Saitama Math. J.
Vol. 25 (2008), 1–26

Certain quotient singularities in dimension four associated with irreducible reflection groups

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(*Received September 14, 2007; Revised May 19, 2008*)

Abstract

We study quotient singularities of certain finite reducible groups in dimension 4 associated with irreducible reflection groups in $\mathrm{GL}(3,\mathbb{C})$. We obtain 6 types of hypersurface singularities, another type of complete intersection singularities, and 2 types of non complete intersection singularities. We also obtain that their singular locuses are of pure dimension 2.

1. Introduction

Let G be a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$, $S=\mathbb{C}[X_1,\dots,X_n]$ be the polynomial ring, and let $R=S^G$ be the invariant subring of S under the natural action of G. We want to study the invariant subring $R=S^G$ and the quotient variety $\mathbb{C}^n/G$. We are interested in the following problems:

(i) To find generators of R and its relations, and to study its properties. In other words, to determine the embedding dimension of $\mathbb{C}^n/G$, and its defining equations.

(ii) The dimension and the structure of the singular locus $\mathrm{Sing}(\mathbb{C}^n/G)$.

About (i), we know that R is Gorenstein, since $G\subset \mathrm{SL}(n,\mathbb{C})$ ([9]). But R may not be a complete intersection. However, if R is a complete intersection, then its embedding dimension is at most $2n-1$ ([4]). About (ii), The dimension of the singular locus of $\mathbb{C}^n/G$ is at most $n-2$. In particular, if $\mathbb{C}^n/G$ is a complete intersection, then the dimension of its singular locus is exactly $n-2$ ([4]).

We need to study finite subgroups of $\mathrm{SL}(n,\mathbb{C})$ before to study invariant subrings $R=S^G$. We adopt the following rough classification of finite subgroups of $\mathrm{SL}(n,\mathbb{C})$.

(A) Abelian groups.

(B) Reducible groups which are not abelian.

2000 Mathematics Subject Classification. 13A50, 14L24, 14L30.

Key words and phrases. quotient singularity, invariant subring, reflection group, reducible group.

(C) Imprimitive groups.

(D) Primitive groups.

For groups of (A), invariant subrings which are complete intersections are completely classified by [10]. And it is known that the generators of R and the relations of R can be calculated since the basic invariants of R (see §2.2) can be calculated ([7]). The dimension of $\mathrm{Sing}(\mathbb{C}^n/G)$ is less than $n-2$ in some cases.

For groups of (B), (C) and (D), the structure of R and $\mathrm{Sing}(\mathbb{C}^n/G)$ are not well studied except the case of $n = 2, 3$.

In the case of $n = 2$, the group of (A) induces the rational double point of type A, the group of (C) induces the RDP of type D, and the group of (D) induces the RDP of type E.

In the case of $n = 3$, the structure of R was obtained from [8], [11]. For every family of (A), (B) and (C), we can find groups which induce each of hypersurfaces, complete intersections and non complete intersections. For the family (D), there exist 8 types of groups. Seven of them induce hypersurfaces, and another induces a complete intersection which is not a hypersurface. $\mathbb{C}^3/G$ is an isolated singularity if and only if G is a group of (A) and 1 is not an eigenvalue of A for every nontrivial element A in G ([11]). Thus $\mathrm{Sing}(\mathbb{C}^3/G)$ is of pure dimension 1 except the above case.

In the case of $n \geq 4$, almost nothing are known yet about (i) except for the family (A). About (ii), it is expected that $\mathrm{Sing}(\mathbb{C}^n/G)$ is of pure dimension $n-2$ except some cases. In fact, if there exists nontrivial A in G such that 1 is the eigenvalue of A whose multiplicity is $n-2$, then the dimension of $\mathrm{Sing}(\mathbb{C}^n/G)$ is $n-2$.

We want to study whether the results like the case $n = 3$ can be obtained in the case $n = 4$. But there are too many types of G. In this article, we treat the groups of type (B1') as follows.

The groups of type (B) is naturally classified as follows:

(B1) A direct sum of an irreducible 3-dimensional representation and a 1-dimensional representation, i.e.,

$$G = \left\{ \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array} \right) \;\middle|\; A' \in G' \right\},$$

where G' is an irreducible group of $\mathrm{GL}(3, \mathbb{C})$ (See §2.1).

(B2) A direct sum of an irreducible 2-dimensional representation and two 1-dimensional representations, i.e. any element of G is the following form:

$$\left(\begin{array}{c|c|c} A' & & \\ \hline & a & \\ \hline & & b \end{array} \right), \quad A' \in \mathrm{GL}(2, \mathbb{C}), \;\; ab \cdot \det A' = 1.$$

(B3) A direct sum of two irreducible 2-dimensional representations, i.e. any element of G is the following form:

$$\left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right), \quad A_1, A_2 \in \mathrm{GL}(2, \mathbb{C}), \quad \det A_1 \cdot \det A_2 = 1.$$

Even if we treat only type (B1), there exist too many groups. But when G' is an irreducible reflection group, the classification of G' is obtained from [5], [1]. In this article, we study $R = S^G$ with its classification, i.e. we study the invariant subrings of the following group G:

(B1')

$$G = \left\{ \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array}\right) \;\middle|\; A' \in G' \right\},$$

where G' is an irreducible reflection group of $\mathrm{GL}(3, \mathbb{C})$.

2. Preliminaries

We use the following notation:

$S = \mathbb{C}[X_1, \ldots, X_n]$	the polynomial ring.
G	a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$.
$R = S^G$	the invariant subring of G.
I_n	the identity of $\mathrm{GL}(n, \mathbb{C})$.
$\lvert G \rvert$	the order of G.

2.1 Some remarks of finite subgroups of $\mathrm{GL}(n, \mathbb{C})$ and invariant subrings of reflection groups

Definition 2.1. (i) G is called *reducible* if there exists a proper G-invariant subspace of $\mathbb{C}^n$. If G is not reducible, we say G is *irreducible*.

(ii) For an irreducible group G, G is called *imprimitive* if there exists a decomposition to vector subspaces $\mathbb{C}^n = W_1 \oplus \cdots \oplus W_r$ $(r \geq 2)$ such that the following condition is satisfied: For any $A \in G$ and $1 \leq i \leq r$, there exists $1 \leq j \leq r$ such that $A(W_i) = W_j$. If G is not imprimitive, we say G is *primitive*.

Definition 2.2. $A \in \mathrm{GL}(n, \mathbb{C})$ is called a *pseudo-reflection* if A has a finite order and if $\mathrm{rank}(I_n - A) = 1$. A finite group generated by pseudo-reflections is called a *reflection group*.

Theorem 2.3. (cf.[5]) The following two statements are equivalent:

(i) G is a refletion group.

(ii) R is a polynomial ring, i.e., R is generated by n elements which are algebraically independent over $\mathbb{C}$.

Furthermore, if G is a reflection group, the degrees of n minimal generators of R are determined uniquely.

Definition 2.4. For a reflection group G, the degrees of n generators of R which are algebraically independent are called the *degrees of* G.

Theorem 2.5. (cf.[5]) Let G be a reflection group, and let $d_1, \dots, d_n$ be the degrees of G. Then,

(i) $|G| = d_1 d_2 \cdots d_n$.

(ii) The number of pseudo-reflections in G is $\sum_{i=1}^{n}(d_i - 1)$.

Definition 2.6. For any linear character $\chi : G \longrightarrow \mathbb{C}^*$ of G, we define

$$R_\chi := \{f \in S \mid A(f) = \chi(A)f, \text{ for all } A \in G\}.$$

R_χ is an R-module. Elements of R_χ are called χ-*invariants.*

From [6] §2, if G is a reflection group, generators of R_χ over R can be obtained as the following way:

For a pseudo-reflection $A \in G$,

$$H_A := \{x \in \mathbb{C}^n \mid Ax = x\}$$

is called a *reflecting hyperplane* of A, and it is a subspace of $\mathbb{C}^n$ of dimension $n-1$. Let $H_1, \dots, H_r$ be the all distinct reflecting hyperplanes associated with G. For $i = 1, \dots, r$, let $f_i = f_i(X_1, \dots, X_n)$ be the linear form defining H_i. f_i is called a *reflecting linear form.* Let

$$C_i := \{A \in G \mid Ax = x, \text{ for all } x \in H_i\}.$$

Then C_i is a cyclic group. Let P_i be a generator of C_i. For $i = 1, \dots, r$, we choose s_i so that s_i is the least non-negative integer satisfying $\chi(P_i) = \det P_i^{s_i}$. Finally let

$$f_\chi := \prod_{i=1}^{r} f_i^{s_i}. \tag{2.7}$$

Then f_χ is a homogeneous polynomial of degree $s_1 + \cdots + s_r$, and does not depend on the choice of P_i. Moreover we have the following.

Theorem 2.8. ([6], Theorem 3.1) If G is a reflection group, then R_χ is a free R-module of rank 1 generated by the above f_χ.

2.2 Basic invariants

We shall explain the way to calculate minimal relations of generators and

Poincaré series of R according to [11].

Definition 2.9. ([11], p.40-p.41) If R can be written as a direct sum

$$R = \mathbb{C}[\xi_1, \dots, \xi_n] \oplus \mathbb{C}[\xi_1, \dots, \xi_n]\eta_1 \oplus \cdots \oplus \mathbb{C}[\xi_1, \dots, \xi_n]\eta_r \tag{2.10}$$

where $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_r$ are homogeneous polynomials of R and $\xi_1, \dots, \xi_n$ are algebraically independent over $\mathbb{C}$, (2.10) is called a *basic decomposition* of R, and $(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_r)$ are called *basic invariants* of R. Then any $f \in R$ can be written as $f = p_0 + p_1\eta_1 + \cdots + p_r\eta_r$ by certain $p_0, \dots, p_r \in \mathbb{C}[\xi_1, \dots, \xi_n]$. This is called the *basic form* of f denoted by

$$\operatorname{bas}(f) = p_0 + p_1\eta_1 + \cdots + p_r\eta_r.$$

Theorem 2.11. ([3], [11] Theorem 20) For any finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, $R = S^G$ has a basic decomposition.

The minimal relations of generators of R can be calculated in the following way([11] p.43):

Let $(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_r)$ be the basic invariants of R where $\xi_1, \dots, \xi_n$ are algebraically independent over $\mathbb{C}$ and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_t$ $(t \le r)$ are minimal generators of R. Let

$$\begin{aligned} \overline{\mathrm{Rel}}(G) &:= \left\{\eta_i\eta_j \;\middle|\; 1 \le i \le t, i \le j \le r\right\} - \left\{\eta_1, \dots, \eta_r\right\}, \\ \mathrm{Rel}(G) &:= \left\{h \in \overline{\mathrm{Rel}}(G) \;\middle|\; h' \nmid h, \text{ for any } h' \in \overline{\mathrm{Rel}}(G) - \{h\}\right\}. \end{aligned}$$

Then we have the following theorem.

Theorem 2.12. The minimal relations of generators of R are $\{h - \operatorname{bas}(h) \mid h \in \mathrm{Rel}(G)\}$.

Furthermore, Poincaré series $P(R, t)$ of R is

$$P(R, t) = \frac{1 + t^{b_1} + \cdots + t^{b_r}}{(1 - t^{d_1}) \cdots (1 - t^{d_n})}, \tag{2.13}$$

where $d_i = \deg \xi_i$ $(1 \le i \le n)$, $b_j = \deg \eta_j$ $(1 \le j \le r)$.

3. Certain reducible groups and their invariant subrings

We use the following notation:

$S' = \mathbb{C}[X_1, \dots, X_{n-1}]$	the polynomial ring, $S' \subset S$.
G'	an irreducible subgroup of $\mathrm{GL}(n-1, \mathbb{C})$, $G' \not\subset \mathrm{SL}(n-1, \mathbb{C})$.
$l = l(G')$	the least positive integer such that $\det(A')^l = 1$ for any $A' \in G'$ (note $l > 1$).

$R' = (S')^{G'}$	the invariant subring of G'.
$\chi(i)$ $(i = 0, 1, 2, \dots)$	the linear character of G' defined by $\chi(i)(A') = (\det A')^i$ $(A' \in G')$.
$R'_{\chi(i)}$ $(i = 0, 1, 2, \dots)$	the minimal R'-submodule of S' which contains all $\chi(i)$-invariants (note $R'_{\chi(0)} = R'$).

We shall study the invariant subring $R = S^G$ of the following group:

$$G = \left\{ \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array} \right) \;\middle|\; A' \in G' \right\} \subset \mathrm{SL}(n, \mathbb{C}).$$

Proposition 3.1. (cf. [8], §1) R is generated over R' by X_n^l, and $R'_{\chi(i)} X_n^i$ $(1 \leq i \leq l-1)$.

Proof. Let $\tilde{R}$ be the ring generated over R' by X_n^l, $R'_{\chi(i)} X_n^i$ $(1 \leq i \leq l-1)$. Clearly, $\tilde{R} \subset R$. Conversely, we shall show $\tilde{R} \supset R$. Let $f \in R$. Then f can be written as $f = \sum_{i=0}^r g_i X_n^i$ by some $g_i \in S'$. Let $A' \in G'$ be any element, and let

$$A = \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array} \right) \in G.$$

Since $A(f) = f$, we have

$$\sum_{i=0}^r A'(g_i) \cdot (\det A')^{-i} X_n^i = \sum_{i=0}^r g_i X_n^i.$$

This means

$$A'(g_i) = (\det A')^i g_i \quad (0 \leq i \leq r),$$

and $g_i \in R'_{\chi(i)}$ since A' is arbitrary. Define non-negative integers k_i and m_i by $0 \leq k_i \leq l-1$ and $i = m_i l + k_i$. Then,

$$\begin{aligned} R'_{\chi(i)} &= \{ f \in S' \mid A'(f) = (\det A')^i f \text{ for all } A' \in G' \} \\ &= \{ f \in S' \mid A'(f) = (\det A')^{m_i l + k_i} f \text{ for all } A' \in G' \} \\ &= \{ f \in S' \mid A'(f) = (\det A')^{k_i} f \text{ for all } A' \in G' \} \\ &= R'_{\chi(k_i)}. \end{aligned}$$

Thus $g_i \in R'_{\chi(k_i)}$. Finally we have

$$f = \sum_{i=0}^r g_i X_n^i = \sum_{i=0}^r g_i X_n^{k_i} \cdot (X_n^l)^{m_i} \in \tilde{R}.$$

$\square$

We assume G' is a reflection group. Then R' is a polynomial ring. Let $y_1, \dots, y_{n-1}$ be homogeneous algebraically independent generators of R' over $\mathbb{C}$, and let $R_0 := \mathbb{C}[y_1, \dots, y_{n-1}, X_n^l]$. Let $f_{\chi(i)}$ be the generator of $R'_{\chi(i)}$ constructed as (2.7). Then we have the following theorem.

Theorem 3.2.

$$R = R_0 \oplus R_0 f_{\chi(1)} X_n \oplus R_0 f_{\chi(2)} X_n^2 \oplus \cdots \oplus R_0 f_{\chi(l-1)} X_n^{l-1}.$$

In other words,

$$(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_r) = (y_1, \dots, y_{n-1}, X_n^l; f_{\chi(1)} X_n, f_{\chi(2)} X_n^2, \dots, f_{\chi(l-1)} X_n^{l-1}) \qquad (r = l-1)$$

are the basic invariants of R.

Proof. We use the same notation in the proof of Proposition 3.1. From Proposition 3.1, $R = R_0[f_{\chi(1)} X_n, \dots, f_{\chi(l-1)} X_n^{l-1}]$. Let $f \in R$. Since $0 \leq k_i \leq l-1$, and $g_i \in R'_{\chi(k_i)}$ can be written as $g_i = h_i f_{\chi(k_i)}$ $(h_i \in R')$,

$$f = \sum_{i=0}^{r} g_i X_n^{k_i} \cdot (X_n^l)^{m_i} = \sum_{i=0}^{r} f_{\chi(k_i)} X_n^{k_i} \cdot h_i \cdot (X_n^l)^{m_i} = \sum_{j=0}^{l-1} \tilde{h_j} f_{\chi(j)} X_n^j$$

for certain $\tilde{h_j} \in R_0$. Thus

$$R = R_0 + R_0 f_{\chi(1)} X_n + \cdots + R_0 f_{\chi(l-1)} X_n^{l-1}.$$

Since $R_0 = \mathbb{C}[y_1, \dots, y_{n-1}, X_n^l]$, and y_j does not contain X_n, we have

$$R_0 + R_0 f_{\chi(1)} X_n + \cdots + R_0 f_{\chi(l-1)} X_n^{l-1} = R_0 \oplus R_0 f_{\chi(1)} X_n \oplus \cdots \oplus R_0 f_{\chi(l-1)} X_n^{l-1}.$$

□

Corollary 3.3. Let $y_n = X_n^l$ and $y_{n+1} = f_{\chi(1)} X_n$. If $f_{\chi(i)} = f_{\chi(1)}^i$ $(1 \leq i \leq l-1)$, then $R = \mathbb{C}[y_1, \dots, y_n, y_{n+1}]$, and the generators have a unique relation of the form

$$y_{n+1}^l - y_n F(y_1, \dots, y_{n-1}) = 0,$$

where $F(y_1, \dots, y_{n-1})$ is a polynomial of $y_1, \dots, y_{n-1}$.

Proof. From Theorem 3.2,

$$R = R_0 \oplus R_0 y_{n+1} \oplus R_0 y_{n+1}^2 \oplus \cdots \oplus R_0 y_{n+1}^{l-1}.$$

Thus $R = \mathbb{C}[y_1, \dots, y_n, y_{n+1}]$ and $\mathrm{Rel}(G) = \{y_{n+1}^l\}$. From §2.2, $y_1, \dots, y_n, y_{n+1}$ have a unique relation $y_{n+1}^l - \mathrm{bas}(y_{n+1}^l) = 0$. Since $f_{\chi(1)}^l \in \mathbb{C}[y_1, \dots, y_{n-1}]$,

$\mathrm{bas}(y_{n+1}^l) = f_{\chi(1)}^l X_n^l$ is a polynomial of the form $y_n F(g_1, \ldots, y_{n-1})$. □

We can calculate the generators, the relations, and Poincaré series $P(R,t)$ of R in the following way:

(1) Calculate the generators $y_1, \ldots, y_{n-1}$ of R'.

(2) Calculate the generator $f_{\chi(i)}$ of the free R'-module $R'_{\chi(i)}$ as (2.7).

(3) Then $(y_1, \ldots, y_{n-1}, X_n^l; f_{\chi(1)}X_n, \ldots, f_{\chi(l-1)}X_n^{l-1})$ are the basic invariants of R from Theorem 3.2. Thus we can calculate the minimal relations and Poincaré series $P(R,t)$ of R as explained in §2.2.

Next, we shall prove the following theorem.

Theorem 3.4. The dimension of the singular locus of $\mathbb{C}^n/G$ is $n-2$.

Proof. Let $F = \{x \in \mathbb{C}^n \mid Ax = x, \text{ for some } A \in G,\ A \neq I_n\}$. Since the singular locus of $\mathbb{C}^n/G$ is F/G (see Theorem 5.1), it is sufficient to show there eixists a linear subspace H of $\mathbb{C}^n$ of dimension $n-2$ such that $H \subset F$. Since G' is a reflection group, G' has pseudo-reflections. Let $A' \in G'$ be a pseudo-reflection, and we put

$$A := \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array}\right) \in G, \qquad H_A := \{x \in \mathbb{C}^n \mid Ax = x\}.$$

Then 1 is the eigenvalue of A whose mutiplicity is $n-2$. Thus $\dim H_A = n-2$. Futhermore, $H_A \subset F$. □

4. Main results

In this section we study the invariant subring of the group

$$G = \left\{ \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array}\right) \;\middle|\; A' \in G' \right\} \subset \mathrm{SL}(4, \mathbb{C})$$

where G' is an irreducible reflection group of $\mathrm{GL}(3, \mathbb{C})$. It is known that G' is conjugate to one of the following groups ([5], [1]).

Group	Order	Number of pseudo-reflections	Degrees
$G(m,p,3)$	$6qm^2$	$3(m+q-1)$	m, $2m$, $3q$
$W(H_3)$	120	15	2, 6, 10
$W(J_3(4))$	336	21	4, 6, 14
$W(L_3)$	648	24	6, 9, 12
$W(M_3)$	1296	33	6, 12, 18
$W(J_3(5))$	2160	45	6, 12, 30

Table 4.1

Here m, p, q are positive integers such that $m > 1$ and $m = pq$. Each group will be explained later.

Remark 4.2. $W(A_3)$ in [1] is conjugate to $G(2, 2, 3)$.

We will calculate the invariant subring $R = S^G$ for each above group. As one of the consequences we obtain the following theorems.

Theorem 4.3. (i) If G' is one of the following groups

$$G(m, m, 3),\ G(2p, p, 3),\ W(H_3),\ W(J_3(4)),\ W(L_3),\ W(J_3(5)),$$

then R is a hypersurface.

(ii) If $G' = G(pq, p, 3)$ where q is an even number such that $q \geq 4$, then R is a complete intersection, and embedding dimension of R is $\mathrm{emb}(R) = 6$.

(iii) If G' is one of the following groups

$$G(pq, p, 3)\ (q \text{ is odd, } q \geq 3),\ W(M_3),$$

then R is not a complete intersection, $\mathrm{emb}(R) = 8$, and the number of relations of generators is 9.

Theorem 4.4. For any G' in the above table, $\mathrm{Sing}(\mathbb{C}^4/G)$ is of pure dimension 2.

Remark 4.5. We note that if $\mathbb{C}^4/G$ is a complete intersection, the above theorem is a special case of [2], exp. 10.

Now, we start calculations of $R = S^G$. We use the notation such as

$$\begin{aligned}
\sum_3 X_1^2 &:= X_1^2 + X_2^2 + X_3^2, \\
\sum_3 X_1^4 X_2^3 X_3^2 &:= X_1^4 X_2^3 X_3^2 + X_1^3 X_2^2 X_3^4 + X_1^2 X_2^4 X_3^3, \\
\sum_6 X_1^4 X_2^3 X_3^2 &:= X_1^4 X_2^3 X_3^2 + X_1^3 X_2^2 X_3^4 + X_1^2 X_2^4 X_3^3 \\
&\quad + X_1^4 X_2^2 X_3^3 + X_1^2 X_2^3 X_3^4 + X_1^3 X_2^4 X_3^2, \\
\prod_3 (\alpha X_1 + \beta X_2 + \gamma X_3) &:= (\alpha X_1 + \beta X_2 + \gamma X_3)(\beta X_1 + \gamma X_2 + \alpha X_3) \\
&\quad \times (\gamma X_1 + \alpha X_2 + \beta X_3).
\end{aligned}$$

4.1 $G' = G(m, p, 3)$ $(m = pq > 1)$

$G(m, p, 3)$ is defined as the group generated by $\mathfrak{S}_3$ and

$$A(m, p, 3) := \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle|\ a_1^m = a_2^m = a_3^m = 1, (a_1 a_2 a_3)^q = 1 \right\},$$

where we regard $\mathfrak{S}_3 \subset \mathrm{GL}(3,\mathbb{C})$ by the natural way. The generators of R' are the following polynomials (see [1]):

$$\begin{aligned} y_1 &= X_1^m + X_2^m + X_3^m, \\ y_2 &= X_1^m X_2^m + X_2^m X_3^m + X_3^m X_1^m, \\ y_3 &= (X_1 X_2 X_3)^q. \end{aligned}$$

We note the polynomials $\sigma_1 = X+Y+Z$, $\sigma_2 = XY+YZ+ZX$, $\sigma_3 = XYZ$ and $\delta = (X-Y)(Y-Z)(Z-X)$ have the relation

$$\delta^2 + 4\sigma_1^3\sigma_3 - \sigma_1^2\sigma_2^2 - 18\sigma_1\sigma_2\sigma_3 + 4\sigma_2^3 + 27\sigma_3^2 = 0. \tag{4.6}$$

4.1.1 The case $m = p$, $q = 1$.

As is explained in [1], $G' = G(m,m,3)$ is generated by the following elements as a reflection group:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m & 0 \\ \zeta_m^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \left(\zeta_m = \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)\right).$$

Thus $l(G') = 2$. The number of pseudo-reflctions of G' is $3m$, and any pseudo-reflection of G' is one of the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_m^i \\ 0 & \zeta_m^{-i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \zeta_m^i \\ 0 & 1 & 0 \\ \zeta_m^{-i} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m^i & 0 \\ \zeta_m^{-i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0 \le i \le m-1). \tag{4.7}$$

The reflecting linear forms of above pseudo-reflections are

$$\zeta_m^i X_2 - X_3,\ \zeta_m^i X_3 - X_1,\ \zeta_m^i X_1 - X_2 \quad (0 \le i \le m-1).$$

By (2.7), we have

$$f_{\chi(1)} = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m).$$

From Corollary 3.3, R is generated by y_1, y_2, y_3, and

$$y_4 = X_4^2,\ y_5 = f_{\chi(1)} X_4 = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m)X_4.$$

From (4.6), we have the following relation:

$$y_5^2 + y_4(4y_1^3 y_3^m - y_1^2 y_2^2 - 18 y_1 y_2 y_3^m + 4y_2^3 + 27 y_3^{2m}) = 0.$$

In other words

$$\begin{aligned} R &= \mathbb{C}[y_1, y_2, y_3, y_4, y_5] \\ &\cong \mathbb{C}[Y_1, Y_2, Y_3, Y_4, Y_5]/(Y_5^2 + Y_4(4Y_1^3Y_3^m - Y_1^2Y_2^2 - 18Y_1Y_2Y_3^m + 4Y_2^3 + 27Y_3^{2m}))). \end{aligned}$$

Thus, R is a hypersurface. From (2.13), Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1 - t^{2(3m+1)}}{(1-t^m)(1-t^{2m})(1-t^3)(1-t^2)(1-t^{3m+1})}.$$

4.1.2 The case $m \neq p$, $q > 1$.

$G' = G(m,p,3)$ is generated by $G(m,m,3)$, and

$$\begin{pmatrix} \zeta_q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are $3(m+q-1)$ pseudo-reflections, and they are of the forms (4.7) or

$$\begin{pmatrix} \zeta_q^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_q^i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_q^i \end{pmatrix} \quad (1 \leq i \leq q-1). \tag{4.8}$$

Let $P_1 = P_1(X_1, X_2, X_3)$ be the product of the all reflecting linear forms obtained from the pseudo-reflections of type (4.7), and let $P_2 = P_2(X_1, X_2, X_3)$ be the product of those of type (4.8). Then

$$P_1 = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m), \ P_2 = X_1X_2X_3.$$

We note that $P_1^2, P_2^q \in \mathbb{C}[y_1, y_2, y_3]$.
(a) The case q is even.

By the method explained in §3, we obtain $f_{\chi(i)}$ and $f_{\chi(i)}X_4^i$ as the following table:

i	$f_{\chi(i)}$	$f_{\chi(i)}X_4^i$	$\deg f_{\chi(i)}X_4^i$
1	P_1P_2	$P_1P_2X_4 =: y_5$	$3m+4$
2	P_2^2	$P_2^2X_4^2 =: y_6$	8
3	$P_1P_2^3$	$P_1P_2^3X_4^3 = y_5y_6$	$3m+12$
4	P_2^4	$P_2^4X_4^4 = y_6^2$	16
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q-2$	P_2^{q-2}	$P_2^{q-2}X_4^{q-2} = y_6^{(q-2)/2}$	$4(q-2)$
$q-1$	$P_1P_2^{q-1}$	$P_1P_2^{q-1}X_4^{q-1} = y_5y_6^{(q-2)/2}$	$3m+4(q-1)$

By Theorem 3.2, R is generated by 1 and $f_{\chi(1)}X_4, f_{\chi(2)}X_4^2, \ldots, f_{\chi(q-1)}X_4^{q-1}$ as R_0-module. From the above table, we can choose y_1, y_2, y_3 and

$$y_4 = X_4^q,$$
$$y_5 = P_1P_2X_4 = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m)X_1X_2X_3X_4,$$
$$y_6 = P_2^2X_4^2 = (X_1X_2X_3X_4)^2.$$

as a system of generators of $\mathbb{C}$-algebra R. From §2.2, we have $\mathrm{Rel}(G) = \{y_5^2, y_6^{q/2}\}$. By (4.6), we conclude that the relations are

$$y_5^2 + y_6(4y_1^3y_3^p - y_1^2y_2^2 - 18y_1y_2y_3^p + 4y_2^3 + 27y_3^{2p}) = 0,$$
$$y_6^{q/2} - y_3y_4 = 0.$$

Thus, R is a complete intersection. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{(1-t^{4q})(1-t^{2(3m+4)})}{(1-t^m)(1-t^{2m})(1-t^q)(1-t^{3q})(1-t^8)(1-t^{3m+4})}.$$

(b) The case q is odd.

The $\chi(i)$-invariant $f_{\chi(i)}$ and $f_{\chi(i)}X_4^i$ are:

i	$f_{\chi(i)}$	$f_{\chi(i)}X_4^i$	$\deg f_{\chi(i)}X_4^i$
1	P_1P_2	$P_1P_2X_4 =: y_5$	$3m+4$
2	P_2^2	$P_2^2X_4^2 =: y_6$	8
3	$P_1P_2^3$	$P_1P_2^3X_4^3 = y_5y_6$	$3m+12$
4	P_2^4	$P_2^4X_4^4 = y_6^2$	16
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q-1$	P_2^{q-1}	$P_2^{q-1}X_4^{q-1} = y_6^{(q-1)/2}$	$4(q-1)$
q	P_1	$P_1X_4^q =: y_7$	$3m+q$
$q+1$	P_2	$P_2X_4^{q+1} =: y_8$	$q+4$
$q+2$	$P_1P_2^2$	$P_1P_2^2X_4^{q+2} = y_6y_7$	$3m+q+8$
$q+3$	P_2^3	$P_2^3X_4^{q+3} = y_6y_8$	$q+12$
$q+4$	$P_1P_2^4$	$P_1P_2^4X_4^{q+4} = y_6^2y_7$	$3m+q+16$
$q+5$	P_2^5	$P_2^5X_4^{q+5} = y_6^2y_8$	$q+20$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2q-1$	$P_1P_2^{q-1}$	$P_1P_2^{q-1}X_4^{2q-1} = y_6^{(q-1)/2}y_7$	$3m+q+4(q-1)$

Thus R is generated by y_1, y_2, y_3, and

$$y_4 = X_4^{2q},$$
$$y_5 = P_1P_2X_4 = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m)X_1X_2X_3X_4,$$
$$y_6 = P_2^2X_4^2 = (X_1X_2X_3X_4)^2,$$
$$y_7 = P_1X_4^q = (X_1^m - X_2^m)(X_2^m - X_3^m)(X_3^m - X_1^m)X_4^q,$$
$$y_8 = P_2X_4^{q+1} = X_1X_2X_3X_4^{q+1}.$$

From §2.2, we have

$$\mathrm{Rel}(G) = \left\{ y_5^2, y_5y_7, y_5y_8, y_5y_6^{(q-1)/2}, y_6^{(q+1)/2}, y_6^{(q-1)/2}y_8, y_7^2, y_7y_8, y_8^2 \right\}.$$

With the help of (4.6) and Theorem 2.12, we easily obtain that the minimal relations are

$$\begin{aligned} &y_5^2 + y_6(4y_1^3y_3^p - y_1^2y_2^2 - 18y_1y_2y_3^p + 4y_2^3 + 27y_3^{2p}) = 0,\\ &y_5y_7 + y_8(4y_1^3y_3^p - y_1^2y_2^2 - 18y_1y_2y_3^p + 4y_2^3 + 27y_3^{2p}) = 0,\\ &y_5y_8 - y_6y_7 = 0,\\ &y_5y_6^{(q-1)/2} - y_3y_7 = 0,\\ &y_6^{(q+1)/2} - y_3y_8 = 0,\\ &y_6^{(q-1)/2}y_8 - y_3y_4 = 0,\\ &y_7^2 - y_4(4y_1^3y_3^p - y_1^2y_2^2 - 18y_1y_2y_3^p + 4y_2^3 + 27y_3^{2p}) = 0,\\ &y_7y_8 - y_4y_5 = 0,\\ &y_8^2 - y_4y_6 = 0. \end{aligned} \tag{4.9}$$

Thus, R is not a complete intersection and $\mathrm{emb}(R) = 8$. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1 + t^{q+4} + t^{3m+q} + t^{3m+4} - t^{4q}(t^4 + t^q + t^{3m} + t^{3m+q+4})}{(1-t^m)(1-t^{2m})(1-t^{3q})(1-t^{2q})(1-t^8)}.$$

4.2 $G' = W(H_3)$

Let $\alpha = \sqrt{5} - 1$, $\beta = \sqrt{5} + 1$. $W(H_3)$ is defined as the group generated by following elements:

$$\begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} -\alpha & 2 & \beta\\ 2 & \beta & -\alpha\\ \beta & -\alpha & 2 \end{pmatrix}. \tag{4.10}$$

Since each determinant of the above generators is equal to -1, we have $l = 2$. For a homogeneous polynomial f, we denote the Reynolds operator by

$$\rho(f) = \frac{1}{|G'|} \sum_{A' \in G'} A'(f).$$

It is convenient to use GAP4 ([12]) to calculate $\rho(f)$, for GAP4 provides all elements of G' from (4.10). As the Table 4.1, the degrees of the generators of R' are 2, 6 and 10. Thus we can choose $\rho(X_1^2)$, $\rho(X_1^6)$, $\rho(X_1^{10})$ as a system of the generators of the $\mathbb{C}$-algebra R'. But to simplify, we take the following polynomials instead:

$$
\begin{aligned}
y_1 &:= 3\rho(X_1^2) = \sum_3 X_1^2, \\
y_2 &:= \alpha\left(\rho(X_1^6) - \frac{7}{48}y_1^3\right) \\
&= \frac{1}{2^4}\left[2\sum_3 X_1^4X_2^2 + (\sqrt{5}-3)\sum_3 X_1^2X_2^4 + 4(1-\sqrt{5})X_1^2X_2^2X_3^2\right], \\
y_3 &:= -\frac{3(\sqrt{5}+5)}{50}\rho(X_1^{10}) + \frac{19(\sqrt{5}+5)}{3200}y_1^5 - \frac{43\sqrt{5}+80}{200}y_1^2y_2 \\
&= \frac{1}{2^7}\left[2\sum_3 X_1^2X_2^8 + (1+\sqrt{5})\sum_3 X_1^6X_2^4 - (14+2\sqrt{5})\sum_3 X_1^6X_2^2X_3^2\right. \\
&\qquad \left. -2\sqrt{5}\sum_3 X_1^4X_2^6 + (15+5\sqrt{5})\sum_3 X_1^4X_2^4X_3^2\right].
\end{aligned}
$$

We can find pseudo-reflections among 120 elements of G', by determine whether $\mathrm{rank}(I_3 - A) = 1$, using GAP4. Thus we find 15 pseudo-reflections, and the product of their forms gives $f_{\chi(1)}$.

$$
\begin{aligned}
f_{\chi(1)} = X_1X_2X_3 &\cdot \prod_3(2X_1+\beta X_2+\alpha X_3)\cdot\prod_3(2X_1+\beta X_2-\alpha X_3) \\
&\times \prod_3(2X_1-\beta X_2+\alpha X_3)\cdot\prod_3(2X_1-\beta X_2-\alpha X_3).
\end{aligned}
$$

By Theorem 3.2, R is generated by y_1, y_2, y_3, and

$$
\begin{aligned}
y_4 &:= X_4^2, \\
y_5 &:= f_{\chi(1)}\cdot X_4 \\
&= X_1X_2X_3X_4\cdot\prod_3(2X_1+\beta X_2+\alpha X_3)\cdot\prod_3(2X_1+\beta X_2-\alpha X_3) \\
&\quad \times \prod_3(2X_1-\beta X_2+\alpha X_3)\cdot\prod_3(2X_1-\beta X_2-\alpha X_3).
\end{aligned}
$$

Note that $f_{\chi(1)}^2 \in R'$. With the help of computer, we eliminate X_1, X_2, X_3, X_4 from y_1, y_2, y_3, y_4, y_5, and we have

$$
\begin{aligned}
y_5^2 + y_4\Big[&2^8(1+\sqrt{5})y_1^6y_2^3 - 2^3(1+3\sqrt{5})y_1^3y_2^4 + 2\cdot 3^3(385-383\sqrt{5})y_2^5 \\
&- 2^{11}y_1^7y_2y_3 + 2^5(45-\sqrt{5})y_1^4y_2^2y_3 - 2^{11}\cdot 3^2\cdot 5(5+3\sqrt{5})y_1y_2^3y_3 \\
&\quad - 2^7(3-\sqrt{5})y_1^5y_3^2 + 2^3\cdot 5^2(4-\sqrt{5})y_1^2y_2y_3^2 + 5^2(5-\sqrt{5})y_3^3\Big] = 0.
\end{aligned}
$$

So, R is a hypersurface. From (2.13), Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1-t^{32}}{(1-t^2)^2(1-t^6)(1-t^{10})(1-t^{16})}.$$

4.3 $G' = W(J_3(4))$

In this case, our calculation proceed similarly as §4.2. Let $\alpha = (1+\sqrt{-7})/2$. $W(J_3(4))$ is defined as the group generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 & \overline{\alpha} \\ -1 & 1 & \overline{\alpha} \\ \alpha & \alpha & 0 \end{pmatrix}.$$

Then $l = 2$, and R' is generated by

$$\begin{aligned}
y_1 &:= 12\rho(X_1^4) = \sum_3 X_1^4 - 3\alpha \sum_3 X_1^2 X_2^2, \\
y_2 &:= \frac{224}{9}\rho(X_1^6) \\
&= 2\sum_3 X_1^6 + 5\alpha \sum_6 X_1^4 X_2^2 - (30+10\sqrt{-7})X_1^2X_2^2X_3^2, \\
y_3 &:= \frac{1}{17}\left(4096\rho(X_1^{14}) - 191 y_1^2 y_2\right) \\
&= 16\alpha \sum_6 X_1^{12}X_2^2 + \frac{1}{2}\left(131 - 49\sqrt{-7}\right)\sum_6 X_1^{10}X_2^4 \\
&\quad + (35+233\sqrt{-7})\sum_3 X_1^{10}X_2^2X_3^2 + \frac{3}{2}\left(-49+\sqrt{-7}\right)\sum_6 X_1^8X_2^6 \\
&\quad + \frac{1}{2}\left(563+159\sqrt{-7}\right)\sum_6 X_1^8X_2^4X_3^2 - (357+273\sqrt{-7})\sum_3 X_1^6X_2^6X_3^2 \\
&\quad + (95+609\sqrt{-7})\sum_3 X_1^6X_2^4X_3^4
\end{aligned}$$

R is generated by y_1, y_2, y_3, and

$$\begin{aligned}
y_4 &:= X_4^2, \\
y_5 &:= 2^6 \cdot 7^3 \cdot \sqrt{7} \cdot X_1X_2X_3X_4(X_1^2 - X_2^2)(X_2^2 - X_3^2)(X_3^2 - X_1^2) \\
&\quad \times \prod_3(\overline{\alpha}X_1 + X_2 + X_3) \cdot \prod_3(\overline{\alpha}X_1 - X_2 + X_3) \cdot \prod_3(\overline{\alpha}X_1 + X_2 - X_3) \\
&\quad \times \prod_3(\overline{\alpha}X_1 - X_2 - X_3).
\end{aligned}$$

The relation is

$$y_5^2 + y_4[320y_1^9y_2 + 272y_1^6y_2^3 - 196y_1^3y_2^5 + 27y_2^7 + 112y_1^7y_3$$

$$+1736y_1^4y_2^2y_3 - 441y_1y_2^4y_3 + 1568y_1^2y_2y_3^2 + 343y_3^3] = 0.$$

Thus, R is a hypersurface. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1-t^{44}}{(1-t^4)(1-t^6)(1-t^{14})(1-t^2)(1-t^{22})}.$$

4.4 $G' = W(L_3)$

This case also similar to §4.2 Let ω be the cubic root of 1. $W(L_3)$ is defined as the group generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} \omega+2 & \omega-1 & \omega-1 \\ \omega-1 & \omega+2 & \omega-1 \\ \omega-1 & \omega-1 & \omega+2 \end{pmatrix}.$$

Then, $l = 3$. R' is generated by

$$\begin{aligned}
y_1 &:= 18\rho(X_1^6) = \sum_3 X_1^6 - 10\sum_3 X_1^3X_2^3, \\
y_2 &:= 6\rho(X_1^6X_2^3) = \sum_3 X_1^6X_2^3 - \sum_3 X_1^3X_2^6, \\
y_3 &:= \frac{81}{155}\rho(X_1^{12}) - \frac{41}{930}y_1^2 \\
&= \sum_6 X_1^9X_2^3 - 4\sum_3 X_1^6X_2^6 + 2\sum_3 X_1^6X_2^3X_3^3.
\end{aligned}$$

The $f_{\chi(i)}$'s are

$$\begin{aligned}
f_{\chi(1)} &= X_1X_2X_3(X_1^3 + X_2^3 + X_3^3 - 3X_1X_2X_3) \\
&\quad \times \prod_3 (X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 - X_2X_3 - X_3X_1), \\
f_{\chi(2)} &= f_{\chi(1)}^2.
\end{aligned}$$

Thus R is generated by y_1, y_2, y_3, and

$$\begin{aligned}
y_4 &:= X_4^3, \\
y_5 &:= X_1X_2X_3X_4(X_1^3 + X_2^3 + X_3^3 - 3X_1X_2X_3) \\
&\quad \times \prod_3 (X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 - X_2X_3 - X_3X_1).
\end{aligned}$$

The relation is

$$4y_5^3 + y_4(y_1^3y_2^2 + 108y_2^4 + 36y_1y_2^2y_3 - y_1^2y_3^2 - 32y_3^3) = 0.$$

Thus, R is a hypersurface. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1-t^{39}}{(1-t^6)(1-t^9)(1-t^{12})(1-t^3)(1-t^{13})}.$$

4.5 $G' = W(M_3)$

$W(M_3)$ is defined as the group generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} \omega+2 & \omega-1 & \omega-1 \\ \omega-1 & \omega+2 & \omega-1 \\ \omega-1 & \omega-1 & \omega+2 \end{pmatrix}.$$

The order of any pseudo-reflection of G' is 2 or 3, and $l = 6$. We can choose the generators of R' as the followings:

$$\begin{aligned} y_1 &:= 18\rho(X_1^6) = \sum_3 X_1^6 - 10\sum_3 X_1^3X_2^3, \\ y_2 &:= \frac{81}{155}\rho(X_1^{12}) - \frac{41}{930}y_1^2 \\ &= \sum_6 X_1^9X_2^3 - 4\sum_3 X_1^6X_2^6 + 2\sum_3 X_1^6X_2^3X_3^3, \\ y_3 &:= \frac{486}{4181}\rho(X_1^{18}) - \frac{1093}{112887}y_1^3 - \frac{3598}{12543}y_1y_2 \\ &= (X_1^3 - X_2^3)^2(X_2^3 - X_3^3)^2(X_3^3 - X_1^3)^2. \end{aligned}$$

Let $P_1 = P_1(X_1, X_2, X_3)$ be the product of the all reflecting linear forms obtained from the pseudo-reflections of order 2, and let $P_2 = P_2(X_1, X_2, X_3)$ be the product of those of order 3. Then

$$\begin{aligned} P_1 &= (X_1^3 - X_2^3)(X_2^3 - X_3^3)(X_3^3 - X_1^3), \\ P_2 &= X_1X_2X_3(X_1^3 + X_2^3 + X_3^3 - 3X_1X_2X_3) \\ &\quad \times \prod_3 (X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 - X_2X_3 - X_3X_1). \end{aligned}$$

We note that $P_1^2, P_2^3 \in \mathbb{C}[y_1, y_2, y_3]$. $f_{\chi(i)}$ and $f_{\chi(i)}X_4^i$ are

i	$f_{\chi(i)}$	$f_{\chi(i)}X_4^i$	$\deg f_{\chi(i)}X_4^i$
1	P_1P_2	$P_1P_2X_4 =: y_5$	22
2	P_2^2	$P_2^2X_4^2 =: y_6$	26
3	P_1	$P_1X_4^3 =: y_7$	12
4	P_2	$P_2X_4^4 =: y_8$	16
5	$P_1P_2^2$	$P_1P_2^2X_4^5 = y_6y_7$	38

R is generated by y_1, y_2, y_3, and

$$y_4 := X_4^6,\ y_5 := P_1P_2X_4,\ y_6 := P_2^2X_4^2,\ y_7 := P_1X_4^3,\ y_8 := P_2X_4^4.$$

From §2.2, we have

$$\mathrm{Rel}(G) = \left\{y_5^2, y_5y_6, y_5y_7, y_5y_8, y_6^2, y_6y_8, y_7^2, y_7y_8, y_8^2\right\}.$$

It is clear that $P_1^2 = y_3$, and using computer, we have

$$P_2^3 = \frac{1}{4}y_1^2y_2 + 8y_2^3 - \frac{1}{4}y_1^3y_3 - 9y_1y_2y_3 - 27y_3^2.$$

By Theorem 2.12, the minimal relations are

$$\begin{aligned}
& y_5^2 - y_3y_6 = 0, \\
& y_5y_6 - \left(\frac{1}{4}y_1^2y_2 + 8y_2^3 - \frac{1}{4}y_1^3y_3 - 9y_1y_2y_3 - 27y_3^2\right) y_7 = 0, \\
& y_5y_7 - y_3y_8 = 0, \\
& y_5y_8 - y_6y_7 = 0, \\
& y_6^2 - \left(\frac{1}{4}y_1^2y_2 + 8y_2^3 - \frac{1}{4}y_1^3y_3 - 9y_1y_2y_3 - 27y_3^2\right) y_8 = 0, \\
& y_6y_8 - \left(\frac{1}{4}y_1^2y_2 + 8y_2^3 - \frac{1}{4}y_1^3y_3 - 9y_1y_2y_3 - 27y_3^2\right) y_4 = 0, \\
& y_7^2 - y_3y_4 = 0, \\
& y_7y_8 - y_4y_5 = 0, \\
& y_8^2 - y_4y_6 = 0.
\end{aligned} \tag{4.11}$$

Thus, R is not a complete intersection, and $\mathrm{emb}(R) = 8$. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1 + t^{12} + t^{16} + t^{22} + t^{26} + t^{38}}{(1-t^6)^2(1-t^{12})(1-t^{18})}.$$

4.6 $G' = W(J_3(5))$

This case is similar to §4.2. Let $\alpha = \sqrt{5}-1$, $\beta = \sqrt{5}+1$. $W(J_3(5))$ is defined as the group generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad -\frac{1}{4}\begin{pmatrix} \alpha & 2\omega^2 & \beta\omega \\ 2\omega & -\beta & \alpha\omega^2 \\ \beta\omega^2 & \alpha\omega & -2 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 2 & \alpha & -\beta \\ \alpha & \beta & 2 \\ -\beta & 2 & -\alpha \end{pmatrix}.$$

Since these determinants are -1, we have $l = 2$. Using GAP4, we can represent 2160 elements of $W(J_3(5))$ by matrices, and can compute $\rho(X_1^k)$. We shall choose the following y_1, y_2, y_3 of degrees $6, 12, 30$ as the generators of R'. Note that we choose somewhat complicated coefficients for y_2, y_3 to simplify the later calculation.

$$
\begin{aligned}
y_1 &:= 16\rho(X_1^6)\\
&= 4\sum_3 X_1^6 + 3\sqrt{5}\left((-1+\sqrt{5})\omega + 1 + \sqrt{5}\right)\sum_3 X_1^4X_2^2\\
&\quad - 3\sqrt{5}\left((1+\sqrt{5})\omega + 2\right)\sum_3 X_1^2X_2^4 + 12\sqrt{5}\left(2\omega + \sqrt{5} + 1\right)X_1^2X_2^2X_3^2,\\
y_2 &:= \left(\sqrt{-15} + 3\sqrt{5} - 5\sqrt{-3} + 5\right)\left(\frac{80}{39}\rho(X_1^{12}) + \frac{37}{12480}y_1^2\right)\\
&= 4\sum_3 X_1^{10}X_2^2 + 4\omega^2\sum_3 X_1^2X_2^{10} + \frac{1}{4}\left((5\sqrt{5}+3)\omega - 5\sqrt{5} + 3\right)\sum_3 X_1^8X_2^4\\
&\quad - \frac{3}{2}\left((3\sqrt{5}-13)\omega + 6\sqrt{5}\right)\sum_3 X_1^8X_2^2X_3^2\\
&\quad - \frac{1}{4}\left(10\sqrt{5}\omega + 5\sqrt{5} + 3\right)\sum_3 X_1^4X_2^8\\
&\quad - \frac{1}{2}\left((5\sqrt{5}+13)\omega + 10\sqrt{5}\right)\sum_3 X_1^6X_2^6\\
&\quad + \frac{1}{2}\left(38\sqrt{5}\omega + 19\sqrt{5} + 21\right)\sum_3 X_1^6X_2^4X_3^2\\
&\quad + \frac{1}{2}\left((-19\sqrt{5}-21)\omega + 19\sqrt{5} - 21\right)\sum_3 X_1^6X_2^2X_3^4\\
&\quad + \frac{5}{2}\left((13\sqrt{5}-27)\omega + 26\sqrt{5}\right)X_1^4X_2^4X_3^4,\\
y_3 &:= \left[\frac{\sqrt{-15} - 15\sqrt{-3} + 3\sqrt{5} + 15}{5^2}\left(-2^{30}\rho(X_1^{30}) + \frac{3^2\cdot 11\cdot 61099}{2^8}y_1^5\right)\right.\\
&\quad + \frac{3\cdot 797\cdot 911}{2^3}\left(\sqrt{-15} + 3\sqrt{-3} + 3\sqrt{5} - 3\right)y_1^3y_2\\
&\quad \left. -\frac{3^2}{2\cdot 5}\left(2287\cdot 7187\sqrt{-15} - 5\cdot 181\cdot 10949\right)y_1y_2^2\right]\cdot\frac{1}{181\cdot 10949}\\
&= 2^8\cdot 3^2\cdot\omega^2\sum_3 X_1^{26}X_2^2X_3^2\\
&\quad - 4\left(53\sqrt{-15} - 237\sqrt{-3} - 159\sqrt{5} - 237\right)\sum_3 X_1^{24}X_2^6\\
&\quad - 12\left(17\sqrt{-15} - 279\sqrt{-3} + 51\sqrt{5} + 279\right)\sum_3 X_1^{24}X_2^4X_3^2\\
&\quad + 24\left(17\sqrt{-15} + 279\right)\sum_3 X_1^{24}X_2^2X_3^4\\
&\quad - \frac{3}{4}\left(1093\sqrt{-15} - 3459\sqrt{-3} + 3279\sqrt{5} + 3459\right)\sum_3 X_1^{22}X_2^8
\end{aligned}
$$

$$
\begin{aligned}
&+ 6\left(835\sqrt{-15} + 4917\right) \sum_{3} X_1^{22} X_2^{6} X_3^{2} \\
&- \frac{9}{2}\left(869\sqrt{-15} - 2205\sqrt{-3} - 2607\sqrt{5} - 2205\right) \sum_{3} X_1^{22} X_2^{4} X_3^{4} \\
&- 3\left(835\sqrt{-15} - 4917\sqrt{-3} + 2505\sqrt{5} + 4917\right) \sum_{3} X_1^{22} X_2^{2} X_3^{6} \\
&+ \frac{3}{2}\left(1093\sqrt{-15} + 3459\right) \sum_{3} X_1^{8} X_2^{22} \\
&- \frac{9}{4}\left(2283\sqrt{-15} + 1229\right) \sum_{3} X_1^{20} X_2^{10} \\
&- \frac{9}{8}\left(161\sqrt{-15} + 17287\sqrt{-3} - 483\sqrt{5} + 17287\right) \sum_{6} X_1^{20} X_2^{8} X_3^{2} \\
&- \frac{9}{4}\left(13221\sqrt{-15} + 25309\sqrt{-3} + 39663\sqrt{5} - 25309\right) \sum_{3} X_1^{20} X_2^{6} X_3^{4} \\
&+ \frac{9}{2}\left(13221\sqrt{-15} - 25309\right) \sum_{3} X_1^{20} X_2^{4} X_3^{6} \\
&+ \frac{9}{8}\left(2283\sqrt{-15} - 1229\sqrt{-3} + 6849\sqrt{5} + 1229\right) \sum_{3} X_1^{10} X_2^{20} \\
&- \frac{1}{8}\left(5333\sqrt{-15} - 15501\sqrt{-3} - 15999\sqrt{5} - 15501\right) \sum_{6} X_1^{18} X_2^{12} \\
&+ \frac{147}{2}\left(311\sqrt{-15} + 159\sqrt{-3} + 933\sqrt{5} - 159\right) \sum_{3} X_1^{18} X_2^{10} X_3^{2} \\
&- \frac{3}{4}\left(37967\sqrt{-15} + 662793\right) \sum_{3} X_1^{18} X_2^{8} X_3^{4} \\
&- \left(83483\sqrt{-15} - 287715\sqrt{-3} - 250449\sqrt{5} - 287715\right) \sum_{3} X_1^{18} X_2^{6} X_3^{6} \\
&+ \frac{3}{8}\left(37967\sqrt{-15} - 662793\sqrt{-3} + 113901\sqrt{5} + 662793\right) \sum_{3} X_1^{18} X_2^{4} X_3^{8} \\
&- 147\left(311\sqrt{-15} - 159\right) \sum_{3} X_1^{18} X_2^{2} X_3^{10} \\
&- 6\left(145\sqrt{-15} - 279\sqrt{-3} + 435\sqrt{5} + 279\right) \sum_{3} X_1^{16} X_2^{14} \\
&+ \frac{3}{4}\left(74645\sqrt{-15} + 8883\right) \sum_{3} X_1^{16} X_2^{12} X_3^{2} \\
&+ \frac{3}{8}\left(135263\sqrt{-15} - 949383\sqrt{-3} - 405789\sqrt{5} - 949383\right) \sum_{6} X_1^{16} X_2^{10} X_3^{4}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{9}{4}\Big(42593\sqrt{-15}-168903\sqrt{-3}+127779\sqrt{5}+168903\Big)\sum_3 X_1^{16}X_2^{8}X_3^{6}\\
&-\frac{9}{2}\left(42593\sqrt{-15}+168903\right)\sum_3 X_1^{16}X_2^{6}X_3^{8}\\
&-\frac{3}{8}\Big(74645\sqrt{-15}-8883\sqrt{-3}+223935\sqrt{5}+8883\Big)\sum_3 X_1^{16}X_2^{2}X_3^{12}\\
&+12\left(145\sqrt{-15}+279\right)\sum_3 X_1^{14}X_2^{16}\\
&+9\Big(1783\sqrt{-15}+11937\sqrt{-3}-5349\sqrt{5}+11937\Big)\sum_3 X_1^{14}X_2^{14}X_3^{2}\\
&+\frac{9}{2}\Big(56385\sqrt{-15}-102631\sqrt{-3}+169155\sqrt{5}+102631\Big)\sum_3 X_1^{14}X_2^{12}X_3^{4}\\
&+30\left(27559\sqrt{-15}+24945\right)\sum_3 X_1^{14}X_2^{10}X_3^{6}\\
&+27\Big(14357\sqrt{-15}+51763\sqrt{-3}-43071\sqrt{5}+51763\Big)\sum_3 X_1^{14}X_2^{8}X_3^{8}\\
&-15\Big(27559\sqrt{-15}-24945\sqrt{-3}+82677\sqrt{5}+24945\Big)\sum_3 X_1^{14}X_2^{6}X_3^{10}\\
&-9\left(56385\sqrt{-15}+102631\right)\sum_3 X_1^{14}X_2^{4}X_3^{12}\\
&+12\Big(10975\sqrt{-15}-123399\sqrt{-3}-32925\sqrt{5}-123399\Big)\sum_3 X_1^{12}X_2^{12}X_3^{6}\\
&+6\Big(41071\sqrt{-15}+160791\sqrt{-3}+123213\sqrt{5}-160791\Big)\sum_3 X_1^{12}X_2^{10}X_3^{8}\\
&-12\left(41071\sqrt{-15}-160791\right)\sum_3 X_1^{12}X_2^{8}X_3^{10}\\
&-21\Big(80083\sqrt{-15}-177627\sqrt{-3}-240249\sqrt{5}-177627\Big)X_1^{10}X_2^{10}X_3^{10}.
\end{aligned}
$$

R is generated by y_1, y_2, y_3, and

$$
\begin{aligned}
y_4 &:= \omega^2 X_4^2,\\
y_5 &:= 2^{12}\cdot 3\cdot X_1X_2X_3X_4\cdot\prod_3(X_1+\omega X_2)\cdot\prod_3(X_1-\omega X_2)\\
&\quad\times\prod_3(X_1+\omega X_2+\gamma X_3)\cdot\prod_3(X_1-\omega X_2+\gamma X_3)\cdot\prod_3(X_1+\omega X_2-\gamma X_3)\\
&\quad\times\prod_3(X_1-\omega X_2-\gamma X_3)\cdot\prod_3(X_1+\beta X_2+\alpha X_3)\cdot\prod_3(X_1-\beta X_2+\alpha X_3)\\
&\quad\times\prod_3(X_1+\beta X_2-\alpha X_3)\cdot\prod_3(X_1-\beta X_2-\alpha X_3)\cdot\prod_3(X_1+\omega^2\alpha X_2+\omega\beta X_3)
\end{aligned}
$$

$$\times \prod_3 (X_1 - \omega^2\alpha X_2 + \omega\beta X_3) \cdot \prod_3 (X_1 + \omega^2\alpha X_2 - \omega\beta X_3)$$
$$\times \prod_3 (X_1 - \omega^2\alpha X_2 - \omega\beta X_3),$$

where

$$\gamma = \frac{1}{4}\omega(\sqrt{-15} - \sqrt{-3} - \sqrt{5} - 3).$$

The following relation can be easily obtained using GAP4, by linear elimination of the monomials $X_1^{i_1} \cdots X_4^{i_4}$ from some monomials $y_1^{j_1} \cdots y_5^{j_5}$ of lower degrees.

$$\begin{aligned}
y_5^2 + y_4 \Big[&\frac{1}{4} \cdot 3^2 \cdot 5^2 \left(11\sqrt{-15} - 45\sqrt{-3} - 33\sqrt{5} - 45\right) y_3^3 \\
&+ 2^3 \cdot 3^3 \cdot 5^2 \left(3\sqrt{-15} - \sqrt{-3} - 9\sqrt{5} - 1\right) y_1 y_2^2 y_3^2 \\
&- 45 \left(13\sqrt{-15} + 5\right) y_1^3 y_2 y_3^2 \\
&+ \frac{1}{4} \left(7\sqrt{-15} + 17\sqrt{-3} + 21\sqrt{5} - 17\right) y_1^5 y_3^2 \\
&+ 2^{10} \cdot 3^5 \left(5\sqrt{-15} + 3\sqrt{-3} + 15\sqrt{5} - 3\right) y_2^5 y_3 \\
&- 144 \left(677\sqrt{-15} + 1035\sqrt{-3} - 2031\sqrt{5} + 1035\right) y_1^2 y_2^4 y_3 \\
&+ 8 \left(659\sqrt{-15} + 219\right) y_1^4 y_2^3 y_3 \\
&- \frac{9}{4} \left(7\sqrt{-15} + 17\sqrt{-3} + 21\sqrt{5} - 17\right) y_1^6 y_2^2 y_3 \\
&- 2^{15} \cdot 3^5 \left(\sqrt{-15} - 7\sqrt{-3} + 3\sqrt{5} + 7\right) y_1 y_2^7 \\
&- 48 \left(6087\sqrt{-15} + 23633\sqrt{-3} - 18261\sqrt{5} + 23633\right) y_1^3 y_2^6 \\
&+ 2^7 \cdot 3^2 \left(5\sqrt{-15} + 3\right) y_1^5 y_2^5 \Big] = 0.
\end{aligned}$$

Thus, R is a hypersurface. Poincaré series $P(R,t)$ is

$$P(R,t) = \frac{1 - t^{92}}{(1-t^6)(1-t^{12})(1-t^{30})(1-t^2)(1-t^{46})}.$$

5. Proof of Theorem 4.4

In this section, we study the singular locus Sing$V$ of the quotient variety $V = \mathbb{C}^4/G$ associated with the invariant subring $R = S^G$ as §4. We note the following theorem.

Theorem 5.1. (cf. [11]) Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a subgroup which contains no pseudo-reflections, and let

$$F = \{x \in \mathbb{C}^n \mid Ax = x, \text{ for some } A \in G, \ A \neq I_n\}.$$

Then the singular locus of $\mathbb{C}^n/G$ is F/G.

We use the following notation:

$$r := \mathrm{emb}(R),\ y_4 := X_4^l,\ z_i := f_{\chi(m_i)},\ y_i := z_i X_4^{m_i}\ (5 \leq i \leq r),$$
$$\tilde{G} := G' \cap \mathrm{SL}(3, \mathbb{C}),\ \tilde{V} := \mathbb{C}^3/\tilde{G},\ \tilde{R} := (S')^{\tilde{G}}.$$

We can write

$$R' = \mathbb{C}[y_1, y_2, y_3],\ \ R = \mathbb{C}[y_1, y_2, y_3, y_4, y_5, \ldots, y_r],$$
$$\tilde{R} = \mathbb{C}[y_1, y_2, y_3, z_5, z_6 \ldots, z_r].$$

By this representation, we consider $V \subset \mathbb{C}^r$ and $\tilde{V} \subset \mathbb{C}^{r-1}$. Let $\varphi : \mathbb{C}^4 \longrightarrow V$ and $\tilde{\varphi} : \mathbb{C}^3 \longrightarrow \tilde{V}$ be the natural surjections.

For $A \in G$, we define A' by

$$A = \left(\begin{array}{c|c} A' & \\ \hline & (\det A')^{-1} \end{array}\right).$$

And we put

$$\mathcal{R}(G) := \{A \in G \mid A' \text{ is a pseudo-reflection in } G'\},$$
$$\mathcal{S}(G) := \{A \in G \mid A' \in \tilde{G},\ \mathrm{rank}(A' - I_3) = 2\}.$$

Note that $\mathcal{R}(G) \neq \emptyset$ and $\mathcal{S}(G) \neq \emptyset$. Since any pseudo-reflection in G' is not contained in $\tilde{G}$, $\mathcal{R}(G) \cap \mathcal{S}(G) = \emptyset$.

We put

$$H_A := \{x \in \mathbb{C}^4 \mid Ax = x\},\ \tilde{H}_A := \{x' \in \mathbb{C}^3 \mid A'x' = x'\}.$$

Then, we have

$$\mathrm{Sing}\tilde{V} = \bigcup_{A \in \mathcal{S}(G)} \tilde{\varphi}(\tilde{H}_A)$$

by Theorem 5.1. For every $A \in \mathcal{S}(G)$, $\tilde{\varphi}(\tilde{H}_A)$ is an irreducible component of $\mathrm{Sing}\tilde{V}$, and $\dim \tilde{\varphi}(\tilde{H}_A) = \dim \tilde{H}_A = 3 - \mathrm{rank}(A' - I_3) = 1$.

Theorem 5.2. (i) For any $A \in \mathcal{S}(G)$, there exists the natural surjection $\tilde{\varphi}(\tilde{H}_A) \times \mathbb{C} \longrightarrow \varphi(H_A)$.

(ii) $\mathrm{Sing}V = \bigcup_{A \in \mathcal{R}(G) \cup \mathcal{S}(G)} \varphi(H_A)$.

(iii) $\mathrm{Sing}V$ is of pure dimension 2.

Proof. (i) Let $A \in \mathcal{S}(G)$ and $\deg z_j = b_j$ $(5 \leq j \leq r)$. We define the morphism

$\psi : \tilde{V} \times \mathbb{C} \longrightarrow V$ by

$$\psi(y_1, y_2, y_3, z_5, \ldots, z_r, s) = (y_1, y_2, y_3, s^l, z_5 s^{b_5}, \ldots, z_r s^{b_r}).$$

Then ψ is surjective, and the following diagram is commutative:

$$\begin{array}{ccc} \tilde{V} \times \mathbb{C} & \xrightarrow{\psi} & V \\ {\scriptstyle \tilde{\varphi} \times \mathrm{id}_{\mathbb{C}}} \uparrow & & \uparrow {\scriptstyle \varphi} \\ \mathbb{C}^3 \times \mathbb{C} & \xrightarrow{=} & \mathbb{C}^4 \end{array}$$

Furthermore, $\psi\left(\tilde{\varphi}(\tilde{H}_A) \times \mathbb{C}\right) = \varphi(H_A)$.

(ii) Clearly, $\mathrm{Sing} V \supset \bigcup_{A \in \mathcal{R}(G) \cup \mathcal{S}(G)} \varphi(H_A)$. Conversely, we shall show $\mathrm{Sing} V \subset \bigcup_{A \in \mathcal{R}(G) \cup \mathcal{S}(G)} \varphi(H_A)$. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ such that $\varphi(x) \in \mathrm{Sing} V$.

First, suppose that $x_4 \neq 0$. Then there exists $A \in G$, $A \neq I_4$ such that $Ax = x$. Since $x_4 \neq 0$, we have $\det A' = 1$, i.e. $A' \in \tilde{G}$. Thus $\varphi(x) \in \varphi(H_B)$ by some $B \in \mathcal{S}(G)$.

Next, suppose that $x_4 = 0$. Let $x' = (x_1, x_2, x_3)$.

Case 1: R is a hypersuface.

As we studied in §4.1, §4.2, §4.3, §4.4 and §4.6, the relation is of the form

$$f := y_5^l + y_4 F(y_1, y_2, y_3),$$

where $F = F(y_1, y_2, y_3)$ is a polynomial of y_1, y_2, y_3. We note that $F = f_{\chi(1)}^l$ is a product of reflecting linear forms of G'. Since

$$\frac{\partial f}{\partial y_4} = F(y_1, y_2, y_3),$$

we have $F(x') = F(y_1(x'), y_2(x'), y_3(x')) = f_{\chi(1)}^l(x') = 0$. Thus there exists a reflecting linear form L of G' such that $L(x') = 0$. We take a pseudo-reflection $A' \in G'$ assciated with L, and we let $A \in \mathcal{R}(G)$ be the element associated with A'. Then $x \in H_A$, thus $\varphi(x) \in \varphi(H_A)$.

Case 2: $G' = G(pq, p, 3)$ (q is even, $q \geq 4$).

The relations are

$$f_1 := y_5^2 + y_6 F(y_1, y_2, y_3), \ f_2 := y_6^{q/2} - y_3 y_4,$$

where $F(y_1, y_2, y_3) = 4y_1^3 y_3^p - y_1^2 y_2^2 - 18 y_1 y_2 y_3^p + 4 y_2^3 + 27 y_3^{2p}$. We note that $F = P_1^2$ and $y_3 = P_2^q$. The Jacobian matrix $J = \left(\dfrac{\partial f_i}{\partial y_j}\right)_{i,j}$ is

$$J = \begin{bmatrix} y_6F_{y_1} & y_6F_{y_2} & y_6F_{y_3} & 0 & 2y_5 & F \\ 0 & 0 & -y_4 & -y_3 & 0 & \frac{q}{2} \cdot y_6^{(q-2)/2} \end{bmatrix}.$$

We have $y_4(x) = y_5(x) = y_6(x) = 0$ from $x_4 = 0$, and $\mathrm{rank}J(\varphi(x)) < \mathrm{emb}(R) - \dim V = 6 - 4 = 2$ where $J(\varphi(x))$ is the Jacobian matrix at $\varphi(x)$. Thus $F(x) = 0$ or $y_3(x) = 0$, i.e. $P_1(x) = 0$ or $P_2(x) = 0$. Both of them, there exists a reflecting linear form L of G' such that $L(x') = 0$. The rest of the proof is the same as Case 1.

Case 3: $G' = G(pq, p, 3)$ (q is odd, $q \geq 3$).

By the relations (4.9), we obtain the Jacobian matrix

$$J = \begin{bmatrix} y_6F_{y_1} & y_6F_{y_2} & y_6F_{y_3} & 0 & 2y_5 & F & 0 & 0 \\ y_8F_{y_1} & y_8F_{y_2} & y_8F_{y_3} & 0 & y_7 & 0 & y_5 & 0 \\ 0 & 0 & 0 & 0 & y_8 & -y_7 & -y_6 & y_5 \\ 0 & 0 & -y_7 & 0 & y_6^{(q-1)/2} & \frac{q-1}{2} \cdot y_5y_6^{(q-3)/2} & -y_3 & 0 \\ 0 & 0 & -y_8 & 0 & 0 & \frac{q+1}{2} \cdot y_6^{(q-1)/2} & 0 & -y_3 \\ 0 & 0 & -y_4 & -y_3 & 0 & \frac{q-1}{2} \cdot y_8y_6^{(q-3)/2} & 0 & y_6^{(q-1)/2} \\ -y_4F_{y_1} & -y_4F_{y_2} & -y_4F_{y_3} & -F & 0 & 0 & 2y_7 & 0 \\ 0 & 0 & 0 & -y_5 & -y_4 & 0 & y_8 & y_7 \\ 0 & 0 & 0 & -y_6 & 0 & -y_4 & 0 & 2y_8 \end{bmatrix},$$

where $F = 4y_1^3y_3^p - y_1^2y_2^2 - 18y_1y_2y_3^p + 4y_2^3 + 27y_3^{2p}$. We have $y_4(x) = y_5(x) = y_6(x) = y_7(x) = y_8(x) = 0$ from $x_4 = 0$, and $\mathrm{rank}J(\varphi(x)) < \mathrm{emb}(R) - \dim V = 8 - 4 = 4$. Thus $F(x) = 0$ or $y_3(x) = 0$. The rest of the proof is the same as Case 2.

Case 4: $G' = W(M_3)$

Recall the relations (4.11). Let $F = F(y_1, y_2, y_3) = \dfrac{1}{4}y_1^2y_2 + 8y_2^3 - \dfrac{1}{4}y_1^3y_3 - 9y_1y_2y_3 - 27y_3^2$. Note that $F = P_2^3$, $y_3 = P_1^2$, furthermore P_1 and P_2 are products of reflecting linear forms of G'. The Jacobian matrix is

$$J = \begin{bmatrix} 0 & 0 & -y_6 & 0 & 2y_5 & -y_3 & 0 & 0 \\ -y_7F_{y_1} & -y_7F_{y_2} & -y_7F_{y_3} & 0 & y_6 & y_5 & -F & 0 \\ 0 & 0 & -y_8 & 0 & y_7 & 0 & y_5 & -y_3 \\ 0 & 0 & 0 & 0 & y_8 & -y_7 & -y_6 & y_5 \\ -y_8F_{y_1} & -y_8F_{y_2} & -y_8F_{y_3} & 0 & 0 & 2y_6 & 0 & F \\ -y_4F_{y_1} & -y_4F_{y_2} & -y_4F_{y_3} & -F & 0 & y_8 & 0 & y_6 \\ 0 & 0 & -y_4 & -y_3 & 0 & 0 & 2y_7 & 0 \\ 0 & 0 & 0 & -y_5 & -y_4 & 0 & y_8 & y_7 \\ 0 & 0 & 0 & -y_6 & 0 & -y_4 & 0 & 2y_8 \end{bmatrix}.$$

We have $y_4(x) = y_5(x) = y_6(x) = y_7(x) = y_8(x) = 0$ from $x_4 = 0$, and $\mathrm{rank}J(\varphi(x)) < \mathrm{emb}(R) - \dim V = 8 - 4 = 4$. Thus $F(x) = 0$ or $y_3(x) = 0$.

The rest of the proof is the same as Case 2.

(iii) For any $A \in \mathcal{R}(G) \cup \mathcal{S}(G)$, $\dim \varphi(H_A) = \dim H_A = 2$. Thus $\mathrm{Sing}V$ is of pure dimension 2 from (ii). $\square$

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