Uniform Sobolev inequalities for second order non-elliptic differential operators

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Interactions between harmonic and geometric analysis
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Outline

1. Elliptic uniform Sobolev inequality
   - The optimal result and proof of Kenig, Ruiz, and Sogge
2. Non-elliptic case
   - Previous result of Kenig, Ruiz, and Sogge and their argument
   - The optimal result and proof (the main result)
3. Application to unique continuation

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Constant coefficient differential operators of second order

- Set $d \geq 3$ and $1 \leq k \leq d$.
- Let Q be a non-degenerate real quadratic form on $\mathbb{R}^d$ given as

$$Q(\xi) = -\xi_1^2 - \cdots - \xi_k^2 + \xi_{k+1}^2 + \cdots + \xi_d^2.$$

- Let $P(D)$ be a second order differential operator defined by

$$P(D) = Q(D) + a \cdot D + b,$$

where $D = -i\nabla = (-i\frac{\partial}{\partial x_1}, \cdots, -i\frac{\partial}{\partial x_d})$, $a \in \mathbb{C}^d$ and $b \in \mathbb{C}$.
- P is elliptic if $k = d$ ($Q(D) = \Delta$).
- P is non-elliptic otherwise ($Q(D) = \Delta_{\mathbb{R}^k} - \Delta_{\mathbb{R}^{d-k}}$).

Uniform Sobolev inequality

- If there exists an absolute constant $C = C(d, k, p, q)$, **independent of $a \in \mathbb{C}^d$ and $b \in \mathbb{C}$**, such that

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|P(D)u\|_{L^p(\mathbb{R}^d)}, \quad \forall u \in W^{2,p}(\mathbb{R}^d),$$

  we call this the **Uniform Sobolev inequality**.

- **Hardy-Littlewood-Sobolev** For $1 < p < q < \infty$,

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|\Delta u\|_{L^p(\mathbb{R}^d)}, \quad \forall u \in \mathcal{S}(\mathbb{R}^d)$$

  holds if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}. \qquad \text{(gap condition)}$$

- By homogeneity, the same gap condition is necessary for the uniform Sobolev inequality.

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Uniform Sobolev inequality: Elliptic case $Q(D) = \Delta$

- When $P(D)$ is elliptic Kenig, Ruiz, and Sogge characterized the optimal range of p and q.

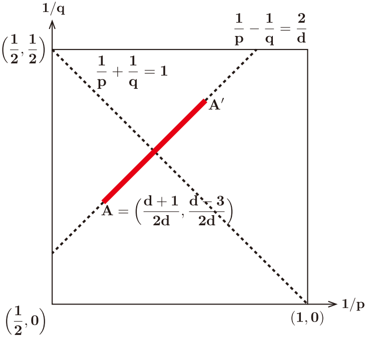
Theorem (Kenig-Ruiz-Sogge, 87')

Let $d \geq 3$ and $Q(D) = \Delta$. Then

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|P(D)u\|_{L^p(\mathbb{R}^d)}$$

holds uniformly in $a \in \mathbb{C}^d$ and $b \in \mathbb{C}$
if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \quad \frac{d+1}{2d} < \frac{1}{p} < \frac{d+3}{2d}.$$



- In the figure the horizontal axis denotes the interval $1/2 \leq 1/p \leq 1$ and the vertical axis denotes $0 \leq 1/q \leq 1/2$.
- If $A = (x, y)$ then $A' = (1-y, 1-x)$ denotes the "dual point" of A, which is symmetric with A with respect to the dual line $\frac{1}{p} + \frac{1}{q} = 1$.

Restriction-extension operator for the sphere

- We call the operator

$$f \to \mathcal{F}^{-1}\Big(\delta(1-|\xi|^2)\widehat{f}(\xi)\Big)(x) \approx \int_{\mathbb{S}^{d-1}} \widehat{f}(\xi)e^{ix\cdot\xi}d\sigma(\xi)$$

  the **restriction-extension operator** defined by the sphere $\mathbb{S}^{d-1}$.
- Here $\delta(1-|\xi|^2)$ is the composition of the δ-distribution with the smooth function $1-|\xi|^2$, and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.
- The operator is the composition of the Fourier restriction and the extension operators associated with the sphere.

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Uniform Sobolev ineq. $\Rightarrow$ restriction-extension estimate

The $L^p - L^q$ elliptic uniform Sobolev inequality implies the following $L^p - L^q$ restriction-extension estimate for the sphere

$$\Big\| \int_{\mathbb{S}^{d-1}} \widehat{f}(\xi) e^{ix\cdot\xi} d\sigma(\xi) \Big\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}.$$

To show this,

1. Assume that the Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^d)} \le C\|(\Delta + 1 \pm i\varepsilon)u\|_{L^p(\mathbb{R}^d)}$$

holds uniformly in all $\varepsilon > 0$.

2. By the Fourier transform we see that the above inequality is equivalent to the multiplier estimates

$$\Big\|\mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{1 - |\xi|^2 \pm i\varepsilon}\Big)\Big\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)},$$

where the constant C is independent of all $\varepsilon > 0$.

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$$\frac{1}{1-|\xi|^2+i\varepsilon} - \frac{1}{1-|\xi|^2-i\varepsilon} = \frac{-2i\varepsilon}{(1-|\xi|^2)^2+\varepsilon^2} \to -2\pi i\delta(1-|\xi|^2)$$

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4. So, the $L^p - L^q$ elliptic uniform Sobolev inequality implies the following $L^p - L^q$ estimate for the restriction-extension operator

$$\left\|\mathcal{F}^{-1}\left(\delta(1-|\xi|^2)\widehat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}.$$

5. Therefore a necessary condition on p and q for the restriction-extension estimate is also necessary for the uniform Sobolev inequality.

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Bochner-Riesz operator of negative order

The Bochner-Riesz operator of order $\alpha > -1$, is defined by

$$\widehat{S^\alpha f}(\xi) = \frac{(1-|\xi|^2)_+^\alpha}{\Gamma(\alpha+1)}\widehat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

By analytic continuation this definition makes sense when $\alpha \leq -1$.

Conjecture ($L^p - L^q$ boundedness of S^α, $\alpha < 0$)

Let $-\frac{d+1}{2} < \alpha < 0$. Then

$$\|S^\alpha\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} < \infty$$

if and only if $\frac{1}{p} - \frac{1}{q} \geq \frac{-2\alpha}{d+1}$, $\frac{1}{p} > \frac{d-1-2\alpha}{2d}$, and $\frac{1}{q} < \frac{d+1+2\alpha}{2d}$.

- Necessity of the conditions are well-known (Börjeson, Carbery, Soria).
- For sufficiency, partial progresses have been made by some mathematicians (Tomas, Stein, Börjeson, Sogge, Carbery, Soria, Bak, Gutierrez, Lee) but the full conjecture still remains open.

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Boundedness of S^{-1} or $(\hat{f}\, d\sigma)^{\vee}$

- When $\alpha = -1$

$$\widehat{S^{-1}f}(\xi) = \delta(1 - |\xi|^2)\hat{f}(\xi) \approx \hat{f}(\xi) d\sigma(\xi),$$

so S^{-1} is the restriction-extension operator for $\mathbb{S}^{d-1}$.

Theorem (Tomas, Stein, Börjeson, Sogge)

$$\left\| \int_{\mathbb{S}^{d-1}} \hat{f}(\xi) e^{ix\cdot\xi} d\sigma(\xi) \right\|_{L^q(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)}$$

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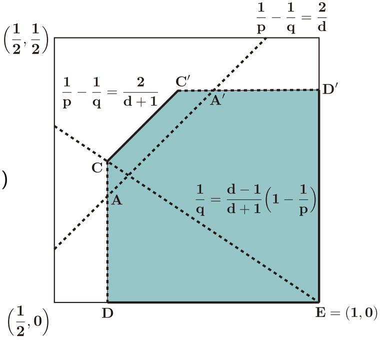


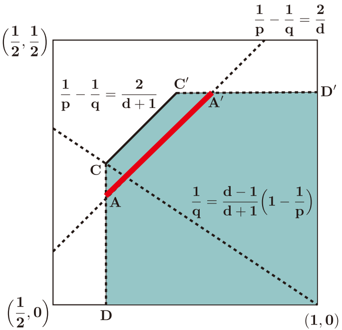
Necessary conditions for the elliptic uniform Sobolev ineq.

- The gap condition

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}.$$

- The pair (p, q) must also satisfy the conditions for the $L^p - L^q$ boundedness of the Bochner-Riesz operator S^{-1} or the restriction-extension operator $(\widehat{f}\,d\sigma)^\vee$, that is,

$$\frac{1}{p} > \frac{d+1}{2d}, \quad \frac{1}{q} < \frac{d-1}{2d}.$$



Argument of Kenig-Ruiz-Sogge for proving elliptic uniform Sobolev inequalities (1/2)

- The main part is establishing the uniform resolvent estimate

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|(\Delta + z)u\|_{L^p(\mathbb{R}^d)}, \quad \forall z \in \mathbb{C}.$$

- The (elliptic) uniform resolvent estimate, combined with the restriction-extension estimate (for the sphere), implies the (elliptic) uniform Sobolev inequality.
- The uniform resolvent estimate is equivalent to

$$\left\|\mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{z - |\xi|^2}\Big)\right\|_q \leq C\|f\|_p, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

- The kernel can be calculated explicitly as

$$K(x) = \Big(\frac{z}{|x|^2}\Big)^{\frac{d-2}{4}} \mathcal{K}_{\frac{d-2}{2}}(\sqrt{z|x|^2}),$$

where $\mathcal{K}_\nu(w) = \int_0^\infty e^{-w\cosh t} \cosh \nu t \, dt$, $w \in \mathbb{C}$, $Re(w) > 0$.

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Argument of Kenig-Ruiz-Sogge (2/2)

- They computed the bound of the kernel by making use of

Theorem (Stein, 86')

Let $d \geq 3$, $1 \leq p \leq 2$, and $\frac{1}{q} \leq \frac{d-1}{d+1}(1 - \frac{1}{p})$. Suppose that ψ is supported away from the diagonal. Then we have

$$\Big\| \int e^{i\lambda|x-y|}\psi(x,y)f(y)dy \Big\|_q \leq C\lambda^{-\frac{d}{q}}\|f\|_p.$$

- By using some quantitative properties of the special function $\mathcal{K}_\nu$ and this oscillatory integral theorem, they obtained

$$\Big\| \mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{z - |\xi|^2}\Big) \Big\|_q = \|K * f\|_q \leq C\|f\|_p, \quad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

for all p and q such that $\frac{1}{p} - \frac{1}{q} = \frac{2}{d}$ and $\frac{d+1}{2d} < \frac{1}{p} < \frac{d+3}{2d}$.

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$$\left\| \int e^{i\lambda|x-y|}\psi(x,y)f(y)dy \right\|_q \leq C\lambda^{-\frac{d}{q}}\|f\|_p.$$

- By using some quantitative properties of the special function $\mathcal{K}_\nu$ and this oscillatory integral theorem, they obtained

$$\left\| \mathcal{F}^{-1}\left(\frac{\widehat{f}(\xi)}{z - |\xi|^2} \right) \right\|_q = \|K * f\|_q \leq C\|f\|_p, \quad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

for all p and q such that $\frac{1}{p} - \frac{1}{q} = \frac{2}{d}$ and $\frac{d+1}{2d} < \frac{1}{p} < \frac{d+3}{2d}$.

Uniform Sobolev inequality: Non-elliptic case $Q(D) \neq \Delta$

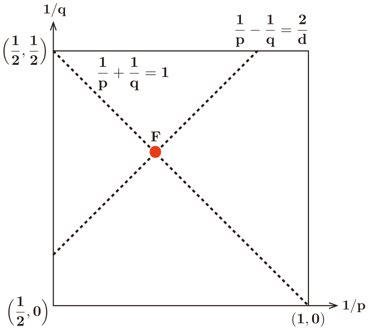
$P(D) = Q(D) + a \cdot D + b$

Theorem (Kenig-Ruiz-Sogge, 87')

Let $d \geq 3$, $Q(D) = \Delta_{\mathbb{R}^k} - \Delta_{\mathbb{R}^{d-k}}$. Then

$$\|u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C \|P(D)u\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

holds uniformly in $a \in \mathbb{C}^d$, $b \in \mathbb{C}$.



Argument of Kenig-Ruiz-Sogge proving a non-elliptic uniform Sobolev inequality (1/3)

- The main part is establishing the non-elliptic uniform resolvent estimate

$$\|u\|_{\frac{2d}{d-2}} \leq C\|(Q(D)+z)u\|_{\frac{2d}{d+2}}, \quad \forall z \in \mathbb{C},$$

because this estimate, combined with Strichartz's restriction-extension estimate for the quadratic surface $\{\xi \in \mathbb{R} : Q(\xi) = \pm 1\}$

$$\Big\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\delta(Q(\xi) \mp 1)d\xi \Big\|_{\frac{2d}{d-2}} \leq C\|f\|_{\frac{2d}{d+2}},$$

implies the (non-elliptic) uniform Sobolev inequality.

- The uniform resolvent estimate is equivalent to

$$\Big\|\mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{Q(\xi)+z}\Big)\Big\|_{\frac{2d}{d-2}} \leq C\|f\|_{\frac{2d}{d+2}}, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

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Argument of Kenig-Ruiz-Sogge (2/3)

- However, for non-elliptic case, there is no such results as Stein's oscillatory integral theorem. Hence, they had to use interpolation along a complex analytic family of distributions. This method is less flexible and restrictive.
- They imbedded the multiplier operator in the following analytic family of operators $\{T_\lambda\}$ in the strip $-d/2 \leq Re(\lambda) \leq 0$, given by the multipliers

$$m_\lambda(\xi) = \frac{e^{\lambda^2}}{\Gamma(\frac{d}{2}+\lambda)}(Q(\xi)+z)^\lambda,$$

so that

$$\|T_\lambda f\|_2 \leq C\|f\|_2, \quad Re(\lambda) = 0,$$
$$\|T_\lambda f\|_\infty \leq C\|f\|_1, \quad Re(\lambda) = -d/2.$$

- The $L^2 - L^2$ bound follows from the Plancherel's theorem because $\|m_\lambda\|_\infty < \infty$ when $Re(\lambda) = 0$.

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Argument of Kenig-Ruiz-Sogge (3/3)

- For the $L^1 - L^\infty$ bound they showed the kernel estimate

$$\|\widehat{m_\lambda}\|_\infty \leq C, \quad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

where $Re(\lambda) = -d/2$ by calculating the kernel

$$\widehat{m_\lambda}(x) = \frac{e^{\lambda^2} 2^{\lambda+1} e^{-\pi i k/2}}{(2\pi)^d \Gamma(-\lambda)\Gamma(\frac{d}{2}+\lambda)} \Big(\frac{z}{Q(x)}\Big)^{\frac{1}{2}(\frac{d}{2}+\lambda)} \mathcal{K}_{\frac{d}{2}+\lambda}\Big(\sqrt{zQ(x)}\Big).$$

- By Stein's analytic interpolation theorem they obtained

$$\Big\|\Big(\frac{\widehat{f}(\xi)}{Q(\xi)+z}\Big)^\vee\Big\|_{\frac{2d}{d-2}} \approx \|T_{-1}f\|_{\frac{2d}{d-2}} \leq C\|f\|_{\frac{2d}{d+2}},$$

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Main result: non-elliptic uniform Sobolev inequatilies

$$P(D) = Q(D) + a \cdot D + b$$

Theorem (Jeong-K.-Lee)

Let $d \geq 3$, $Q(D) = \Delta_{\mathbb{R}^k} - \Delta_{\mathbb{R}^{d-k}}$. Then

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C\|P(D)u\|_{L^p(\mathbb{R}^d)}$$

holds uniformly in $a \in \mathbb{C}^d$ and $b \in \mathbb{C}$ **if and only if** *$1/p - 1/q = 2/d$ and*

$$\frac{d}{2(d-1)} < \frac{1}{p} < \frac{d^2+2d-4}{2d(d-1)}.$$

At the critical points B and B' we have the restricted weak type bounds

$$\|u\|_{L^{q,\infty}(\mathbb{R}^d)} \leq C\|P(D)u\|_{L^{p,1}(\mathbb{R}^d)}.$$

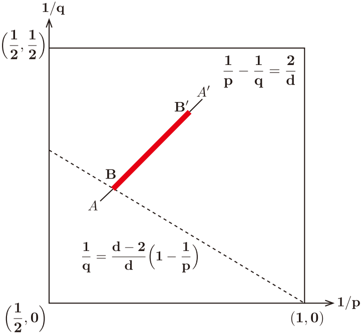

Figure: The optimal $(\frac{1}{p}, \frac{1}{q})$-range for the uniform Sobolev inequalities when $Q(D) \neq \Delta$.

Idea of proof

- Our method proving the non-elliptic uniform Sobolev inequalities is different from that of Kenig, Ruiz and Sogge.
- Their idea is based on interpolation along a complex analytic family of distributions for which $L^2 - L^2$ and $L^1 - L^\infty$ bounds are relatively easier to obtain from computations of kernel.
- Instead, we directly analyze the associated multiplier operator in the frequency domain, whose singularity lies on the surface given by the quadratic form $Q(\xi)$.
- We decompose the multiplier dyadically away from its singularity by taking into account the distance to the surface.
- This approach is rather typical in the study of boundedness of Bochner-Riesz type operators and of inhomogeneous Strichartz estimates.
- In this manner, all the pairs of (p, q) for which the non-elliptic uniform Sobolev inequalities are completely characterized.

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Uniform Sobolev ⇒ Restriction-extension estimate

Similarly as in the elliptic case, the non-elliptic Sobolev inequality implies the restriction-extension estimate for the quadratic surface $\Sigma_\pm = \{\xi \in \mathbb{R}^d : Q(\xi) = \pm 1\}$

$$\Big\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\delta(Q(\xi) \mp 1) d\xi \Big\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}. \qquad (*)$$

1. Assume the non-elliptic uniform Sobolev (or resolvent) inequality

$$\|u\|_{L^q(\mathbb{R}^d)} \le C\|(Q(D) - 1 \pm i\varepsilon)u\|_{L^p(\mathbb{R}^d)}.$$

2. As before in the elliptic case, the multiplier estimates

$$\Big\| \Big(\frac{\widehat{f}(\xi)}{Q(\xi) - 1 - i\varepsilon} - \frac{\widehat{f}(\xi)}{Q(\xi) - 1 + i\varepsilon} \Big)^\vee \Big\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

must also be true uniformly in $\varepsilon > 0$. So, taking limit $\varepsilon \to 0$ we have the restriction-extension estimate $(*)$.

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The cone multiplier operator of negative order

The cone multiplier operator of order $\mu > -1$, is defined by

$$\widehat{C^\mu f}(\xi) = \frac{\phi(\xi_d)}{\Gamma(\alpha+1)}(1 - |\tilde{\xi}|^2/\xi_d^2)_+^\mu \widehat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\xi = (\tilde{\xi}, \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $\phi \in C_0^\infty(1,2)$. This definition makes sense when $\mu \leq -1$ by analytic continuation.

Conjecture ($L^p - L^q$ boundedness of C^μ, $\mu < 0$)

Let $-\frac{d}{2} < \mu < 0$. Then

$$\|C^\mu\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} < \infty$$

if and only if $\frac{1}{p} - \frac{1}{q} \geq \frac{-2\alpha}{d}$, $\frac{1}{p} > \frac{d-2-2\alpha}{2(d-1)}$, and $\frac{1}{q} < \frac{d+2\alpha}{2(d-1)}$.

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Boundedness of C^{-1}

When $\mu = -1$

$$\widehat{C^{-1}f}(\xi) \approx \phi(\xi_d)\delta(\xi_d^2 - |\tilde{\xi}|^2)\widehat{f}(\xi), \quad \xi = (\tilde{\xi}, \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

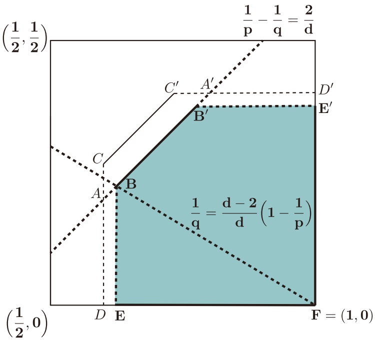
so C^{-1} is the restriction-extension operator for the conic surface $\{\xi \in \mathbb{R}^d : \xi_d = |\tilde{\xi}| \in [1,2]\}$.

Theorem (Lee, 2003)

If $d \geq 3$,

$$\|C^{-1}f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

if and only if $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{d}$, $\frac{1}{p} > \frac{d}{2(d-1)}$ *and* $\frac{1}{q} < \frac{d-2}{2(d-1)}$.



Necessary conditions for non-elliptic uniform Sobolev ineq.

1. The non-elliptic uniform Sobolev inequality fails unless the restriction-extension estimate holds:

$$\Big\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\delta(Q(\xi) \mp 1) d\xi \Big\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

2. By scaling $\xi \to |\rho|^{-1/2}\xi$, $\rho \in \mathbb{R} \setminus \{0\}$, this is equivalent to

$$\Big\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\delta(Q(\xi) - \rho) d\xi \Big\|_{L^q(\mathbb{R}^d)} \lesssim |\rho|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{d})} \|f\|_{L^p(\mathbb{R}^d)}.$$

3. But the uniform Sobolev inequality holds only when $\frac{1}{p} - \frac{1}{q} = \frac{2}{d}$. So, if $Q(\xi) = -\xi_1^2 - \cdots - \xi_{d-1}^2 + \xi_d^2$, $|\rho| \ll 1$, and $\widehat{f}$ is supported away from the zero, then the above restriction-extension operator looks like the cone multiplier operator of order -1.

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$$\left\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\delta(Q(\xi) - \rho)d\xi \right\|_{L^q(\mathbb{R}^d)} \lesssim |\rho|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q}-\frac{2}{d})}\|f\|_{L^p(\mathbb{R}^d)}.$$

3. But the uniform Sobolev inequality holds only when $\frac{1}{p} - \frac{1}{q} = \frac{2}{d}$. So, if $Q(\xi) = -\xi_1^2 - \cdots - \xi_{d-1}^2 + \xi_d^2$, $|\rho| \ll 1$, and $\widehat{f}$ is supported away from the zero, then the above restriction-extension operator looks like the cone multiplier operator of order -1.

Necessary conditions for non-elliptic uniform Sobolev ineq.

4. Indeed, a similar argument as in the cone multiplier operator shows that the condition

$$\frac{d}{2(d-1)} < \frac{1}{p} < \frac{d^2+2d-4}{2d(d-1)}$$

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By a standard reduction argument, the proof of non-elliptic uniform Sobolev inequality follows from the two steps:

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- Uniform resolvent estimate

$$\|u\|_{L^q(\mathbb{R}^d)} \le C\|(Q(D)+z)u\|_{L^p(\mathbb{R}^d)}, \quad \forall z \in \mathbb{C}.$$

• When $Q(D) = \partial_1^2 - \partial_2^2 - \cdots - \partial_d^2 = \Box$, and $z = 1$, the resolvent estimate is the inhomogeneous Strichartz estimate for the Klein-Gordon equation.

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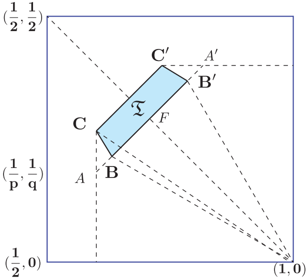
Estimate for the restriction-extension operator

- $d \geq 3$
- $B = \left(\frac{d}{2(d-1)}, \frac{(d-2)^2}{2d(d-1)}\right)$ $C = \left(\frac{d+1}{2d}, \frac{(d-1)^2}{2d(d+1)}\right)$
- $\mathfrak{T}$: the closed trapezoid with B, B', C', C from which B, B', C, C' are removed



Theorem (Jeong-K.-Lee)

(i) *If* $(1/p, 1/q) \in \mathfrak{T}$ *then*

$$\left\| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) \delta(Q(\xi) \mp 1) d\xi \right\|_q \lesssim \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

(ii) *If* $(1/p, 1/q)$ *is one of* B, B', C, C', *then we have* $L^{p,1} - L^{q,\infty}$ *estimate.*

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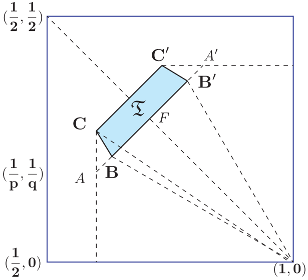

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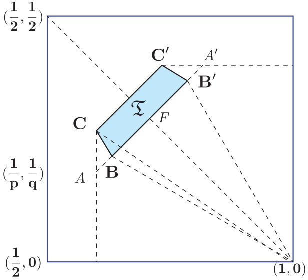

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Proof of restriction-extention estimates: reduction

- By duality and real interpolation, it is enough to prove the $L^{p,1} - L^{q,\infty}$ estimate when $(1/p, 1/q) = B$ or C.
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$$\|\mathcal{F}^{-1}\big(\delta(Q \pm 1)\widehat{P_j f}\big)\|_{q,\infty} \lesssim \|P_j f\|_{p,1}$$

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$$\eta = (\tilde{\eta}, \eta_d) = (\eta_1, \eta', \eta'', \eta_d) \in \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{d-k-1} \times \mathbb{R} = \mathbb{R}^d.$$

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- Now the restriction-extension estimate is reduced to showing
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$\exists\ \psi \in \mathcal{S}(\mathbb{R})$ *with* $\widehat{\psi}$ *supported in* $[-2,-1/2]\cup[1/2,2]$ *s.t.*
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Proof of rest.-ext. estimates: key estimates for T_ℓ

- $L^2 - L^q$ estimates:

$$\|T_\ell f\|_{\frac{2(d+1)}{d-1}} \lesssim |\rho|^{-\frac{1}{2(d+1)}} 2^{-\frac{\ell}{2}} \|f\|_2, \quad \|T_\ell f\|_{\frac{2d}{d-2}} \lesssim 2^{-\frac{\ell}{2}} \|f\|_2$$

- $L^1 - L^\infty$ estimates:

$$\|T_\ell f\|_\infty \lesssim |\rho|^{-\frac{1}{2}} 2^{\frac{\ell(d-1)}{2}} \|f\|_1, \quad \|T_\ell f\|_\infty \lesssim 2^{\frac{\ell(d-2)}{2}} \|f\|_1$$

Proposition

(i) *For $1 \le p \le 2$ and $\frac{1}{q} = \frac{d-1}{d+1}(1 - \frac{1}{p})$, (the green line in the figure)*

$$\|T_\ell f\|_q \lesssim |\rho|^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})} 2^{\ell(\frac{d}{p} - \frac{d+1}{2})} \|f\|_p.$$

(ii) *For $1 \le p \le 2$ and $\frac{1}{q} = \frac{d-2}{d}(1 - \frac{1}{p})$, (the blue line in the figure)*

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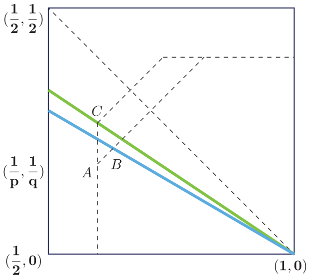


Proof of rest.-ext. estimates: summation over ℓ

Lemma (Summation in Lorentz space)

Let $\epsilon_0, \epsilon_1 > 0$, *and let* $\{T_\ell : \ell \in \mathbb{Z}\}$ *be a sequence of linear operators satisfying*

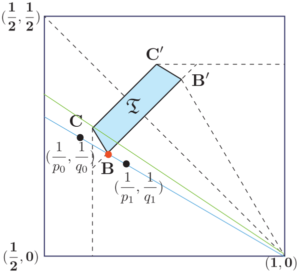
$$\begin{aligned} \|T_\ell f\|_{q_0} &\leq M_0 2^{-\epsilon_0 \ell} \|f\|_{p_0}, \\ \|T_\ell f\|_{q_1} &\leq M_1 2^{\epsilon_1 \ell} \|f\|_{p_1}. \end{aligned}$$

Then for $\theta = \epsilon_1/(\epsilon_0 + \epsilon_1)$,

$$\Big\| \sum_{\ell \in \mathbb{Z}} T_\ell f \Big\|_{q,\infty} \leq C M_0^\theta M_1^{1-\theta} \|f\|_{p,1},$$

where

$$\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$



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- Similarly, we have $L^{p,1} - L^{q,\infty}$ estimates for $(1/p, 1/q) = C$.
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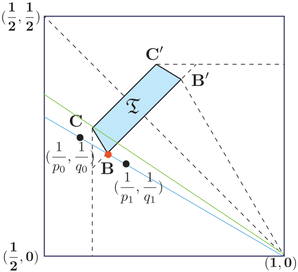

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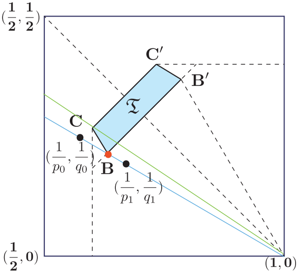

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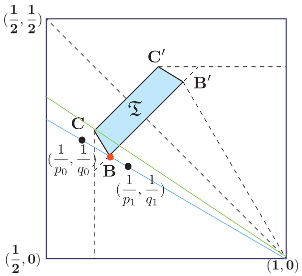

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Proof of rest.-ext. estimates: proof of the $L^2 - L^q$ estimate

- Consider the evolution operator $U_\rho(t)$ given by

$$U_\rho(t)g(\tilde{x}) = \int_{\mathbb{R}^{d-1}} e^{i(\tilde{x}\cdot\tilde{\eta} + t\mathcal{G}_\rho(\tilde{\eta}))}\tilde{\chi}(\tilde{\eta})\,\widehat{g}(\tilde{\eta})d\tilde{\eta}, \quad \tilde{\chi} \in C_c^\infty(\mathcal{D}).$$

Lemma (Stationary phase estimates)

There is a constant C, independent of ρ, such that

$$\left| \int e^{i(\tilde{x}\cdot\tilde{\eta} + x_d\mathcal{G}_\rho(\tilde{\eta}))}\chi(\tilde{\eta})d\tilde{\eta} \right| \leq C(1+|t||\rho|)^{-\frac{1}{2}}(1+|t|)^{-\frac{d-2}{2}}.$$

- The standard TT^*-method and the above lemma imply

$$\|U_\rho(t)g(\tilde{x})\|_{L_{t,\tilde{x}}^{\frac{2(\sigma+1)}{\sigma}}(\mathbb{R}^d)} \leq C|\rho|^{\frac{1}{2(\sigma+1)}(\frac{d-2}{2}-\sigma)}\|g\|_{L^2(\mathbb{R}^{d-1})}$$

for $\frac{d-2}{2} \leq \sigma \leq \frac{d-1}{2}$.

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$$\widehat{\mathcal{T}^\rho_\lambda f}(\eta) = \tilde{\chi}(\tilde{\eta})\psi\big(\lambda^{-1}(\eta_d - \mathcal{G}_\rho(\tilde{\eta}))\big)\widehat{f}(\eta).$$

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Proof of rest.-ext. estimates: proof of the $L^1 - L^\infty$ estimate

Lemma (Estimate for the kernel of $\mathcal{T}_\lambda^\rho$)

For every $\rho \neq 0$ and $0 < \lambda \lesssim 1$, let $\mathcal{K}_\lambda^\rho$ be defined by

$$\mathcal{K}_\lambda^\rho(x) = \int_{\mathbb{R}^d} \psi\big(\lambda^{-1}(\eta_d - \mathcal{G}_\rho(\tilde{\eta}))\big)\tilde{\chi}(\tilde{\eta})e^{ix\cdot\eta}d\eta,$$

where $\tilde{\chi}$ is a smooth function supported on $\mathcal{D}$. Suppose $\widehat{\psi}$ is supported on $\{t : 1/2 \leq |t| \leq 2\}$. Then $\mathcal{K}_\lambda^\rho$ is supported in the set $\{x \in \mathbb{R}^d : |x_d| \sim \lambda^{-1}\}$ and

$$|\mathcal{K}_\lambda^\rho(x)| \leq C\lambda^{\frac{d}{2}} \min(1, \lambda^{\frac{1}{2}}|\rho|^{-\frac{1}{2}}).$$

- Without cancellation property the best possible bound is

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Proof of the Lemma.

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$$\begin{aligned}\mathcal{K}_\lambda^\rho(x) &= \int e^{ix_d\eta_d}\psi(\lambda^{-1}\eta_d)d\eta_d \int_{\mathbb{R}^{d-1}} e^{i(\tilde{x}\cdot\tilde{\eta}+x_d\mathcal{G}_\rho(\tilde{\eta}))}\tilde{\chi}(\tilde{\eta})d\tilde{\eta} \\ &= \lambda\widehat{\psi}(\lambda x_d)\int_{\mathbb{R}^{d-1}} e^{i(\tilde{x}\cdot\tilde{\eta}+x_d\mathcal{G}_\rho(\tilde{\eta}))}\tilde{\chi}(\tilde{\eta})d\tilde{\eta}\end{aligned}$$

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- Now we use the stationary phase lemma to get

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Application

Corollary (Carleman inequalities for non-elliptic operators)

Let p, q, and $P(D) = Q(D) + a \cdot D + b$ be as in the (eliptic or non-elliptic) uniform Sobolev inequality. Then we have the Carleman inequlity:

$$\|e^{tv\cdot x}u\|_{L^q(\mathbb{R}^d)} \le C\|e^{tv\cdot x}P(D)u\|_{L^p(\mathbb{R}^d)}, \qquad \forall u \in C_c^\infty(\mathbb{R}^d).$$

Here, the constant C is independent of $t \in \mathbb{R}$, $v \in \mathbb{R}^d$, $a \in \mathbb{C}^d$, and $b \in C$.

($\because$) Replace $u(x)$ by $e^{-tv\cdot x}u(x)$. Then the above inequality is equivalent to

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Corollary (global unique continuation for non-elliptic operators)

Let $\frac{2d(d-1)}{d^2+2d-4} < p < \frac{2(d-1)}{d}$ *and let* $P(D)$ *be as before. Suppose the three conditions:*

- $V \in L^{d/2}(\mathbb{R}^d)$,
- $u \in W^{2,p}(\mathbb{R}^d)$ *is supported in a half space,*
- $|P(D)u| \leq |Vu|$.

Then $u = 0$ *on the whole space* $\mathbb{R}^d$.

To see this,

• Let C be the constant of the Carleman inequality.

• Because $V \in L^{d/2}(\mathbb{R}^d)$, we can find a $\delta > 0$ such that

$$\|V\|_{L^{d/2}(S)} \leq 1/2C$$

whenever S is **any** "strip" in $\mathbb{R}^d$ which is congruent to $\mathbb{R}^{d-1} \times [0, \delta]$.

• By translation, we may assume that $u = 0$ on the half space

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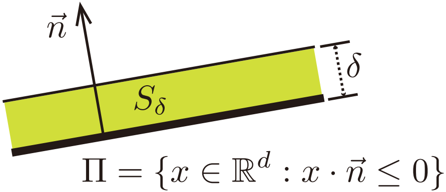
whenever S is **any** "strip" in $\mathbb{R}^d$ which is congruent to $\mathbb{R}^{d-1} \times [0, \delta]$.
• By translation, we may assume that $u = 0$ on the half space

$$\Pi = \{x \in \mathbb{R}^d : x \cdot \vec{n} \leq 0\}$$

for some unit vector $\vec{n}$.



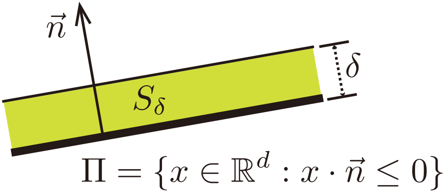
- By inductive argument, it is enough to show that $u = 0$ on the strip

$$S_\delta = \{x \in \mathbb{R}^d : 0 \leq x \cdot \vec{n} \leq \delta\}.$$

- Let q be such that $1/p - 1/q = 2/d$ and let $t > 0$.

$$\begin{aligned} \|e^{-tx\cdot\vec{n}}u\|_{L^q(S_\delta)} &\leq C\|e^{-tx\cdot\vec{n}}P(D)u\|_{L^p(\mathbb{R}^d)} \\ &\leq C\|e^{-tx\cdot\vec{n}}Vu\|_{L^p(S_\delta)} + C\|e^{-tx\cdot\vec{n}}P(D)u\|_{L^p(\mathbb{R}^d\setminus S_\delta)} \\ &\leq C\|V\|_{L^{d/2}(S_\delta)}\|e^{-tx\cdot\vec{n}}u\|_{L^q(S_\delta)} + Ce^{-t\delta}\|P(D)u\|_{L^p(\mathbb{R}^d\setminus S_\delta)}. \end{aligned}$$

- Hence we have $\|e^{t(\delta - x\cdot\vec{n})}u\|_{L^q(S_\delta)} \leq 2C\|P(D)\|_{L^p(\mathbb{R}^d\setminus S_\delta)}$ uniformly in $t > 0$. This is impossible unless $u = 0$ on S_δ.



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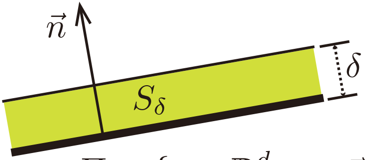

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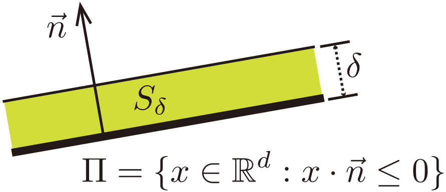

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Thank you for your attention!