The Fefferman-Stein type inequality for strong maximal operator in the heigher dimensions

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Introduction 1

The purpose of this talk is to develop a theory of weights for strong maximal operator in the heigher dimensions and to present an elementary proof of the endpoint estimate for the strong maximal operator.

We first fix some notations.

By weights we will always mean non-negative and locally integrable functions on $\mathbb{R}^d$.

Given a measurable set $E \subset \mathbb{R}^d$ and a weight w, $w(E) = \int_E w\,dx$, $|E|$ denotes the Lebesgue measure of E and $\mathbf{1}_E$ denotes the characteristic function of E.

Let $0 < p \le \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(\mathbb{R}^d, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(\mathbb{R}^d,w)} = \left(\int_{\mathbb{R}^d} |f|^p w\,dx\right)^{\frac{1}{p}}. \tag{A}$$

Introduction 2

For a locally integrable function f on $\mathbb{R}^d$, we define the Hardy-Littlewood maximal operator $\mathfrak{M}$ by

$$\mathfrak{M}f(x) = \sup_{Q \in \mathcal{Q}} \mathbf{1}_Q(x) ⨍_Q |f| \, dy, \tag{A}$$

where $\mathcal{Q}$ is the set of all cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes and the barred integral $⨍_Q f \, dy$ stands for the usual integral average of f over Q.

For a locally integrable function f on $\mathbb{R}^d$, we define the strong maximal operator $\mathfrak{M}_d$ by

$$\mathfrak{M}_d f(x) = \sup_{R \in \mathcal{R}_d} \mathbf{1}_R(x) ⨍_R |f| \, dy, \tag{B}$$

where $\mathcal{R}_d$ is the set of all rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes.

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It is well known that

$$w(\{x \in \mathbb{R}^d : \mathfrak{M}f(x) > t\}) \leq \frac{C}{t}\|f\|_{L^1(\mathbb{R}^d, \mathfrak{M}w)}, \quad t > 0, \tag{A}$$

holds for arbitrary weight w and, by interpolation, that

$$\|\mathfrak{M}f\|_{L^p(\mathbb{R}^d, w)} \leq C\|f\|_{L^p(\mathbb{R}^d, \mathfrak{M}w)}, \quad p > 1, \tag{B}$$

holds for arbitrary weight w.

These are called the Fefferman-Stein inequality.

Introduction 4

There is a problem in the book[1]:

Problem 1.1

Does the analogue of the Fefferman-Stein inequality hold for the strong maximal operator, i.e.

$$\|\mathfrak{M}_d f\|_{L^p(\mathbb{R}^d,w)} \leq C\|f\|_{L^p(\mathbb{R}^d,\mathfrak{M}_d w)}, \quad p > 1, \tag{1.1}$$

for arbitrary $w(x) \geq 0$?

Concerning Problem 1.1 it is known that by Lin[2] (for $d = 2$) and by Pérez[3] (for $d \geq 2$), (1.1) holds for all $p > 1$ if $w \in A_\infty^*$.

[1] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Math. Stud., **116** (1985).

[2] Kai-Ching Lin, Ph.D. University of California, Los Angeles 1984 United States. Dissertation: Harmonic Analysis on the Bidisc.

[3] C. Pérez, *A remark on weighted inequalities for general maximal operators*, Proc. Amer. Math. Soc., **119** (1993), no. 4, 1121–1126.

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We say that w belongs to the class A_p^* whenever

$$[w]_{A_p^*} = \sup_{R \in \mathcal{R}_d} \fint_R w\, dx \left(\fint_R w^{-1/(p-1)}\, dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

$$[w]_{A_1^*} = \sup_{R \in \mathcal{R}_d} \frac{\fint_R w\, dx}{\operatorname{ess\,inf}_{x \in R} w(x)} < \infty.$$

It follows by Hölder's inequality that the A_p^* classes are increasing, that is, for $1 \le p \le q < \infty$ we have $A_p^* \subset A_q^*$. Thus one defines

$$A_\infty^* = \bigcup_{p>1} A_p^*. \tag{A}$$

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The endpoint behavior of $\mathfrak{M}_d$ close to L^1 is given by Mitsis[4] (for $d = 2$) and Luque and Parissis[5] (for $d \geq 2$). That is, for $t > 0$,

$$w(\{x \in \mathbb{R}^d : \mathfrak{M}_d f(x) > t\}) \leq C \int_{\mathbb{R}^d} \frac{|f|}{t} \left(1 + \left(\log^+ \frac{|f|}{t}\right)^{d-1}\right) \mathfrak{M}_d w \, dx \tag{A}$$

holds for any $w \in A_\infty^*$, where $\log^+ t = \max(0, \log t)$.

[4] T. Mitsis, *The weighted weak type inequality for the strong maximal function*, J. Fourier Anal. Appl. **12** (2006), no. 6, 645–652.

[5] T. Luque and I. Parissis, *The endpoint Fefferman-Stein inequality for the strong maximal function*, J. Funct. Anal. **266** (2014), no. 1, 199–212.

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Concerning Problem 1.1 we established the following.

Theorem 1.2 (with H. Saito)

Let w be any weight on $\mathbb{R}^2$ and set $W = \mathfrak{M}_2\mathfrak{M}w$. Then, for $t > 0$,

$$w(\{x \in \mathbb{R}^2 : \mathfrak{M}_2 f(x) > t\}) \leq C \int_{\mathbb{R}^2} \frac{|f|}{t} \left(1 + \log^+ \frac{|f|}{t}\right) W\, dx \tag{A}$$

holds, where the constant $C > 0$ does not depend on w and f.

In this talk we consider a weaker results of Theorem 1.2 in the heigher dimensions.

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Let $c = 1, 2, \ldots, d$.
We say that the set of rectangles in $\mathbb{R}^d$ have the complexity c whenever the sidelengths of the its element R are exactly α_1 or α_2 or $\ldots$ or α_c for varying $\alpha_1, \alpha_2, \ldots, \alpha_c > 0$.
That is, the set of rectangles with complexity c is the c-parameter family of rectangles.
For a locally integrable function f on $\mathbb{R}^d$, we define the strong maximal operator $\mathfrak{M}_c$ by

$$\mathfrak{M}_c f(x) = \sup_{R \in \mathcal{R}_c} \mathbf{1}_R(x) \fint_R |f| \, dy, \tag{A}$$

where $\mathcal{R}_c$ is the set of all rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes and having the complexity c.
We notice that $\mathfrak{M}_1$ is the Hardy-Littlewood maximal operator $\mathfrak{M}$.

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Theorem 1.3

Let $c = 1, 2, \ldots, d$. Let w be any weight on $\mathbb{R}^d$ and set $W = \mathfrak{M}_c \mathfrak{M}_{c-1} \cdots \mathfrak{M}_1 w$. Then, for $p > 1$,

$$w(\{x \in \mathbb{R}^d : \mathfrak{M}_c f(x) > t\})^{\frac{1}{p}} \leq \frac{C}{t} \|f\|_{L^p(\mathbb{R}^d, W)}, \quad t > 0, \tag{A}$$

holds, where the constant $C > 0$ does not depend on w and f.

Corollary 1.4

Let $c = 1, 2, \ldots, d$. Let w be any weight on $\mathbb{R}^d$ and set $W = \mathfrak{M}_c \mathfrak{M}_{c-1} \cdots \mathfrak{M}_1 w$. Then, for $p > 1$,

$$\|\mathfrak{M}_c f\|_{L^p(\mathbb{R}^d, w)} \leq C \|f\|_{L^p(\mathbb{R}^d, W)} \tag{B}$$

holds, where the constant $C > 0$ does not depend on w and f.

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Probably, the endpoint Fefferman-Stein inequality for the strong maximal operator with compositions of some maximal operators hold in the heigher dimensions, but, I can not prove it until now. Further refinement of the known proofs for the boundedness of the strong maximal operator would be needed.

In this talk we will present an elementary proof of the endpoint estimate for the strong maximal operator. Our method used is a covering lemma for rectangles due to Robert Fefferman and Jill Pipher[6]

[6] R. Fefferman and J. Pipher, *A covering lemma for rectangles in* $\mathbb{R}^n$, Proc. Amer. Math. Soc., **133** (2005), no. 11, 3235–3241.

Theorem

In what follows we shall prove the following theorem, which is originally due to in the paper[7].

Theorem 2.1

Let $c = 1, 2, \ldots, d$. Then

$$|\{x \in \mathbb{R}^d : \mathfrak{M}_c f(x) > t\}| \le C \int_{\mathbb{R}^d} \frac{|f|}{t} \left(1 + \log^+ \frac{|f|}{t}\right)^{c-1} dx, \quad t > 0, \tag{A}$$

holds, where the constant $C > 0$ does not depend on f.

[7] B. Jessen, J. Marcinkiewicz and A. Zygmund *Note on the differentiability of multiple integrals*, Fund. Math. **25** (1935), 217–234.

Proof 1

We denote by P_i, $i = 1, 2, \ldots, d$, the projection on the x_i-axis.
First, we notice that the theorem holds for $c = 1$.
We assume that the theorem holds for $c = m - 1$ and then we shall prove it for $c = m$.
With a standard argument, we may assume that the basis $\mathcal{R}_m$ is the set of all dyadic rectangles (cartesian products of dyadic intervals).
By allowing a multiple constant $d!$, we further assume that, when $R \in \mathcal{R}_m$, the sidelengths $|P_i(R)|$ decrease and, for some fixed $\hat{m}$,

$$|P_1(R)| = |P_2(R)| = \cdots = |P_{\hat{m}}(R)| > |P_{\hat{m}+1}(R)|. \tag{A}$$

Fix $t > 0$ and given the finite collection of dyadic rectangles $\{R_i\}_{i=1}^{M} \subset \mathcal{R}_m$ such that

$$⨍_{R_i} |f| \, dy > t, \quad i = 1, 2, \ldots, M. \tag{2.1}$$

It suffices to estimate $\left|\bigcup_{i=1}^{M} R_i\right|$.

Proof 2

First relabel if necessary so that the R_i are ordered so that their long sidelengths $|P_1(R_i)|$ decrease.
We now give a selection procedure to find subcollection $\{\widetilde{R}_i\}_{i=1}^N \subset \{R_i\}_{i=1}^M$.
Take $\widetilde{R}_1 = R_1$. Suppose have now chosen the rectangles $\widetilde{R}_1, \widetilde{R}_2, \dots, \widetilde{R}_{i-1}$.
We select $\widetilde{R}_i$ to be the first rectangle R_k occurring after $\widetilde{R}_{i-1}$ so that

$$\int_{R_k} \exp\left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1\, dy < |R_k|. \tag{A}$$

Thus, we see that

$$\int_{\widetilde{R}_i} \exp\left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1\, dy < |\widetilde{R}_i|, \quad i = 2, 3, \dots, N. \tag{2.2}$$

Proof 3

We claim that

$$\bigcup_{i=1}^{M} R_i \subset \left\{ x \in \mathbb{R}^d : \mathfrak{M}_{m-1} \left[\exp\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i} \right)^{\frac{1}{m-1}} - 1 \right](x) \geq 1 \right\}. \quad (2.3)$$

Proof 4

Indeed, choose any point x inside a rectangle R_j that is not one of the selected rectangles $\widetilde{R}_i$.
Then, there exists a unique $J \leq N$ such that

$$\int_{R_j} \exp\left(\sum_{i=1}^{J} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{1}{m-1}} - 1\, dy \geq |R_j|. \tag{A}$$

Since, $|P_l(\widetilde{R}_i)| \geq |P_l(R_j)|$ for $l = 1, 2, \ldots, \hat{m}$ and $i = 1, 2, \ldots, J$, we have that

$$P_l(\widetilde{R}_i) \cap P_l(R_j) = P_l(R_j) \text{ when } \widetilde{R}_i \cap R_j \neq \emptyset, \tag{B}$$

where we have used the property of the dyadic intervals.

Proof 5

It follows from Fubini's theorem that

$$\int_{R_j} \exp\left(\sum_{i=1}^{J} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{1}{m-1}} - 1\, dy_1 dy_2 \cdots dy_d$$

$$= |P_1(R_j)|^{\hat{m}} \int_{R'_j} \exp\left(\sum_{i=1}^{J} \mathbf{1}_{\widetilde{R}'_i}\right)^{\frac{1}{m-1}} - 1\, dy_{\hat{m}+1} dy_{\hat{m}+2} \cdots dy_d,$$

where, for $R \in \mathcal{R}_m$,

$$R' = \prod_{l=\hat{m}+1}^{d} P_l(R). \tag{A}$$

Thus,

$$\int_{R'_j} \exp\left(\sum_{i=1}^{J} \mathbf{1}_{\widetilde{R}'_i}\right)^{\frac{1}{m-1}} - 1\, dy_{\hat{m}+1} dy_{\hat{m}+2} \cdots dy_d \geq |R'_j|. \tag{B}$$

Proof 6

Thanks to the fact that $|P_{\hat{m}+1}(R_j)| < |P_{\hat{m}}(R_j)|$, this implies that

$$\int_R \exp\left(\sum_{i=1}^{J} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{1}{m-1}} - 1\,dy \geq |R|, \tag{C}$$

where R is a unique dyadic rectangle containing x and satisfies

$$|P_1(R)| = |P_2(R)| = \cdots = |P_{\hat{m}}(R)| = |P_{\hat{m}+1}(R_j)|. \tag{D}$$

This proves (2.3), because such R should belong to $\mathcal{R}_{m-1}$.

Proof 7

It follows from (2.3) and our assumption that

$$\begin{aligned}\left|\bigcup_{i=1}^{M} R_i\right| &\le \left|\left\{x \in \mathbb{R}^d :\, \mathfrak{M}_{m-1}\left[\exp\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{1}{m-1}} - 1\right](x) \ge 1\right\}\right| \\ &\le C \int_{\mathbb{R}^d} \left(\exp\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{1}{m-1}} - 1\right)\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{m-2}{m-1}} dx \\ &= C \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{k+m-2}{m-1}} dx.\end{aligned}$$

Proof 8

We use an elementary inequality:

$$\left(\sum_{i=1}^{\infty} a_i\right)^s \leq s \sum_{i=1}^{\infty} a_i \left(\sum_{j=1}^{i} a_j\right)^{s-1}, \quad s > 1, \tag{2.4}$$

where $\{a_i\}_{i=1}^{\infty}$ is a sequence of summable nonnegative reals.
Then, for $k > 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right)^{\frac{k+m-2}{m-1}} dx &\leq \frac{k+m-2}{m-1} \sum_{i=1}^{N} \int_{\widetilde{R}_i} \left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{k-1}{m-1}} dx \\
&\leq k \sum_{i=1}^{N} \int_{\widetilde{R}_i} \left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{k-1}{m-1}} dx.
\end{aligned}$$

Proof 9

Inserting this estimate and changing the order of sums, we obtain

$$\left|\bigcup_{i=1}^{M} R_i\right| \leq C \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right) dx + C \sum_{i=1}^{N} \int_{\widetilde{R}_i} \exp\left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1 \, dx$$

$$\leq C \sum_{i=1}^{N} |\widetilde{R}_i|,$$

Proof 10

where we have used (2.2) and (2.5).

$$\int_{\widetilde{R}_i} \exp\left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1\, dy < |\widetilde{R}_i|, \quad i = 2, 3, \ldots, N. \tag{2.2}$$

$$\begin{aligned} &\int_{\widetilde{R}_i} \exp\left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1\, dx \qquad (2.5)\\ &\le \int_{\widetilde{R}_i} \exp\left[\left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} + 1\right] - 1\, dx \\ &\le e \int_{\widetilde{R}_i} \exp\left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j}\right)^{\frac{1}{m-1}} - 1\, dx + (e-1)|\widetilde{R}_i|. \end{aligned}$$

Proof 11

We shall evaluate the quantity:

$$(\mathrm{i}) = \sum_{i=1}^{N} |\widetilde{R}_i|.$$

By (2.1) we have that

$$\begin{aligned}(\mathrm{i}) &\le \sum_{i=1}^{N} \int_{\widetilde{R}_i} \frac{|f|}{t}\,dy \\ &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i} \right) \cdot \frac{|f|}{t}\,dx.\end{aligned}$$

$$⨍_{R_i} |f|\,dy > t, \quad i = 1, 2, \ldots, M. \tag{2.1}$$

Proof 12

We now employ the following inequality:
For $a \geq 0$, let

$$\phi(a) = \int_0^a \exp s^{\frac{1}{m-1}}\, ds. \tag{A}$$

Then we easily see that, by noticing $\phi(a) > a$,

$$ab \leq \phi(a) + b(\log^+ b)^{m-1}, \quad a, b > 0. \tag{B}$$

Proof 13

Choosing δ_0 small enough determined later, we obtain

$$
\begin{aligned}
\text{(i)} \le \delta_0 \int_{\mathbb{R}^d} \phi\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right) dx \\
+ \int_{\mathbb{R}^d} \frac{|f|}{t}\left(1+\log^{+}\frac{|f|}{\delta_0 t}\right)^{m-1} dx.
\end{aligned}
$$

We have to evaluate the quantity:

$$
\text{(ii)} = \int_{\mathbb{R}^d} \phi\left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i}\right) dx.
$$

Proof 14

There holds

$$\begin{aligned}\phi(a) &= \int_0^a \exp s^{\frac{1}{m-1}}\, ds \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^a s^{\frac{k}{m-1}}\, ds \\ &= a + \sum_{k=1}^{\infty} \frac{m-1}{(k+m-1)k!} a^{\frac{k+m-1}{m-1}},\end{aligned}$$

which entails

$$\text{(ii)} = \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i} \right) dx + \sum_{k=1}^{\infty} \frac{m-1}{(k+m-1)k!} \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i} \right)^{\frac{k+m-1}{m-1}} dx. \tag{A}$$

Proof 15

It follows from (2.4)[8] that, for $k > 0$,

$$\frac{m-1}{(k+m-1)k!}\int_{\mathbb{R}^d}\left(\sum_{i=1}^{N}\mathbf{1}_{\widetilde{R}_i}\right)^{\frac{k+m-1}{m-1}}dx$$
$$\le \frac{1}{k!}\sum_{i=1}^{N}\int_{\widetilde{R}_i}\left(\sum_{j=1}^{i}\mathbf{1}_{\widetilde{R}_j}\right)^{\frac{k}{m-1}}dx.$$

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$$\left(\sum_{i=1}^{\infty}a_i\right)^{s}\le s\sum_{i=1}^{\infty}a_i\left(\sum_{j=1}^{i}a_j\right)^{s-1},\quad s>1, \tag{2.4}$$

Proof 16

Inserting this estimate and changing the order of sums, we obtain

$$
\begin{aligned}
\text{(ii)} &\le \int_{\mathbb{R}^d} \left(\sum_{i=1}^{N} \mathbf{1}_{\widetilde{R}_i} \right) dx + \sum_{i=1}^{N} \int_{\widetilde{R}_i} \exp \left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j} \right)^{\frac{1}{m-1}} - 1 \, dx \\
&\le C_0(\mathrm{i}),
\end{aligned}
$$

where we have used (2.2) and (2.5).

$$
\int_{\widetilde{R}_i} \exp \left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j} \right)^{\frac{1}{m-1}} - 1 \, dy < |\widetilde{R}_i|, \quad i = 2, 3, \ldots, N. \tag{2.2}
$$

$$
\int_{\widetilde{R}_i} \exp \left(\sum_{j=1}^{i} \mathbf{1}_{\widetilde{R}_j} \right)^{\frac{1}{m-1}} - 1 \, dx \le e \int_{\widetilde{R}_i} \exp \left(\sum_{j=1}^{i-1} \mathbf{1}_{\widetilde{R}_j} \right)^{\frac{1}{m-1}} - 1 \, dx + (e-1)|\widetilde{R}_i|. \tag{2.5}
$$

Proof 17

If we choose δ_0 so that $C_0\delta_0 = \frac{1}{2}$, we obtain

$$\text{(i)} \le C \int_{\mathbb{R}^d} \frac{|f|}{t} \left(1 + \log^+ \frac{|f|}{t}\right)^{m-1} dx. \tag{A}$$

This completes the proof.