The trilinear restriction estimate with sharp dependence on the transversality

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ICMAT

- Given a function $f \in L^p(\mathbb{R}^d)$, does it make sense to restrict its Fourier Transform to a hypersurface?
- Well, where does the Fourier Transform of a function in $L^p(\mathbb{R}^d)$ live?
- $$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } 1 \leq p \leq 2.$$

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- Therefore, the Fourier Transform of a function in L^p with $1 \leq p \leq 2$ lives in $L^{p'}$. The L^p spaces are defined modulo zero measure sets.
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- Therefore, the Fourier Transform of a function in L^p with $1 \leq p \leq 2$ lives in $L^{p'}$. The L^p spaces are defined modulo zero measure sets.
- But $\widehat{f}$ may live in a subset of $L^{p'}$ sufficiently good so that the restriction makes sense.

We have, for example:

Theorem (Riemann–Lebesgue)

Let $f \in L^1(\mathbb{R}^d)$, then $\widehat{f}$ lives on the space of continuous functions that tend to zero at infinity.

The continuity ensures that the restriction makes sense and we have for a zero measure set S,

$$\|\widehat{f}\|_{L^\infty(S)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

- On the other hand, given $g \in L^2(\mathbb{R}^d)$, there exists $f \in L^2(\mathbb{R}^d)$ such that $\widehat{f} = g$. Indeed, the Fourier transform is a bijection from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The restriction can not make sense for functions in $L^2(\mathbb{R}^d)$.

- What is going on for the cases $f \in L^p(\mathbb{R}^d)$ with $1 < p < 2$? Well, we can not expect continuity as in the case of $L^1(\mathbb{R}^d)$ and the restriction to an arbitrary zero measure set may not make sense.

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- What is going on for the cases $f \in L^p(\mathbb{R}^d)$ with $1 < p < 2$? Well, we can not expect continuity as in the case of $L^1(\mathbb{R}^d)$ and the restriction to an arbitrary zero measure set may not make sense.

If we require some curvature to surface, things change.

Conjecture

Let $S = \mathbb{S}^{d-1}$, *if* $1 \leq p, q \leq \infty$ *then*

$$\|\widehat{f}\|_{L^q(S)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

for every function and constant C *independent of* f, *if and only if*

$$p < \frac{2d}{d+1} \qquad \textit{and} \qquad p' \geq \frac{d+1}{d-1}q.$$

More generally, for every compact surface with boundary, with non-vanishing gaussian curvature, it is conjectured the same necessary and sufficient conditions.

The Conjecture for dimension $d = 2$ was proven by Fefferman and Zygmund in the seventies.

It is open for dimension $d \geq 3$. Nevertheless, a lot of partial results have been achieved.

The boundedness of

$$\begin{array}{cccc} T: & L^p(\mathbb{R}^d) & \longrightarrow & L^q(S) \\ & f & \mapsto & \widehat{f}|_S \end{array}$$

is equivalent to the boundedness of the adjoint

$$\begin{array}{cccc} T^*: & L^{q'}(S) & \longrightarrow & L^{p'}(\mathbb{R}^d) \\ & g & \mapsto & \widehat{gd\sigma} \end{array}$$

To see that the adjoint is that one, we just use Parseval:

$$\langle Tf, g\rangle = \int_S \widehat{f}(\xi)\overline{g}(\xi)d\sigma(\xi) = \int_{\mathbb{R}^d} f(x)\overline{\int_S e^{2\pi i x\xi} g(\xi)d\sigma(\xi)}dx = \langle f, T^*g\rangle.$$

Therefore our problem is equivalent to study

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leq C\|g\|_{L^{q'}(S)}.$$

I will focus on the following tools in the restriction theory:

i) **Induction on scales**
Enlarge the scale for which an estimate is valid.

ii) L^4 **Orthorgonality**
Under assumption on S_1, S_2

$$\Big\|\Big(\sum_n \widehat{f_1\chi_{\tau_n}d\sigma}\Big)\Big(\sum_{n'} \widehat{f_2\chi_{\tau_{n'}}d\sigma}\Big)\Big\|^2_{L^2(B_R)} \sim \sum_n \sum_{n'} \|\widehat{f_1\chi_{\tau_n}d\sigma}\,\widehat{f_2\chi_{\tau_{n'}}d\sigma}\|^2_{L^2(B_R)}$$

iii) **Transversality**
Under assumption on S_n

$$\|\prod_{n=1} \widehat{f_n d\sigma}\|_{L^{p'}(B_R)} \lesssim \prod_{n=1} \|f_n\|_{L^2}$$

Our result, as we will see, links the three tools.

Why is the case of dimension $d = 2$ easy ?
i) The sum function of (a quarter of) $\mathbb{S}^1$,

$$\begin{array}{rccc} f: & \mathbb{S}^1 \times \mathbb{S}^1 & \longrightarrow & \mathbb{R}^2 \\ & (a, b) & \mapsto & a + b \end{array}$$

is injective.
It implies good orthogonality
ii) Transversality
If S_1 and S_2 are θ-arcs whose separation is comparable to θ, then

$$\| \prod_{n=1}^{2} \widehat{f_n d\sigma} \|_{L^2} \lesssim \theta^{-\frac{1}{2}} \prod_{n=1}^{2} \|f_n\|_{L^2}$$

In this case, as

$$p = \frac{4}{3} \quad \rightarrow \quad p' = 4 = 2 \times 2$$

we can use easily transversality and orthogonality.

In dimension $d = 3$ the objective is to push up p as close as possible to $\frac{3}{2}$.

$p < \frac{4}{3} = 1.33\ldots$	Tomas	1975	Measure decay
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Case $q = 2$. The relevant for dispersive equations.
The proof relies basically in the following measure asymptotic behaviour

Proposition

If $d\sigma$ is the surface measure of the sphere, then

$$\widehat{d\sigma}(x) = C\frac{e^{2\pi i|x|}}{|x|^{(d-1)/2}} + C\frac{e^{-2\pi i|x|}}{|x|^{(d-1)/2}} + O(|x|^{-d/2}) \text{ when } |x| \to \infty.$$

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Conjecture (Kakeya Conjecture)

Let $p = \frac{d}{d-1}$, *and let* $\{T_{w_i}\}_{w_i}$ *be any collection of* $R \times R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}}$ *tubes, whose orientations* w *are* $R^{-\frac{1}{2}}$ *separated . Then,*

$$\|\sum_{i=1}^{N} \chi_{T_{w_i}}\|_{L^p(\mathbb{R}^d)} \leq C_d R^{\epsilon}(NR^{\frac{d+1}{2}})^{\frac{1}{p}} \quad \textit{for every} \quad \epsilon > 0.$$

That is, the tubes behave as disjoint (up to an epsilon). The conjecture remains open in dimensions $d \geq 3$. The case $d = 2$ was proven by Córdoba.

In higher dimensions there have been some partial results.
We have the trivial bounds

$$\|\sum_{i=w}^{N} \chi_{T_w}\|_{L^\infty(\mathbb{R}^d)} \leq C_d N.$$

If we interpolate it with the conjecture, we get

$$\|\sum_{i=w}^{N} \chi_{T_w}\|_{L^p(\mathbb{R}^d)} \leq C_d R^\epsilon (NR^{\frac{d+1}{2}})^{\frac{1}{p}} N^{1-\frac{d}{p(d-1)}}$$

for $p \geq \frac{d}{d-1}$.
In dimension $d = 3$, the best result asserts

$$\|\sum_{i=w}^{N} \chi_{T_w}\|_{L^{\frac{5}{3}}(\mathbb{R}^d)} \leq C_d R^\epsilon (NR^2)^{\frac{3}{5}} N^{\frac{1}{10}}$$

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Let f_w be the characteristic function of a cap in $\mathbb{S}^{d-1}$ with radio equals to $\frac{1}{100\sqrt{R}}$ and centered at $w \in \mathbb{S}^{d-1}$. The Fourier Transform is

$$\widehat{f_w d\sigma}(x) = \int_{|w-\theta| \leq \frac{1}{100\sqrt{R}}} e^{2\pi i x \cdot \theta} d\theta.$$

Observe that the cap is contained in a disc of dimensions $1/R \times 1/\sqrt{R} \times \cdots \times 1/\sqrt{R}$. Let T_w be the tube

$$T_w := \left\{ x \in \mathbb{R}^d \; : \; |x \cdot w| \leq \frac{R}{100}, \; |x - w(x \cdot w)| \leq \frac{\sqrt{R}}{100} \right\}.$$

If $x \in T_w$ and $\theta, w \in \mathbb{S}^{d-1}$ obey $|w - \theta| \leq \frac{1}{\sqrt{R}}$, then $|x \cdot (w - \theta)| \leq \frac{1}{100}$. Hence, for $x \in T_w$,

$$\begin{aligned} |\widehat{f_w d\sigma}(x)| &= \left| \int_{|w-\theta| \leq \frac{1}{\sqrt{R}}} e^{2\pi i x \theta} d\theta \right| = \left| \int_{|w-\theta| \leq \frac{1}{\sqrt{R}}} e^{2\pi i x(\theta - w)} d\theta \right| \\ &\sim \left| \int_{|w-\theta| \leq \frac{1}{\sqrt{R}}} d\theta \right| \sim R^{-(d-1)/2}. \end{aligned}$$

Informally, in general we have

$$\widehat{fd\sigma} = \sum_{j,m} c_{j,m}\phi_{j,m}$$

where the coefficients obey $\|\{c_{j,m}\}_{j,m}\|_{\ell^2} \leq \|f\|_{L^2}$, the functions $\phi_{j,m}$ are adapted to a tube in the direction w_j and position a_m and Fourier supported in the cap centered at w_j.

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Asuming that $|a_m - a_{m'}| \sim 1$, it is an estimate of the form

$$\|\widehat{f\chi_{B_{\frac{1}{100}}(a_m)}d\sigma}\,\widehat{f\chi_{B_{\frac{1}{100}}(a_{m'})}d\sigma}\|_{L^q(\mathbb{R}^{d+1})} \lesssim \|f\chi_{B_{\frac{1}{100}}(a_m)}\|_{L^p(\mathbb{R}^d)}\|f\chi_{B_{\frac{1}{100}}(a_{m'})}\|_{L^p(\mathbb{R}^d)}.$$

Of course, by Cauchy–Schwarz, a linear estimate

$$\|\widehat{f\chi_{B_{\frac{1}{100}}(a_m)}d\sigma}\|_{L^{2q}(\mathbb{R}^{d+1})} \lesssim \|f\chi_{B_{\frac{1}{100}}(a_m)}\|_{L^p(\mathbb{R}^d)}.$$

implies the bilinear one.

Tao–Vargas–Vega proved that the bilinear estimate also implies the linear one. The key of the proof is the Whitney decomposition, which is very useful when restriction estimates depends only on the distance.

The point of the bilinear estimates is that the distance gives some help in order to prove restriction estimates. Tubes pointing out in separated directions.

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Why are multilinear estimates interesting?

Let, for example in $\mathbb{R}^d$ when S_i is the hyperplane with normal vector e_i passing through the origin for $i = 1, \cdots, d$. We have then

$$\widehat{f_1 d\sigma}(x) = \int e^{2\pi i \sum_{j>1} \xi_j \cdot x_j} f_i(\xi_2, \cdots, \xi_d) d\xi = \widehat{f_1}(x_2, \cdots, x_d),$$

$$\ldots$$

$$\widehat{f_i d\sigma}(x) = \int e^{2\pi i \sum_{j \neq i} \xi_j \cdot x_j} f_i(\xi_1, \cdots, \xi_{i-1}, \xi_{i+1}, \cdots, \xi_d) d\xi = \widehat{f_i}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d),$$

$$\ldots$$

$$\widehat{f_d d\sigma}(x) = \int e^{2\pi i \sum_{j<d} \xi_j \cdot x_j} f_i(\xi_1, \cdots, \xi_{d-1}) d\xi = \widehat{f_D}(x_1, \cdots, x_{d-1}).$$

We have clearly that $\|\widehat{f_i d\sigma}\|_{L^q} = \infty$ for every $q < \infty$ and $1 \leq i \leq d$. However we have

$$\|\prod_{i=1}^{d} \widehat{f_i d\sigma}\|_{\frac{2}{d-1}} \leq C \prod_{i=1}^{d} \|f_i\|_{L^2(S)}.$$

Let's see it, for the sake of simplicity, in the case of dimension $d = 3$

$$
\begin{aligned}
\|\widehat{f_1}\widehat{f_2}\widehat{f_3}\|_{L^1} &= \int\int\int |\widehat{f_1 d\sigma}(x)\widehat{f_2 d\sigma}(x)\widehat{f_3 d\sigma}(x)| dx_1 dx_2 dx_3 \\
&= \int\int\int |\widehat{f_1}(x_2,x_3)\widehat{f_2}(x_1,x_2)\widehat{f_3}(x_1,x_2)| dx_1 dx_2 dx_3 \\
&= \int\int |\widehat{f_1}(x_2,x_3)| \int |\widehat{f_2}(x_1,x_2)\widehat{f_3}(x_1,x_2)| dx_1 dx_2 dx_3 \\
&\leq \int\int |\widehat{f_1}(x_2,x_3)| (\int |\widehat{f_2}(x_1,x_3)|^2 dx_1)^{\frac{1}{2}} (\int |\widehat{f_3}(x_1,x_2)|^2 dx_1)^{\frac{1}{2}} dx_2 dx_3 \\
&= \int (\int |\widehat{f_2}(x_1,x_3)|^2 dx_1)^{\frac{1}{2}} \int |\widehat{f_1}(x_2,x_3)| (\int |\widehat{f_3}(x_1,x_2)|^2 dx_1)^{\frac{1}{2}} dx_2 dx_3 \\
&\leq \int (\int |\widehat{f_2}(x_1,x_3)|^2 dx_1)^{\frac{1}{2}} (\int |\widehat{f_1}(x_2,x_3)|^2 dx_2)^{\frac{1}{2}} (\int\int |\widehat{f_3}(x_1,x_2)|^2 dx_1 dx_2)^{\frac{1}{2}} dx_3 \\
&= (\int\int |\widehat{f_3}(x_1,x_2)|^2 dx_1 dx_2)^{\frac{1}{2}} \int (\int |\widehat{f_2}(x_1,x_3)|^2 dx_1)^{\frac{1}{2}} (\int |\widehat{f_1}(x_2,x_3)|^2 dx_2)^{\frac{1}{2}} dx_3 \\
&\leq (\int\int |\widehat{f_3}(x_1,x_2)|^2 dx_1 dx_2)^{\frac{1}{2}} (\int\int |\widehat{f_2}(x_1,x_3)|^2 dx_1 dx_3)^{\frac{1}{2}} (\int |\widehat{f_1}(x_2,x_3)|^2 dx_2 dx_3)^{\frac{1}{2}} \\
&= \|\widehat{f_1}\|_{L^2(S_1)} \|\widehat{f_2}\|_{L^2(S_2)} \|\widehat{f_3}\|_{L^2(S_3)} \\
&= \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)} \|f_3\|_{L^2(S_3)}.
\end{aligned}
$$

Theorem (Bennett–Carbery–Tao)

(d=3, paraboloid case) Let $S_n = \{(\xi,\tau) : \xi \in \operatorname{supp} f_n,\ |\xi|^2 = \tau\}$ *($n = 1, 2, 3$) satisfy*

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \sim 1$$

for all choices $\xi_n \in S_n$*, where* $n(\xi_n)$ *is the normal vector to* S_n *in* ξ_n*. Then there exist constants* C *and* κ *such that*

$$\| \prod_{n=1}^{3} \widehat{f_n d\sigma} \|_{L^1(B_R)} \leq C(\log_2 R)^\kappa \prod_{n=1}^{3} \|f_n\|_{L^2}$$

for all $R > 0$.

Their theorem applies to more general surfaces. The curvature does not appear, just transversality.

Roughly speaking the Bourgain-Guth argument to deduce linear estimates from trilinear ones follows the following idea:
if

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \sim 1$$

is because S_1, S_2 and S_3 live in a small neighborhood of the vertices of a triangle with area 1.

If we are not in that case, S_1, S_2 and S_3 are close to be aligned, and we have, as in the case of two dimensions, good orthogonality properties. Dichotomy between transversality and orthogonality.

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Theorem

Let $S_n = \{(\xi,\tau):\ \xi \in \operatorname{supp} f_n,\ |\xi|^2 = \tau\}$ *(*$n = 1, 2, 3$*) satisfy*

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \geq \theta$$

for all choices $\xi_n \in S_n$. *Then there exist constants* C *and* κ *such that*

$$\| \prod_{n=1}^{3} \widehat{f_n d\sigma} \|_{L^1(B_R)} \leq \theta^{-\frac{1}{2}} C (\log_2 R)^{\kappa} \prod_{n=1}^{3} \|f_n\|_{L^2}$$

for all $R > 0$.

Refined orthogonality:
Consider the cases when $S_1, S_2, S_3 \in [0,2]^2$ are
i)

$$d(S_1,S_2) \sim |S_1| \sim |S_2| \sim 2^{-j} \quad d(S_3,S_1) \sim 1 \text{ and } \angle(S_1,S_2,S_3) \sim 1$$

ii)

$$d(S_1,S_2) \sim d(S_1,S_3) \sim d(S_2,S_3) \sim 1 \text{ and } \begin{array}{l} S_1,S_2,S_3 \subset \text{ strip of width } 2^{-t} \\ \angle(S_1,S_2,S_3) \sim 2^{-t} \end{array}$$

τ^j_k:= the square with length side 2^{-j} whose left-down vertex is placed in the point k. $t^j_{w,m}$:= the strip in the plane of width 2^{-j} which passes through $m \in [0,2]^2$ and in the direction w.

Definition

Let $S_1, S_2, S_3 \subset S$ and the transversality condition holds. We write $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$ if we can find $(r, j, t, w, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{S}^1 \times \mathbb{R}^2$, $r \leq j$, such that, perhaps reordering the S_n, we have

$$S_1 \subset \{(\xi, \tfrac{1}{2}|\xi|^2) : \ \xi \in \tau^j_k \cap t^{j+t}_{w,m}\},$$
$$S_2 \subset \{(\xi, \tfrac{1}{2}|\xi|^2) : \ \xi \in \tau^j_{k'} \cap t^{j+t}_{w,m}\},$$
$$S_3 \subset \{(\xi, \tfrac{1}{2}|\xi|^2) : \ \xi \in \tau^r_{k''} \cap t^{r+t}_{w',m'}\},$$

for some k, k', k'', w' such that $m \in \tau^j_k$, $d(\tau^j_k, \tau^j_{k'}) \sim 2^{-j}$, $d(\tau^j_{k'}, \tau^r_{k''}) \sim d(\tau^j_k, \tau^r_{k''}) \sim 2^{-r}$, $|w - w'| \sim 2^{-t}$ and

$$\theta \sim 2^{-j} 2^{-r} 2^{-t}.$$

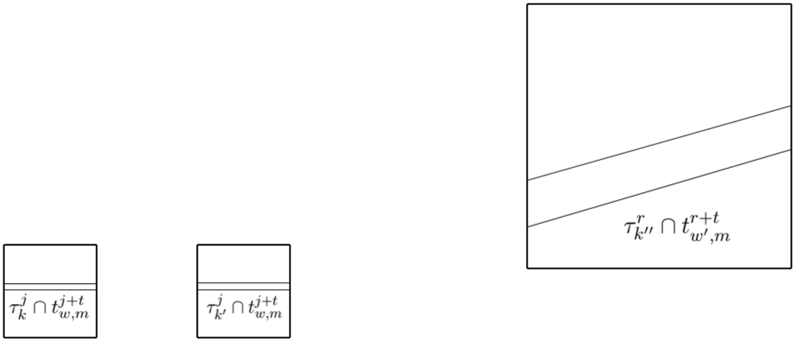

Figure : Example of S_1, S_2, S_3 with $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$.

Proposition

Let $\tau_k^j, \tau_{k'}^j$ be such that $d(\tau_k^j, \tau_{k'}^j) = 2^{-r} \gtrsim 2^{-j}$, then for every $m \in \tau_k^j$, $m' \in \tau_{k'}^j$, $w, w' \in \mathbb{S}^1$ with $|w - w'| \lesssim 2^{-t}$, we have

$$\int \phi_{\mathfrak{P}(j,t,w,m)} \left| \mathfrak{R}^* f_1 \chi_{\tau_k^j \cap t_{w,m}^{j+t}} \mathfrak{R}^* f_2 \chi_{\tau_{k'}^j \cap t_{w',m}^{j+t} \cap t_{w'^\perp,m'}^{\mathbf{2}j-r}} \right|^2$$

$$\lesssim \sum_{\substack{\alpha,\ \alpha':\ \alpha \in 2^{-(j+\mathbf{2}t)}\mathbb{Z}w \\ \alpha' \in 2^{-(\mathbf{2}j-r+\mathbf{2}t)}\mathbb{Z}w'}} \int \phi_{\mathfrak{P}(j,t,w,m)} \left| \mathfrak{R}^* f_1 \chi_{\tau_k^j \cap t_{w,m}^{j+t} \cap t_{w^\perp,\alpha}^{j+\mathbf{2}t}} \mathfrak{R}^* f_2 \chi_{\tau_{k'}^j \cap t_{w',m}^{j+t} \cap t_{w'^\perp,m'}^{\mathbf{2}j-r} \cap t_{w'^\perp,\alpha'}^{\mathbf{2}j-r+\mathbf{2}t}} \right|^2.$$

Proposition

Let $\tau_{k'}^j, \tau_{k''}^r$ be such that $d(\tau_{k'}^j, \tau_{k''}^r) \sim 2^{-r}$, then for every $w, w' \in \mathbb{S}^1$ with $|w - w'| \lesssim 2^{-t}$, $m \in \tau_{k'}^j$, we have

$$\int \phi_{\mathfrak{P}(j,t,w,m)} \left| \mathfrak{R}^* f_1 \chi_{\tau_{k'}^j \cap t_{w,m}^{j+t}} \mathfrak{R}^* f_2 \chi_{\tau_{k''}^r \cap t_{w',m}^{r+t}} \right|^2$$

$$\lesssim \sum_{\substack{\alpha,\ w'',\alpha':\ \alpha \in 2^{-(j+\mathbf{2}t)}\mathbb{Z}w, \\ w'' \in \mathbb{S}^{\mathbf{1}}_{j+t-r}, \\ \alpha' \in 2^{-(\mathbf{2}j-r+\mathbf{2}t)}\mathbb{Z}w''}} \int \phi_{\mathfrak{P}(j,t,w,m)} \left| \mathfrak{R}^* f_1 \chi_{\tau_{k'}^j \cap t_{w,m}^{j+t} \cap t_{w^\perp,\alpha}^{j+\mathbf{2}t}} \mathfrak{R}^* f_2 \chi_{\tau_{k''}^r \cap t_{w',m}^{r+t} \cap \varrho_{w'',m}^{j+t-r} \cap t_{w''^\perp,\alpha'}^{\mathbf{2}j-r+\mathbf{2}t}} \right|^2.$$

The orthogonality implies

$$\int_{\mathfrak{P}(j,t,w,m)} \left|\mathfrak{R}^* f_1 \mathfrak{R}^* f_2 \mathfrak{R}^* f_3\right| \lesssim \int_{\mathfrak{P}(j,t,w,m)} \Big(\sum_{\alpha} \Big|\mathfrak{R}^* f_1 \chi_{\tau_k^j \cap t_{w,m}^{j+t} \cap t_{w^\perp,\alpha}^{j+2t}} \Big|^2 \Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\alpha'} \Big|\mathfrak{R}^* f_2 \chi_{\tau_{k'}^j \cap t_{w,m}^{j+t} \cap t_{w^\perp,\alpha'}^{j+2t}} \Big|^2 \Big)^{\frac{1}{2}}$$

$$\Big(\sum_{w'',\alpha''} \Big|\mathfrak{R}^* f_3 \chi_{\tau_{k''}^r \cap t_{w',m}^{r+t} \cap \varrho_{w'',m}^{j+t-r} \cap t_{w''^\perp,\alpha''}^{2j-r+2t}} \Big|^2 \Big)^{\frac{1}{2}}$$

$$\lesssim 2^{\frac{j+r+t}{2}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.$$

If $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$, then by definition,

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \sim 2^{-(j+r+t)}$$

for all choices $\xi_n \in S_n$.
We can invoke Guth's Kakeya multilinear estimate with the transversality condition to perform an induction on scales.

Theorem (Guth)

If $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$*, then*

$$\int_{\mathbb{R}^3} \prod_{n=1}^{3} \Big(\sum_{k \in 2^{-\lambda}\mathbb{Z}^2 \cap \operatorname{supp} f_n} \mu_{\mathring{\tau}_k^\lambda} * \chi_{\mathring{\tau}_k^\lambda} \Big)^{\frac{1}{2}}$$

$$\lesssim 2^{\frac{j+r+t}{2}} 2^{3\lambda} \prod_{n=1}^{3} \Big(\sum_{k \in 2^{-\lambda}\mathbb{Z}^2 \cap \operatorname{supp} f_n} \|\mu_{\mathring{\tau}_k^\lambda}\| \Big)^{\frac{1}{2}}$$

for all finite measure $\mu_{\mathring{\tau}_k^\lambda}$.

Definition

We denote by $\mathcal{K}(\lambda)$ the smallest constant C such that

$$\int_{\mathfrak{P}(j,t,w,m)[\lambda]} |\mathfrak{R}^* f_1 \mathfrak{R}^* f_2 \mathfrak{R}^* f_3| \leq C 2^{\frac{j+r+t}{2}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}$$

for every f_1, f_2, f_3 with $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$.

Proposition

$\mathcal{K}(1) \lesssim 1$

Proposition

$\mathcal{K}(2\lambda) \lesssim \mathcal{K}(\lambda)$.

For a triple $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$ we have proven the result. But the hypothesis

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \gtrsim \theta$$

does not imply $(S_1, S_2, S_3) \sim (r, j, t, w, m, \theta)$.

Lemma

Let S_1, S_2, S_3 *satisfy*

$$|n(\xi_1) \wedge n(\xi_2) \wedge n(\xi_3)| \gtrsim \theta$$

for all choices $\xi_n \in S_n$. *Then, there exists a collection* $\{S_{1,i}, S_{2,i}, S_{3,i}, r_i, j_i, t_i, w_i, m_i\}_{i=1}$ *such that*
i) It is a partition

$$S_1 \times S_2 \times S_3 = \bigcup_{i \le C(\theta)} S_{1,i} \times S_{2,i} \times S_{3,i}.$$

ii) $(S_{1,i}, S_{2,i}, S_{3,i}) \sim (r_i, j_i, t_i, w_i, m_i, \theta')$ *triangle type with* $\theta \lesssim 2^{-(j+r+t)}$.
iii)

$$\sum_i \|f_1\|_{L^2(S_{1,i})} \|f_2\|_{L^2(S_{3,i})} \|f_3\|_{L^2(S_{3,i})} \le C(\log_2 \theta^{-1}) \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)} \|f_3\|_{L^2(S_3)}$$

ありがとうございました