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The Lifespan of Solutions to Quasilinear Hyperbolic Systems of $n \times n$ in the Critical Case

Akira Hoshiga and Taiichi Nose

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Abstract

In this paper, we discuss the Cauchy problem of the quasilinear hyperbolic systems of $n \times n$. We estimate the lifespan of the classical solutions to this problem in the critical case. This is a generalization of the results which have been showed in [2] and [3] when $n = 1$ and $n = 2$, respectively.

1. Introduction

We consider the following quasilinear system of first order;

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \qquad (x,t) \in \mathbf{R} \times (0,\infty), \tag{1.1}$$

where $u = u(x,t) = {}^t(u_1, \cdots, u_n)$ is the unknown vector function and $A(u)$ is an $n \times n$ matrix with smooth elements a_{ij} $(i,j = 1, \cdots, n)$. We assume that $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{1.2}$$

in a neighbourhood of $u = 0$. This assumption means that the system (1.1) is strictly hyperbolic in the neighbourhood of $u = 0$. Let $l_i(u)$ and $r_i(u)$ be left and right eigenvectors, respectively, corresponding to $\lambda_i(u)$, *i.e.*,

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \text{and} \quad A(u)r_i(u) = \lambda_i(u)r_i(u) \qquad (i = 1, \cdots, n).$$

Note that all $\lambda_i(u), l_i(u)$ and $r_i(u)$ have the same regularity as $a_{ij}(u)$. Without loss of generality, we may assume that

$$l_i(u)r_j(u) = \delta_{ij} \quad \text{and} \quad {}^t r_i(u)r_i(u) = 1 \qquad (i,j = 1, \cdots, n) \tag{1.3}$$

hold for any $u \in \mathbf{R}^n$, where δ_{ij} is Kronecker's delta.

We prescribe the initial condition

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$$u(x,0)=\varepsilon\varphi(x), \qquad x\in\mathbf{R}, \tag{1.4}$$

where ε is a small positive parameter. We assume that $\varphi(x) = {}^t(\varphi_1(x),\cdots,\varphi_n(x))$ is a C^1 vector valued function and satisfies

$$\sup_{x\in\mathbf{R}}\{(1+|x|)^{1+\mu}(|\varphi(x)|+|\varphi'(x)|)\}<\infty \tag{1.5}$$

for some constant $\mu>0$. In this paper, we shall study the existence and the blow-up of C^1 solutions to the Cauchy problem (1.1) and (1.4).

Li Ta-tsien, Zou Yi and Kong De-xing studied the Cauchy problem (1.1) and (1.4) in [2]. To state their results we need the concept of the weak linear degeneracy.

Definition. We call the solution of the following ordinary differential equation the i-th *characteristic trajectory* $u=u^i(s)$ passing through $u=0$;

$$\frac{du(s)}{ds}=r_i(u(s)) \quad \text{for small} \quad |s|, \tag{1.6}$$

$$u(0)=0.$$

Then we say that the i-th characteristic $\lambda_i(u)$ $(i=1,\cdots,n)$ is *weakly linearly degenerate*, if we have

$$\lambda_i(u^i(s))\equiv\lambda_i(0) \qquad \text{for small} \quad |s|,$$

namely,

$$r_i\cdot\nabla\lambda_i(u^i(s))\equiv 0 \qquad \text{for small} \quad |s|.$$

If all characteristics are weakly linearly degenerate, we say that the system (1.1) is weakly linearly degenerate.

Now we state the results presented in [2]. They proved that if the system (1.1) is weakly linearly degenerate, there exists a global C^1 solution to (1.1) and (1.4) for sufficiently small ε.

On the other hand, if the system (1.1) is not weakly linearly degenerate, we can define a nonempty set $J\subset\{1,\cdots,n\}$ such that $\lambda_j(u)$ is not weakly linearly degenerate if and only if $j\in J$.

For each $j\in J$, either there exists an integer $\alpha_j\geq 0$ such that

$$\frac{d^l}{ds^l}(r_j\cdot\nabla\lambda_j(u^j(s)))\bigg|_{s=0}=0\ (l=1,\cdots,\alpha_j) \text{ and } \frac{d^{\alpha_j+1}}{ds^{\alpha_j+1}}(r_j\cdot\nabla\lambda_j(u^j(s)))\bigg|_{s=0}\neq 0$$

or

$$\frac{d^l}{ds^l}(r_j\cdot\nabla\lambda_j(u^j(s)))\bigg|_{s=0}=0 \qquad \text{for any} \quad l=0,1,2,\cdots, \tag{1.7}$$

where $u = u^{(i)}(s)$ is defined by (1.6). In the case (1.7) holds, we define $\alpha_j = +\infty$. Furthermore, set

$$\alpha = \min\{\alpha_j \mid j \in J\}.$$

Then, they also proved that if the system (1.1) is not weakly linearly degenerate and α is finite, then there exist an ε_0 and constants c and C such that

$$\frac{c}{\varepsilon^{\alpha+1}} \leq T(\varepsilon) \leq \frac{C}{\varepsilon^{\alpha+1}}$$

holds for $0 < \varepsilon < \varepsilon_0$. Here $T(\varepsilon)$ is the *lifespan* which is defined by the supremum over all $T > 0$ such that a C^1 solutions to (1.1) and (1.4) exists in $\mathbf{R} \times [0, T)$.

Therefore, we are interested in the critical case, namely, the case where the system (1.1) is not weakly linearly degenerate and $\alpha = \infty$. Since the critical case for single equation and 2×2 system have been studied in [2] and [3], our aim in this paper is to obtain upper and lower bounds of the lifespan in the critical case for the $n \times n$ system.

To state our results, we introduce a function $F(p) \in C^\infty[-M, M]$;

$$F(p) = \begin{cases} \exp\left(-\dfrac{1}{a(|p|)}\right) & 0 < |p| \leq M, \\ 0 & p = 0 \end{cases} \tag{1.8}$$

for some $M > 0$. Here $a(\cdot)$ satisfies the following assumptions;

$$\begin{array}{ll} \text{(i)} & F(\cdot) \in C^\infty[-M, M]. \\ \text{(ii)} & a(0) = 0. \\ \text{(iii)} & a'(y) > 0 \ \text{ in } \ (0, M) \qquad y = |p|. \\ \text{(iv)} & \text{For any } A, B \text{ and } \mu > 0, \text{ there exists an } \varepsilon_0 > 0 \text{ such that} \\ & \dfrac{a(A\varepsilon + B\varepsilon^2)}{a(A\varepsilon - B\varepsilon^2)} \leq 1 + \mu \quad \text{for} \ \ 0 < \varepsilon \leq \varepsilon_0. \end{array} \tag{1.9}$$

We have to assume (iv) to derive the blow-up of the solutions, because we deal with $n \times n$ systems. Typical examples of $a(\cdot)$ satisfying the assumption (1.9) are

(1) $a(y) = y^r \qquad (r > 0)$

(2) $$a(y) = \begin{cases} \exp\left(-\dfrac{1}{y^r}\right) & y > 0, \\ 0 & y = 0. \end{cases} \qquad (0 < r < 1)$$

Now we state our theorem.

Theorem. *Assume that (1.2) and (1.5) hold. Let* $u = u^i(s)$ *be the* i*-th characteristic trajectory passing through* $u = 0$ $(i = 1, \cdots, n)$. *Assume that there exists* $i_0 \in \{1, \cdots, n\}$ *such that* $l_{i_0}(0)\varphi(x) \not\equiv 0$ *and*

$$r_{i_0} \cdot \nabla\lambda_{i_0}(u^{i_0}(s)) = -F(u^{i_0}(s)), \tag{1.10}$$

where F *is the one defined in (1.8) and (1.9). Furthermore, assume that if* $i \neq i_0$,

$$\left|\frac{d^l}{ds^l}(r_i \cdot \nabla\lambda_i(u^i(s)))\right| \leq C\left|\frac{d^l}{ds^l}F(u^i(s))\right| \tag{1.11}$$

holds for any $l = 0, 1, 2, \cdots$. *Then there exist an* $\varepsilon_1 > 0$ *and constants* c_1, c_2, C_1 *and* $C_2 > 0$ *such that*

$$c_1 \exp\left(\frac{1}{a(c_2\varepsilon)}\right) \leq T(\varepsilon) \leq C_1 \exp\left(\frac{1}{a(C_2\varepsilon)}\right) \tag{1.12}$$

holds for $0 < \varepsilon < \varepsilon_1$.

Remark. Our proof of theorem essentially follows the method used in [3] for the case of 2×2 systems. However, as is mentioned in the second remark of Lemma 3 in [3], we have a problem to apply the method to $n \times n$ system. We succesfully overcome the difficulty by using another norm $\tilde{V}_1$ which will be defined in section 3.

2. Preliminaries

Since we are assuming the system (1.1) is strictly hyperbolic, if the matrix $A(u)$ in system (1.1) consists of C^k elements $a_{ij}(u)$, there exists an invertible C^{k+1} transformation $u = u(\hat{u})$ such that $u(0) = 0$ and in $\hat{u}$-space the i-th characteristic trajectory passing through $\hat{u} = 0$ coincides with the $\hat{u}_i$-axis near the origin. This means that

$$\hat{r}_i(\hat{u}_i e_i) \equiv e_i, \qquad i = 1, \cdots, n \tag{2.1}$$

holds for sufficiently small $|\hat{u}_i|$. Here e_i $(i = 1, \cdots, n)$ stands for usual unit vectors. This transformation is called to be the *normalized transformation* and the corresponding variables $\hat{u} = {}^t(\hat{u}_1, \cdots, \hat{u}_n)$ are called to be the *normalized coordinates*. The existence of the normalized transformation was shown in [1]. In the normalized coordinates, we can rewrite system (1.1) as

$$\frac{\partial \hat{u}}{\partial t} + \hat{A}(\hat{u})\frac{\partial \hat{u}}{\partial x} = 0 \tag{2.2}$$

with

$$\hat{A}(\hat{u}) = \left(\frac{\partial u}{\partial \hat{u}}\right)^{-1} A(u(\hat{u})) \left(\frac{\partial u}{\partial \hat{u}}\right),$$

where $(\partial u/\partial \hat{u})$ is the Jacobi matrix. Denote the i-th eigenvalue of $\hat{A}(\hat{u})$ by $\hat{\lambda}_i(\hat{u})$. Then we have $\hat{\lambda}_i(\hat{u}) = \lambda_i(u(\hat{u}))$, $(i = 1, \cdots, n)$. From now on, we omit the sysmbol $\wedge$ and treat the transformed system (2.2).

Now we introduce important notations. Let

$$v_i = l_i(u)u \tag{2.3}$$

and

$$w_i = l_i(u)\frac{\partial u}{\partial x}, \tag{2.4}$$

where $l_i(u)$ is the i-th left eigenvector $(i = 1, \cdots, n)$. By (1.3) we have

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.5}$$

and

$$\frac{\partial u}{\partial x} = \sum_{k=1}^{n} w_k r_k(u). \tag{2.6}$$

Hence, we find that $\partial u/\partial x$ blows up if and only if w_i bolows up for some i. Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.7}$$

be the directional derivative along the i-th characteristic. Then we have (cf.[2])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k, \qquad i = 1, \cdots, n, \tag{2.8}$$

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k, \qquad i = 1, \cdots, n, \tag{2.9}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u), \tag{2.10}$$

$$\gamma_{ijk}(u) = (\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_i(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik}. \tag{2.11}$$

By (2.10), (2.11) and (2.1) we have

$$\beta_{iji}(u) \equiv 0 \qquad \text{for } \ i, j = 1, \cdots, n, \tag{2.12}$$

$$\gamma_{ijj}(u) \equiv 0 \quad \text{for } i,j=1,\cdots,n \text{ and } \quad i \neq j, \tag{2.13}$$

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u)r_i(u) \quad \text{for } i,j=1,\cdots,n, \tag{2.14}$$

$$\beta_{ijj}(u_je_j) \equiv 0 \quad \text{for } |u_j| \leq M \quad \text{and} \quad i,j=1,\cdots,n. \tag{2.15}$$

Let $a > 0$ and $y \geq 0$ be costants. On the existence domain of C^1 solution, we denote the i-th characteristic passing through a point $(y, y/a)$ by $x = \tilde{x}_i(y,t)$. Namely

$$\frac{d\tilde{x}_i(y,t)}{dt} = \lambda_i(u(\tilde{x}_i(y,t),t)), \tag{2.16}$$

$$\tilde{x}_i\left(y, \frac{y}{a}\right) = y.$$

Differentiating (2.16) with respect to y, we get

$$\frac{d}{dt}\left(\frac{\partial\tilde{x}_i(y,t)}{\partial y}\right) = \nabla\lambda_i(u(\tilde{x}_i(y,t),t))\frac{\partial u}{\partial x}(\tilde{x}_i(y,t),t)\frac{\partial\tilde{x}_i(y,t)}{\partial y}. \tag{2.17}$$

Let $p_i(x,t)$ be definded by

$$p_i(\tilde{x}_i(y,t),t) = v_i(\tilde{x}_i(y,t),t)\frac{\partial\tilde{x}_i(y,t)}{\partial y}, \tag{2.18}$$

then we have

$$\frac{dp_i}{d_it} = \sum_{j,k=1}^{n}\tilde{\beta}_{ijk}(u)\frac{\partial\tilde{x}_i(y,t)}{\partial y}v_jw_k, \tag{2.19}$$

where

$$\tilde{\beta}_{ijk}(u) = \beta_{ijk}(u) + \nabla\lambda_i(u)r_k(u)\delta_{ij}. \tag{2.20}$$

By (2.12) and (2.1) we have

$$\tilde{\beta}_{ijj}(u_je_j) \equiv 0 \quad \text{for } |u_j| \leq M \quad \text{and} \quad i \neq j, \tag{2.21}$$

$$\tilde{\beta}_{iii}(u) \equiv \nabla\lambda_i(u)r_i(u) \quad \text{for } i,j=1,\cdots,n. \tag{2.22}$$

Moreover, by (2.14), (2.22), (1.10) and (1.11) we have

$$\gamma_{i_0i_0i_0}(u_{i_0}e_{i_0}) = -\tilde{\beta}_{i_0i_0i_0}(u_{i_0}e_{i_0}) = F(u_{i_0}) \quad \text{for} \quad |u_{i_0}| \leq M, \tag{2.23}$$

$$|\gamma_{iii}(u_ie_i)|,\ |\tilde{\beta}_{iii}(u_ie_i)| \leq F(u_i) \quad \text{for} \quad |u_i| \leq M \quad \text{and} \quad i \neq i_0. \tag{2.24}$$

3. The proof of Theorem

We will prove (1.12), by dividing the argument into two parts; the lower bound part and the upper bound part.

Firstly we prove the lower bound part. By the existence and uniqueness of local C^1 solution to the Cauchy problem (1.1) and (1.4) (cf.[2]), in order to prove Theorem it suffices to establish a uniform *a priori* estimates on the C^0 norm of u and $\partial u/\partial x$ on the existence domain of the C^1 solution $u = u(x,t)$. Without loss of generality we may assume

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.1}$$

In fact under the following invertible transformation;

$$\bar{t} = t, \quad \bar{x} = x - (\lambda_1(0) - 1)t,$$

the system (1.1) is rewritten as

$$\frac{\partial u}{\partial \bar{t}} + (A(u) - (\lambda_1(0) - 1)E)\frac{\partial u}{\partial \bar{x}} = 0.$$

It is easy to see that (3.1) holds for this system.

By (3.1), we find that for each small $\delta_0 > 0$ there exists a $\delta > 0$ such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq 4\delta_0 \qquad \text{for} \quad |u|, |v| \leq \delta, \ i = 1, \cdots, n-1 \tag{3.2}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2} \qquad \text{for} \quad |u|, |v| \leq \delta, \ i = 1, \cdots, n. \tag{3.3}$$

By (3.1) and (3.2) we find that

$$0 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{3.4}$$

holds as long as $|u(x,t)| \leq \delta$. We introduce some regions in the first quadrant of (x,t)-plane. For fixed $T > 0$, let

$$D_+^T = \{(x,t) \,|\, 0 \leq t \leq T, x \geq (\lambda_n(0) + \delta_0)t\}, \tag{3.5}$$
$$D_-^T = \{(x,t) \,|\, 0 \leq t \leq T, x \leq (\lambda_1(0) - \delta_0)t\}, \tag{3.6}$$
$$D^T = \{(x,t) \,|\, 0 \leq t \leq T, (\lambda_1 - \delta_0)t \leq x \leq (\lambda_n(0) + \delta_0)t\} \tag{3.7}$$

and

$$\begin{aligned} D_i^T = \{(x,t) \mid 0 \leq t \leq T, -[\delta_0 + \eta(\lambda_i(0) - \lambda_1(0))]t \leq \\ x - \lambda_i(0)t \leq [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]t\}, \end{aligned} \tag{3.8}$$

for $i = 1, \cdots, n$. Here we have taken $\eta > 0$ so small that

$$D_i^T \cap D_j^T = \phi \ \text{ if } i \neq j \tag{3.9}$$

and

$$\bigcup_{i=1}^{n} D_i^T \subset D^T \tag{3.10}$$

hold. Then we get the following lemma.

Lemma 1. *Let $T > 0$ be a costant. Then, there exist constans c and C independent of T such that in the domain $D^T \setminus D_i^T$,*

$$ct \le |x - \lambda_i(0)t| \le Ct \quad \text{and} \quad cx \le |x - \lambda_i(0)t| \le Cx \tag{3.11}$$

hold for $i = 1, \cdots, n$.

Proof: By (3.7) and (3.8) we easily have (3.11).

In the same manner as [2], we define some norms;

$$V(D_\pm^T) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} v_i(x,t)\|_{L^\infty(D_\pm^T)}, \tag{3.12}$$

$$W(D_\pm^T) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} w_i(x,t)\|_{L^\infty(D_\pm^T)}, \tag{3.13}$$

$$V_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in D^T\setminus D_i^T} (1+|x-\lambda_i(0)t|)^{1+\mu}|v_i(x,t)|, \tag{3.14}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in D^T\setminus D_i^T} (1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(x,t)|, \tag{3.15}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in D^T\setminus D_i^T} (1+|x-\lambda_i(0)t|)^{1+\mu}|w_i(x,t)|, \tag{3.16}$$

$$\tilde{V}_1(T) = \max_{\substack{i=1,\cdots,n\\ j\neq i}} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |v_i(x,t)|dt, \tag{3.17}$$

$$\tilde{W}_1(T) = \max_{\substack{i=1,\cdots,n\\ j\neq i}} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |w_i(x,t)|dt, \tag{3.18}$$

$$V_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |v_i(x,t)|dx, \tag{3.19}$$

$$W_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |w_i(x,t)|dx, \tag{3.20}$$

$$V_\infty(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in \mathbf{R}\times[0,T]} |v_i(x,t)|, \tag{3.21}$$

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in \mathbf{R}\times[0,T]} |u_i(x,t)|, \tag{3.22}$$

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{(x,t)\in \mathbf{R}\times[0,T]} |w_i(x,t)|, \tag{3.23}$$

where $\tilde{c}_j$ stands for any j-th characteristic in D_i^T and where $D_i^T(t)$ $(t \ge 0)$ denotes the t-section of D_i^T;

$$D_i^T(t) = \{x \,|\, (x,t) \in D_i^T\}.$$

Then we have the following lemma which we can show by the same manner as Lemma 3.2 and Lemma 3.3 in [2].

Lemma 2. *Let $T > 0$ be a constant. Then, there exist positive constants $k_1, k_2, k_3, k_4 > 0$ and $\tilde{\varepsilon}_0 > 0$ such that if a C^1 solution to (1.1) and (1.4) exists in $\mathbf{R} \times [0,T]$, then*

$$V(D_\pm^T),\ W(D_\pm^T) \le k_1\varepsilon, \tag{3.24}$$

$$W_\infty^c(T) \le k_2\varepsilon, \tag{3.25}$$

$$W_1(T),\ \tilde{W}_1(T) \le k_3\varepsilon, \tag{3.26}$$

$$V_\infty(T),\ U_\infty(T) \le k_4\varepsilon \tag{3.27}$$

hold for $0 < \varepsilon < \tilde{\varepsilon}_0$.

Note that if we take $\tilde{\varepsilon}_0$ to be $\tilde{\varepsilon}_0 < \delta/k_4$, where δ is the one in (3.2), then (3.27) admits (3.4) as long as C^1 solutions exist.

Using Lemma 1 and Lemma 2 we can prove

Lemma 3. *Let the assumption in Theorem be fulfilled. Let $T > 0$ be a constant satisfying*

$$T \exp\left(-\frac{1}{a(k_4\varepsilon)}\right) \le 1. \tag{3.28}$$

And assume that there exists a solution u to (1.1) and (1.4) in $C^1(\mathbf{R} \times [0,T])$. Then, there exist positive constants k_5, k_6 and $\tilde{\varepsilon}_1 > 0$ such that

$$V_\infty^c(T),\ U_\infty^c(T) \le k_5\varepsilon, \tag{3.29}$$

$$V_1(T),\ \tilde{V}_1(T) \le k_6\varepsilon, \tag{3.30}$$

$$W_\infty(T) \le k_7\varepsilon \tag{3.31}$$

hold for $0 < \varepsilon < \tilde{\varepsilon}_1$.

Remark. Lemma 2 and Lemma 3 lead that the lower bound part of (1.12) hold for $0 < \varepsilon < \tilde{\varepsilon}_1$.

Proof of Lemma 3: We first estimate

$$\tilde{V}_1(T) = \max_{\substack{i=1,\cdots,n \\ j\neq i}} \sup_{\tilde{c}_j} \int_{\tilde{c}_j} |v_i(x,t)|\, dt. \tag{3.32}$$

Fix a j-th characteristic $\tilde{c}_j \in D_i^T$ and denote

$$\tilde{c}_j : x = x_j(t) \quad (t_1 \le t \le t_2), \tag{3.33}$$

where $0 \le t_1 < t_2 \le T$ and $(x_j(t_1), t_1)$, $(x_j(t_2), t_2) \in \partial D_i^T$. Denote the i-th characteristic passing through $(0,0)$ by C_i^0 and let C_i^0 intersect with $\tilde{c}_j$ the point $(x(t_0), t_0)\,(t_1 < t_0 < t_2)$. Passing through any given point $(x,t) = (x_j(t), t)$ on $\tilde{c}_j$, we draw the i-th characteristic $\xi = \tilde{x}_i(y,s)$ which intersects one of the boundaries of D^T, say, $x = (\lambda_n(0) + \delta_0)t$ (resp. $x = (\lambda_1(0) - \delta_0)t$) at a point $A_y = (y, y/(\lambda_n(0) + \delta_0))$ (resp. $B_y = (y, y/(\lambda_1(0) - \delta_0))$) if $t_0 \le t \le t_2$ (resp. $t_1 \le t \le t_0$).

Then we have

$$\tilde{x}_i(y,t) = x_j(t) \tag{3.34}$$

which gives a one-to-one correspondence $t = t(y)$ between the segment $\overline{OA_{y_2}}$ (resp. $\overline{B_{y_1}O}$) and $\tilde{c}_j(t_0 \le t \le t_2)$ (resp. $\tilde{c}_j(t_1 \le t \le t_0)$). Thus, the integral in $\tilde{c}_j$ with respect to t can be reduced to the integral with respect to y. Differentiating (3.34) with respect to t, we have

$$dt = \frac{1}{\lambda_j(u(\tilde{x}_i(y,t),t)) - \lambda_i(u(\tilde{x}_i(y,t),t))} \frac{\partial \tilde{x}_i(y,t)}{\partial y}\, dy, \tag{3.35}$$

in which $t = t(y)$. Then, noting (3.2) we find that in order to estimate

$$\begin{aligned}
\int_{\tilde{c}_j} |v_i(x,t)|\, dt &= \int_{t_1}^{t_0} |v_i(x_j(t),t)|\, dt + \int t_0{}^{t_2} |v_i(x_j(t),t)|\, dt \\
&= \int_{t_1}^{t_0} |v_i(\tilde{x}_i(y,t),t)|\, dt + \int_{t_0}^{t_2} |v_i(\tilde{x}_i(y,t),t)|\, dt \\
&\le C \left\{ \int_0^{y_1} |p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\, dy + \int_0^{y_2} |p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\, dy \right\},
\end{aligned}$$

it suffices to estimate

$$\int_0^{y_1} |p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\, dy \qquad \text{and} \qquad \int_0^{y_2} |p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\, dy,$$

where $p_i(\tilde{x}_i(y,t),t)$ is defined by (2.18).

We now estimate

$$\int_0^{y_2} |p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\, dy.$$

Integrating (2.19) along $\xi = \tilde{x}_i(y,s)$ with $a = \lambda_n(0) + \delta_0$, we have

$$\begin{aligned}
p_i(\tilde{x}_i(y,t),t)|_{t=t(y)} &= v_i\left(y, \frac{y}{\lambda_n(0)+\delta_0}\right)\left(1 - \frac{\lambda_i(u(y, y/(\lambda_n(0)+\delta_0)))}{\lambda_n(0)+\delta_0}\right) \\
&\quad + \int_{y/(\lambda_n(0)+\delta_0)}^{t(y)} \sum_{k\ne i,\ k\ne j} \tilde{\beta}_{ijk}(u) \frac{\partial \tilde{x}_i(y,s)}{\partial y} v_j w_k(\tilde{x}_i(y,s),s)\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{y/(\lambda_n(0)+\delta_0)}^{t(y)}\sum_{j=1}^{n}(\tilde{\beta}_{ijj}(u)-\tilde{\beta}_{ijj}(u_je_j))\frac{\partial\tilde{x}_i(y,s)}{\partial y}v_jw_j(\tilde{x}_i(y,s),s)\,ds\\
&+\int_{y/(\lambda_n(0)+\delta_0)}^{t(y)}\tilde{\beta}_{iii}(u_ie_i)\frac{\partial\tilde{x}_i(y,s)}{\partial y}v_iw_i(\tilde{x}_i(y,s),s)\,ds, \qquad (3.36)
\end{aligned}
$$

where we use (2.21). By Hadamard's formula, we have

$$
\beta_{ijj}(u)-\beta_{ijj}(u_je_j)=u_l\int_0^1\sum_{l\neq j}\frac{\partial\beta_{ijj}}{\partial u_l}(\tau u_1,\cdots,\tau u_{j-1},u_j,\tau u_{j+1},\cdots,\tau u_n)\,d\tau.
$$

Then using Lemma 1 we have

$$
\begin{aligned}
&\int_0^{y_2}|p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\,dy\\
\leq&\int_0^{y_2}\left|v_i\Big(y,\frac{y}{\lambda_n(0)+\delta_0}\Big)\right|dy\\
&+C\Big\{W_\infty^c(T)V_\infty^c(T)\int_0^{y_2}\int_{y/(\lambda_n(0)+\delta_0)}^{t(y)}(1+s)^{-(1+\mu)}(1+|\tilde{x}_i(y,s)|)^{-(1+\mu)}\frac{\partial\tilde{x}_i(y,s)}{\partial y}\,ds\,dy\\
&+W_\infty^c(T)\sum_{j=1}^{n}\int_0^{y_2}\int_{(\tilde{x}_i(y,s),s)\in D_j^T}(1+s)^{-(1+\mu)}|v_j(\tilde{x}_i(y,s),s)|\frac{\partial\tilde{x}_i(y,s)}{\partial y}\,ds\,dy\\
&+V_\infty^c(T)\sum_{k=1}^{n}\int_0^{y_2}\int_{(\tilde{x}_i(y,s),s)\in D_k^T}(1+s)^{-(1+\mu)}|w_k(\tilde{x}_i(y,s),s)|\frac{\partial\tilde{x}_i(y,s)}{\partial y}\,ds\,dy\Big\}\\
&+\int_0^{y_2}\int_{y/(\lambda_n(0)+\delta_0)}^{t(y)}\left|\tilde{\beta}_{iii}(u_ie_i)\frac{\partial\tilde{x}_i(y,s)}{\partial y}v_iw_i(\tilde{x}_i(y,s),s)\right|ds\,dy.
\end{aligned}
$$

Introducing the trasformation

$$
\begin{cases} x=\tilde{x}_i(y,s),\\ s=s,\end{cases}
$$

we have the area element

$$
ds\,dx=\frac{\partial\tilde{x}_i(y,s)}{\partial y}ds\,dy.
$$

Hence, by (3.25), (3.26), (3.27), (3.28), (1.11) and (1.8) we have

$$
\begin{aligned}
&\int_0^{y_2}|p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\,dy\\
\leq\;& C\Big\{k_1\varepsilon+k_2\varepsilon V_\infty^c(T)+k_2\varepsilon V_1(T)+k_3\varepsilon V_\infty^c(T)
\end{aligned}
$$

$$+T \sup_{(\xi,s)\in\mathbf{R}\times[0,T]} \exp\left(-\frac{1}{a(|u_i(\xi,s)|)}\right) V_\infty(T)(W^c_\infty(T)+W_1(T))\Big\}$$

$$\le C\left\{k_1\varepsilon+k_2\varepsilon V^c_\infty(T)+k_2\varepsilon V_1(T)+k_3\varepsilon V^c_\infty(T)+T\exp\left(-\frac{1}{a(k_4\varepsilon)}\right)k_4(k_2+k_3)\varepsilon^2\right\}$$

$$\le C\left\{k_1\varepsilon+k_2\varepsilon V^c_\infty(T)+k_2\varepsilon V_1(T)+k_3\varepsilon V^c_\infty(T)+k_4(k_2+k_3)\varepsilon^2\right\}.$$

In a similar manner, we can obtain the same estimate for $\displaystyle\int_0^{y_1}|p_i(\tilde{x}_i(y,t),t)|_{t=t(y)}\,dy$. Thus we have

$$\tilde{V}_1(T)\le C\left\{k_1\varepsilon+k_2\varepsilon V^c_\infty(T)+k_2\varepsilon V_1(T)+k_3\varepsilon V^c_\infty(T)+k_4(k_2+k_3)\varepsilon^2\right\}. \tag{3.37}$$

Moreover, we have (cf.[2])

$$V_1(T)\le C\left\{k_1\varepsilon+k_2\varepsilon V^c_\infty(T)+k_2\varepsilon V_1(T)+k_3\varepsilon V^c_\infty(T)+k_4(k_2+k_3)\varepsilon^2\right\}. \tag{3.38}$$

We next prove

$$U^c_\infty(T)\le CV^c_\infty(T).$$

We fix point a $(x,t)\in D^T\setminus D^T_i$ and estimate $(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(x,t)|$. In the case of $(x,t)\notin\bigcup_{k=1}^n D^T_k$, by (3.11) and (2.5) we have

$$\begin{aligned}(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(x,t)| &\le C\sum_{k=1}^n(1+t)^{1+\mu}|v_k(x,t)|\\ &\le C\sum_{k=1}^n(1+|x-\lambda_k(0)t|)^{1+\mu}|v_k(x,t)|\\ &\le CV^c_\infty(T).\end{aligned}$$

On the other hand, in the case of $(x,t)\in D^T_j$ (for some $j\ne i$), we find that $(x,t)\notin\bigcup_{k\ne j} D^T_k$. Moreover, (2.5) and (2.1) lead to

$$\begin{aligned}u_i(x,t) &= {}^tu(x,t)e_i\\ &=\sum_{k=1}^n v_k(x,t)\,{}^tr_k(u)e_i\\ &=\sum_{k\ne j} v_k(x,t)\,{}^tr_k(u)e_i+v_j({}^tr_j(u)-{}^tr_j(u_je_j))e_i.\end{aligned}$$

By Hadamard's formula, we have

$$r_j(u) - r_j(u_j e_j) = u_k \int_0^1 \sum_{k\neq j} \frac{\partial r_j}{\partial u_k}(su_1, \cdots, su_{j-1}, u_j, su_{j+1}, \cdots, su_n)\, ds.$$

Hence, noting $(x,t) \notin D_i^T$, we have

$$(1 + |x - \lambda_i(0)t|)^{1+\mu}|u_i(x,t)| \le C(V_\infty^c(T) + V_\infty U_\infty^c(T)).$$

Combining the above facts, we have

$$\begin{aligned} U_\infty^c(T) &\le C(V_\infty^c(T) + V_\infty U_\infty^c(T)) \\ &\le C(V_\infty^c(T) + k_4\varepsilon U_\infty^c(T). \end{aligned}$$

Hence, if we take ε_1' to be $Ck_4\varepsilon_1' \le 1/2$,

$$U_\infty^c(T) \le 2CV_\infty^c(T)$$

holds for $0 < \varepsilon < \varepsilon_1'$.

Thus we have only to estimate $V_\infty^c(T)$.

For $(x,t) \in D^T \setminus D_i^T$, by the definition of D_i^T, without loss of generalities, we may suppose that

$$x - \lambda_i(0)t > [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]t, \tag{3.39}$$

which implies $i < n$. Then, the i-th characteristic passing through (x,t) must intersect with D_+^T at a point (y, t_0). Let us denote this characteristic by $\xi = x_i(s; x, t)$, then we have

$$\frac{dx_i}{ds} = \lambda_i(u(x_i(s;x,t),s)), \quad t_0 \le s \le T, \tag{3.40}$$

$$x_i(t_0; x, t) = y, \tag{3.41}$$

where $t_0 = y/(\lambda_n(0) + \delta_0)$. Integrating (3.40) from t_0 to t and using (3.3) we find

$$x - \left(\lambda_i(0) + \frac{\delta_0}{2}\right) t \le y - \left(\lambda_i(0) + \frac{\delta_0}{2}\right) t_0.$$

Since $y = (\lambda_n(0) + \delta_0)t_0$, noting (3.39) and $t \ge t_0$, we have

$$\eta t \le t_0 \le t. \tag{3.42}$$

Integrating (2.8) along $\xi = x_i(s; x, t)$ from t_0 to t and multiplying $(1 + |x - \lambda_i(0)t|)^{1+\mu}$ to both sides, we obtain

$$(1 + |x - \lambda_i(0)t|)^{1+\mu} v_i(x,t)$$

$$
\begin{aligned}
&= (1+|x-\lambda_i(0)t|)^{1+\mu} v_i(y,t_0) + (1+|x-\lambda_i(0)t|)^{1+\mu} \\
&\quad \times\Big(\int_{t_0}^t \sum_{\substack{k\neq i\\ k\neq j}} \beta_{ijk}(u) v_j w_k \, ds + \int_{t_0}^t \sum_{j=1}^n \beta_{ijj}(u) v_j w_j \, ds\Big), \qquad (3.43)
\end{aligned}
$$

where we use (2.12). Since $(x,t),\ (y,t_0) \in D^T \setminus D_i^T$, Lemma 1, (3.42) and Lemma 2 lead to

$$
\begin{aligned}
(1+|x-\lambda_i(0)t|)^{1+\mu}|v_i(y,t_0)| &\le C(1+t)^{1+\mu}|v_i(y,t_0)| \\
&\le C(1+t_0)^{1+\mu}|v_i(y,t_0)| \\
&\le C(1+|y-\lambda_i(0)t_0|)^{1+\mu}|v_i(y,t_0)| \\
&\le CV(D_+^T) \\
&\le Ck_1\varepsilon \qquad (3.44)
\end{aligned}
$$

for $0<\varepsilon<\tilde{\varepsilon}_0$. Now we estimate the second term of (3.43). Since $(x_i(s;x,t),s) \in D^T \setminus D_i^T$, (3.42), Lemma 1 and Lemma 2 lead to

$$
\begin{aligned}
&(1+|x-\lambda_i(0)t|)^{1+\mu} \int_{t_0}^t \sum_{\substack{k\neq i\\ k\neq j}} |\beta_{ijk}(u) v_j w_k(x_i(s;x,t),s)| \, ds \\
&\le C\Big\{ V_\infty^c(T) W_\infty^c(T) \int_{t_0}^t (1+s)^{-(1+\mu)}\, ds \\
&\quad + W_\infty^c(T) \sum_{j=1}^n \int_{(x_i(s;x,t),s)\in D_j^T} |v_j|\, ds + V_\infty^c(T) \sum_{k=1}^n \int_{(x_i(s;x,t),s)\in D_k^T} |w_k|\, ds \Big\} \\
&\le C\Big\{ V_\infty^c(T)(Ck_2+k_3)\varepsilon + \tilde{V}_1(T)k_2\varepsilon \Big\} \qquad (3.45)
\end{aligned}
$$

for $0<\varepsilon<\tilde{\varepsilon}_0$. On the other hand, noting $\beta_{ijj}(u_je_j)\equiv 0$ and using Hadamard's formula, we have

$$
\begin{aligned}
&(1+|x-\lambda_i(0)t|)^{1+\mu} \int_{t_0}^t \sum_{j=1}^n |\beta_{ijj}(u) v_j w_j(x_i(s;x,t),s)| \, ds \\
&\le C(1+t)^{1+\mu} \int_{t_0}^t \sum_{j=1}^n |(\beta_{ijj}(u)-\beta_{ijj}(u_je_j)) v_j w_j| \, ds \\
&\le C(1+t)^{1+\mu} \int_{t_0}^t \sum_{j=1}^n \sum_{l\neq j} |u_l(x_i(s;x,t),s) v_j w_j| \, ds \\
&\le C \sup_{(y,s)\in D^T\setminus D_l^T} (1+s)^{1+\mu}(u_l(y,s)) V_\infty(T) \int_{(x_i,s)\in D^T\setminus D_l^T} |w_j|\, ds \\
&\quad + U_\infty \times \sup_{(y,s)\in D^T\setminus D_j^T} (1+s)^{1+\mu}(v_j(y,s)) \sum_{j=1}^n \int_{(x_i,s)\in D^T\setminus D_j^T} |w_j|\, ds
\end{aligned}
$$

$$\begin{aligned}
&\le C(U^c_\infty(T)V_\infty(T)\tilde{W}_1(T) + U_\infty(T)V^c_\infty(T) \times CW^c_\infty(T)) \\
&\le CV^c_\infty(T)(V_\infty(T)\tilde{W}_1(T) + CW^c_\infty(T)U_\infty(T)) \\
&\le CV^c_\infty(T)k_4(ck_2+k_3)\varepsilon^2 \qquad (3.46)
\end{aligned}$$

for $0 < \varepsilon < \tilde{\varepsilon}_0$. Combining (3.43), (3.44), (3.45) and (3.46), we get

$$\begin{aligned}
&V^c_\infty(T) \\
&\le C\left\{k_1\varepsilon + (Ck_2+k_3)\varepsilon V^c_\infty(T) + k_2\varepsilon\tilde{V}_1(T) + k_4(Ck_2+k_3)\varepsilon^2 V^c_\infty(T)\right\}. \qquad (3.47)
\end{aligned}$$

By the continuity of the norms, we may assume that there exists $T_0 \le T$ such that

$$V^c_\infty(T_0) \le 2k_5\varepsilon \qquad (3.48)$$

$$V_1(T_0),\ \tilde{V}_1(T_0) \le 2k_6\varepsilon. \qquad (3.49)$$

Substituting (3.48) and (3.49) into the right-hand side of (3.37), (3.38) and (3.47) (in which we take $T = T_0$), we get

$$\begin{aligned}
\tilde{V}_1(T_0),\ V_1(T_0) &\le C\left\{k_1\varepsilon + 2k_2k_5\varepsilon^2 + 2k_2k_6\varepsilon^2 + 2k_3k_5\varepsilon^2 + k_4(k_2+k_3)\varepsilon^2\right\}, \\
V^c_\infty(T_0) &\le C\left\{k_1\varepsilon + 2k_5(Ck_2+k_3)\varepsilon^2 + 2k_2k_6\varepsilon^2 + 2k_4k_5(Ck_2+k_3)\varepsilon^3\right\}.
\end{aligned}$$

Hence, if we take k_5, k_6 and $\tilde{\varepsilon}_1$ to be

$$\begin{aligned}
2k_5(Ck_2+k_3)\tilde{\varepsilon}_1 + 2k_2k_6\tilde{\varepsilon}_1 + 2k_4k_5(Ck_2+k_3)\tilde{\varepsilon}_1 &\le Ck_1, \\
2k_2k_5\tilde{\varepsilon}_1 + 2k_2k_6\tilde{\varepsilon}_1 + 2k_3k_5\tilde{\varepsilon}_1 + k_4(k_2+k_3)\tilde{\varepsilon}_1 &\le Ck_1
\end{aligned}$$

and

$$\begin{aligned}
2Ck_1 &\le k_5, \\
2Ck_1 &\le k_6, \qquad (3.50)
\end{aligned}$$

we have

$$\begin{aligned}
V^c_\infty(T_0) &\le k_5\varepsilon, \\
V_1(T_0),\ \tilde{V}_1(T_0) &\le k_6\varepsilon
\end{aligned}$$

for $0 < \varepsilon < \tilde{\varepsilon}_1$. Moreover by (3.50), taking k_5 bigger if we need, we have $U^c_\infty(T), V^c_\infty(T) \le k_5\varepsilon$ for $\varepsilon < \tilde{\varepsilon}$. The continuation method (3.29) and (3.30) admits for any $T_0 \le T$, espacially for $T_0 = T$. This implies (3.29) and (3.30).

Now we prove (3.31). Integrating (2.9) along $\xi = x_i(s; x, t)$ from t_0 to t ($t \le T' \le T$), we have

$$w_i(x,t) = w_i(y,t_0) + \sum_{j\ne k}\int_{t_0}^{t} \gamma_{ijk}(u)w_jw_k(x_i(s;x,t),s)\,ds$$

$$
\begin{aligned}
&+\int_{t_0}^{t}(\gamma_{iii}-\gamma_{iii}(u_ie_i))w_i^2(x_i(s;x,t),s)\,ds\\
&+\int_{t_0}^{t}\gamma_{iii}(u_ie_i)w_i^2(x_i(s;x,t),s)\,ds, \qquad (3.51)
\end{aligned}
$$

where we use (2.13). In the same manner as the proof of (3.29), the followings can be seen.

$$|w_i(y,t_0)| \le W(D_{\pm}^{T'}) \le k_1\varepsilon.$$

$$
\begin{aligned}
\sum_{j\neq k}\int_{t_0}^{t}|\gamma_{ijk}w_jw_k(x_i(s;x,t),s)|\,ds &\le CW_\infty(T')W_\infty^c(T')\int_{t_0}^{t}\frac{1}{(1+s)^{1+\mu}}\,ds\\
&\le Ck_2\varepsilon W_\infty(T'). \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t_0}^{t}|\gamma_{iii}(u)-\gamma_{iii}(u_ie_i)|w_i^2(x_i(s;x,t),s)\,ds\\
&=\int_{t_0}^{t}\left|u_j(x_i(s;x,t),s)\int_0^1\frac{\partial\gamma_{iii}}{\partial u_j}\,d\tau\right|w_i^2(x_i(s;x,t),s)\,ds\\
&\le C\left\{V_\infty^c(T')(W_\infty(T'))^2+V_\infty(T')(W_\infty^c(T'))^2\right\}\int_{t_0}^{t}\left(\frac{1}{(1+s)^{1+\mu}}\right)^2ds\\
&\le C\left\{k_5\varepsilon(W_\infty(T'))^2+Ck_2^2k_4\varepsilon^2\right\}. \qquad (3.53)
\end{aligned}
$$

By (1.11), (1.8), (3.28) and Lemma 2, we find that

$$
\begin{aligned}
&\int_{t_0}^{t}|\gamma_{iii}(u_ie_i)|w_i^2(x_i(s;x,t),s)\,ds\\
&\le T'\sup_{(\xi,s)\in\mathbf{R}\times[0,T']}\exp\left(-\frac{1}{a(|u_i(\xi,s)|)}\right)(W_\infty^c(T'))^2+C(W_\infty^c(T'))^2\\
&\le T'\exp\left(-\frac{1}{a(k_4\varepsilon)}\right)(W_\infty(T'))^2+Ck_2^2\varepsilon^2\\
&\le (W_\infty(T'))^2+Ck_2^2\varepsilon^2 \qquad (3.54)
\end{aligned}
$$

holds. Thus (3.51)$-$(3.54) imply

$$W_\infty(T')\le C\left\{k_1\varepsilon+k_2^2(1+k_4)\varepsilon^2+k_2\varepsilon W_\infty(T')+(1+k_5\varepsilon)(W_\infty(T'))^2\right\}$$

for any $T'\le T$. The continuity of the norms, we may assume that there exists $T'\le T$ such that $W_\infty(T')\le 2k_7\varepsilon$. Then we have

$$W_\infty(T')\le C\left\{k_1\varepsilon+k_2^2(1+k_4)\varepsilon^2+2k_2k_7\varepsilon^2+4(1+k_5\varepsilon)k_7^2\varepsilon^2\right\}.$$

Hence, if we take k_7 and $\tilde{\varepsilon}_1$ to be

$$k_2^2(1+k_4)\tilde{\varepsilon}_1 + 2k_2k_7\tilde{\varepsilon}_1 + 4(1+k_5\varepsilon_1)k_7^2\tilde{\varepsilon}_1 \le Ck_1$$

and

$$2Ck_1 \le k_7,$$

we have

$$W_\infty(T') \le k_7\varepsilon$$

for $0 < \varepsilon < \tilde{\varepsilon}_1$. By the continuation method, (3.31) holds for any $T' \le T$, especially for $T' = T$. This completes the proof of Lemma 3.

In the remainder of this section we concentrate on proving the upper bound part of (1.12). Since we are considering the transformed problem, the initial condition (1.4) should be rewritten as

$$u(x,0) = \varepsilon\bar{\varphi}(x) + O(\varepsilon^2). \tag{3.55}$$

For the simplicity, we assume $i_0 = 1$ in Theorem. Since $\bar{\varphi}_1(x)$ satisfies (1.5) and does not vanish identically, there exists a point $x_0 \in \mathbf{R}$ such that $\bar{\varphi}_1(x_0) \neq 0$ and $\bar{\varphi}_1'(x_0) > 0$. We assume $\bar{\varphi}_1(x_0) > 0$, for example. Let $x = x_1(x_0, t)$ be the first characteristic passing through $(x_0, 0)$. Then there exists a $t_0 > 0$ depending on x_0 such that

$$(x_1(x_0,t),t) \in D_1^T \quad \text{for} \quad t_0 \le t. \tag{3.56}$$

In fact, noting (3.40), we have

$$x_1(t) - x_0 = \int_0^t \lambda_1(u(x_1(s;x,t),s))\,ds.$$

Thus (3.3) implies,

$$x_0 + \left(\lambda_1(0) - \frac{\delta_0}{2}\right)t \le x_1(t) \le x_0 + \left(\lambda_1(0) + \frac{\delta_0}{2}\right)t.$$

If we take t_0 to be

$$-2x_0 \le \delta_0 t_0 \quad \text{and} \quad x_0 \le \left\{\frac{\delta_0}{2} + \eta(\lambda_2 - \lambda_1(0))\right\}t_0,$$

we find that $(x_1(x_0,t),t) \in D_1^T$ for $t \ge t_0$.

There exists an $\varepsilon_* > 0$, such that Lemma 2, Lemma 3 and

$$t_0 \le \exp\left(\frac{1}{a(k_4\varepsilon)}\right) \tag{3.57}$$

hold for $0 < \varepsilon < \varepsilon_*$. Then we obtain

Lemma 4. *Let $x = x_1(x_0, t)$ be the same to the above one. Then there exists a constant $B > 0$ and $\tilde{\varepsilon}_2$ depending on t_0 such that*

$$|u_1(x_1(x_0,t),t) - \varepsilon\bar{\varphi}_1(x_0)| \le B\varepsilon^2 \tag{3.58}$$

holds for $0 < \varepsilon < \tilde{\varepsilon}_2$ and $0 \le t < T(\varepsilon)$.

Proof: By (2.8), (2.12), (3.56), Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
|u_1(x_1(x_0,t),t) - v_1(x_0,0)| &\le \int_0^t \sum_{k\neq 1} |\beta_{1jk} v_j w_k(x_1(x_0,s),s)|\,ds \\
&\le \int_0^{t_0} \sum_{k\neq 1} |\beta_{1jk} v_j w_k(x_1(x_0,s),s)|\,ds \\
&\quad + \int_{t_0}^{t} \sum_{k\neq 1} |\beta_{1jk} v_j w_k(x_1(x_0,s),s)|\,ds \\
&\le C\left\{V_\infty(t_0)W_\infty(t_0)t_0 + V_\infty(t)W_\infty^c(t)\right\} \\
&\le C\left\{k_4k_7t_0 + k_2k_4\right\}\varepsilon^2 \qquad (3.59)
\end{aligned}$$

for $0 < \varepsilon < \varepsilon_*$. On the other hand, by (2.1) we find that

$$\begin{aligned}
u_1(x,t) - v_1(x,t) &= {}^t u(x,t)\cdot e_1 - v_1(x,t) \\
&= \sum_{k=1}^n v_k(x,t)\,{}^t r_k(u)\cdot e_1 - v_1(x,t) \\
&= \sum_{k=1}^n v_k(x,t)({}^t r_k(u) - {}^t r_k(u_k e_k))\cdot e_1 \\
&= \sum_{k\neq j} v_k(x,t)u_j(x,t)\int_0^1 \frac{\partial\, {}^t r_k}{\partial u_j}\,ds\cdot e_1
\end{aligned}$$

holds. Thus we have

$$|u_1(x,t) - v_1(x,t)| \le CV_\infty(t)U_\infty(t) \le Ck_4^2\varepsilon^2. \tag{3.60}$$

Combining (3.59) and (3.60) we obtain

$$\begin{aligned}
|u_1(x_1(x_0,t),t) - \varepsilon\bar{\varphi}_1(x_0)| &\le |u_1(x_1(x_0,t),t) - v_1(x_1(x_0,t),t)| \\
&\quad + |v_1(x_1(x_0,t),t) - v_1(x_0,0)| + |v_1(x_0,0) - \varepsilon\bar{\varphi}_1(x_0)| \\
&\le C\left(2k_4^2 + k_4k_7t_0 + k_2k_4\right)\varepsilon^2.
\end{aligned}$$

Hence, we get (3.58) for $0 < \varepsilon < \tilde{\varepsilon}_2$ if we set $\tilde{\varepsilon}_2 = \varepsilon_*$ and $B = C(2k_4^2 + k_4k_7t_0 +$

$k_2 k_4$).

From now on we set

$$T_0 \equiv \exp\left(\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0)+B\varepsilon^2)}\right) - 1 \quad (\geq t_0)$$

and assume that the C^1 solution to (1.1) and (1.4) exists for $t \geq T_0$. By using a notation $w_1(t) = w_1(x_1(x_0,t),t)$, (2.9) is rewritten as

$$\begin{aligned} \frac{d}{dt} w_1(t) &= \gamma_{111}(u_1 e_1) w_1(t)^2 + (\gamma_{111}(u) - \gamma_{111}(u_1 e_1)) w_1(t)^2 \\ &\quad + \sum_{j \neq k} \gamma_{1jk}(u) w_j w_k(t). \end{aligned} \tag{3.61}$$

By Lemma 4, we find that

$$u_1(x_1(x_0,t),t) \geq \varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2 > 0$$

holds for $t \geq T$ and $\varepsilon < \tilde{\varepsilon}_2$. Hence, (1.9-iii) implies

$$\gamma_{111}(u_1 e_1) = \exp\left(-\frac{1}{a(u_1(x_1(x_0,t),t))}\right) \geq \exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0)-B\varepsilon^2)}\right). \tag{3.62}$$

for $t \geq T$ and $\varepsilon < \tilde{\varepsilon}_2$. Since $(x_1(x_0,t),t) \in D_1^T$, (3.56) and Lemma 3 lead to

$$\begin{aligned} |\gamma_{111}(u) - \gamma_{111}(u_1 e_1)| &= \left| u_j(x_1(x_0,t),t) \sum_{j\neq 1} \int_0^1 \frac{\partial \gamma_{111}}{\partial u_j}(u_1, \tau u_j)\, d\tau \right| \\ &\leq C(1+t)^{-1-\mu} U_\infty^c(t) \\ &\leq C\varepsilon(1+T_0)^{-1-\mu} \\ &\leq C\varepsilon \exp\left(-\frac{1+\mu}{a(\varepsilon\bar{\varphi}_1(x_0)+B\varepsilon^2)}\right). \end{aligned}$$

Moreover, by (1.9-iv), there exists an $\bar{\varepsilon}_0 > 0$ depending on $\bar{\varphi}_1(x_0)$ and B such that

$$\frac{a(\varepsilon\bar{\varphi}_1(x_0)+B\varepsilon^2)}{a(\varepsilon\bar{\varphi}_1(x_0)-B\varepsilon^2)} \leq 1+\mu,$$

i.e.,

$$\exp\left(-\frac{1+\mu}{a(\varepsilon\bar{\varphi}_1(x_0)+B\varepsilon^2)}\right) \leq \exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0)-B\varepsilon^2)}\right)$$

holds for $0 < \varepsilon < \bar{\varepsilon}_0$. Thus we have

$$
\begin{aligned}
|\gamma_{111}(u) - \gamma_{111}(u_1 e_1)| &\leq C\varepsilon \exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) \\
&\leq \frac{1}{4}\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) \quad \text{for } \ \varepsilon < \frac{1}{4}\,. \qquad (3.63)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\left|\sum_{j\neq k}\gamma_{1jk}w_j w_k(t)\right| &\leq C(1+t)^{-1-\mu}W^c_\infty(t)|w_1(t)| \\
&\leq C\varepsilon(1+T_0)^{-1-\mu}|w_1(t)| \\
&\leq C\varepsilon\exp\left(-\frac{1+\mu}{a(\varepsilon\bar{\varphi}_1(x_0) + B\varepsilon^2)}\right)(1+w_1(t)^2).
\end{aligned}
$$

Applying (1.9-iv) with replacing μ to $\mu/2$, we find that there exists an $\bar{\varepsilon}_1 > 0$ such that

$$
\exp\left(-\frac{1+\mu/2}{a(\bar{\varphi}_1(x_0) + B\varepsilon^2)}\right) \leq \exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right)
$$

and

$$
\exp\left(-\frac{\mu/2}{a(\varepsilon\bar{\varphi}_1(x_0) + B\varepsilon^2)}\right) \leq C\varepsilon^2
$$

hold for $0 < \varepsilon < \bar{\varepsilon}_1$. Thus we obtain

$$
\begin{aligned}
&\left|\sum_{j\neq k}\gamma_{1jk}w_j w_k(t)\right| \\
&\leq C\varepsilon\exp\left(-\frac{1+\mu/2}{a(\varepsilon\bar{\varphi}_1(x_0) + B\varepsilon^2)}\right)\exp\left(-\frac{\mu/2}{a(\varepsilon\bar{\varphi}_1(x_0) + B\varepsilon^2)}\right)(1+w_1(t)^2) \\
&\leq C\varepsilon^3\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right)(1+w_1(t)^2) \\
&\leq \frac{1}{4}\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right)w_1(t)^2 + C\varepsilon^3\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) \qquad (3.64)
\end{aligned}
$$

for $\varepsilon < \left(\frac{1}{4C}\right)^{\frac{1}{3}}$. Substituting (3.62), (3.63) and (3.64) into (3.61) we have

$$
\begin{aligned}
\frac{d}{dt}w_1(t) \geq\ &\frac{1}{2}\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right)w_1(t)^2 \\
&-C\varepsilon^3\exp\left(-\frac{1}{a(\varepsilon\bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) \qquad (3.65)
\end{aligned}
$$

holds for $t \geq T_0$.

Now to derive an estimate for w_1 at $t = T_0$, we introduce a norm;

$$W^1_\infty(T) \equiv \sup_{0 \leq t \leq T} |w_1(t) - w_1(x_0, 0)|.$$

Then we can prove

Lemma 5. *There exist constants k_8 and $\tilde{\varepsilon}_3 > 0$ such that*

$$W^1_\infty(T) \leq k_8 \varepsilon^2 \tag{3.66}$$

holds for $0 < \varepsilon < \tilde{\varepsilon}_3$ and $T \leq T_0$.

Proof: Integrating (3.61) from 0 to t ($t \leq T \leq T_0$) we have

$$\begin{aligned}
&|w_1(t) - w_1(x_0, 0)| \\
&\leq \sum_{j \neq k} \int_0^t |\gamma_{1jk}(u) w_j w_k(s)|\, ds + \int_0^t |\gamma_{111}(u) - \gamma_{111}(u_1 e_1)| w_1(s)^2\, ds \\
&\quad + \int_0^t \gamma_{111}(u_1 e_1) w_1(s)^2\, ds \\
&\leq C W^c_\infty(t) \left(\sup_{0 \leq s \leq t} |w_1(s) - w_1(x_0, 0)| + |w_1(x_0, 0)| \right) \int_0^t \frac{1}{(1+s)^{1+\mu}}\, ds \\
&\quad + \int_0^t \left| u_2(s) \int_0^1 \frac{\partial \gamma_{111}}{\partial u_2}\, d\tau \right| (|w_1(s) - w_1(x_0, 0)|^2 + |w_1(x_0, s)|^2)\, ds \\
&\quad + \int_0^t \exp\left(-\frac{1}{a(|u_1(s)|)} \right) (|w_1(s) - w_1(x_0, 0)|^2 + |w_1(x_0, 0)|^2)\, ds.
\end{aligned} \tag{3.67}$$

Moreover, by the initial condition, we have

$$|w_1(x_0, 0)| \leq C\varepsilon$$

and

$$|u_1(s)| \leq \varepsilon \bar{\varphi}_1(x_0) + B\varepsilon^2.$$

Thus it follows from (3.67), (3.56), Lemma 2 and Lemma 3 that

$$\begin{aligned}
W^c_\infty(T) &\leq C \left\{ k_2 \varepsilon (W^1_\infty(T) + \varepsilon) + (t_0 k_4 + k_5)((W^1_\infty(T))^2 + \varepsilon^2)\varepsilon \right\} \\
&\quad + T \exp\left(-\frac{1}{a(\varepsilon \bar{\varphi}_1(x_0) + B\varepsilon^2)} \right) ((W^1_\infty(T))^2 + \varepsilon^2) \quad \text{for } 0 < \varepsilon < \tilde{\varepsilon}_2.
\end{aligned}$$

By $W^1_\infty(0) = 0$, (3.66) holds for sufficiently small T. Hence there exists $T\,(\leq T_0)$ such that $W^1_\infty(T) \leq 2k_8\varepsilon^2$ holds. Then we get

$$W^1_\infty(T) \leq C \left\{ (k_2 + 1)\varepsilon^2 + k_2 k_8 \varepsilon^3 + (t_0 k_4 + k_5)(k_8^2 \varepsilon + 1)\varepsilon^3 + k_8^2 \varepsilon^4 \right\}.$$

Therefore if we take k_8 and $\tilde{\varepsilon}_3$ to be

$$k_2 k_8 \tilde{\varepsilon}_3 + (t_0 k_4 + k_5)(k_8^2 \tilde{\varepsilon}_3 + 1)\tilde{\varepsilon}_3 + k_8^2 \tilde{\varepsilon}_3^2 \leq k_2 + 1$$

and

$$2C(k_2 + 1) \leq k_8,$$

then, we obtain

$$W_\infty^1(T) \leq k_8 \varepsilon^2$$

for $0 < \varepsilon < \tilde{\varepsilon}_3$. By continuation method, we find that (3.66) holds for any $T \leq T_0$.

Proof of (1.12): In the same manner as (3.60), we have

$$|w_1(x_0, 0) - \varepsilon \bar{\varphi}_1'(x_0)| \leq C\varepsilon^2.$$

Thus Lemma 5 implies

$$|w_1(t) - \varepsilon \bar{\varphi}_1'(x_0)| \leq C\varepsilon^2 \quad \text{for} \quad t \leq T_0,$$

especially,

$$w_1(T_0) \geq \varepsilon \bar{\varphi}_1'(x_0) - C\varepsilon^2 > 0$$

for $0 < \varepsilon < \tilde{\varepsilon}_3$. Using (3.65), we find that $w_1'(t) > 0$ for $t \geq T_0$ and that $w_1(t) > \varepsilon \bar{\varphi}_1'(x_0) - C\varepsilon^2$ for $t \geq T_0$. Thus if we take a $\varepsilon_1 > 0$ smaller than $\tilde{\varepsilon}_0$, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\varepsilon}_3$, $\bar{\varepsilon}_0$ and $\bar{\varepsilon}_1$, we have

$$\begin{aligned} \frac{d}{dt} w_1(t) &> \frac{1}{4} \exp\left(-\frac{1}{a(\varepsilon \bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) w_1(t)^2, \quad t \geq T_0, \\ w_1(T_0) &> \frac{1}{2} \varepsilon \bar{\varphi}_1(x_0) \end{aligned}$$

for $0 < \varepsilon < \varepsilon_1$. Hence, the fundamental comparison theorem implies

$$\begin{aligned} w_1(t) &> \frac{w_1(T_0)}{1 - \frac{1}{4} w_1(T_0) \exp\left(-\frac{1}{a(\varepsilon \bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) t} \\ &> \frac{\frac{1}{2} \varepsilon \bar{\varphi}_1'(x_0)}{1 - \frac{1}{8} \varepsilon \bar{\varphi}_1'(x_0) \exp\left(-\frac{1}{a(\varepsilon \bar{\varphi}_1(x_0) - B\varepsilon^2)}\right) t} \end{aligned}$$

for $0 < \varepsilon < \varepsilon_1$. Since we assume $\varphi_1'(x_0) > 0$, we find that

$$T(\varepsilon) \leq \frac{C}{\varepsilon} \exp\left(\frac{1}{a(\varepsilon \bar{\varphi}_1(x_0) - B\varepsilon^2)}\right)$$

$$\leq C \exp\left(\frac{1}{a(\frac{1}{2}\varepsilon\bar{\varphi}_1(x_0))}\right)$$

hold for $0 < \varepsilon < \varepsilon_1$. This implies the upper bound part of (1.12).

Summing up the arguments in this section, we have found constans c_1, c_2, C_1, C_2 and ε_1 which satisfy

$$c_1 \exp\left(\frac{1}{a(c_2\varepsilon)}\right) \leq T(\varepsilon) \leq C_1 \exp\left(\frac{1}{a(C_2\varepsilon)}\right)$$

for $0 < \varepsilon < \varepsilon_1$. This completes the proof of (1.12).

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Akira Hoshiga and Taiichi Nose,
Faculty of Engineering,
Shizuoka University,
Hamamatsu 432–8561, Japan
e-mail: tsahosh@ipc.shizuoka.ac.jp