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Bi-Lipschitz trivial quasi-homogeneous stratifications

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Abstract

We call a stratification quasi-homogeneous when it is invariant by a certain class of action. The paper gives criteria for a quasi-homogeneous stratification to fulfill the (w)-condition of Kuo-Verdier or to be bi-Lipschitz trivial. We also give some explicit conditions on the weights to ensure some stability of the volume of quasi-homogeneous families of germs.

0. Introduction and known results

The notion of stratification is one of the fundamental concepts in algebraic geometry and singularity theory. The idea to decompose a singular topological space X into non singular parts which we call the strata, satisfying some regularity conditions, goes back to Whitney's and Thom's works.

One of the most famous regularity conditions for a stratification is the (w)-condition of Kuo-Verdier. It is defined in the following way. First, given E and F two vector subspaces of $\mathbb{R}^n$ we define $\delta(E;F) = \sup_{u\in E,|u|=1} d(u;F)$, where $|.|$ is the Euclidian norm and d the Euclidian distance. Then:

Definition 0.1. (Cf. [13] and [7]) We recall that a pair (X,Y) of submanifolds of $\mathbb{R}^n$, with $Y \subset \overline{X} - X$, is said to be a (w)**-stratification** (or **to satisfy the** (w)**-condition**) at $y_0 \in Y$ if there is a real constant $C > 0$ and a neighborhood U of y_0 such that

$$\delta(T_yY, T_xX) \leq C|x-y|,$$

for all $y \in U \cap Y$ and all $x \in U \cap X$.

Using Hironaka's resolution of singularities, Verdier established that every analytic variety admits a (w)-stratification. For differentiable stratifications, the (w)-condition implies local topological triviality along strata [13]. T. Fukui and L. Paunescu [5] found some conditions on the equations defining the strata to show that a given family is topologically trivial. They gave a weighted version of (w) regularity and of Kuo's ratio test, which provides weaker conditions for

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topological stability.

More recently, the attention of geometers started to focus on the metric properties of singularities. T. Mostowski introduced the notion of Lipschitz stratification to establish a bi-Lipschitz version of the Thom-Mather isotopy theorem and proved its existence for complex analytic sets. His work has then been extended to the real setting by A. Parusiński [11, 12]. In [15], the second author has given a semi-algebraic bi-Lipschitz isotopy theorem for sets which are definable in an o-minimal structure. In [3, 4] Lev Birbrair has investigated the problem of classification of semi-algebraic sets, up to bi-Lipschitz equivalence.

Quasi-homogeneous stratifications have been studied by several authors in the past. In the algebraic setting, they occur naturally. The idea is, given a stratification which is preserved by a certain class of action (see section 1 below), to provide some explicit criteria on the weights of the action in order to ensure a certain stability in the geometry. Usually, the above mentioned regularity conditions on stratifications are difficult to check explicitly since this requires some technical computations.

K. Bekka in [1] gives sufficient conditions for a quasi-homogeneous stratification to satisfy his (c)-condition. In [2], the authors show that a quasi-homogeneous complex surface singularity is metrically conical (i.e., bi-Lipschitz equivalent to a metric cone) only if its two lowest weights are equal. The problem of bi-Lipschitz stability of quasi-homogeneous germs is studied in [6] from the point of view of sufficiency of jets. They give some interesting explicit criteria for bi-Lipschitz determinacy of quasi-homogeneous germs. Our approach is similar but we are interested in stratification theory. The interest of quasi-homogeneous stratification is that the conditions which are generally required to establish stability are very elegant and explicit.

In this paper, we give some conditions for a quasi-homogeneous set stratified by two strata $(X;Y)$ to be bi-Lipschitz trivial along Y (Theorem 3.3). The significant advantage of such a criterion is to be very explicit and easy to check (see example 3.1).

For this purpose we start by giving a condition sufficient to ensure the Kuo-Verdier regularity for quasi-homogeneous stratifications (Theorem 2.2). This estimation may also be used to ensure stronger conditions such as the strong Kuo-Verdier regularity condition for instance. We give in section 3 some consequences of this fact on the volume of quasi-homogeneous stratified families. We provide a condition on the weights of the action which ensures the stability of the volume of the germs of the singularities. We are able for instance to determine whenever the density of a set is constant along a stratum. We provide an explicit example (see example 2.2).

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1. Definitions and notations

Definition 1.1. Let V be a finite dimensional real vector space. A **$\mathbb{R}$-action** α on V is a map given by:

$$\alpha \ : \ (\mathbb{R} \setminus \{0\}) \times V \ \longrightarrow \ V$$

$$(t, x) = (t, (x_1, \cdots, x_n)) \ \mapsto \ t.x = (t^{a_1} x_1, \cdots, t^{a_n} x_n)$$

with $a_1, a_2, \ldots, a_n$ integers, called the **weights** of the action α.

Definition 1.2. Let Σ be a stratification of a set A and let α be an action defined on A. We say that Σ is **invariant** by α if for any nonzero real number t we have $t.S \subseteq S$, $\forall S \in \Sigma$.

We also say that Σ is a **quasi-homogeneous stratification**.

We fix two integers k and n with $k \leq n$.

Notations and setting. Throughout this paper, the letter C stands for various positive constants. By stratification, we will mean a finite partition of a set into smooth manifolds (at least C^2).

The letter A will stand for a subset of $\mathbb{R}^n$ containing $Y := \mathbb{R}^k \times 0$. We set $X := A \setminus Y$ and will assume that X is at least C^2. We fix an action α of weights $a = (a_1, \cdots, a_n)$ preserving X and Y.

Important assumption: *We will always assume that* $a_i \leq 0$, $\forall i \leq k$ *and* $a_i > 0$, $\forall i > k$.

In this setting, we may define a positive continuous function $\boldsymbol{\rho_\alpha}$ by:

$$\rho_\alpha : \ \mathbb{R}^n \ \to \ \mathbb{R}$$

$$\rho_\alpha(x) \ = \rho_\alpha(x_1, \ldots, x_n) = \big(\sum_{i>k} \mid x_i \mid^{\frac{m}{a_i}} \ \big)^{\frac{1}{m}},$$

where $m = l.c.m.\ of\ \{a_i \mid \ k < i \leq n\}$.

As $a_i > 0$ for $i > k$, we see that ρ_α is a differentiable function outside $\mathbb{R}^k \times \{0\}$. Moreover clearly we have:

$$\rho_\alpha(t.x) = |t|\,\rho_\alpha(x) \qquad \forall t \in \mathbb{R} \setminus \{0\}.$$

We will use the following notations:

$$S = \operatorname*{Sup}_{i>k} a_i \ ; \quad I = \operatorname*{Inf}_{i>k} a_i \quad \textit{and} \quad S_Y = \operatorname*{Sup}_{i\le k} a_i.$$

2. Quasi-homogeneous (w) stratifications

Now, we are going to give some sufficient conditions for a couple of strata to satisfy the (w)-condition in the case of a quasi-homogeneous stratifications. This estimation will be used also in the next section to obtain a criterion for the condition (w) with exponent k.

We recall that we assume $a_i \ge 0$ for all $i > k$ and $a_i < 0$ for all $i \le k$.

Lemma 2.1. *Assume that there exists a neighborhood U of the origin such that*

$$\delta(Y, T_q X) \le c < 1, \tag{2.1}$$

for all $q \in U$. Then there exists a positive constant C such that for any q in U,

$$\delta(Y, T_q X) \le \ C\ \rho_\alpha(q)^{I - S_Y}.$$

Proof. It follows from the definition of δ that for each point $p \in X$ such that $\rho_\alpha(p) = \varepsilon$, we can find u^j in T_pX such that $\pi(u^j) = e_j = (0, \dots, 0, 1, 0, \dots, 0)$, where the j^{th} coordinate is 1, with $j \le k$ (where π is the orthogonal projection onto Y) and such that $|u^j|$ is bounded by a constant C independent of p. Let $q \in X$ and $t = \frac{\varepsilon}{\rho_\alpha(q)}$. Then $\rho_\alpha(t.q) = \varepsilon$, by definition of ρ_α. Note that as $a_i \le 0$ for $i \in \{1, \dots, k\}$ the point $t.q$ stays in a relatively compact neighborhood of the origin. Therefore (2.1) is fulfilled at $t.q$.

Hence we can choose $u^j = (u^j_1, \dots, u^j_n) \in T_{t.q}X$ satisfying

$$\pi(u^j) = e_j$$

and $|u^j| \le C$.

Remark that $\frac{1}{t}.u^j \in T_qX$ because X is invariant by α. Note also that the norm of $\frac{1}{t}.u^j$ is greater than or equal to $\frac{1}{t^{a_j}}$. By definition of δ, this implies that:

$$\delta(Y, T_qX) \le \max_{1\le j\le k} \frac{|\frac{1}{t}.u^j - \frac{1}{t}.e_j|}{(\frac{1}{t})^{a_j}}$$

and

$$\frac{|\frac{1}{t}.u - \frac{1}{t}.e_j|}{(\frac{1}{t})^{a_j}} = \frac{|(0, \dots, 0, (\frac{1}{t})^{a_{k+1}}\ u_{k+1}, \dots, (\frac{1}{t})^{a_n}\ u_n)|}{(\frac{1}{t})^{a_j}}$$

$$= (\frac{1}{t})^{I-a_j} \left|(0,\dots,0,(\frac{1}{t})^{a_{k+1}-I}\ u_{k+1},\dots,(\frac{1}{t})^{a_n-I}\ u_n)\right|$$
$$\leq C\ (\frac{1}{t})^{I-a_j}$$
$$\leq C\ (\frac{1}{t})^{I-S_Y}$$

(because $\frac{1}{t}$ is small for q in the neighborhood of zero).

Hence we can conclude that $\delta(Y, T_qX) \leq C\ \rho_\alpha(q)^{I-S_Y}$. □

Theorem 2.2. *In the situation of Lemma 2.1, if $I - S_Y \geq S$, then*

$$\delta(Y, T_qX) \leq C\ \rho_\alpha(q)^S,$$

on a neighborhood of the origin. In particular the pair (X, Y) is a (w) stratification.

Proof. If $I - S_Y \geq S$, then for q close to the stratum Y,

$$\rho_\alpha(q)^{I-S_Y}\ \leq\ \rho_\alpha(q)^S.$$

So, using Lemma 2.1:

$$\delta(Y, T_qX) \leq C\ \rho_\alpha(q)^S$$

But by definition of ρ_α:

$$\rho_\alpha(q)^S\ = \Big(\sum_{i>k} \mid x_i \mid^{\frac{m}{a_i}}\Big)^{\frac{S}{m}} \leq \sum_{i>k} \mid x_i \mid^{\frac{S}{a_i}} \leq \sum_{i>k}\ \mid x_i \mid .$$

Hence $(X; Y)$ satisfies the (w)-condition at the origin. □

Example 2.1. Let $V = \{(x, y, u, v) \subset \mathbb{R}^4 \mid\ y^4 = x^5u^2v^4 + x^2\ \}$, then Sing $V = \{0\} \times \{0\} \times \mathbb{R}^2$.

Stratify V with $X = Reg\,V$ and $Y = Sing\,V$. Define an action α by $t.(x, y, u, v) = (t^4x, t^2y, t^{-2}u, t^{-2}v)$.

Then, we have $S = 4, I = 2, S_Y = -2$. The pair $(X; Y)$ is invariant by α and the condition of Theorem 2.2 holds. We know that: $\delta(Y, T_qX) = 1 \ \Leftrightarrow\ \frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, with $f = y^4 - x^5u^2v^4 - x^2$. We have near the origin:

$$\frac{\partial f}{\partial y} = 0\ \Rightarrow\ y = 0, \text{ and } \frac{\partial f}{\partial x} = 0\ \Rightarrow\ x = 0.$$

This means that there exists a positive constant ε such that $\delta(Y, T_qX) < 1$, for all q such that $\rho_\alpha(q) = \varepsilon$. Hence the pair of (X, Y) is a (w)-regular stratification.

We end this section by giving an application of the above theorem to the volume of quasi-homogenous families. We will not use this theorem in the next section but we mention it since it is interesting by itself.

2.1 On the volume of quasi-homogeneous families

Assume that E is a subanalytic set. We denote by $B(y;r)$ the open ball of radius r centered at y.

We denote by $\mathcal{H}^l$ the l-dimensional Haussdorf measure. Let:

$$\psi(E;y;r) = \mathcal{H}^l(E \cap B(y;r)),$$

where l is the dimension of E at y. We call the following number **density** of E at y:

$$\lim_{r\to 0} \frac{\psi(E;y;r)}{r^l}.$$

One can prove that this limit exists for any subanalytic set (see [8]).

In [9] J.-M. Lion and J.-P. Rolin show that for any $y \in E$, the function ψ has a Puiseux expansion in r and $\ln r$. More precisely, the exists a positive integer q such that we have:

$$\psi(E;y;r) = \sum_{j=0}^{k} \sum_{p\in \frac{1}{q}\mathbb{N}} a_{p,j} r^p (\ln r)^j \tag{2.2}$$

where for each j the coefficient at $(\ln r)^j$ is a convergent fractional power series. By definition the **order** of the monomial $r^p(\ln r)^j$ is p.

It is known that $\ln t$ does not appear in the term of order l. The density is thus the coefficient of term of order l (which may be zero).

On the other hand, the second author proved in [14] that if the Kuo-Verdier condition is satisfied at some orders, the expansion of the volume is preserved at a corresponding order when the parameters vary (see [14], Theorem 3.1.3). We use below the results of this section on quasi-homogeneous stratified families to get a sufficient condition for the expansion of the volume of the fibers of such families to be stable.

Theorem 2.3. *Let $\nu > 1$. Assume that E is a closed subanalytic subset of $\mathbb{R}^n$ stratified by a quasi-homogeneous subanalytic stratification $\Sigma = \{Y, X_1, \dots, X_m\}$. Assume that (2.1) holds for any couple $(Y;X_i)$. If*

$$I - S_Y \geq \nu S,$$

then the expansion (2.2) of $\psi(E;y;r)$ is constant up the order $(\nu + l - 1)$ along Y where $l = \dim E$.

In particular, if $I - S_Y > S$ then the density of E is constant along Y.

Proof. Applying Theorem 2.2 to $(Y; X_i)$ we get:

$$\delta(Y, T_q X_i) \leq C\rho_\alpha(q)^{\nu S},$$

which means that we have:

$$\delta(Y, T_q X_i) \leq C|q - \pi(q)|^\nu,$$

for q in a neighborhood of the origin. This is the Kuo-Verdier condition "with exponent ν".

We denote by E_y the germ of E at y. For simplicity we do a translation i.e. we will consider E_y as a germ at the origin by identifying y with 0. By Theorem 3.2.2 of [14] we get a family of homeomorphisms $h_y : E_y \to E_0$ (for $y \in Y$ sufficiently close to zero) which is a $(\nu - 1)$ approximation of the identity, which means that for q in the ball of radius r:

$$|h_y(q) - q| \leq Cr^{\nu-1},$$

for some positive constant C.

But now by Theorem 3.1.3 of [14] we see that this implies that the expansion of $\psi(E; y; r)$ is constant up to the order $\nu + l - 1$. The last statement of the theorem also ensues since the term of order l is, by definition, the density. ☐

Example 2.2. Let us consider an example which is similar to the previous one but stratified differently.

Let

$$E = \{(x, y, z, u) \subset \mathbb{R}^4 \mid \ y^2 = x^4z^2u^2 + x^3 \ \},$$

then Sing $E = \{0\} \times \{0\} \times \mathbb{R}^2$. Define the action α by $t.(x, y, u, v) = (t^4x, t^6y, t^3z, t^{-5}u)$.

Denote by Y the u-axis and by X_1 the complement of Y in the $(z; u)$ plane. Stratify the set E by $\Sigma := \{Y, X_1, X_2\}$ where $X_2 := E\backslash(Y \cup X_1)$. This constitutes a quasi-homogeneous stratification of E and we have $S = 6, I = 3, S_Y = -5$.

Moreover, observe that in this setting we have:

$$I - S_Y > S.$$

On the other hand, condition (2.1) is also satisfied for the same reasons as in example 2.1.

Thus, the last sentence of Theorem 2.3 yields that the density of E is constant along Y around the origin.

3. Bi-Lipschitz triviality of quasi-homogeneous stratifications

In this section, we give sufficient conditions on the weights for the set A to be bi-Lipschitz trivial along Y. We will use a similar method to the one used in section 2. We are still in the situation described in the first section. We denote by π the orthogonal projection onto Y.

Proposition 3.1. *Assume that (2.1) holds and that A is closed. There exist a neighborhood of zero and a constant C such that for all couples $(q; q') \in X \times X$ in this neighborhood satisfying*

$$| \rho_\alpha(q)^S - \rho_\alpha(q')^S | < \frac{1}{2}\, \rho_\alpha(q)^S \tag{3.1}$$

and for each unit vector v in T_qX normal to Y, we can find a vector v' in $T_{q'}X$, normal to Y and such that:

$$| v - v' | \leq C \, \frac{| q - q' |}{\rho_\alpha(q)^{(2S-I)}}.$$

Proof. Thanks to condition (2.1), the vector space $Y^\perp$ is transverse to T_qX. Thus, the distribution $Y^\perp \cap T_qX$ is smooth on X, sufficiently close to zero. In, particular it is a locally Lipschitz distribution.

This means in particular that for $(q; q')$ in the set

$$V = \{p \in X \ | \ \frac{\varepsilon}{2} \leq \rho_\alpha(p)^S \leq 2\varepsilon \},$$

for ε small enough, and $v \in T_qX$ normal to Y we can easily construct a vector v' in $T_{q'}X$ fulfilling the desired properties.

Let $q \in X$ and $t = \frac{\varepsilon^{\frac{1}{S}}}{\rho_\alpha(q)}$, and let $v \in T_qX$, with $|v| = 1, \pi(v) = 0$. Let $q' \in X$ such that $| \rho_\alpha(q')^S - \rho_\alpha(q)^S | < \frac{1}{2}\rho_\alpha(q)^S$. As by (3.1), $t.q$ and $t.q'$ belong to V, we can find a vector w in $T_{t.q'}X$ such that

$$| t.v - w | \leq C \, | t.q - t.q' | | t.v | .$$

Let $v' = \frac{1}{t}.w$ and remark that $| t.v | \leq C\rho_\alpha(q)^{-S}$.

So

$$| v - v' | \leq | v - \frac{1}{t}.w | \leq C\big(\frac{1}{t}\big)^I \, | t.v - w |,$$

since v and w are normal to Y. This implies that:

$$\begin{aligned} | v - v' | &\leq C\big(\frac{1}{t}\big)^I \, | t.q - t.q' | | t.v | \\ &\leq C\big(\frac{1}{t}\big)^{I-S} \, | q - q' | \, \rho_\alpha(q)^{-S} \\ &\leq C\frac{| q - q' |}{\rho_\alpha(q)^{2S-I}}. \end{aligned}$$

□

Proposition 3.2. *Suppose that* (2.1) *holds in a neighborhood of the origin and that A is closed. Then there exist a neighborhood of the origin and a constant C such that for each couple $(q;q')$ in $X \times X$ and for each vector e_j of the canonical basis of Y, there exist a vector v_j in T_qX and a vector w_j in $T_{q'}X$ such that for each couple $(q;q')$ satisfying (3.1):*

$$\mid v_j - e_j \mid \leq C\rho_\alpha(q)^{I-a_j},$$

$$\mid v_j - w_j \mid \leq C \mid q - q' \mid \rho_\alpha(q)^{I-S-a_j}.$$

Proof. We use the same method as in the proof of Proposition 3.1.
Let $V = \{q \in X \mid \frac{\varepsilon}{2} \leq \rho_\alpha(q)^S \leq 2\varepsilon\}$ and set $t = \frac{\varepsilon^{\frac{1}{S}}}{\rho_\alpha(q)}$. Then $t.q$ and $t.q'$ belong to V.
As in the proof of Lemma 2.1, we can choose a vector w of $T_{t.q}X$ whose norm is bounded by a constant independent of q and such that $\pi(w) = e_j$.

Now, we have:

$$\begin{aligned} \frac{\left|\frac{1}{t}.w - \frac{1}{t}.e_j\right|}{(\frac{1}{t})^{a_j}} &\leq C\,(\frac{1}{t})^{I-a_j} \\ &\leq C\rho_\alpha(q)^{I-a_j}. \end{aligned}$$

As the set V is smooth and $t.q' \in V$, we can choose a vector $w' \in T_{tq'}X$ such that $\pi(w') = e_j$ and $\mid w - w' \mid \leq C \mid t.q - t.q' \mid$, for a constant C independent of q and q'.

And by a similar computation as in the proof of Proposition 3.1 we get:

$$\frac{\left|\frac{1}{t}.w - \frac{1}{t}.w'\right|}{(\frac{1}{t})^{a_j}} \leq C\,\frac{\mid q - q' \mid}{\rho_\alpha(q)^{I-S+a_j}}.$$

Hence it is enough to set

$$v_j = \frac{\frac{1}{t}.w}{(\frac{1}{t})^{a_j}} \quad \text{and} \quad w_j = \frac{\frac{1}{t}.w'}{(\frac{1}{t})^{a_j}}.$$

□

We are now ready to prove the main theorem of this paper.

Theorem 3.3. *Suppose that A is locally closed at the origin and that* (2.1) *holds in a neighborhood of the origin. If $(2I - S_Y) \geq 2S$, then the set A is locally bi-Lipschitz trivial along Y at the origin.*

Proof. It is enough to show that on a neighborhood of the origin, we can find a constant C satisfying:

$$| (P_q - P_{q'})\pi | \leq C \ | q - q' |,$$

for all couple $(q; q')$, if π is the orthogonal projection on Y and P_q is the ortogonal projection on T_qX. This inequality clearly enables us to extend a basis of Y to a Lipschitz family of stratified vector fields whose respective flows will provide the desired trivialization (see [12]).

We begin by considering the case where $(q; q')$ satisfies $| \rho_\alpha(q)^S - \rho_\alpha(q')^S | \leq \frac{1}{2} \rho_\alpha(q)^S$.

Let $1 \leq j \leq k$. By Proposition 3.2, there exist a vector v_j and a vector w_j such that:

$$| v_j - e_j | \leq C \rho_\alpha(q)^{I - S_Y}$$

and

$$| v_j - w_j | \leq C \ | q - q' | .\rho_\alpha(q)^{I-S-S_Y}.$$

But since $2I - S_Y \geq 2S$, we get:

$$| v_j - e_j | \leq C \rho_\alpha(q)^{2S-I} \tag{3.2}$$

and

$$| v_j - w_j | \leq C \ | q - q' | \ \rho_\alpha(q)^{S-I} \tag{3.3}$$

By Proposition 3.1, if $u \in T_qX$ is a unit vector normal to Y in T_qX, there exists a vector u' in $T_{q'}X$ also normal to Y such that

$$| u - u' | \leq \frac{| q - q' |}{\rho_\alpha(q)^{2S-I}}.$$

Together with (3.2) and (3.3) this implies that:

$$\delta(T_qX, T_{q'}X) \leq C \frac{| q - q' |}{\rho_\alpha(q)^{2S-I}},$$

which amounts to:

$$| P_q - P_{q'} | \leq C \frac{| q - q' |}{\rho_\alpha(q)^{2S-I}}. \tag{3.4}$$

Now:

$$\begin{aligned} | (P_q - P_{q'})(e_j) | &\leq C \ | (P_q - P_{q'})(v_j - e_j) | + | (P_q - P_{q'})(v_j) | \\ &\leq C \ | P_q - P_{q'} | . | v_j - e_j | + | P_{q'}^{\perp}(v_j) |, \ \text{ for } v_j \in T_qX \\ &\leq C \frac{| q - q' |}{\rho_\alpha(q)^{2S-I}} . \rho_\alpha(q)^{2S-I} + |v_j - w_j |, \ \text{ by (3.2) and (3.4)} \end{aligned}$$

$$\leq C \ \mid q - q' \mid \qquad \text{(by (3.3))}.$$

It remains the case where

$$\mid \rho_\alpha(q)^S - \rho_\alpha(q')^S \mid \geq \frac{1}{2} \max(\rho_\alpha(q)^S ; \rho_\alpha(q')^S).$$

Assume for simplicity that we have $\rho_\alpha(q) \geq \rho_\alpha(q')$.
Observe as ρ_α^S is Lipschitz we have:

$$C|q - q'| \geq |\rho_\alpha(q)^S - \rho_\alpha(q')^S| \geq \frac{\rho_\alpha(q)^S}{2}. \tag{3.5}$$

Since $2I - S_Y \geq 2S$, we have $I - S_Y \geq S$ and according to Theorem 2.2 we get:

$$\mid P_q^\perp \pi \mid \leq C \rho_\alpha(q)^S.$$

Then:

$$\begin{aligned}
\mid (P_q - P_{q'})\pi \mid &= \mid (P_q^\perp - P_{q'}^\perp)\pi \mid \\
&\leq \mid P_q^\perp \pi \mid \ + \ \mid P_{q'}^\perp \pi \mid \\
&\leq \ C \ \rho_\alpha(q)^S \ + \ C\rho_\alpha(q')^S \\
&\leq \ C \ \rho_\alpha(q)^S \\
&\leq \ C \ \mid q - q' \mid \qquad \text{(by (3.5))},
\end{aligned}$$

which completes the proof. ☐

Example 3.1. Let $A = \{(x, y, u, v) \subset \mathbb{R}^4 \mid \ y^4 = x^6u^2v^4 + x^3 \}$, then Sing $A = \{0\} \times \{0\} \times \mathbb{R}^2$. We can stratify A by $X = RegA$ and $Y = SingA$.

We define an action α by $t.(x, y, u, v) = (t^4x, t^3y, t^{-2}u, t^{-2}v)$.
The pair (X, Y) is invariant by α and the conditions of Theorem 3.3 in this case: $6 - (-2) \geq 8$ holds. The same computation as in Example 2.1 shows that there exists ε such that $\delta(Y, T_qX) < c < 1$ for all q sufficiently close to the origin and such that $\rho_\alpha(q) = \varepsilon$.
Therefore according to Theorem 3.3 the stratification $\Sigma = (X; Y)$ of A is bi-Lipschitz trivial along Y.

Remark 3.1. Theorem 3.3 provides bi-Lipschitz triviality along Y locally at the origin. The proof actually also establishes it all along Y (in a sufficiently small neighborhood of Y), but the obtained Lipschitz constants are local and in general may be bounded only on compact subsets of Y.

The criterion provided by Theorem 3.3 cannot be used for a stratification having more than two strata. We end this paper by showing this by means of a counterexample.

Example 3.2. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x;y;z;u) = z^3y^5 - z^8x^2u^2 + x^4$. The set A constituted by the zero locus of this polynomial is invariant under the action $t.q := (t^2x; ty; tz; t^{-2}u)$. The singular locus of this map is given by the $(y;u)$-plane and the $(z;u)$-plane. Thus we can stratify this set by three strata: let Y be the u-axis, X_1 the complement of Y in the two latter 2-dimensional planes and X_2 the complement of $X_1 \cup Y$ in A.

Condition (2.1) is obviously satisfied by X_1. A straightforward computation of derivative yields that it holds as well for X_2, at any point of Y. Notice that we have: $2I - S_Y \geq 2S$.

However, we can see that this family is *not* bi-Lipschitz trivial along Y. Indeed at any point $q := (0;0;z_0;u_0)$ with z_0 nonzero, the corresponding fiber of the family is a family of cusps (along the z-axis) with order of contact $\frac{5}{4}$, if u_0 is zero, and $\frac{5}{2}$ if u_0 is nonzero. For each u_0, the z-axis is the set of points at which the fiber is not a Lipschitz manifold and hence has to be preserved by any bi-Lipschitz trivialization of the family and the order of contact of the cusps should be preserved as well.

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