Lorentz extension including improvement of some inequaties

Shohei Nakamura (joint work with Professors Neal Bez, Sanghyuk Lee and Yoshihiro Sawano)

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Brascamp-Lieb inequality

Let $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be a linear surjective map and $p_j \in [1, \infty]$ for each $1 \le j \le m$. Then we consider the following inequality:

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j(L_j(x))dx \le \mathbf{BL}(\vec{L}, \vec{p}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}, \quad (f_j : \mathbb{R}^{d_j} \to \mathbb{R}_{\ge 0}) \qquad (1)$$

where $\mathbf{BL}(\vec{L}, \vec{p}) \in [0, \infty]$ denotes a best constant of the inequality.

Example 1 (Loomis-Whitney)

Let $d = 3$, $d_j = 2$ and $L_j : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection, for example, $L_1(x_1, x_2, x_3) = (x_2, x_3)$. Then $\mathbf{BL}(\vec{L}, \vec{p}) < \infty$ if and only if $p_j = 2$ for all j. In other words, the following inequality holds if and only if $p_j = 2$:

$$\int\int\int_{\mathbb{R}^3} f_1(x_2, x_3) f_2(x_1, x_3) f_3(x_1, x_2) dx \le \prod_{j=1}^{3} \|f_j\|_{L^{p_j}}.$$

Brascamp-Lieb inequality

An unified treatment of such inequality is studied by J. Bennet, A. Carbery, M. Christ and T. Tao. In particular, we here introduce the following:

Theorem 2 (J. Bennet, A. Carbery, M. Christ and T. Tao)

The Brascamp-Lieb constant $\mathbf{BL}(\vec{L}, \vec{p})$ *is finite if and only if following two conditions:*

- *(Scaling condition)*

$$\sum_{j=1}^{m} \frac{d_j}{p_j} = d;$$

- *(Dimension condition) For all subspace* $V \subset \mathbb{R}^d$*,*

$$\mathrm{dim}(V) \leq \sum_{j=1}^{m} \frac{1}{p_j} \mathrm{dim}(L_j V).$$

First problem

Let us consider the Lorentz extension of Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j(L_j(x))dx \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,r_j}(\mathbb{R}^{d_j})}, \tag{2}$$

where $\{r_j\}_{j=1}^m \subset [1, \infty]$ and $L^{p,r}$ denotes the usual Lorentz space. In an appendix of P. A. Perry's paper, M. Christ gave the sufficient condition.

Theorem 3 (M. Christ)

Let L_j be linear surjective, $p_j \in (1, \infty)$ and $r_j \in [1, \infty]$. Assume the scaling condition and the subcritical dimension condition which means $\mathrm{dim}(V) < \sum_{j=1}^m \frac{1}{p_j} \mathrm{dim}(L_j V)$ for all nonzero proper subspace V. Then the Lorentz extension (2) *holds whenever*

$$\sum_{j=1}^{m} \frac{1}{r_j} \geq 1.$$

First problem

Is the condition $\sum_{j=1}^m 1/r_j \geq 1$ sharp or not?? To this problem, we obtain the following.

Theorem 4

Let L_j be a linear map, $p_j \in (1,\infty)$ and $r_j \in [1,\infty]$. If Lorentz extension (2) *holds, then we have*

1. *(Scaling and Dimension conditions)*

$$\sum_{j=1}^m \frac{d_j}{p_j} = d, \quad \dim(V) \leq \sum_{j=1}^m \frac{1}{p_j} \dim(L_j V);$$

2. *(Lorentz exponent)*

$$\sum_{j=1}^m \frac{1}{r_j} \geq 1.$$

Counterexample for Loomis-Whitney

Scaling condition and Dimension condition can be proved from the same manner as the case of Lubesgue space. Assume

$$\int\int\int_{\mathbb{R}^3} f_1(x_2,x_3)f_2(x_1,x_3)f_3(x_1,x_2)dx \le \prod_{j=1}^{3} \|f_j\|_{L^{2,r_j}}.$$

Take any sequences $\{a_j^i\}_{j=1}^{\infty}$ for $i=1,2,3$ and let, for example

$$f_1(x_2,x_3) = \sum_{j=1}^{\infty} a_j^1 \chi_{[2^{j-1},2^j)}(x_2)\chi_{[2^{j-1},2^j)}(x_3).$$

Then from the detailed calculation,

$$\|f_1\|_{L^{2,r_1}} \lesssim \left(\sum_{j=1}^{\infty}[a_j^1 2^j]^{r_1}\right)^{\frac{1}{r_1}}.$$

Continue

For the left-hand side,

$$\int\int\int_{\mathbb{R}^3} f_1(x_2,x_3)f_2(x_1,x_3)f_3(x_1,x_2)dx \gtrsim \sum_{j=1}^{\infty}(\prod_{i=1}^{3} a_j^i 2^j).$$

So, the Lorentz extension of Loomis-Whitney implies

$$\sum_{j=1}^{\infty}(\prod_{i=1}^{3} a_j^i 2^j) \lesssim \prod_{i=1}^{3}\left(\sum_{j=1}^{\infty}[a_j^i 2^j]^{r_i}\right)^{\frac{1}{r_i}},$$

which gives, by letting $a_j^i 2^j = b_j^i$, Hölder's inequality for sequence

$$\sum_{j=1}^{\infty}(\prod_{i=1}^{3} b_j^i) \lesssim \prod_{i=1}^{3}\left(\sum_{j=1}^{\infty}[b_j^i]^{r_i}\right)^{\frac{1}{r_i}}.$$

So, $\sum_{i=1}^{3} 1/r_i \geq 1$.

General case

Assume

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j(L_j(x))dx \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,r_j}(\mathbb{R}^{d_j})},$$

where $(p_j)_{j=1}^m$ satisfy the scaling. Again take any $(a_k^j)_{k=1}^\infty$ for $j = 1, \dots, m$ and let

$$f_j(x_j) = \sum_{k=1}^{\infty} a_k^j \chi_{L_j([2^{k-1},2^k)^d)}(x_j), \quad x_j \in \mathbb{R}^{d_j}.$$

Thanks to the linearity and surjectivity of L_j, involving the maximal operator or scaling argument, we have

$$\|f_j\|_{L^{p_j,r_j}(\mathbb{R}^{d_j})} \leq C_j \left(\sum_{k=1}^{\infty} [a_k^j 2^{\frac{d_j}{p_j}k}]^{r_j} \right)^{\frac{1}{r_j}}.$$

Continue

On the other hand, from the surjectivity of L_j and scaling condition of $(p_j)_j$,

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j(L_j(x))dx \geq C \sum_{k=1}^{\infty} (\prod_{j=1}^{m} a_k^j) 2^{kd} = \sum_{k=1}^{\infty} (\prod_{j=1}^{m} a_k^j 2^{\frac{d_j}{p_j} k}).$$

So, again replacing $a_k^j 2^{\frac{d_j}{p_j} k} = b_k^j$,

$$\sum_{k=1}^{\infty} (\prod_{j=1}^{m} b_k^j) \leq C \left(\sum_{k=1}^{\infty} [b_k^j]^{r_j} \right)^{\frac{1}{r_j}},$$

which implies $\sum_{j=1}^{m} 1/r_j \geq 1$.

Remark on the critical case

The above results does not touch on the critical case: there exists a subspace $V \subset \mathbb{R}^d$ such that $\dim(V) = \sum_{j=1}^m \frac{1}{p_j} \dim(L_j V)$. Indeed, the Loomis-Whitney is this case. To this inequality, the stronger necessary condition is needed for the Lorentz extension. That is, if

$$\int\int\int_{\mathbb{R}^3} f_1(x_2,x_3) f_2(x_1,x_3) f_3(x_1,x_2)\, dx \le \prod_{j=1}^3 \|f_j\|_{L^{2,r_j}},$$

then $\{r_j\}_{j=1}^3$ must satisfy

$$\sum_{j=1}^3 \frac{1}{r_j} \ge \frac{d}{d-1} = \frac{3}{2} > 1.$$

So, the Christ's sufficient condition $\sum_{j=1}^m 1/r_j \ge 1$ is not enough and hence, such critical cases are still unknown.

Lorentz improvement of Multilinear Hardy-Littlewood-Sobolev

Similar Lorentz extension (refinement) is available for multilinear Hardy-Littlewood-Sobolev, due to M. Christ (1985):

$$\int_{\mathbb{R}^d} \prod_{1\le i<j\le d} |x_i - x_j|^{-\gamma} \prod_{k=1}^{d} f_k(x_k) dx_1 \cdots dx_d \le C \prod_{k=1}^{d} \|f_k\|_{L^{p,d}}, \quad (3)$$

where p and γ satisfy $\gamma < 2/d$, $1 \le p < d$ and $1/p + \gamma(d-1)/2 = 1$. The Lorentz exponents $r_1 = r_2 = \cdots = r_d = d$ is satisfying

$$\sum_{j=1}^{d} \frac{1}{r_j} = \sum_{j=1}^{d} \frac{1}{d} = 1.$$

Using this improvement, J. Bennet, N. Bez, S. Gutiérretz and S. Lee reproved (improved) the known Strichatrz estimate for the kinetic transport equation in terms of Lorentz spaces.

Kinetic transport equation

The solution of the Kinetic transport equation:

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = 0, \quad F(0,x,v) = f(x,v)$$

where $(t,x,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, can be written by $F(t,x,v) = f(x-tv,v)$.
The Strichartz type estimate for the velocity average:

$$\|\rho f\|_{L^q_t L^p_x} \leq C\|f\|_{L^a_{x,v}}, \quad \rho f(t,x) = \int_{\mathbb{R}^d} f(x-tv,v)dv. \tag{4}$$

The necessary conditions for (4), for example Keel-Tao's paper:

$$\frac{2}{q} + \frac{d}{p} = d, \quad a = \frac{2p}{p+1}, \quad p \leq \frac{d+1}{d-1}. \tag{5}$$

Strichartz estimate for the velocity average

Theorem 5

Let parameters q, p, a satisfy (5)*.*

1. *(Castella and Perthama (1996), Keel-Tao(1998), E. Ovcharov(2011))* *When $1 \leq p < \frac{d+1}{d-1}$, the Strichartz estimate* (4) *holds.*
2. *(Guo and Peng(2007), J. Bennet, N. Bez, S. Gutiérretz and S. Lee (2014))* *For the end-point $p = \frac{d+1}{d-1}$, the Strichartz estimate* (4) *fails.*

Dual version of the Strichartz estimate

The dual version of the Strichartz estimate:

$$\|\rho^* g\|_{L^{\sigma(d+1)}_{x,v}} \leq C\|g\|_{L_t^{q(\sigma)} L_x^{\frac{\sigma(d+1)}{2}}}, \quad \rho^* g(x,v) = \int_{\mathbb{R}} g(t, x+tv)dt, \qquad (6)$$

where

$$\sigma \geq 1, \quad \frac{1}{q(\sigma)} + \frac{d}{(d+1)\sigma} = 1.$$

The case $\sigma = 1$ is corresponding to the end-point $p = \frac{d+1}{d-1}$. So, (6) holds as long as $\sigma > 1$ and (6) with $\sigma = 1$ fails.

Lorentz refinement of the Strichartz estimate

J. Bennet, N. Bez, S. Gutiérretz and S. Lee reproved the dual version Strichartz estimate (6) following the multilinear approach, multilinear Hardy-Littlewood-Sobolev (3).
In there, it was pointed out that the Strichartz estimate (6) can be improved in terms of Lorentz language:

$$\|\rho^* g\|_{L^{\sigma(d+1)}_{x,v}} \le C\|g\|_{L_t^{q(\sigma),r} L_x^{\frac{\sigma(d+1)}{2}}}, \tag{7}$$

with $r = d+1$ and $\sigma > 1$. Since $q(\sigma) \le d+1$, this improves (6).
Is the Lorentz exponent in (7): $r = d+1$ sharp or not??

Result

We realized that the exponent $r = d + 1$ can be improved further.

Theorem 6

Let $\sigma > 1$ and $q(\sigma)$ be determined by $\frac{1}{q(\sigma)} + \frac{d}{\sigma(d+1)} = 1$. Then the Lorentz improvement

$$\|\rho^* g\|_{L^{\sigma(d+1)}_{x,v}} \leq C\|g\|_{L_t^{q(\sigma),r} L_x^{\frac{\sigma(d+1)}{2}}},$$

holds if and only if

$$r \leq \sigma(d+1).$$

Counterexample

Let us see that $r \leq \sigma(d+1)$ is necessary. But, the split is same as before. Again, we take any sequence $(a_N)_{N=1}^{\infty}$ and let

$$g(x,t) = \chi_{[0,1)^d}(x)\left(a_0\chi_{[0,1)}(t) + \sum_{N=1}^{\infty} a_N \chi_{[2^{N-1},2^N)}(t)\right).$$

Then carefully calculation and the scaling shows

$$\|\rho^* g\|_{L^{(d+1)\sigma}_{x,v}} \geq C\left(\sum_{N=1}^{\infty}[a_N 2^N]^{(d+1)\sigma}2^{-N}\right)^{\frac{1}{(d+1)\sigma}} = \left(\sum_{N=1}^{\infty}[a_N 2^{\frac{N}{q(\sigma)}}]^{(d+1)\sigma}\right)^{\frac{1}{(d+1)}$$

and

$$\|g\|_{L_t^{q(\sigma),r}L_x^{\frac{\sigma(d+1)}{2}}} \sim \left(\sum_{N=1}^{\infty}[a_N 2^{\frac{N}{q(\sigma)}}]^r\right)^{\frac{1}{r}}.$$

Orthonormal Strichartz estimate for Schrödinger

For $q, r \geq 2$ such that $\frac{1}{q} = \frac{d}{2}(\frac{1}{2} - \frac{1}{r})$ and suitable β, we consider the following orthonormal (orthogonal) Strichartz inequality

$$\left\| \left(\sum_k |e^{it\Delta} f_k|^2 \right)^{\frac{1}{2}} \right\|_{q,r} \leq C \left(\sum_k \|f_k\|_2^\beta \right)^{\frac{1}{\beta}} \tag{8}$$

for any orthogonal system $(f_k)_k$ in L^2.
From the triangle inequality and the one function Strichartz estimate, (8) with $\beta = 2$ follows:

$$\left\| \left(\sum_k |e^{it\Delta} f_k|^2 \right)^{\frac{1}{2}} \right\|_{q,r} \leq C \left(\sum_k \|f_k\|_2^2 \right)^{\frac{1}{2}}.$$

So, the problem is how large can we take the exponent β??

Recent development

Theorem 7 (Frank-Lewin-Lieb-Seiringer(2013), Frank-Sabin(2014))

Let q, r be as before.

- *(Necessary condition) If the orthonormal Strichartz* (8) *holds with β, then it must be*
$$\beta \leq \frac{4r}{r+2}.$$
- *(Sufficient condition) Assume $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) < \frac{1}{2} - \frac{1}{q}$. Then the orthonormal Strichatrz* (8) *holds for $\beta = \frac{4r}{r+2}$.*
- *Further, on the end-point: $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) = \frac{1}{2} - \frac{1}{q}$, the orthonormal Strichatrz* (8) *with $\beta = \frac{4r}{r+2}$ fails.*

Two remarks

- On the region $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) > \frac{1}{2} - \frac{1}{q}$, Frank and Sabin very recently gave some thoughts which we will introduce later.
- For the failure of the end-point $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) = \frac{1}{2} - \frac{1}{q}$ with $\beta = \frac{4r}{r+2}$, Frank, Lewin, Lieb and Seiringer found that the failure is coming from the logarithmically divergent. So, they expected the weak type estimate:

$$\left\| \left(\sum_k |e^{it\Delta} f_k|^2 \right)^{\frac{1}{2}} \right\|_{q,r} \leq C \, \|\{\|f_k\|_2\}_k\|_{\ell^{\frac{4r}{r+2},1}},$$

where $\ell^{p,r}$ denotes the Lorentz space for the sequence.

Result for the orthonormal Strichartz on the region $\frac{1}{2}(\frac{1}{2}-\frac{1}{r}) > \frac{1}{2}-\frac{1}{q}$.

Theorem 8

Let q, r satisfy the scaling condition.

- *(Necessary condition) If the orthonormal Strichartz* (8) *holds with β, then it must be*

$$\beta \leq \min\left(\frac{4r}{r+2}, q\right).$$

- *(Sufficient condition) On the region $\frac{1}{2}(\frac{1}{2}-\frac{1}{r}) > \frac{1}{2}-\frac{1}{q}$, the orthonormal Strichartz* (8) *holds with β as long as*

$$\beta < q.$$

Some remarks

- The same results were obtained by Frank and Sabin very recently.
- The advantage of our approach to this problem is on its proof. That is, we can show the result even more directly as we will see.
- The problem is almost solved for Schrödinger equation. That is, the remaining problem is whether the orthonormal Strichartz (8) with $\beta = q$ holds or not on $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) > \frac{1}{2} - \frac{1}{q}$??
- If the conjecture by Frank-Lewin-Lieb-Seiringer: the weak type orthonormal Strichartz on $\frac{1}{2}(\frac{1}{2} - \frac{1}{r}) = \frac{1}{2} - \frac{1}{q}$ is true, then the orthonormal Strichartz (8) with $\beta = q$ on the desired range follows from the real interpolation.

Counterexample for $\beta \le q$

Let us see the necessity of $\beta \le q$. Take any sequence $\{\lambda_j\}_j \in \ell^\beta$ and $\phi \in \mathcal{S}$ such that $\mathrm{supp}(\hat{\phi}) \subset B(0,1)$ with $\|\phi\|_2 = 1$. And further, choose $\{v_j\}_j \subset \mathbb{R}^d$ so that $B(v_j,100) \cap B(v_k,100) = \emptyset$. Let us put

$$f_j(x) = \lambda_j e^{-i2^j\Delta}[\phi \cdot e^{i\cdot v_j}](x).$$

The point is as follows:

1. The system $\{f_j\}_j$ is orthogonal in L^2 since $\mathrm{supp}(\hat{f}_j) \subset B(v_j,1)$.
2. We have

$$|e^{it\Delta}f_j(x)| = \lambda_j|e^{i(t-2^j)\Delta}\phi(x+2(t-2^j)v_j)|,$$

   and if $\delta > 0$ is small,

$$\|e^{i(t-2^j)\Delta}\phi\|_{L^r_x} \sim_\delta \|\phi\|_{L^r_x}, \quad (|t-2^j| \le \delta).$$

Continue

With these observations in mind,

$$
\begin{aligned}
\left\| \left(\sum_k |e^{it\Delta} f_k|^2 \right)^{\frac{1}{2}} \right\|_{q,r} &\geq \left(\sum_j \int_{2^j}^{2^j+\delta} \left[\int_{\mathbb{R}^d} \left(\sum_k |e^{it\Delta} f_k(x)|^2 \right)^{\frac{r}{2}} dx \right]^{\frac{q}{r}} dt \right)^{\frac{1}{q}} \\
&\geq \left(\sum_j \int_{2^j}^{2^j+\delta} \left[\int_{\mathbb{R}^d} \left(|e^{it\Delta} f_j(x)|^2 \right)^{\frac{r}{2}} dx \right]^{\frac{q}{r}} dt \right)^{\frac{1}{q}} \\
&\geq \left(\sum_j \lambda_j^q \int_{2^j}^{2^j+\delta} \| e^{i(t-2^j)\Delta} \phi \|_{L_x^r}^q \, dt \right)^{\frac{1}{q}} \\
&\sim \left(\sum_j \lambda_j^q \int_{2^j}^{2^j+\delta} \| \phi \|_{L_x^r}^q \, dt \right)^{\frac{1}{q}} \sim_{\delta,\phi} \left(\sum_j \lambda_j^q \right)^{\frac{1}{q}}
\end{aligned}
$$

Continue and the necessity of $\beta \le \frac{4r}{r+2}$

Meanwhile, since $\|\phi\|_2 = 1$, $\|f_k\|_2 = \lambda_k$ and hence,

$$\left(\sum_k \|f_k\|_2^\beta\right)^{\frac{1}{\beta}} = \left(\sum_k \lambda_k^\beta\right)^{\frac{1}{\beta}}.$$

(Outline of the necessity of $\beta \le 4r/(r+2)$)
Take $\psi \in C_c^\infty(B(0,1))$ and large $R > 0$. For $v \in B(0,1) \cap R^{-1}\mathbb{Z}^d$, put

$$\hat{f}_v(\xi) = R^{\frac{d}{2}} \phi(R(\xi - v)).$$

Then it follows that

$$|e^{it\Delta} f_v(x)| \ge CR^{-\frac{d}{2}} \chi_{[-R,R]\times B(0,R)}(x,t)$$

for all v. Applying this estimate to ONS, we can obtain $\beta \le \frac{4r}{r+2}$.

Frank-Lewin-Lieb-Seiringer's conjecture for $d = 1$

We get the negative result to the conjecture at least $d = 1$.

Theorem 9

Let $d = 1$. Then weak type orthonormal Strichartz fails.

Our approach is reducing the problem to Kinetic transport equation. To this end, it would be nice to rewrite the orthonormal Strichartz in the following way:

$$\Big\| \sum_k |e^{it\Delta} f_k|^2 \Big\|_{q,p} \le C \Big(\sum_k \|f_k\|_2^{2\alpha} \Big)^{\frac{1}{\alpha}}, \tag{9}$$

where the parameters q, p, α are determined by suitable replacement of original parameters q, r, β; $q/2 \mapsto q$, $r/2 \mapsto p$ and $\beta/2 \mapsto \alpha$. Note:

$$\frac{1}{q} = \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big) \Leftrightarrow \frac{2}{q} + \frac{d}{p} = d, \quad \beta = \frac{4r}{r+2} \Leftrightarrow \alpha = \frac{2p}{p+1}.$$

Relation between orthonormal Strichartz and kinetic transport equation

Theorem 10 (Sabin)

Let q, p satisfy scaling condition ($\frac{2}{q} + \frac{d}{p} = d$) and $\alpha = \frac{2p}{p+1}$. If orthonormal Strichartz (9) holds, then for same q, p, α, we have

$$\|\rho f\|_{L^q_t L^p_x} \le C \|f\|_{L^\alpha_{x,v}}.$$

It is not difficult to obtain the analogous of this relation on the Lorentz setting. That is, for q, p in the above and $r \in [1, \infty]$,

$$\left\| \sum_k |e^{it\Delta} f_k|^2 \right\|_{q,p} \le C \left\| \{\|f_k\|_2^2\}_k \right\|_{\ell^{\frac{2p}{p+1}, r}} \Rightarrow \|\rho f\|_{L^q_t L^p_x} \le C \|f\|_{L^{\frac{2p}{p+1}, r}_{x,v}}$$

Reduction

On the (q, p, α) language, Frank-Lewin-Lieb-Seiringer's weak type conjecture can be translated the following form:

$$\left\| \sum_k |e^{it\Delta} f_k|^2 \right\|_{L_t^{\frac{d+1}{d}}, L_x^{\frac{d+1}{d-1}}} \leq C \left\| \{ \|f_k\|_2^2 \}_k \right\|_{\ell^{\frac{d+1}{d}, 1}}, \tag{10}$$

From the previous observation, to show the failure of (10), it suffices to show the failure of

$$\|\rho f\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}}} \leq C \|f\|_{L_{x,v}^{\frac{d+1}{d}, 1}},$$

which is the Lorentz end-point Strichartz estimate for velocity average.

Lemma 11

Let $d = 1$. Then the following Strichartz estimate fails:

$$\|\rho f\|_{L_t^2 L_x^\infty} \leq C \|f\|_{L_{x,v}^{2,1}}.$$

Sketch of Proof

Take any small $\delta > 0$ and denote the δ-fat Kakeya set by $E_\delta \subset \mathbb{R}^2$. Then the geometrical observation tells us that

$$\sup_{x \in \mathbb{R}} \rho\chi_{E_\delta}(x,t) \geq \frac{1}{(1+t^2)^{\frac{1}{2}}}, \quad (r \in \mathbb{R}),$$

which gives a lower bound:

$$\|\rho\chi_{E_\delta}\|_{L_t^2 L_x^\infty} \geq \|(1+t^2)^{\frac{1}{2}}\|_{L_t^2} > 0.$$

Meanwhile, the right-hand side

$$\|\chi_{E_\delta}\|_{L^{2,1}_{x,v}} = |E_\delta|^{\frac{1}{2}} \to 0,$$

since the Lorentz norm does not effect for the characteristic function.

Thank you for listening!!