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On Heegaard splittings of link exteriors

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Abstract

The tunnel number of knots directly gives the Heegaard genus of their exteriors. For the link case, if we admit in addition splittings of link exteriors into two compression bodies, things become more complicated. In this paper we introduce a concept of *types* of Heegaard splittings for compact orientable 3-manifolds and give relations between these types. We also discuss in detail the 2-component link case using examples.

1. Introduction

Let S be a connected closed orientable surface. A *compression body* H is a 3-manifold obtained by attaching 2-handles to $S \times [0,1]$ on $S \times \{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component corresponding to $S \times \{1\}$ of ∂H is denoted by $\partial_+ H$ and $\partial H - \partial_+ H$ is denoted by $\partial_- H$. A compression body H is called a *handlebody* if $\partial_- H = \emptyset$.

Suppose a compact 3-manifold M is the union of two compression bodies H_1 and H_2 attached along their common boundary $S = \partial_+ H_1 = \partial_+ H_2$, we call the decomposition $M = H_1 \cup_S H_2$ a *Heegaard splitting* of M and S a *Heegaard surface* of M. The *Heegaard genus of* a 3-manifold M, denoted by $g(M)$, is the minimal genus of Heegaard surfaces of M. The genus of a surface S is denoted by $g(S)$. If $g(S) = n$, $g(\partial_- H_1) = m$, and $g(\partial_- H_2) = l$, then we call a Heegaard splitting $M = H_1 \cup_S H_2$ a *Heegaard splitting of type* $(n; m, l)$ or a *type* $(n; m, l)$ *splitting*. If H_1 (resp. H_2) is a handlebody, we define the type to be $(n; 0, l)$ (resp. $(n; m, 0)$). We call a Heegaard surface of a type $(n; m, l)$ splitting simply *type* $(n; m, l)$ *splitting surface*. For example, tunnel number one link exteriors have type $(2; 2, 0)$ splittings.

For an arbitrary compact 3-manifold M, if M has a type $(n; m, l)$ splitting, then we can obtain a type $(n+1; m, l)$ splitting by a stabilization. Moreover, the following proposition is known as a *boundary stabilization* (see [6]).

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Proposition 1.1. *If M has a type $(n; m, l)$ splitting $M = H_1 \cup H_2$ so that $g(\partial_- H_1) = m$, $g(\partial_- H_2) = l$, and $\partial_- H_2$ has a genus k component, then M has also a type $(n + k; m + k, l - k)$ splitting.*

Let M be a closed orientable 3-manifold with torus boundary components. Then we can consider M to be the link exterior $E(L)$ for some link L in some 3-manifold N. We get the following corollary by an application of Proposition 1.1 to the link exteriors.

Corollary 1.2. *If the link exterior $E(L)$ has a type $(n; m, l)$ decomposition for a $(m+l)$-component link L, then $E(L)$ has also a type $(n+k; m+k, l-k)$ splitting for any k satisfying $0 \leq k \leq l$.*

We can restate the definition of the tunnel number of a link using this type. An m-component link L in a 3-manifold is called a *tunnel number n link* if the exterior $E(L)$ of L has a type $(n + 1; m, 0)$ splitting, but does not have a type $(n; m, 0)$ splitting. We use $t(L)$ to denote the tunnel number of L. Next we consider the relation between tunnel numbers and types of Heegaard splittings. If a link exterior $E(L)$ has a type $(n; m, l)$ splitting, then the tunnel number of L is at most $n + l - 1$ $(l \leq m \leq n)$. Hence we obtain the following corollary.

Corollary 1.3. *If L is a k-component link, then*

$$g(E(L)) - 1 \leq t(L) \leq g(E(L)) + \frac{k}{2} - 1.$$

In this paper, we discuss 2-component link exteriors. For a 2-component link L, if $E(L)$ has a type $(g; 2, 0)$ (resp. $(g; 1, 1)$) splitting, then by a boundary stabilization (Proposition 1.1), $E(L)$ has also a type $(g+1; 1, 1)$ (resp. $(g+1; 2, 0)$). We consider the existence of a 2-component link exterior which does not have any type $(g; 2, 0)$ (resp. $(g; 1, 1)$) splittings but have a type $(g; 1, 1)$ (resp. $(g; 2, 0)$) splitting.

Kobayashi [4] showed that for a link L which is a connected sum of a $(4, 3)$ torus knot and a Hopf link, the link exterior has a type $(2; 1, 1)$ splitting, but does not have any Heegaard splitting of type $(2; 2, 0)$. Note that by the same way as [4], we can obtain the same result for a connected sum of a (p, q) torus knot and a Hopf link, where $p > q \geq 3$. In section 2, we show other examples.

In section 3, we show the following theorems, using distances of Heegaard splittings.

Theorem 1.4. *For any integer $g > 1$, there exists infinitely many link exteriors which have type $(g; 1, 1)$ splittings, but do not have any Heegaard splittings of type $(g; 2, 0)$.*

Theorem 1.5. *For any integer $g > 1$, there exists infinitely many link exteriors* *which have type* $(g; 2, 0)$ *splittings, but do not have any Heegaard splittings of type* $(g; 1, 1)$.

Theorem 1.6. *For any integer* $g > 1$, *there exists infinitely many link exteriors which have both type* $(g+1; 2, 0)$ *and* $(g+1; 1, 1)$ *splittings but do not have any Heegaard splittings of type* $(g; 2, 0)$ *and* $(g; 1, 1)$.

Let M be a 3-manifold with a torus boundary component T. We can obtain a 3-manifold by gluing a solid torus V to M along T in such a way that a slope γ on T bounds a meridian disk in V. This operation is called a γ-*Dehn filling* and the Dehn filled manifold is denoted by $M(\gamma)$. Note that $g(M(\gamma)) \leq g(M)$. It is also shown in [8] that $g(M)-1 \leq g(M(\gamma)) \leq g(M)$ for a cylindrical 3-manifold M with an incompressible torus boundary component T with all but finitely many γ.

For a tunnel number one link exterior, i.e., a link exterior which has a type $(2; 2, 0)$ splitting, we find a sufficient condition for admitting a type $(2; 1, 1)$ splitting.

Theorem 1.7. *Let* L *be a tunnel number one link. If a Dehn filling for the exterior* $E(L)$ *along one boundary component yields a solid torus, then* $E(L)$ *has type* $(2; 1, 1)$ *splitting.*

2. Examples

In this section, we show some examples of links in S^3 satisfying Theorem 1.4 and Theorem 1.6.

Example 2.1. *Let* $L = M(b; (a_1, b_1), (2, 1), (a_2, b_2), (2, 1))$ *be a 2-component Montesinos link with 4 branches. Then* $E(L)$ *has a type* $(2; 1, 1)$ *splitting, and does not have any type* $(2; 2, 0)$ *splitting.*

We choose an arc τ_i which is the core of the rational tangle $\frac{b_i}{a_i}$ ($i = 1$ and 2). Then by Figure 2, we can see $E(L)$ has a type $(2; 1, 1)$ splitting. By the determination of tunnel number one Montesinos links [7], L is not a tunnel number one link. Hence $E(L)$ does not have any type $(2; 2, 0)$ splitting.

3. Proofs of Theorem 1.4, 1.5, and 1.6

The idea of these proofs is suggested by T. Kobayashi. The similar methods are also used in [3] and [5].

Let S be a closed, orientable, genus g surface. The *curve complex* $C(S)$ is the complex whose verticies are the isotopy classes of essential simple closed curves in S, and where distinct verticies $x_0, x_1, \ldots, x_k$ determine k-simplex of $C(S)$ if they are represented by pairwise disjoint simple closed curves in S.

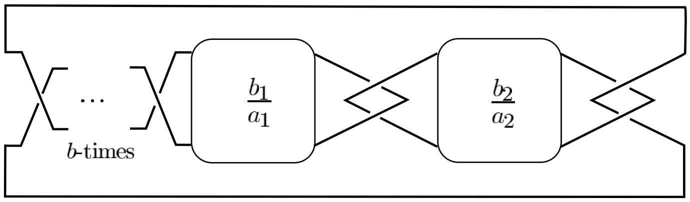

Figure 1

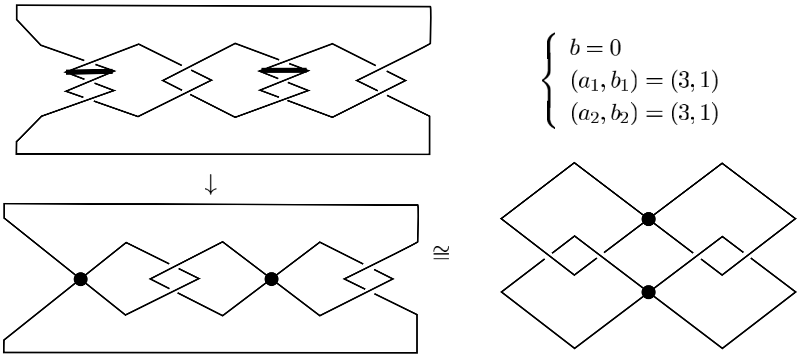

Figure 2

For given vertices l_1, l_2 in $C(S)$, the *distance* $d(l_1, l_2)$ is the geodesic distance of $C(S)$: the number of edges in the shortest path from l_1 to l_2. This definition extends to a definition of distances between subsets A and B of $C(S)$ by defining $d(A, B) = min\{d(a, b) \mid a \in A, b \in B\}$. Let H be a compression body and $S = \partial_+ H$. The *compression body set* $\mathbf{K}(H)$ corresponding to H is a subcomplex of $C(S)$ consisting of verticies which bound disks in H. Let M be an arbitrary 3-manifold. For a Heegaard splitting $M = H_1 \cup_S H_2$, the *distance of the splitting* is $d(S) = d(\mathbf{K}(H_1), \mathbf{K}(H_2))$. See [2] for details.

Theorem 3.1 ([10, Corollary 4.7])**.** *Suppose P and Q are both Heegaard surfaces for the compact orientable 3-manifold M. Then either $d(P) \leq 2g(Q)$, or Q is isotopic to P or a stabilization or a boundary stabilization of P.*

From Theorem 3.1, we obtain the following corollary. Note the genus increases with a stabilization or a boundary stabilization. Hence, if two Heegaard surfaces have same genus, then one is not isotopic to a stabilization nor a boundary stabilization of the other.

Corollary 3.2. *If $H_1 \cup_S H_2$ and $H_1' \cup_{S'} H_2'$ are genus g Heegaard splittings of M and $d(S) > 2g$ then S' is isotopic to S.*

Theorem 1.4 (resp. 1.5) follows from Corollary 3.2 and Proposition 3.3 (resp.

3.4) below.

Proposition 3.3. *For any integer $g > 1$, there exists infinitely many link exteriors each of which has a type $(g; 2, 0)$ splitting surface S with $d(S) > 2g$.*

Proposition 3.4. *For any integer $g > 1$, there exists infinitely many link exteriors each of which has a type $(g; 1, 1)$ splitting surface S with $d(S) > 2g$.*

For the proof of Proposition 3.3 and Proposition 3.4, first we prepare a knot whose exterior has a Heegaard splitting with high distance.

Theorem 3.5 ([5]). *For any pair of integers $g > 1$ and $n > 0$, there is a knot K in S^3 and a genus g splitting of $E(K)$ having a distance greater than n.*

Next, from the knot obtained in Theorem 3.5, by adding one more component we will construct a 2-component link whose exterior has a Heegaard splitting with high distance.

Let K be a knot in S^3 which satisfies Theorem 3.5. Let $H_1 \cup_S H_2$ be a genus g Heegaard splitting of $E(K)$ with $d(S) > n$. We can choose a knot K' in H_1 so that $\mathrm{Cl}(H_1 - N(K'))$ become a compression body, where $\mathrm{Cl}(\cdot)$ means the closure. Let $H'_1 = \mathrm{Cl}(H_1 - N(K'))$ and $H'_2 = H_2$. Then $H'_1 \cup_{S'} H'_2$ is a genus g Heegaard splitting of $E(K \cup K')$. Since $\mathbf{K}(H'_i) \subset \mathbf{K}(H_i)$, $d(S') \leq d(S)$. Then we obtain the following corollaries.

Corollary 3.6. *For any pair of integers $g > 1$ and $n > 0$, there is a link $L \subset S^3$ and a $(g; 2, 0)$ splitting of $E(L)$ having distance greater than n.*

Corollary 3.7. *For any pair of integers $g > 1$ and $n > 0$, there is a link $L \subset S^3$ and a $(g; 1, 1)$ splitting of $E(L)$ having distance greater than n.*

Now we prove Proposition 3.3.

Proof of Proposition 3.3. Let $N_0 = 2g$. For an integer k, we will define N_k and D_k inductively as follows. For an integer N_{k-1}, by Corollary 3.6, there is a link $L_k \subset S^3$ and a type $(g; 1, 1)$ splitting surface S_k with $d(S_k) > N_{k-1}$. Let $N_k = d(S_k)$. Note that $N_k > N_{k-1}$. S_k is a type $(2; 1, 1)$ splitting surface for $E(L_k)$ with $d(S_k) = N_k > 4$. Moreover, if $k \neq k'$, since $N_k \neq N_{k'}$, $E(L_k)$ and $E(L_{k'})$ are not homeomorphic by Corollary 3.2. □

By the same way as this proof, we can show Proposition 3.4 from Corollary 3.7. Moreover we obtain the following Corollary 3.8 from Theorem 3.5.

Corollary 3.8. *For any integer $g > 1$, there exists infinitely many knots in S^3 which exteriors have Heegaard genus g.*

Let M_1 and M_2 be compact orientable 3-manifolds. We denoted by $M_1 \sharp M_2$ the connected sum of M_1 and M_2. From Heegaard splittings of M_1 and M_2, we can obtain a Heegaard splitting of $M_1\sharp M_2$ as follows. We consider $M_1\sharp M_2$ is the union of $\mathrm{Cl}(M_1-B_1)$ and $\mathrm{Cl}(M_2-B_2)$. We take 3-balls B_1 and B_2 so that each B_i meets the Heegaard surface S_i in a disk D_i. Then $\mathrm{Cl}(S_1-D_1)\cup\mathrm{Cl}(S_2-D_2)$ gives a Heegaard splitting of $M_1\sharp M_2$. Hence we obtain that $g(M_1\sharp M_2)\leq g(M_1)+g(M_2)$. Haken [1] also showed the following theorem.

Theorem 3.9 ([1]).

$$g(M_1\sharp M_2)=g(M_1)+g(M_2).$$

Now we prove Theorem 1.6.

Proof of Theorem 1.6. Let $K\subset S^3$ be a knot whose exterior has Heegaard genus g. By Corollary 3.8, there exist infinitely many such knots. Let L be a splitting union of K and the trivial knot. Then from connected sum of a genus g splitting of $E(K)$ and a genus one splitting of the trivial knot exterior which is the solid torus, we obtain both type $(g+1;2,0)$ and $(g+1;1,1)$ splittings of $E(L)$. However, by Theorem 3.9, $E(L)$ does not have any Heegaard splitting of type $(g;2,0)$ and $(g;1,1)$. □

4. Proof of Theorem 1.7

Let M be a tunnel number one link exterior, and $M=H_1\cup H_2$ be a genus 2 Heegaard splitting, where H_1 is a handlebody, and H_2 is a compression body.

Proof. Suppose a Dehn filled manifold $M(\gamma)$ along a slope γ is homeomorphic to a solid torus. Let F be the torus component of ∂_-H_2 which contains γ, and F' the other component of ∂_-H_2. Then we may identify H_2 with $F\times I\sharp_\partial F'\times I$, where $I=[0,1]$, and $\sharp_\partial$ means a boundary connected sum along a disk in $F\times\{1\}$ and a disk in $F'\times\{1\}$. Here we may assume that $\gamma\times\{1\}\cap F'\times I=\emptyset$. We obtain a genus 2 Heegaard splitting, say $H_1\cup H_2(\gamma)$, of the solid torus $M(\gamma)$. By [9], any genus 2 Heegaard splitting of a solid torus is stabilized. On the other hand, it is elementally to show that any non-separating proper disk in $H_2(\gamma)$ is properly isotopic to a disk obtain from $\gamma\times I$ by adding a meridian disk of a attached solid torus. These imply that there is a proper disk D in H_1 such that $\partial D\cap(\gamma\times\{1\})=\{*\}$. We can take a disk $D'\subset F$ so that $D'\times\{1\}\cap F'\times I=\emptyset$, and $D'\cap\gamma=\gamma_1$ is an arc with $\partial D\cap(\gamma_1\times\{1\})=\emptyset$. Let $I_-=[0,\frac{1}{2}]$, and $I_+=[\frac{1}{2},1]$. Let $\hat{H}_1=H_1\cup(D'\times I_+)\cup(F\times I_-)$, $\hat{H}_2=\mathrm{Cl}(M-\hat{H}_1)=(\mathrm{Cl}(F-D')\times I_+)\sharp_\partial F'\times I$. Note that $\hat{H}_1\cup\hat{H}_2$ is a boundary stabilization of $H_1\cup H_2$, and is a type $(3;1,1)$ splitting. Let $\gamma_2=\mathrm{Cl}(\gamma-\gamma_1)$. Disks D in $\hat{H}_1$ and $\gamma_2\times I_+$ in $\hat{H}_2$ intersect in one point. Hence the Heegaard splitting $M=\hat{H}_1\cup\hat{H}_2$ is stabilized. By destabilizing $\hat{H}_1\cup\hat{H}_2$, we obtain a type $(2;1,1)$ splitting of M. □

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