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The projective characterization of elliptic plane curves which have one place at infinity

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Abstract

In this paper we consider smooth affine elliptic plane curves having one place at infinity. We identify them with elliptic projective plane curves having only one cusp as their singular points and meeting with the line at infinity only at the cusp. We characterize such curves by the self–intersection number of the strict transform of them via the minimal embedded resolution of their cusp. Furthermore, we prove that the self–intersection number of them is the maximum value among those of all the elliptic plane curves having only one cusp.

1. Introduction

Let C be a curve on $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. A singular point of C is said to be a *cusp* if it is a locally irreducible singular point. We say that C is *cuspidal* (resp. *unicuspidal*) if C has only cusps (resp. one cusp) as its singular points. Suppose C is unicuspidal. We denote by C' the strict transform of C via the minimal embedded resolution of the cusp of C. In view of [1, 9], we say that C is of *Abhyankar–Moh–Suzuki type* (*AMS type*, for short) if there exists a line L such that $C \cap L = \{\text{the cusp}\}$. By regarding L as the line at infinity, we identify such curves with smooth affine plane curves having one place at infinity. In [11], it was proved that a rational unicuspidal plane curve C is of AMS type if and only if $(C')^2 \geq 2$. In a similar manner to that as in the rational case, we can characterize elliptic unicuspidal plane curves C of AMS type by $(C')^2$. Namely, the purpose of this paper is to prove the following:

Theorem 1. *If C is an elliptic unicuspidal plane curve, then $(C')^2 \leq 6$. The equality holds if and only if C is of AMS type.*

We next determine the maximum value of $(C')^2$ for elliptic unicuspidal plane curves of non–AMS type.

Theorem 2. *If C is an elliptic unicuspidal plane curve of non–AMS type, then $(C')^2 \leq 3$. The equality holds if and only if there exist an irreducible conic*

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$C_2 \subset \mathbf{P}^2$ and a birational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ such that $C \cap C_2 = \{\text{the cusp}\}$, $f|_{\mathbf{P}^2 \setminus C_2} \in \mathrm{Aut}(\mathbf{P}^2 \setminus C_2)$ and $f(C)$ is a smooth cubic curve. Furthermore, we can show the existence of such curves C with $(C')^2 = 3$.

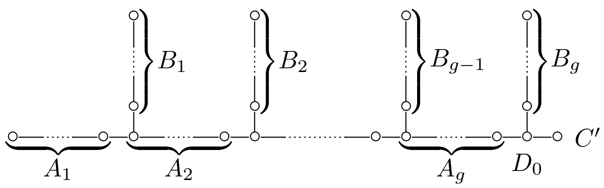
In Section 2, we prove Theorem 1 except the "if" part. After resolving the cusp of an elliptic unicuspidal plane curve C, we perform additional blowings–up to make $\Phi_{|C'|}$ an elliptic fibration. Then we analyze its structure and determine possible types of its singular fibers. We show that if $(C')^2 = 6$, then the fibration has a singular fiber of type II^*. In Section 3, we prove the "if" part of Theorem 1 by using the following property of elliptic unicuspidal plane curves of AMS type.

Theorem 3 ([2, Theorem 8.7], cf. [7]). *Let C be an elliptic unicuspidal plane curve of AMS type and L a line such that $C \cap L = \{\text{the cusp}\}$. Then there exists a birational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ such that $f|_{\mathbf{P}^2 \setminus L} \in \mathrm{Aut}(\mathbf{P}^2 \setminus L)$ and $f(C)$ is a smooth cubic curve.*

In Section 4, we prove Theorem 2. Similar to Section 2, we consider the elliptic fibration associated with $|C'|$. We prove that if $(C')^2 = 3$, then the fibration has a singular fiber of type I_4^*. In order to show the existence of curves C with $(C')^2 = 3$, we give the defining equation of the cubic pencil associated with the elliptic fibration.

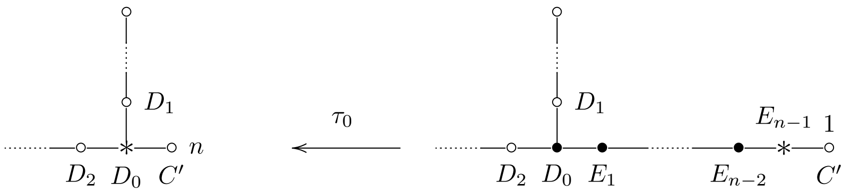
2. Proof of Theorem 1

Let C be an elliptic unicuspidal plane curve and P the cusp of C. Let $\sigma : V \to \mathbf{P}^2$ denote the minimal embedded resolution of P. That is, σ is the composite of the shortest sequence of blowings–up such that the strict transform C' of C intersects $\sigma^{-1}(P)$ transversally. The dual graph of $D := \sigma^{-1}(C)$ has the following shape, where $g \geq 1$ and all A_i, B_i are not empty.



Here D_0 is the exceptional curve of the last blowing–up and A_1 contains that of the first one. The morphism σ contracts $A_g + D_0 + B_g$ to a (-1)–curve E, $A_{g-1} + E + B_{g-1}$ to a (-1)–curve and so on. Every irreducible component E of A_i and B_i is a smooth rational curve with $E^2 < -1$. Each A_i contains an irreducible component E such that $E^2 < -2$. Cf. [3, 6]. We give weights to $A_1, \ldots, A_g, B_1, \ldots, B_g$ in the usual way. We also give the graphs $A_1, \ldots, A_g$ (resp. $B_1, \ldots, B_g$) the direction from the left–hand side to the right (resp. from the bottom to the top) in the above figure. Let D_1 (resp. D_2) denote the first irreducible component of B_g (resp. the last one of A_g) with respect to the direction.

Suppose that $n := (C')^2 \geq 3$. Perform $(n-1)$–times of blowings–up $\tau_0 : W_0 \to V$ over $C' \cap D_0$ in the following way, where $*$ (resp. $\bullet$) denotes a (-1)–curve (resp. (-2)–curve) and E_i is the exceptional curve of the i–th blowing–up. We use the same notation C'.



On W_0, there exists an exact sequence:

$$0 \longrightarrow H^0(\mathcal{O}_{W_0}) \longrightarrow H^0(\mathcal{O}_{W_0}(C')) \longrightarrow H^0(\mathcal{O}_{C'}(C')) \longrightarrow H^1(\mathcal{O}_{W_0}) = 0.$$

Because $(C')^2 = 1$ on W_0, we have $h^0(\mathcal{O}_{C'}(C')) = 1$. This means that $|C'|$ is a pencil on W_0 having a single base point Q. We have either $\{Q\} = C' \cap E_{n-1}$ or $Q \in C' \setminus E_{n-1}$. Let $\tau_1 : W \to W_0$ denote the blowing–up at Q and E_n its exceptional curve. Put $\tau = \tau_0 \circ \tau_1$. The pencil $|C'|$ does not have base points on W. The morphism $\Phi_{|C'|} : W \to \mathbf{P}^1$ is an elliptic fibration. The curve C' is a non-singular fiber. The curve E_n is a 1–section of $\Phi_{|C'|}$. Namely, we have $E_n C' = 1$. If $Q \in C' \setminus E_{n-1}$, then E_{n-1} is also a 1–section. Put $F_0 := \tau^{-1}(D) - E_n - C'$ if $\{Q\} = C' \cap E_{n-1}$, $F_0 := \tau^{-1}(D) - E_n - E_{n-1} - C'$ otherwise. The divisor F_0 is contained in a fiber $\overline{F}_0$ of $\Phi_{|C'|}$.

Lemma 4. *Let F be a fiber of $\Phi_{|C'|}$ other than $\overline{F}_0$. If $\{Q\} = C' \cap E_{n-1}$, then F is irreducible. Otherwise F contains at most two irreducible components.*

Proof. If $\{Q\} = C' \cap E_{n-1}$, then $W \setminus (F_0 \cup E_n \cup C') = \mathbf{P}^2 \setminus C$. The latter does not contain a complete curve. This implies that each irreducible component of F intersects E_n. Since E_n is a 1–section, F must be irreducible. Similarly, if $Q \in C' \setminus E_{n-1}$, each irreducible component of F intersects E_{n-1} or E_n. It follows that F has at most two irreducible components. □

Let $\varphi : W \to X$ be successive contractions of (-1)–curves in the singular fibers of $\Phi_{|C'|}$ such that the fibration $p := \Phi_{|C'|} \circ \varphi^{-1} : X \to \mathbf{P}^1$ is relatively minimal. We will use the following facts about relatively minimal elliptic rational surfaces ([8, Lemma 2.7, 10.2, Theorem 10.3]).

Lemma 5. *The following assertions hold on X.*

(i) *Every 1–section is a (-1)–curve.*

(ii) *The Picard number $\rho(X)$ of X is equal to* 10.

(iii) $\sum_F (r(F)-1) \le 8$, *where* F *runs over all fibers of* p *and* $r(F)$ *denotes the number of irreducible components of* F.

We prove the following three lemmas by using the above Lemmas together with the list of all possible singular fibers of relatively minimal elliptic fibrations given by Kodaira ([5, Theorem 6.2]).

Lemma 6. *The following assertions hold.*

(i) φ *does not contract any irreducible curve which is not contained in* $\overline{F}_0$.

(ii) *If* $\{Q\} = C' \cap E_{n-1}$ *(resp.* $Q \in C' \setminus E_{n-1}$*), then* $\varphi(D_0), \varphi(E_1), \ldots, \varphi(E_{n-1})$ *(resp.* $\varphi(D_0), \varphi(E_1), \ldots, \varphi(E_{n-2})$*) are* (-2)*–curves.*

(iii) φ *does not contract* D_1, D_2.

Proof. The assertion (i) follows from Lemma 4 and Lemma 5 (i). The assertion (iii) follows from (ii). We show (ii). We only prove the assertion for the case in which $\{Q\} = C' \cap E_{n-1}$. The morphism φ does not contract E_{n-1}. If φ contracts a curve meeting with E_{n-1} then $\varphi(E_{n-1})^2 > -2$. It follows that $\varphi(E_{n-1})$ coincides with $\varphi(\overline{F}_0)$ and has a double point. We have $0 = \varphi(E_{n-1})^2 \ge E_{n-1}^2 + 4 = 2$, which is a contradiction. Thus φ does not contract E_{n-2}. So, the fiber $\varphi(\overline{F}_0)$ is reducible. We infer that $\varphi(\overline{F}_0)$ consists of (-2)–curves. It turns out that φ does not contract $D_0, E_1, \ldots, E_{n-3}$. □

Lemma 7. *The following assertions hold.*

(i) *The morphism* φ *is not the identity.*

(ii) *If* $\{Q\} = C' \cap E_{n-1}$ *(resp.* $Q \in C' \setminus E_{n-1}$*), then* $r(\overline{F}_0) - r(F_0) = 1$ *(resp.* $r(\overline{F}_0) - r(F_0) = 2$*).*

(iii) $r(\varphi(\overline{F}_0)) = 9$.

Proof. (i) The morphism φ is not the identity because F_0 contains an irreducible component E with $E^2 \le -3$.

(ii) Since F_0 does not contain a (-1)–curve, we have $1 \le r(\overline{F}_0) - r(F_0)$. The number of the blowings–down of φ is equal to $\rho(W) - \rho(X) = r(D) + n - 10$. By Lemma 5 (iii), we get $9 \ge r(\varphi(\overline{F}_0)) = r(\overline{F}_0) - (r(D) + n - 10)$. Thus $r(\overline{F}_0) \le r(D) + n - 1$. Since $r(F_0) = r(D) + n - 2$ (resp. $r(F_0) = r(D) + n - 3$), we have $1 \le r(\overline{F}_0) - r(F_0) \le 1$ (resp. $1 \le r(\overline{F}_0) - r(F_0) \le 2$). Suppose that $Q \in C' \setminus E_{n-1}$ and $r(\overline{F}_0) - r(F_0) = 1$. Let E_0 denote the (-1)–curve which is contracted by the first blowing–down of φ. Because $F_0 \cap E_n = \emptyset$, E_0 must intersect E_n, which contradicts Lemma 5 (i).

(iii) We have $r(\overline{F}_0) = r(D) + n - 1$. Thus $r(\varphi(\overline{F}_0)) = r(\overline{F}_0) - (r(D) + n - 10) = 9$. □

Let E_0 denote the (-1)–curve which is contracted by the first blowing–down of φ. If $\{Q\} = C' \cap E_{n-1}$ then $\overline{F}_0 = F_0 + E_0$. If $Q \in C' \setminus E_{n-1}$, write $\overline{F}_0$ as $\overline{F}_0 = F_0 + E_0 + E'_0$, where E'_0 is an irreducible curve. The curve E'_0 must intersect E_n, because $F_0 + E_0$ does not. By Lemma 5 (i), φ does not contract E'_0.

Lemma 8. *The following assertions hold.*

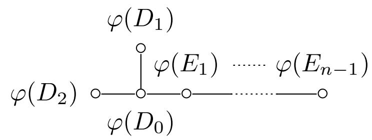
(i) $\varphi(\overline{F}_0)$ *is of type* II^* *or* I_4^*. *We have* $E_0F_0 = E'_0F_0 = 1$.

(ii) *If* $\{Q\} = C' \cap E_{n-1}$, *then* $n = 6$. *The fiber* $\varphi(\overline{F}_0)$ *is of type* II^*.

(iii) *If* $Q \in C' \setminus E_{n-1}$, *then* $n = 3$. *The fiber* $\varphi(\overline{F}_0)$ *is of type* I_4^*.

Proof. (i) By Lemma 7 (iii), the fiber $\varphi(\overline{F}_0)$ is of type II^*, I_4^* or I_9. By Lemma 6, the curve $\varphi(D_0)$ is a branching component of $\varphi(\overline{F}_0)$. Namely, $\varphi(D_0)(\varphi(\overline{F}_0) - \varphi(D_0)) \geq 3$. Thus $\varphi(\overline{F}_0)$ is not of type I_9. It follows that $E_0F_0 = E'_0F_0 = 1$.

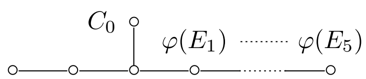
(ii) By Lemma 6 and Lemma 7 (iii), we have $n \leq 7$. The dual graph of $\varphi(F_0)$ contains the following graph.



Suppose that $\varphi(E_i)$ is a branching component of $\varphi(\overline{F}_0)$ for some $i < n$. Let E denote the irreducible component of $\varphi(\overline{F}_0 - E_i - D_0)$ meeting with $\varphi(E_i)$. Let E' denote the strict transform of E via φ. Since $E' \neq E_0$, we have $E' \subset F_0$. Because E' does not intersect E_i, φ must contract a curve meeting with E_i, which is a contradiction. Hence $\varphi(E_1 + \cdots + E_{n-1})$ does not contain a branching component. Since $n \geq 3$, $\varphi(\overline{F}_0)$ must be of type II^*. Because $\varphi(E_{n-1})$ intersects the section, the coefficient of it in $p^*(p(\varphi(\overline{F}_0)))$ is equal to 1. This shows $n = 6$.

(iii) Since φ does not contract E'_0, we have $n \leq 7$ by Lemma 6 and Lemma 7 (iii). If $\varphi(\overline{F}_0)$ is of type II^*, then $\varphi(E_n)$ must intersect $\varphi(E_{n-2})$, which is impossible. Thus $\varphi(\overline{F}_0)$ is of type I_4^*. The dual graph of $\varphi(\overline{F}_0 - E'_0)$ contains the above graph with $\varphi(E_{n-1})$ being replaced with $\varphi(E_{n-2})$. The divisor $\varphi(E_1 + \cdots + E_{n-2})$ contains at most one branching component of $\varphi(\overline{F}_0)$. If it does, then $n = 7$ and E'_0 must intersect E_4. Otherwise we have $n = 3$. Suppose $n = 7$. Let $\psi : X \to \mathbf{P}^2$ denote the contraction of $\varphi(E_7) + \varphi(E_6) + \varphi(\overline{F}_0) - \varphi(E'_0) - \varphi(D_1)$. Then we have $\psi(\varphi(D_1))^2 = 0$, which is absurd. □

Suppose $n = 6$. Let C_0 be an irreducible component of $\varphi(\overline{F}_0)$ whose position in the dual graph of $\varphi(\overline{F}_0)$ is illustrated in the following figure.

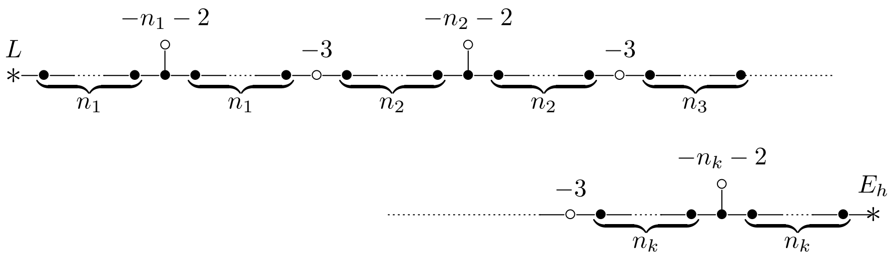


Let $\psi : X \to \mathbf{P}^2$ denote the contraction of $\varphi(E_6) + \varphi(\overline{F}_0) - C_0$. We have $\psi(C_0)^2 = 1$. Put $C_1 = \sigma(\tau(E_0))$, $h = \psi \circ \varphi \circ \tau^{-1} \circ \sigma^{-1} : \mathbf{P}^2 \to \mathbf{P}^2$. The morphism $h|_{\mathbf{P}^2 \setminus C_1} : \mathbf{P}^2 \setminus C_1 \to \mathbf{P}^2 \setminus \psi(C_0)$ is an isomorphism. Since $\mathbf{Z}/(\deg C_1)\mathbf{Z} \cong H_1(\mathbf{P}^2 \setminus C_1, \mathbf{Z}) \cong H_1(\mathbf{P}^2 \setminus \psi(C_0), \mathbf{Z}) \cong \{0\}$, we conclude that C_1 is a line (cf. [4, Proposition 4.1.3]). We completed the proof of Theorem 1 except the "if" part.

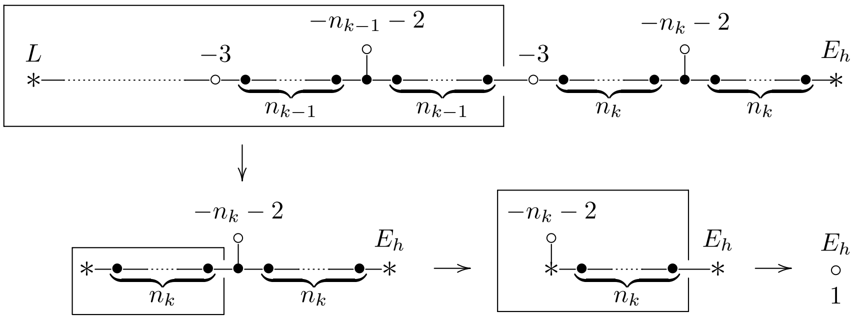
3. Proof of Theorem 1 — continued

We use the following theorem to prove the "if" part of Theorem 1.

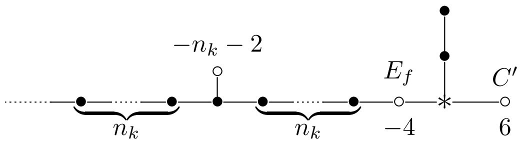
Theorem 9 ([10, Lemma 4.4]). *Let $L \subset \mathbf{P}^2$ be a line and $h : \mathbf{P}^2 \to \mathbf{P}^2$ a birational map such that $h \notin \operatorname{Aut} \mathbf{P}^2$ and $h|_{\mathbf{P}^2 \setminus L} \in \operatorname{Aut}(\mathbf{P}^2 \setminus L)$. Let $\sigma_h : V \to \mathbf{P}^2$ denote the minimal resolution of the base points of h. Then the weighted dual graph of $\sigma_h^{-1}(L)$ has the following shape, where $k \geq 1$, $n_i \geq 0$, E_h is the exceptional curve of the last blowing–up.*



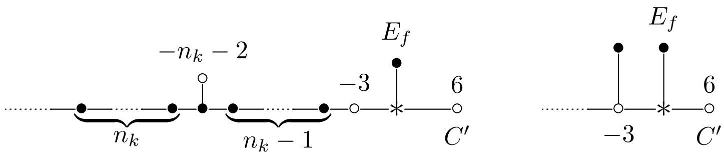
The morphism $h \circ \sigma_h$ contracts the above graph in the following way, where each marked subgraph is contracted to a point. After the contraction, $h \circ \sigma_h$ maps the image of E_h under the contraction to L.



Let C be an elliptic unicuspidal plane curve of AMS type. Let L and f have the same meaning as in Theorem 3. We apply Theorem 9 to L and f. On $\mathbf{P}^2$, f^{-1} has a single base point $Q \in L$. Suppose that $Q \in L \setminus f(C)$. Then the center of every blowing–up of $f \circ \sigma_f : V \to \mathbf{P}^2$ is not on $f(C)$. This means that the strict transform of $f(C)$ via $f \circ \sigma_f$ intersects E_f in the same way as $f(C)$ does L. Let $\sigma' : V' \to V$ denote the 3–times of blowings–up such that the strict transform C' of $f(C)$ on V' intersects $(\sigma_f \circ \sigma')^{-1}(L)$ transversally. We have $(C')^2 = 6$. The center of every blowing–up of $\sigma_f \circ \sigma'$ is on C. Thus $\sigma_f \circ \sigma'$ is the minimal embedded resolution of the cusp of C. The following figure illustrates the weighted dual graph of $(\sigma_f \circ \sigma')^{-1}(C)$ near C'.



Suppose that $\{Q\} = L \cap f(C)$. The center of every blowing–up of $f \circ \sigma_f : V \to \mathbf{P}^2$ except the first two is not on $f(C)$. By arguments similar to those in the previous case, we conclude that $(C')^2 = 6$, where C' is the strict transform of C via the minimal embedded resolution $\sigma : V \to \mathbf{P}^2$ of the cusp. The following figure illustrates the weighted dual graph of $\sigma^{-1}(C)$ near C'.

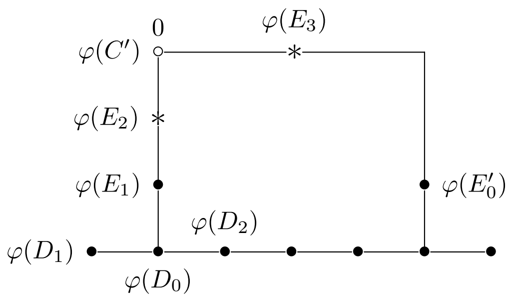


Case: $n_k > 0$

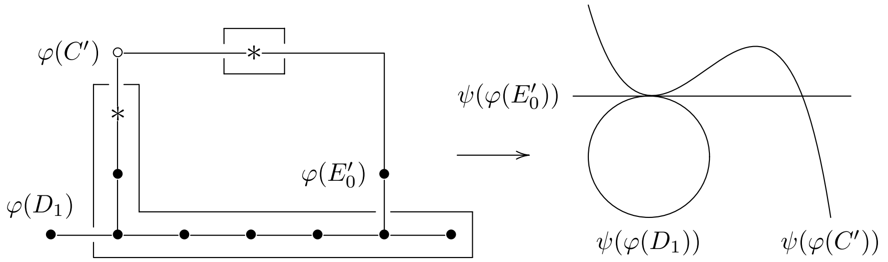
Case: $n_k = 0$

4. Proof of Theorem 2

Let the notation and the assumptions be as in Section 2, where we assumed that $n \geq 3$. Assume in addition that $n \leq 5$. By Lemma 8, we must have $n = 3$. This proves the first inequality. Moreover $\varphi(\overline{F}_0)$ is of type I_4^*. Since the coefficient of $\varphi(E_0')$ in $p^*(p(\varphi(\overline{F}_0)))$ is equal to one, it follows that the weighted dual graph of $\varphi(\overline{F}_0 + E_2 + E_3 + E_0' + C')$ has the following shape.



We perform the blowings–down $\psi : X \to \mathbf{P}^2$ as illustrated in the following figure, where each marked subgraph is contracted to a point.

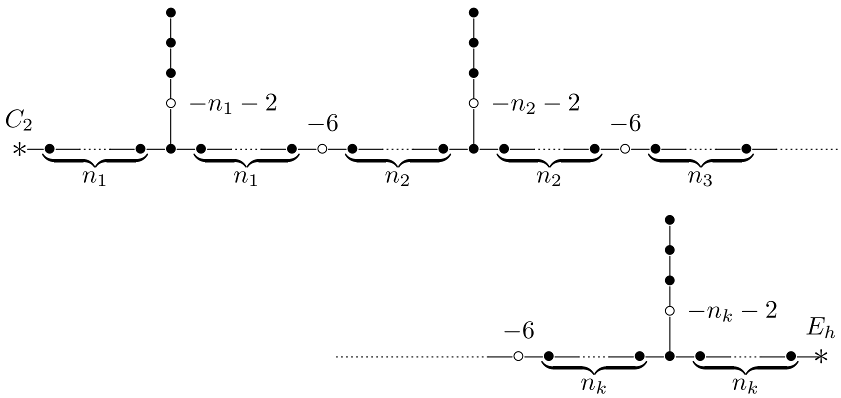


The curve $L := \psi(\varphi(E_0'))$ is a line. The curves $C_2 := \psi(\varphi(D_1))$, $C_3 := \psi(\varphi(C'))$ are a conic and a cubic curve, respectively. We have

$$(*) \qquad C_3 \cdot L = R_1 + 2R_2 \text{ and } C_3 \cdot C_2 = 6R_2$$

for some distinct points R_1, R_2. Put $C_1 = \sigma(\tau(E_0))$, $f = \psi \circ \varphi \circ \tau^{-1} \circ \sigma^{-1}$. The morphism $f|_{\mathbf{P}^2 \setminus C_1} : \mathbf{P}^2 \setminus C_1 \to \mathbf{P}^2 \setminus C_2$ is an isomorphism. This means that C_1 is an irreducible conic. Hence the "only if" part is proved. We prove the converse.

Lemma 10 (cf. [10, Lemma 4.4]). *Let $C_2 \subset \mathbf{P}^2$ be an irreducible conic and $h : \mathbf{P}^2 \to \mathbf{P}^2$ a birational map such that $h \notin \operatorname{Aut} \mathbf{P}^2$ and $h|_{\mathbf{P}^2 \setminus C_2} \in \operatorname{Aut}(\mathbf{P}^2 \setminus C_2)$. Let $\sigma_h : V \to \mathbf{P}^2$ denote the minimal resolution of the base points of h. Then the weighted dual graph of $\sigma_h^{-1}(C_2)$ has the following shape, where $k \geq 1$, $n_i \geq 0$, E_h is the exceptional curve of the last blowing–up. Starting with the contraction of C_2, $h \circ \sigma_h$ contracts the graph and maps the image of E_h under the contraction to C_2.*



Proof. We only give a sketch because the proof is similar to that of [10, Lemma 4.4]. Let $\sigma_i : V_i \to V_{i-1}$ denote the i–th blowing–up of σ_h, Q_{i-1} its center and E_i its exceptional curve ($i = 1, \ldots, r$, $V_0 = \mathbf{P}^2$). We use the same symbols for the strict transforms of C_2 and E_i. We use the following facts ([10, Lemma 2.4, Remark 2.5]).

Lemma 11. *The following assertions hold for $i < r$.*

(i) *We have $Q_i \in E_i$. On V_r, we have $E_i^2 \le -2$ and $C_2^2 = -1$.*

(ii) *If $C_2^2 \ge 0$ (resp. $C_2^2 = -1$) on V_i, then $Q_i \in E_i \cap C_2$ (resp. $Q_i \in E_i \setminus C_2$).*

(iii) *$h \circ \sigma_h$ first contracts C_2, then contracts $E_1, \dots, E_{r-1}$, not necessarily in this order. It does not contract E_r.*

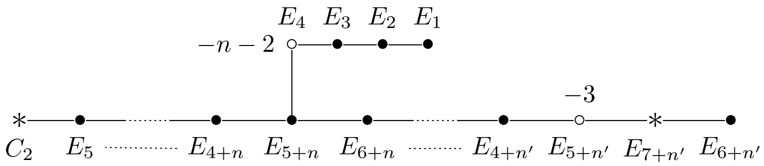
We have $r \ge 5$, $Q_i \in E_i \cap C_2$ for $i < 5$ and $Q_i \in E_i \setminus C_2$ for $i \ge 5$. If $r = 5$, then $h \circ \sigma_h$ cannot contract the remaining curves after the contraction of C_2. Thus $r \ge 6$. For the same reason, we infer $Q_5 \in E_5 \setminus E_4$ if $r = 6$. We have $k = 1$, $n_1 = 0$ in this case.

Suppose $r \ge 7$. We have either $\{Q_5\} = E_5 \cap E_4$ or $Q_5 \in E_5 \setminus E_4$. Suppose $\{Q_5\} = E_5 \cap E_4$. If $Q_6 \in E_6 \cap E_5$, then $E_5^2 \le -3$ on V_r. This means that none of $E_1, \dots, E_{r-1}$ is a (-1)–curve after the contraction of C_2. Thus $Q_6 \in E_6 \setminus E_5$. Let Γ_i denote the preimage of C_2 under $\sigma_1 \circ \cdots \circ \sigma_i$. Suppose that Q_i is a node of Γ_i for all $i \ge 5$. For the same reason as above, we have $Q_i \in E_i \cap E_4$ for all $i \ge 5$. After the contraction of $C_2, E_5, \dots, E_{r-1}$, $h \circ \sigma_h$ cannot contract the remaining curves. Hence there exists $n \ge 1$ such that Q_{5+i} is a node of Γ_{5+i} for $i = 0, \dots, n-1$ but Q_{5+n} is not. We set $n = 0$ when $Q_5 \in E_5 \setminus E_4$.

Suppose that Q_{6+i} is not a node of Γ_{6+i} for all $i \ge n$. We infer that $h \circ \sigma_h$ contracts $C_2, E_5, \dots, E_{r-1}, E_4, E_3, E_2, E_1$ in this order. After the contraction of E_{r-1}, E_4 must be a (-1)–curve. Thus $r = 2n + 6$. We have $k = 1$, $n_1 = n$ in this case.

Suppose the contrary. There exists $n' \ge n$ such that $Q_{6+n'}$ is a node of $\Gamma_{6+n'}$ but Q_{6+i} is not for $i = n, \dots, n'-1$. If $n' = n$, then $E_{5+n}^2 \le -3$ on V_r. We infer that $h \circ \sigma_h$ cannot contract E_{5+n} in this case. Thus $n' > n$.

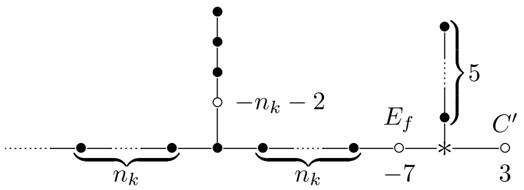
The weighted dual graph of $\Gamma_{7+n'}$ has the following shape.



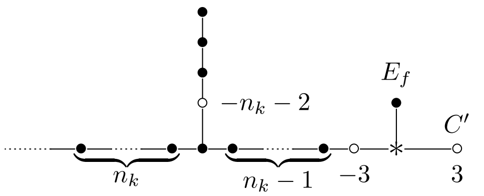
The remaining blowings–up do not change curves other than $E_{5+n'}$, $E_{6+n'}$ and $E_{7+n'}$. It follows that $h \circ \sigma_h$ first contracts $C_2, E_5, \dots, E_{4+n'}$ in this order. After that, E_4 must be a (-1)–curve. Thus $n' = 2n + 1$. If $r = 7 + n'$, then $E_3^2 = E_{5+n'}^2 = -1$ after the contraction of E_4, which is absurd. For the same reason, we have $r \ge 10 + n'$ and $Q_{i+n'} \in E_{i+n'} \cap E_{5+n'}$ for $i = 7, 8, 9$. If $r = 10 + n'$, then $h \circ \sigma_h$ cannot contract $E_{9+n'}$. If $Q_{10+n'} \in E_{10+n'} \cap E_{5+n'}$, then $h \circ \sigma_h$ cannot contract $E_{5+n'}$. Thus $Q_{10+n'} \in E_{10+n'} \setminus E_{5+n'}$. The situation on $V_{10+n'}$

is similar to that on V_5. The curve $E_{5+n'}$ plays the role of C_2, $E_{10+n'}$ does that of E_5 and so on. □

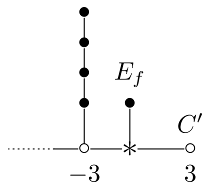
Let C_2 and f be as in Theorem 2. By applying Lemma 10 to C_2 and f, we determine the minimal embedded resolution $\sigma : V' \to \mathbf{P}^2$ of the cusp of C. On $\mathbf{P}^2$, f^{-1} has a single base point $Q \in C_2$. We have either $\{Q\} = C_2 \cap f(C)$ or $Q \in C_2 \setminus f(C)$. By the same arguments as in Section 3, one can prove that the weighted dual graph of $\sigma^{-1}(C)$ near C' has the following shape. In particular, we have $(C')^2 = 3$.



Case: $Q \in C_2 \setminus f(C)$



Case: $\{Q\} = C_2 \cap f(C)$, $n_k > 0$



Case: $\{Q\} = C_2 \cap f(C)$, $n_k = 0$

Finally, we show the existence of curves C with $(C')^2 = 3$. We start with a line L, a smooth conic C_2 and a smooth cubic C_3 meeting each other as in $(*)$. The next lemma shows the existence of such curves.

Lemma 12. *Let L, C_2, C_3 be plane curves defined by the equations: $L : x = 0$, $C_2 : f_2 : \ xz - y^2 = 0$, $C_3 : (ax + 2by - z)f_2 + x^3 = 0$, $(a + b^2)^2 \neq -4$. Then they are smooth, $C_3 \cdot L = R_1 + 2R_2$ and $C_3 \cdot C_2 = 6R_2$, where $R_1 = (0, 1, 2b)$, $R_2 = (0, 0, 1)$.*

Proof. One can show that C_3 is projectively equivalent to the curve defined by $y^2 z = x^3 + (a + b^2)x^2 z - xz^2$. It follows that C_3 is smooth if and only if the right–hand side does not have a multiple component. □

For given k, $n_1, \ldots, n_k$ and $Q \in C_2$, let h be a birational map given in Lemma 10 with the base point Q. Let C denote the strict transform of C_3 via h. Then C is an elliptic unicuspidal plane curve satisfying the condition $(C')^2 = 3$.

Example. *Let h be the birational map given in Lemma 10 with $k = 1$, $n_k = 0$.*

We may assume that the base point of h is $(0,0,1)$ and C_2 is defined by $f_2 := xz - y^2 = 0$. Then $h = h^{-1} = (xf_2^2, -f_3f_2, f_5)$, where $f_3 = (cx+y)f_2 + x^3$, $f_5 = (2x^2(cx+y) + (c^2x + 2cy + z)f_2)f_2 + x^5$. Let C_3 be a smooth cubic meeting with C_2 at a single point Q and C the strict transform of C_3 via h.

(i) $Q = (0,0,1)$. *The curve C is a quintic having five double points at $(0,0,1)$ and is defined by $axf_2^2 - 2bf_3f_2 - f_5 + x^3f_2 = 0$, $(a+b^2)^2 \neq -4$.*

(ii) $Q \neq (0,0,1)$. *The curve C is of degree fifteen and has six singular points at $(0,0,1)$ of multiplicity six. It is defined by $(af_5 - 2bf_3f_2 - xf_2^2)f_2^5 + f_5^3 = 0$, $(a+b^2)^2 \neq -4$.*

Proof. Let L_x, L_y and L_z denote the lines defined by $x = 0$, $y = 0$ and $z = 0$, respectively. Put $g = h^{-1}$. We have $g^*(L_x) = L_x + 2C_2$, $g^*(L_y) = N + C_2$ and $g^*(L_z) = C_5$. Here N (resp. C_5) is a nodal cubic (unicuspidal quintic) such that C_5 has six double points at $P_1 = (0,0,1)$ and that $NC_2 = 6P_1$, $C_5L_x = 4P_1 + P_2$, $C_5C_2 = 10P_1$, $C_5N = 15P_1$, where $P_2 \in L_x \setminus \{P_1\}$. One can show that N is defined by $f_3 = 0$ and has the parameterization $\mathbf{P}^1 \ni (s,t) \mapsto (st^2, s^2t(s-ct), s(s-ct)^2 - t^3) \in N$. By using the latter, one can deduce that C_5 is defined by $f_5 = 0$. Since $zf_2^{12} | f_5 \circ g$, one has $g = (xf_2^2, -f_3f_2, f_5) = h$. If $Q = (0,0,1)$, then C is defined by $f \circ g / f_2^5 = 0$, where $f = (ax + 2by - z)f_2 + x^3$, $(a+b^2)^2 \neq -4$. Otherwise we may assume $Q = (1,0,0)$. The curve C is defined by $f(f_5, -f_3f_2, xf_2^2) = 0$. □

Remark. *We remark the following facts about the curves in Lemma 12.*

(i) *Let L, C_2 and C_3 be a line, a smooth conic and a smooth cubic, respectively. If $C_3 \cdot L = R_1 + 2R_2$ and $C_3 \cdot C_2 = 6R_2$ for some distinct points R_1, R_2, then there exist $a, b \in \mathbf{C}$ with $(a+b^2)^2 \neq -4$ such that they are projectively equivalent to the curves given in Lemma 12.*

(ii) *(F. Sakai) Let R_1 be a flex of a given smooth cubic C_3. Let $R_2 \neq R_1$ be a point at which a line passing through R_1 is tangent to C_3. Then there exists a unique smooth conic meeting with C_3 only at R_2. Conversely, if a smooth conic meets with C_3 only at a single point P, then the line tangent to C_3 at P passes through a flex ($\neq P$). Thus there are exactly 27 such conics for a given smooth cubic.*

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