O(1) = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}^1 : |x| = t,\ \lambda \le t \le 2\lambda\}.$$

For each $q \geq 1$ and for any $\epsilon > 0$, $\lambda \geq 1$, there exists a constant $C > 0$ such that

$$\|\widehat{g}\|_{L^q(d\mu)} \le C\, C_\mu^{\frac{1}{q}} \lambda^{\kappa(\alpha,q,d)+\epsilon} \|g\|_{L^2(\mathbb{R}^2)}, \tag{5}$$

Theorem (Erdoğan, 2004)

For $d = 2$,

$$\kappa(\alpha,q,2) = \begin{cases} \max(1-\frac{\alpha}{q},\ \frac{3}{4}-\frac{\alpha-1}{2q},\ 1-\frac{\alpha}{4}), & \text{if } 1 \le \alpha \le 2, \\ \max(1-\frac{\alpha}{q},\ \frac{3}{4}-\frac{3-2\alpha}{2q},\ \frac{3}{2}-\frac{\alpha}{2}), & \text{if } 2 < \alpha \le 3. \end{cases}$$

Theorem (Cho-H-Lee)

If $d \geq 3$, (5) holds for

$$\kappa(\alpha, q, d) = \begin{cases} \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4}, \frac{3d+1}{8} - \frac{\alpha}{4}\}, & \text{if } 0 < \alpha \leq 1, \\ \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4} + \frac{1-\alpha}{2q}, \frac{3d+1}{8} - \frac{\alpha}{4}\}, & \text{if } 1 < \alpha \leq d, \\ \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4} + \frac{d+1-2\alpha}{2q}, \frac{d+1}{2} - \frac{\alpha}{2}\}, & \text{if } d < \alpha \leq d+1. \end{cases}$$

- Necessary condition:

$$\kappa(\alpha, q, d) = \begin{cases} \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4}\}, & \text{if } 0 < \alpha \leq 1, \\ \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4} + \frac{1-\alpha}{2q}, \frac{d+2}{4} - \frac{\alpha}{4}\}, & \text{if } 1 < \alpha \leq d, \\ \max\{\frac{d}{2} - \frac{\alpha}{q}, \frac{d+1}{4} + \frac{d+1-2\alpha}{2q}, \frac{d+1}{2} - \frac{\alpha}{2}\}, & \text{if } d < \alpha \leq d+1. \end{cases}$$

- When $d = 3$, sharp for all α.
- When $d > 3$, sharp only for $d < \alpha \leq d+1$ (or large q).
- For bilienar estimate, we use induction on scale argument together with wave packet decomposition.

Key estimate

For a function f which is supported away from the origin we define the angular support $\mathcal{A}supp\, f$ by

$$\mathcal{A}supp\, f = \left\{ \frac{\xi}{|\xi|} : \xi \in \mathrm{supp}\, f \right\}.$$

The following may be regarded as a generalization of Wolff's bilinear restriction estimate for the cone.

Theorem

Let $R \gg 1$ and let μ be an α-dimensional measure supported in $\overline{B(0,1)}$. Suppose that f and g are supported in $\Gamma_R(1)$ and

$$\mathrm{dist}(\mathcal{A}supp\, f, \mathcal{A}supp\, g) \geq \frac{1}{100}.$$

For $2 \leq q \leq \infty$, there is a constant $C = C(\beta, n)$ such that

$$\left(\int |\widehat{f}\ \widehat{g}|^{\frac{q}{2}} d\mu \right)^{\frac{2}{q}} \leq C R^{2\beta} \langle \mu \rangle_{\alpha}^{\frac{2}{q}} \|f\|_2 \|g\|_2$$

for any $\beta > \beta(\alpha, q) := \max\{\frac{n}{2} - \frac{\alpha}{q}, \frac{3n+1-2\alpha}{8}\}$.

Suppose that f is supported in $\Gamma_R(1)$ and the diameter of $\mathcal{A}supp\, f$ is $O(R^{-1/2})$. For $q \geq 2$, there exists a constant $C > 0$ such that

$$\|\widehat{f}\|_{L^q(d\mu)} \leq C \langle \mu \rangle_\alpha^{\frac{1}{q}} R^{\beta_\circ(\alpha,q)} \|f\|_2,$$

where

$$\beta_\circ(\alpha, q) = \begin{cases} (n+1)/4 + (n+1-2\alpha)/2q, & \text{if } n < \alpha \leq n+1, \\ (n+1)/4 + (1-\alpha)/2q, & \text{if } 1 \leq \alpha \leq n, \\ (n+1)/4, & \text{if } 0 < \alpha \leq 1. \end{cases}$$

Application: fractal Strichartz estimate

The wave equation defined in $\mathbb{R}^d \times \mathbb{R}$:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \\ u(x,0) = f, \;\; \partial_t u(x,0) = g. \end{cases}$$

- Strichartz estimate:

$$\|u\|_{L_t^q(\mathbb{R},\, L_x^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}};$$

for $s \geq 0, 2 \leq q, r < \infty$,

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s, \;\; \frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}.$$

Here $\dot{H}^s$ is the homogeneous L^2 Sobolev space of order s.

Theorem (Cho-H-Lee)

Let $d \geq 3$. Suppose that u is a solution to the wave equation above. Then

$$\|u\|_{L^q(d\mu)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}},$$

holds with

$$s > \begin{cases} \kappa(\alpha, q, d), & \text{if } 2 \leq q \leq \infty, \\ \kappa(\alpha, 2, d), & \text{if } 1 \leq q \leq 2. \end{cases}$$

- Wolff: $d = 2$, $\alpha \in (1,3)$, $s > \max(\frac{3}{4}, 1 - \frac{\alpha}{4}, 1 - \frac{\alpha}{q})$.
- Erdoğan: $d = 2$, $\alpha \in (1,3)$, $s > \kappa(\alpha, q, 2)$. (Sharp result.)
- Oberlin: $d \geq 3$, $\alpha \in (1, d+1)$, $s > (n-1)/2$ and $q < \alpha$.

The soulution $u(x,t)$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|)\widehat{f}(\xi)d\xi + \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sin(t|\xi|)\frac{\widehat{g}(\xi)}{|\xi|}d\xi.$$

Using Littlewood-Paley operator P_j, we write

$$u(x,t) = P_{\leq 0}(u(\cdot,t))(x) + \sum_{j\geq 1} P_j(u(\cdot,t))(x).$$

- Use Cauchy-Schwarz inequality and Plancherel theorem, $|P_{\leq 0}(u(\cdot,t))(x)| \lesssim \|f\|_2 + \|g\|_2$.
  Use the fact that μ is supported in $\overline{B(0,1)}$,

$$\|P_{\leq 0}(u(\cdot,t))(x)\|_{L^q(d\mu)} \lesssim \|f\|_2 + \|g\|_2.$$

- Use (5), support condition on μ, Planchrel theorem,

$$\|P_j(u(\cdot,t))(x)\|_{L^q(d\mu)} \lesssim 2^{\kappa(\alpha,q,d)j}\|f\|_2 + 2^{(\kappa(\alpha,q,d)-1)j}\|g\|_2.$$

To clarify this, it suffices to show that

$$\|e^{it\sqrt{-\Delta}}P_j h\|_{L^q(d\mu)} \le 2^{\kappa(\alpha,q,d)j}\|h\|_2.$$

$(\widehat{h}(\xi) = \widehat{f}(\xi)$ or $\widehat{h}(\xi) = \widehat{g}(\xi)/|\xi|.)$ Here, $e^{it\sqrt{-\Delta}}f := \int e^{i(x\cdot\xi - t|\xi|)}\widehat{f}(\xi)d\xi$.

Use the smooth function η satisfying $\eta \sim 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}\widehat{\eta} \subset B(0,\frac{1}{2})$, we have

$$\|e^{it\sqrt{-\Delta}}P_j h\|_{L^q(d\mu)} \sim \|\eta\, e^{it\sqrt{-\Delta}}P_j h\|_{L^q(d\mu)}.$$

The space time Fourier transform of $\eta\, e^{it\sqrt{-\Delta}}P_j f(x)$ is supported in $\Gamma_{2^j}(1)$. Using (5) and Plancherel theorem, we get

$$\|e^{it\sqrt{-\Delta}}P_j h\|_{L^q(d\mu)} \lesssim 2^{\kappa(\alpha,q,d)j}\|\eta\, e^{it\sqrt{-\Delta}}P_j h\|_2 \lesssim 2^{\kappa(\alpha,q,d)j}\|h\|_2.$$

Hence

$$\sum_{j\ge 1}\|P_j(u(\cdot,t))\|_{L^q(d\mu)} \lesssim \sum_{j\ge 1}2^{-\epsilon j}2^{\kappa j}\|f\|_2 + \sum_{j\ge 1}2^{-\epsilon j}2^{(\kappa-1)j}\|g\|_2$$

Thank you for your attention.