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Stability of standing waves for nonlinear Schrödinger equation with attractive delta potential and repulsive nonlinearity

Masahiro Kaminaga and Masahito Ohta

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Abstract

We study the orbital stability of standing wave solutions for a nonlinear Schrödinger equation with an attractive delta potential and a repulsive power nonlinearity in one space dimension.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation with a delta potential:

$$i\partial_t u = -\partial_x^2 u + \gamma\delta(x)u + \alpha|u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1}$$

where $\gamma \in \mathbb{R}$, $\alpha = \pm 1$, $1 < p < \infty$, and $\delta(x)$ is the delta measure at the origin. The equations of the form (1) arise in a wide variety of physical models with a point defect on the line (see, e.g., [5, 8, 9, 10, 11] and references therein). The formal expression $-\partial_x^2 + \gamma\delta(x)$ in (1) is formulated as a linear operator A_γ or H_γ associated with a quadratic form a_γ on $H^1(\mathbb{R})$:

$$a_\gamma(u,v) = \Re\left\{\int_{\mathbb{R}} \partial_x u(x)\overline{\partial_x v(x)}\,dx + \gamma u(0)\overline{v(0)}\right\}, \quad u,v \in H^1(\mathbb{R}).$$

Remark that $H^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$. The linear operator $A_\gamma : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is defined by

$$\langle A_\gamma u, v\rangle = a_\gamma(u,v), \quad u,v \in H^1(\mathbb{R}).$$

Moreover, we define a linear operator H_γ in $L^2(\mathbb{R})$ by $H_\gamma v = -\partial_x^2 v$ for $v \in D(H_\gamma)$ with the domain

$$D(H_\gamma) = \{v \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) : \partial_x v(+0) - \partial_x v(-0) = \gamma v(0)\}.$$

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Then, H_γ is a self-adjoint operator in $L^2(\mathbb{R})$, and satisfies

$$(H_\gamma u, v)_{L^2} = a_\gamma(u, v), \quad u, v \in D(H_\gamma).$$

The following spectral properties of H_γ are known: $\sigma_{\mathrm{ess}}(H_\gamma) = \sigma_{\mathrm{ac}}(H_\gamma) = [0, \infty)$, $\sigma_{\mathrm{sc}}(H_\gamma) = \emptyset$. If $\gamma \ge 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \emptyset$. If $\gamma < 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \{-\gamma^2/4\}$ with its positive normalized eigenfuction $(|\gamma|/2)^{1/2} e^{-|\gamma||x|/2}$ (see [1, Chapter I.3] for details).

In this paper, we consider the case where $\gamma < 0$ and $\alpha = 1$ (attractive potential and repulsive nonlinearity), and study the structure and the orbital stability of standing wave solutions $e^{i\omega t}\varphi_\omega(x)$ for (1), where $\omega \in \mathbb{R}$ and $\varphi_\omega \in H^1(\mathbb{R})$ is a positive solution of the stationary problem:

$$A_\gamma \varphi + \omega\varphi + \alpha|\varphi|^{p-1}\varphi = 0 \quad \text{in } \ H^{-1}(\mathbb{R}). \tag{2}$$

The well-posedness of the Cauchy problem for (1) in the energy space $H^1(\mathbb{R})$ follows from an abstract result in Cazenave [3] (see Theorem 3.7.1 and Corollary 3.3.11 in [3], and also Section 2 of [7]).

Proposition 1. *For any $u_0 \in H^1(\mathbb{R})$ there exist $T^* = T^*(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T^*), H^1(\mathbb{R}))$ of (1) with $u(0) = u_0$ such that $\lim_{t \to T^*} \|u(t)\|_{H^1} = \infty$ if $T^* < \infty$. Moreover, $u(t)$ satisfies the conservation of charge and energy:*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0)$$

for all $t \in [0, T^*)$, where the energy E is defined by

$$E(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{\gamma}{2}|v(0)|^2 + \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

for $v \in H^1(\mathbb{R})$.

The stability of standing waves is defined as follows.

Definition. We say that a standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is *stable in* $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\eta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ and $\|u_0 - \varphi_\omega\|_{H^1} < \eta$, then the solution $u(t)$ of (1) with $u(0) = u_0$ exists for all $t \ge 0$ and satisfies

$$\sup_{t \ge 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\varphi_\omega\|_{H^1} < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi_\omega$ is said to be *unstable in* $H^1(\mathbb{R})$. Moreover, $e^{i\omega t}\varphi_\omega$ is said to be *stable in* $H^1_{\mathrm{rad}}(\mathbb{R})$ if the condition $u_0 \in H^1(\mathbb{R})$ is replaced by $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R})$ in the above definition of the stability in $H^1(\mathbb{R})$.

Before we state our main results, we recall some known results for the case where $\gamma \in \mathbb{R}$ and $\alpha = -1$ (attractive nonlinearity). When $\gamma \in \mathbb{R}$, $\alpha = -1$ and

$\omega > \gamma^2/4$, the stationary problem (2) has a unique positive solution in $H^1(\mathbb{R})$. The positive solution φ_ω of (2) is given by

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)} \left\{\cosh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + b_\gamma(\omega)\right)\right\}^{-2/(p-1)}$$

for $x \in \mathbb{R}$, where $b_\gamma(\omega) = \tanh^{-1}(-\frac{\gamma}{2\sqrt{\omega}})$ (see [5, 6, 7]).

For the stability of standing wave solutions $e^{i\omega t}\varphi_\omega$ of (1), the case where $\gamma < 0$ and $\alpha = -1$ (attractive potential and attractive nonlinearity) was first studied by Goodman, Holmes and Weinstein [8] for the special case $p = 3$, and then by Fukuizumi, Ohta and Ozawa [7] for general case $1 < p < \infty$. The following is proved in [7]. If $1 < p \le 5$, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \infty)$. If $p > 5$, there exists $\omega_1^* = \omega_1^*(\gamma, p) \in (\gamma^2/4, \infty)$ such that $e^{i\omega t}\varphi_\omega$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \omega_1^*)$, and is unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1^*, \infty)$.

The case where $\gamma > 0$ and $\alpha = -1$ (repulsive potential and attractive nonlinearity) was studied by Fukuizumi and Jeanjean [6] for radial case, and by Le Coz, Fukuizumi, Fibich, Ksherim and Sivan [12] for general case. The following is proved in [6]. If $1 < p \le 3$, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is stable in $H^1_{\rm rad}(\mathbb{R})$ for any $\omega \in (\gamma^2/4, \infty)$. If $3 < p < 5$, there exists $\omega_2^* = \omega_2^*(\gamma, p) \in (\gamma^2/4, \infty)$ such that $e^{i\omega t}\varphi_\omega$ is stable in $H^1_{\rm rad}(\mathbb{R})$ for any $\omega \in (\omega_2^*, \infty)$, and is unstable for any $\omega \in (\gamma^2/4, \omega_2^*)$. If $p \ge 5$, $e^{i\omega t}\varphi_\omega$ is unstable for any $\omega \in (\gamma^2/4, \infty)$. While, it is proved in [12] that if $1 < p \le 3$ and $\omega \in (\gamma^2/4, \infty)$ or if $3 < p < 5$ and $\omega \in (\omega_2^*, \infty)$, then the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is unstable in $H^1(\mathbb{R})$.

Remark that for the case where $\gamma = 0$ and $\alpha = -1$ (attractive nonlinearity without potential), it is well-known that the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable for any $\omega \in (0, \infty)$ if $1 < p < 5$, and it is unstable for any $\omega \in (0, \infty)$ if $p \ge 5$ (see [2, 4, 3]).

We now state our main results for the case $\gamma < 0$ and $\alpha = 1$ (attractive potential and repulsive nonlinearity).

Theorem 1. *Let $\gamma < 0$, $\alpha = 1$, $1 < p < \infty$ and $0 < \omega < \gamma^2/4$. Then, the stationary problem* (2) *has a unique positive solution $\varphi_\omega \in H^1(\mathbb{R})$ given by*

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)} \left\{\sinh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + c_\gamma(\omega)\right)\right\}^{-2/(p-1)} \tag{3}$$

for $x \in \mathbb{R}$, where $c_\gamma(\omega) = \tanh^{-1}(2\sqrt{\omega}/|\gamma|)$. Moreover, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) *is stable in $H^1(\mathbb{R})$.*

Theorem 2. *Let $\gamma < 0$, $\alpha = 1$, $\omega = 0$ and $1 < p < 5$. Then, the stationary problem* (2) *has a unique positive solution $\varphi_0 \in H^1(\mathbb{R})$ given by*

$$\varphi_0(x) = \left(\frac{2(p+1)\gamma^2}{\{4+(p-1)|\gamma||x|\}^2} \right)^{1/(p-1)} \tag{4}$$

for $x \in \mathbb{R}$. Moreover, the stationary solution φ_0 of (1) *is stable in $H^1(\mathbb{R})$.*

Remark. We do not consider the case $\omega \notin (0, \gamma^2/4)$ or $\omega = 0$ and $p \geq 5$ in Theorems 1 and 2. In Section 2, we prove that there are no nontrivial solutions of (2) in $H^1(\mathbb{R})$ for these cases (see Propositions 2, 3 and 5).

The plan of this paper is as follows. In Section 2, we study the structure of solutions of the stationary problem (2). In Section 3, we prove Theorem 1 by the method of Cazenave and Lions [4] (see Section III of [4] in particular). In Section 4, we prove Theorem 2 by modifying the argument in Section 3.

2. Stationary problem

First, we define the action S_ω and the set $\mathcal{A}_\omega$ of the nontrivial solutions for the stationary problem (2) as follows.

$$\begin{aligned} S_\omega(v) &= \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 + \frac{\gamma}{2}|v(0)|^2 + \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}, \\ \mathcal{A}_\omega &= \{u \in H^1(\mathbb{R}) : S_\omega'(u) = 0,\ u \neq 0\}. \end{aligned}$$

The following regularity result for solutions of (2) is known.

Lemma 1. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\varphi \in \mathcal{A}_\omega$. Then, φ satisfies the following.*

$$\varphi \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}), \tag{5}$$

$$-\varphi''(x) + \omega\varphi(x) + \alpha|\varphi(x)|^{p-1}\varphi(x) = 0, \quad x \in \mathbb{R} \setminus \{0\}, \tag{6}$$

$$\varphi'(+0) - \varphi'(-0) = \gamma\varphi(0), \tag{7}$$

$$\lim_{x\to\pm\infty} \varphi(x) = 0, \quad \lim_{x\to\pm\infty} \varphi'(x) = 0, \tag{8}$$

$$|\varphi'(x)|^2 = \omega|\varphi(x)|^2 + \frac{2\alpha}{p+1}|\varphi(x)|^{p+1}, \quad x \in \mathbb{R} \setminus \{0\}. \tag{9}$$

For the proof of Lemma 1, see those of Lemma 3.2 in [7] and of Lemma 25 in [6].

Proposition 2. *Let $1 < p < \infty$, $\gamma < 0$ and $\alpha > 0$. If $\omega \geq \gamma^2/4$, then $\mathcal{A}_\omega$ is empty.*

Proof. Suppose that there exists $\varphi \in \mathcal{A}_\omega$. Then, we have

$$\|\partial_x\varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 - |\gamma||\varphi(0)|^2 + \alpha\|\varphi\|_{L^{p+1}}^{p+1} = \frac{d}{d\lambda}S(\lambda\varphi)|_{\lambda=1} = 0.$$

Moreover, since the first eigenvalue of H_γ is $-\gamma^2/4$, i.e.,

$$\inf\{\|\partial_x v\|_{L^2}^2 - |\gamma||v(0)|^2 : v \in H^1(\mathbb{R}),\ \|v\|_{L^2} = 1\} = -\frac{\gamma^2}{4},$$

we have

$$\begin{aligned} 0 &= \|\partial_x \varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2 - |\gamma||\varphi(0)|^2 + \alpha\|\varphi\|_{L^{p+1}}^{p+1} \\ &\geq (\omega - \gamma^2/4)\|\varphi\|_{L^2}^2 + \alpha\|\varphi\|_{L^{p+1}}^{p+1} \geq \alpha\|\varphi\|_{L^{p+1}}^{p+1} > 0. \end{aligned}$$

This is a contradiction. Hence, $\mathcal{A}_\omega$ is empty. ☐

Lemma 2. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{R}$. Let φ be a nontrivial solution of* (5)–(9). *Then, $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$.*

Proof. Suppose that there exists $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = 0$. If $x_0 > 0$, then by (9) we have $\varphi'(x_0) = 0$. By the uniqueness of solutions of the Cauchy problem for (6), we see that $\varphi(x) = 0$ for all $x \in (0, \infty)$, and by (7) we have $\varphi(0) = \varphi'(0) = 0$. For the case $x_0 \leq 0$, we see that $\varphi(0) = \varphi'(0) = 0$ in the same way. Thus, by the uniqueness of solutions of the Cauchy problem for (6), we see that $\varphi(x) = 0$ for all $x \in \mathbb{R}$. Since φ is a nontrivial solution, this is a contradition. Hence, $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$. ☐

Lemma 3. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{R}$. Let φ be a nontrivial solution of* (5)–(9). *Then we have either (i) or (ii):*
(i) *$\Im\varphi(x) = 0$ for all $x \in \mathbb{R}$,*
(ii) *there exists $c \in \mathbb{R}$ such that $\Re\varphi(x) = c\Im\varphi(x)$ for all $x \in \mathbb{R}$.*

Proof. We put $u = \Re\varphi$ and $v = \Im\varphi$. Then, (u, v) satisfies

$$\begin{cases} -u''(x) + \omega u(x) + \alpha|\varphi(x)|^{p-1}u(x) = 0, \\ -v''(x) + \omega v(x) + \alpha|\varphi(x)|^{p-1}v(x) = 0 \end{cases}$$

for all $x \in \mathbb{R} \setminus \{0\}$. Thus, we have $(u'(x)v(x) - u(x)v'(x))' = 0$ for $x \in \mathbb{R} \setminus \{0\}$. Moreover, by (8), we have

$$u'(x)v(x) = u(x)v'(x) \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\}. \tag{10}$$

If there exists $x_0 \in \mathbb{R}$ such that $v(x_0) = 0$, then by (10) and Lemma 2, we have $v'(x_0) = 0$. Then, as in the proof of Lemma 2, we see that $v(x) = 0$ for all $x \in \mathbb{R}$. That is, we have the case (i). Otherwise, $v(x) \neq 0$ for all $x \in \mathbb{R}$. Then, by (10), we have

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = 0$$

for all $x \in \mathbb{R} \setminus \{0\}$, which implies (ii). ☐

Proposition 3. *Let $1 < p < \infty$, $\gamma < 0$ and $\alpha > 0$. If $\omega < 0$, then $\mathcal{A}_\omega$ is empty.*

Proof. Suppose that there exists $\varphi \in \mathcal{A}_\omega$. Then, by (8) in Lemma 1, there exists $L > 0$ such that

$$\frac{\alpha}{p+1}|\varphi(x)|^{p-1} \le \frac{|\omega|}{4}$$

for all $|x| \ge L$. Moreover, by (9) in Lemma 1 and by Lemma 2, we have

$$|\varphi'(x)|^2 = |\varphi(x)|^2 \left(\omega + \frac{2\alpha}{p+1}|\varphi(x)|^{p-1}\right) \le -\frac{|\omega|}{2}|\varphi(x)|^2 < 0$$

for $|x| \ge L$. This is a contradiction. Hence, $\mathcal{A}_\omega$ is empty. □

Proposition 4. *Let $1 < p < \infty$, $\gamma < 0$, $\alpha = 1$ and $0 < \omega < \gamma^2/4$. Then, $\mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$, where φ_ω is defined by* (3).

Proof. By direct computations, we see that $\varphi_\omega \in \mathcal{A}_\omega$, and we have $\{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\} \subset \mathcal{A}_\omega$. Next, let $\varphi \in \mathcal{A}_\omega$. Then, by Lemma 3, there exist $\theta \in \mathbb{R}$ and a real-valued function w such that $\varphi(x) = e^{i\theta}w(x)$ for all $x \in \mathbb{R}$. Moreover, w satisfies (5)–(9). By the phase plane analysis on the (w, w')-plane, we see that either $w = \varphi_\omega$ or $w = -\varphi_\omega$. This proves $\mathcal{A}_\omega \subset \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$. □

Proposition 5. *Let $\gamma < 0$ and $\alpha = 1$. If $1 < p < 5$, then $\mathcal{A}_0 = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$, where φ_0 is defined by* (3). *If $p \ge 5$, then the set $\mathcal{A}_0$ is empty.*

Proof. By direct computations, we see that φ_0 satisfies (5)–(9) with $\omega = 0$ for any $1 < p < \infty$. If $1 < p < 5$, then $\varphi_0 \in H^1(\mathbb{R})$ and we see that $\mathcal{A}_0 = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$ in the same way as in the proof of Proposition 4. While, if $p \ge 5$, then $\varphi_0 \notin L^2(\mathbb{R})$ and we see that $\mathcal{A}_0$ is empty. □

3. Proof of Theorem 1

In this section, we always assume $1 < p < \infty$, $\gamma < 0$, $\alpha = 1$ and $0 < \omega < \gamma^2/4$. We put

$$d_\omega = \inf\{S_\omega(v) : v \in H^1(\mathbb{R})\},$$
$$\mathcal{M}_\omega = \{u \in H^1(\mathbb{R}) : S_\omega(u) = d_\omega\}.$$

Lemma 4. *$-\infty < d_\omega < 0$ and $\mathcal{M}_\omega \subset \mathcal{A}_\omega$.*

Proof. We first prove $d_\omega > -\infty$. By the Sobolev and Hölder inequalities, there exist positive constants C_1 and C_2 such that

$$|\gamma||v(0)|^2 \le \frac{1}{2}\|\partial_x v\|_{L^2}^2 + C_1\|v\|_{L^2(-1,1)}^2 \le \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1} + C_2$$

for any $v \in H^1(\mathbb{R})$. Thus, we have

$$E(v) \geq \frac{1}{4}\|\partial_x v\|_{L^2}^2 + \frac{1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} - \frac{C_2}{2} \tag{11}$$

for $v \in H^1(\mathbb{R})$, which implies $d \geq -C_2/2$. Thus, $d_\omega > -\infty$.

Next, we prove $d_\omega < 0$. We put $\Phi(x) = e^{-|\gamma||x|/2}$. Since Φ is an eigenfunction of H_ω corresponding to the first eigenvalue $-\gamma^2/4$, we have

$$\begin{aligned} d_\omega \leq S_\omega(\lambda\Phi) &= \frac{\lambda^2}{2}(\|\partial_x\Phi\|_{L^2}^2 + \gamma|\Phi(0)|^2 + \omega\|\Phi\|_{L^2}^2) + \frac{\lambda^{p+1}}{p+1}\|\Phi\|_{L^{p+1}}^{p+1} \\ &= \frac{\lambda^2}{2}\left(\omega - \frac{\gamma^2}{4}\right)\|\Phi\|_{L^2}^2 + \frac{\lambda^{p+1}}{p+1}\|\Phi\|_{L^{p+1}}^{p+1} < 0 \end{aligned}$$

for sufficiently small $\lambda > 0$. Thus, $d_\omega < 0$.

Finally, we prove $\mathcal{M}_\omega \subset \mathcal{A}_\omega$. Let $w \in \mathcal{M}_\omega$. Then, we have $S_\omega'(w) = 0$. Moreover, since $S_\omega(w) = d_\omega < 0$, we have $w \neq 0$. Thus, we see that $w \in \mathcal{A}_\omega$. This proves $\mathcal{M}_\omega \subset \mathcal{A}_\omega$. □

Lemma 5. *Let $\{v_n\} \subset H^1(\mathbb{R})$ and $S_\omega(v_n) \to d_\omega$. Then, there exist a subsequence $\{v_{n'}\}$ and $w \in \mathcal{M}_\omega$ such that $v_{n'} \to w$ in $H^1(\mathbb{R})$.*

Proof. By (11), we see that $\{v_n\}$ is bounded in $H^1(\mathbb{R})$. Thus, there exist a subsequence of $\{v_n\}$ (we denote it by the same letter) and $w \in H^1(\mathbb{R})$ such that $v_n \rightharpoonup w$ weakly in $H^1(\mathbb{R})$. Moreover, since the embedding $H^1(-1,1) \hookrightarrow C[-1,1]$ is compact, we see that $v_n(0) \to w(0)$. Thus, we have

$$d_\omega \leq S_\omega(w) \leq \liminf_{n\to\infty} S_\omega(v_n) = d_\omega,$$

which implies that $w \in \mathcal{M}_\omega$ and $v_n \to w$ in $H^1(\mathbb{R})$. □

Lemma 6. $\mathcal{M}_\omega = \mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$.

Proof. By Lemmas 4 and 5, we have $\emptyset \neq \mathcal{M}_\omega \subset \mathcal{A}_\omega$. Moreover, by Proposition 4, we have $\mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$, which implies $\mathcal{M}_\omega = \mathcal{A}_\omega$. □

Now we give the proof of Theorem 1.

Proof of Theorem 1. We prove this by contradiction. Suppose that $e^{i\omega t}\varphi_\omega$ is not stable in $H^1(\mathbb{R})$. Then, there exist a constant $\varepsilon_0 > 0$, a sequence $\{u_n(t)\}$ of solutions of (1) and a sequence $\{t_n\}$ in $(0, \infty)$ such that

$$\|u_n(0) - \varphi_\omega\|_{H^1} \to 0, \tag{12}$$

$$\inf_{\theta\in\mathbb{R}} \|u_n(t_n) - e^{i\theta}\varphi_\omega\|_{H^1} = \varepsilon_0 \tag{13}$$

By (12) and the conservation of charge and energy, we have

$$S_\omega(u_n(t_n)) = S_\omega(u_n(0)) \to S_\omega(\varphi_\omega) = d_\omega.$$

By Lemmas 5 and 6, there exist a subsequence $\{u_{n'}(t_{n'})\}$ and $\theta_0 \in \mathbb{R}$ such that $u_{n'}(t_{n'}) \to e^{i\theta_0}\varphi_\omega$ in $H^1(\mathbb{R})$. This contradicts (13). Hence, $e^{i\omega t}\varphi_\omega$ is stable in $H^1(\mathbb{R})$. □

4. Proof of Theorem 2

In this section, we always assume $1 < p < 5$, $\gamma < 0$ and $\alpha = 1$. We put

$$\begin{aligned} X &= \{v \in L^{p+1}(\mathbb{R}) : \partial_x v \in L^2(\mathbb{R})\}, \\ d &= \inf\{E(v) : v \in X\}, \\ \mathcal{M} &= \{\varphi \in X : E(\varphi) = d\}, \\ \mathcal{A} &= \{\varphi \in X : E'(\varphi) = 0,\ \varphi \neq 0\}. \end{aligned}$$

Lemma 7. *$-\infty < d < 0$ and $\mathcal{M} \subset \mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$.*

Proof. The facts $-\infty < d < 0$ and $\mathcal{M} \subset \mathcal{A}$ can be proved in the same way as in the proof of Lemma 4. Remark that the inequality (11) holds true for $v \in X$. If $\varphi \in \mathcal{A}$, then we see that φ satisfies (5)–(9) with $\omega = 0$. Then, as in the proof of Proposition 5, we have $\mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$. □

Lemma 8. *Let $\{v_n\} \subset X$ and $E(v_n) \to d$. Then, there exist a subsequence $\{v_{n'}\}$ and $w \in \mathcal{M}$ such that $v_{n'} \to w$ in X.*

Proof. By (11), we see that $\{v_n\}$ is bounded in X. Since X is reflexive, there exist a subsequence of $\{v_n\}$ (we denote it by the same letter) and $w \in X$ such that $v_n \rightharpoonup w$ weakly in X. Then, we have $v_n(0) \to w(0)$, $v_n \rightharpoonup w$ weakly in $L^{p+1}(\mathbb{R})$, and $\partial_x v_n \rightharpoonup \partial_x w$ weakly in $L^2(\mathbb{R})$. Thus, we have

$$d \le E(w) \le \liminf_{n\to\infty} E(v_n) = d,$$

which implies that $w \in \mathcal{M}$, $\|v_n\|_{L^{p+1}} \to \|w\|_{L^{p+1}}$ and $\|\partial_x v_n\|_{L^2} \to \|\partial_x w\|_{L^2}$. Hence, $v_n \to w$ in X. □

Lemma 9. *Let $\{v_n\} \subset H^1(\mathbb{R})$, $E(v_n) \to E(\varphi_0)$ and $\|v_n\|_{L^2} \to \|\varphi_0\|_{L^2}$. Then, there exist a subsequence $\{v_{n'}\}$ and $\theta_0 \in \mathbb{R}$ such that $v_{n'} \to e^{i\theta_0}\varphi_0$ in $H^1(\mathbb{R})$.*

Proof. By Lemmas 7 and 8, we see that $\mathcal{M} = \mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$ and $d = E(\varphi_0)$. Thus, by Lemma 8, there exist a subsequence of $\{v_n\}$ (we denote it by the same letter) and $\theta_0 \in \mathbb{R}$ such that $v_n \to e^{i\theta_0}\varphi_0$ in X, $v_n \rightharpoonup e^{i\theta_0}\varphi_0$ weakly in $L^2(\mathbb{R})$. By the weakly lower semicontinuity of norm and by our assumption, we have

$$\|e^{i\theta_0}\varphi_0\|_{L^2} \le \liminf_{n\to\infty} \|v_n\|_{L^2} = \|\varphi_0\|_{L^2} = \|e^{i\theta_0}\varphi_0\|_{L^2},$$

which implies that $v_n \to e^{i\theta_0}\varphi_0$ in $L^2(\mathbb{R})$. Hence, $v_n \to e^{i\theta_0}\varphi_0$ in $H^1(\mathbb{R})$. □

Now we give the proof of Theorem 2.

Proof of Theorem 2. We prove this by contradiction. Suppose that φ_0 is not stable in $H^1(\mathbb{R})$. Then, there exist a constant $\varepsilon_0 > 0$, a sequence $\{u_n(t)\}$ of solutions of (1) and a sequence $\{t_n\}$ in $(0, \infty)$ such that

$$\|u_n(0) - \varphi_0\|_{H^1} \to 0, \tag{14}$$

$$\inf_{\theta\in\mathbb{R}} \|u_n(t_n) - e^{i\theta}\varphi_0\|_{H^1} = \varepsilon_0 \tag{15}$$

By (14) and the conservation of charge and energy, we have

$$\begin{aligned} &\|u_n(t_n)\|_{L^2} = \|u_n(0)\|_{L^2} \to \|\varphi_0\|_{L^2}, \\ &E(u_n(t_n)) = E(u_n(0)) \to E(\varphi_0). \end{aligned}$$

By Lemma 9, there exist a subsequence $\{u_{n'}(t_{n'})\}$ and $\theta_0 \in \mathbb{R}$ such that $u_{n'}(t_{n'}) \to e^{i\theta_0}\varphi_0$ in $H^1(\mathbb{R})$. This contradicts (15). Hence, φ_0 is stable in $H^1(\mathbb{R})$. □

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Masahiro Kaminaga
Department of Electrical Engineering and Information Technology,
Tohoku Gakuin University, Tagajo, 985–8537, Japan
e-mail: kaminaga@tjcc.tohoku-gakuin.ac.jp

Masahito Ohta
Department of Mathematics, Faculty of Science,
Saitama University, Saitama, 338–8570, Japan
e-mail: mohta@mail.saitama-u.ac.jp