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Geometry and computation of 2-Weierstrass points on Kuribayashi quartic curves

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Abstract

In this paper, we study the geometry of the 2-Weierstrass points on the Kuribayashi quartic curves:

$$C_a : \; x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + x^2z^2) = 0 \quad (a \neq 1, \pm 2).$$

The 2-Weierstrass points on C_a are divided into flexes and sextactic points. It is known that the symmetric group S_4 acts on C_a (See [8]). Using the S_4-action, we classify the 2-Weierstrass points on C_a.

1. Introduction

We consider a 1-parameter family of smooth quartic curves $C_a \subset \mathbf{P}^2(\mathbf{C})$ (non-hyperelliptic curves of genus 3):

$$C_a : \; x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + x^2z^2) = 0 \quad (a \neq 1, \pm 2).$$

We call these quartic curves C_a *Kuribayashi quartic curves*. It is known that the Weierstrass points on a smooth quartic curve are nothing but flexes. In [8], it was shown that C_a has 12 hyperflexes (resp. 24 ordinary flexes) if $a = 0, 3$ (resp. otherwise). The symmetric group S_4 acts on the Kuribayashi curves C_a.

Definition 1. Define the projective transformation group G to be the group generated by the three elements

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The group G is isomorphic to the symmetric group S_4. Indeed, G acts on the set of four points $O_1 = (-1 : 1 : 1)$, $O_2 = (1 : -1 : 1)$, $O_3 = (-1 : -1 : 1)$, $O_4 = (1 : 1 : 1)$, as the permutations $\sigma \to (12)$, $\tau \to (13)$, $\rho \to (14)$. It turns out that G also acts on C_a. Thus we can regard as $G \subset \mathrm{Aut}(C_a)$.

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It was proved by Kuribayashi-Sekita [8] that $G_a \cong C_{a'}$ if and only if $a' = a$ except if $a = (-3 \pm 3\sqrt{-7})/2$. The group G acts on Weierstrass points and on 2-Weierstrass points on C_a. So we can discuss the structure of G-orbits of the 2-Weierstrass points on C_a (See [2], for Weierstrass points).

Definition 2. A smooth, but not a flex point P on a plane curve C is called a *sextactic point* if there exists an irreducible conic D which meets C with contact order $m \geq 6$. Such a conic D (unique, if exists) is called the *sextactic conic*.

Furthermore, for a sextactic point P, the positive integer $m - 5$ is called the *sextactic order*. We say that P is *s-sextactic* if $s = m - 5 \geq 1$.

Geometrically, a 2-Weierstrass point on a smooth quartic curve is either a flex or a sextactic point (See Section 2). The purpose of this paper is to prove the following

Theorem. *The G-orbits of the 2-Weierstrass points on Kuribayashi curves C_a are classified as follows. We divide the set of 2-Weierstrass points on C_a into the subset of flexes and the subset of sextactic points.*

Table 1 G-orbits of flexes

a	ordinary flexes	hyperflexes
$0, 3$		1 orb. of 12 pts
otherwise	1 orb. of 24 pts	

Table 2 G-orbits of sextactic points

a	1-sextactic pts	2-sextactic pts	3-sextactic pts
$0, 3$	2 orb. of 12 pts 1 orb. of 24 pts		
14	3 orb. of 12 pts 1 orb. of 24 pts		1 orb. of 8 pts
P(a)=0	2 orb. of 12 pts 1 orb. of 24 pts		1 orb. of 12 pts
Q(a)=0	3 orb. of 12 pts	1 orb. of 24 pts	
otherwise	3 orb. of 12 pts 2 orb. of 24 pts		

Here, we set

$$P(a) = a^3 + 68a^2 - 91a + 98,$$
$$Q(a) = 33a^4 - 186a^3 + 205a^2 + 364a + 196.$$

We refer to Section 4, for the location and the detailed structure of the 2-Weierstrass points. We list the numbers of the 2-Weierstrass points.

Corollary. *The numbers of* 2-*Weierstrass points on* C_a *with respect to their types are given in the following table.*

Table 3 Number of 2-Weierstrass points

a	ordinary flexes	hyper-flexes	1-sextac. pts	2-sextac. pts	3-sextac. pts
0,3	0	12	48	0	0
14	24	0	60	0	8
P(a)=0	24	0	48	0	12
Q(a)=0	24	0	36	24	0
otherwise	24	0	84	0	0

In Section 2, we recall basic facts on 2-Weierstrass points, Wronskian forms of quadratic differentials on a smooth quartic curve and the multiplicities of zeros of polynomials. In Section 3, we discuss the G-action on the Kuribayashi curves C_a. In Section 4, we complete the proof of Theorem. In Section 5, we numerically compute some of the 2-Weierstrass points for the cases in which $P(a) = 0$ and $Q(a) = 0$. We use the computer softwares Mathematica and Maple to perform the computations. We refer to Alwaleed [1] for detailed computations and further discussions.

2. Preliminaries

2.1 2-Weierstrass points

Let C be a smooth quartic curve. The 2-Weierstrass points on C are defined by the orders of quadratic differential forms $\omega \in H^0(C, (\Omega^1)^2)$. Since $(\Omega^1)^2 \cong \mathcal{O}_C(2)$, we have $\dim H^0(C, (\Omega^1)^2) = 6$. Take a point $P \in C$. Let $\{\omega_1, \dots, \omega_6\}$ be a basis of $H^0(C, (\Omega^1)^2)$ so that $\mathrm{ord}_P(\omega_1) < \dots < \mathrm{ord}_P(\omega_6)$. Letting $n_i = \mathrm{ord}_P(\omega_i) + 1$, the sequence $\{n_1, \dots, n_6\}$ is called the 2-*gap sequence* of P. The quantity

$$w^{(2)}(P) = \sum_{i=1}^{6} (n_i - i)$$

is called the 2-*weight* of P. We say that P is a 2-*Weierstrass point* if $w^{(2)}(P) > 0$. Geometrically, P is a 2-Weierstrass point if and only if there is a unique conic D with $I_P(C, D) = n_6 - 1 \geq 6$. We infer that either $D = 2L$ (P is a flex and L is the tangent line at P) or D is an irreducible conic (P is not a flex). In the latter case, the point P is a sextactic point. We denote by $W_2(C)$ (resp. $W_1(C)$) the set of 2-Weierstrass points (resp. Weierstrass points) on C.

Lemma 1. *Let* C *be a smooth quartic curve. Then we have*

(i) $W_1(C) = \{\textit{flexes}\}$, $\quad W_2(C) = \{\textit{flexes}\} \cup \{\textit{sextactic points}\}$.

(ii) *The possible* 2*-gap sequences of* $P \in W_2(C)$ *are listed in the following table.*

$w^{(2)}(P)$	2-*gap sequence*	geometry
1	$\{1,2,3,4,5,7\}$	*ordinary flex*
5	$\{1,2,3,5,6,9\}$	*hyperflex*
1	$\{1,2,3,4,5,7\}$	1-*sextactic pt*
2	$\{1,2,3,4,5,8\}$	2-*sextactic pt*
3	$\{1,2,3,4,5,9\}$	3-*sextactic pt*

(iii) *We have*

$$\sum_{P\in C} w^{(2)}(P) = 108.$$

Proof. For (iii), we refer to [6, 9]. ☐

2.2 Wronskians

In order to compute 2-Weierstrass points, one can use the Wronskian form. For a basis $\{\omega_1,\dots,\omega_6\}$ of the space $H^0(C,(\Omega^1)^2)$, one can define the Wronskian form $\Omega = W(\omega_1,\dots,\omega_6) \in H^0(C,(\Omega^1)^{27})$. Then the order of zeros of Ω at P gives us the 2-weight $w^{(2)}(P)$. Cf. [6, 9]. Letting $f(x,y)=0$ be the affine equation of C, we can use the basis:

$$\{\frac{1}{f_y^2}(dx)^2,\ \frac{x}{f_y^2}(dx)^2,\ \frac{y}{f_y^2}(dx)^2,\ \frac{x^2}{f_y^2}(dx)^2,\ \frac{xy}{f_y^2}(dx)^2,\ \frac{y^2}{f_y^2}(dx)^2\}.$$

By computation, we have

Lemma 2. (i) $\Omega = W(1,x,y,x^2,xy,y^2)(dx)^{27}/f_y^{12}$,

(ii) $W(1,x,y,x^2,xy,y^2) = 4y''\left[45y^{(4)}y^{(3)}y'' - 9y^{(5)}(y'')^2 - 40(y^{(3)})^3\right]$.

Remark 1. One can compute the term $y^{(k)}$ by the implicit differentiation. For instance, as is well known, we have

$$y'' = (f_{x^2}f_y^2 - 2f_{xy}f_xf_y + f_{y^2}f_x^2)/f_y^3.$$

2.3 Multiplicities of zeros

Definition 3. Let $p(x)$ be a polynomial. Write $p(x) = c\prod_{i=1}^k (x-\alpha_i)^{m_i}$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. We may arrange as $m_1 \geq m_2 \geq \ldots \geq m_k$. We set

(i) $T(p) = (m_1,\dots,m_k)$,

(ii) $r(p) = \max\{m_i\}$,

(iii) $s(p) = \sum_{i=1}^k (m_i - 1)$.

We call $T(p)$ the *type* of p. We use abbreviations such as $1_n = \overbrace{1,\dots,1}^{n}$ and $2_n = \overbrace{2,\dots,2}^{n}$.

As for the invariants $r(p)$, $s(p)$, we use the following well known facts.

Lemma 3. *Let $p(x)$ be a polynomial.*

(i) *If $V(p, p', \dots, p^{(r-1)}) \neq \emptyset$ and $V(p, p', \dots, p^{(r)}) = \emptyset$, then $r = r(p)$.*

(ii) *Let $R^{(i)}$ be the i-th subresultant of $p(x)$ and $p'(x)$. If $R^{(1)} = \dots = R^{(s)} = 0, R^{(s+1)} \neq 0$. Then we have $s = s(p)$.*

We regard a polynomial $p(x, a) \in \mathbf{C}[x, a]$ as a 1-parameter family of polynomials depending on the value a. We sometimes write as $p_a(x) = p(x, a)$. Consider the ideal $I_k = (p, p', \dots, p^{(k)})$, where the $p^{(i)}$ denotes the i-th differentiation by the variable x. By using the Groebner basis methods (Cf. [3], Chap. 3), we can compute the ideal $J_k = I_k \cap \mathbf{C}[a]$ in $\mathbf{C}[a]$. If $a_0 \in V(J_{r-1})$, $a_0 \notin V(J_r)$, then we infer that $r(p_{a_0}) = r$. Also if $R^{(1)}(a_0) = \dots = R^{(s)}(a_0) = 0$, $R^{(s+1)}(a_0) \neq 0$, then we conclude that $s(p_{a_0}) = s$.

3. G-action on 2-Weierstrass points

We now study the G-action on the Kuribasyashi curves C_a. We use the affine equation of C_a:

$$f(x, y) = x^4 + y^4 + 1 + a(x^2y^2 + x^2 + y^2) = 0.$$

As we have seen in Section 1, a projective transformation group $G \cong S_4$ acts on C_a. For a point P on C_a, let G_P denote the *stabilizer* of P in G. It is well known that G_P is a cyclic group. Let $\mathrm{Orb}(P)$ be the *orbit* of the point P. We denote by $X(C_a)$ the set of the points $P \in C_a$ such that $|G_P| > 1$. We also write $X_i(C_a) = \{P \in C_a \mid |G_P| = i\}$.

Lemma 4. *We have $X(C_a) = X_2(C_a) \cup X_3(C_a)$.*

Proof. Since $G \cong S_4$, the orders of elements in G are 1,2,3 and 4. It suffices to see that $X_4(C_a) = \emptyset$. We show that the group G_P does not contain elements of order four for any point $P \in C_a$. The element

$$\varphi = \rho\tau\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

has order four. We denote by $\mathrm{Fix}(\varphi, \mathbf{P}^2)$ the set of fixed points of φ in $\mathbf{P}^2$. We see that $\mathrm{Fix}(\varphi, \mathbf{P}^2)$ consists of three points $\{(0 : 1 : 0), (1 : 0 : \pm i)\}$, which are disjoint from C_a for $a \neq 2$. Since $G \cong S_4$, all elements of order 4 are conjugate with each other in G, we conclude that G_P for $P \in C_a$ cannot contain elements of order four. ◻

In the symmetric group S_4, there exist 9 elements of order two, which are divided into two conjugacy classes:

$$\{(12),(13),(14),(23),(24),(34)\},\quad \{(12)(34),(13)(24),(14)(23)\}.$$

Using the isomorphism $G \cong S_4$ given in Section 1, we have the correspondences: $\sigma \leftrightarrow (12)$ and $\psi = (\sigma\tau\rho)^2 \leftrightarrow (13)(24)$.

For σ, we have $\mathrm{Fix}(\sigma, \mathbf{P}^2) = \{(-1:1:0)\} \cup \{\text{the line } x = y\}$. Thus $\mathrm{Fix}(\sigma, \mathbf{P}^2) \cap C_a$ consists of four points $\{(\alpha : \alpha : 1)\}$, where the α are the four distinct roots of the equation: $(2+a)x^4 + 2ax^2 + 1 = 0$. Note that $\alpha \neq \pm 1$, since $a \neq -1$. Note also that

$$\mathrm{Disc}((2+a)x^4 + 2ax^2 + 1; x) = 256(a+1)^2(a-2)^2(a+2) \neq 0.$$

We have

$$\mathrm{Orb}((\alpha:\alpha:1)) = \left\{ \begin{array}{llll} (\alpha:\alpha:1), & (\alpha:-\alpha:1), & (-\alpha:-\alpha:1), & (-\alpha:\alpha:1) \\ (1:1/\alpha:1), & (1:-1/\alpha:1), & (-1:1/\alpha:1), & (-1:-1/\alpha:1) \\ (1/\alpha:1:1), & (1/\alpha:-1:1), & (-1/\alpha:1:1), & (-1/\alpha:-1:1) \end{array} \right\}.$$

We see that $G_{(\alpha:\alpha:1)} = \{1, \sigma\}$. For the element

$$\psi = (\sigma\tau\rho)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we have $\mathrm{Fix}(\psi, \mathbf{P}^2) = \{(0:1:0)\} \cup \{\text{the line } y = 0\}$. So it follows that $\mathrm{Fix}(\psi, \mathbf{P}^2) \cap C_a = \{(\beta:0:1)\}$, where the β are the four distinct roots of the equation: $x^4 + ax^2 + 1 = 0$. Note that $\beta \neq 0, \pm 1$, since $a \neq -2$. We have

$$\mathrm{Orb}((\beta:0:1)) = \left\{ \begin{array}{llll} (\beta:0:1), & (1/\beta:0:1), & (-\beta:0:1), & (-1/\beta:0:1) \\ (0:\beta:1), & (0:1/\beta:1), & (0:-\beta:1), & (0:-1/\beta:1) \\ (\beta:1:0), & (1:\beta:0), & (\beta:-1:0) & (-1:\beta:0) \end{array} \right\}.$$

We see that $G_{(\beta:0:1)} = \{1, \psi\}$.

Summarizing, we obtain

Proposition 1. *We have*

(i) $X_2(C_a) = \mathrm{Orb}((\alpha_1:\alpha_1:1)) \cup \mathrm{Orb}((\alpha_2:\alpha_2:1)) \cup \mathrm{Orb}((\beta:0:1))$, *where the α_1 and the α_2 are two distinct roots ($\alpha_1 \neq -\alpha_2$) of the equation: $u(x) = (2+a)x^4 + 2ax^2 + 1 = 0$. The β is a root of the equation:*

$e(x) = x^4 + ax^2 + 1 = 0.$

(ii) $X_2(C_a) \subset W_2(C_a)$,

$$\text{(iii) } X_2(C_a) \cap W_1(C_a) = \begin{cases} \mathrm{Orb}((\beta : 0 : 1)) & \textit{if } a = 0, \\ \mathrm{Orb}((i : i : 1)) & \textit{if } a = 3, \\ \emptyset & \textit{otherwise,} \end{cases}$$

where $\beta = (1+i)/\sqrt{2}$.

Proof. (ii) We can apply Duma's criterion in [5], Satz 6.3. (iii) Let $H(x, y, z)$ be the Hessian of the curve C_a. By computation, we obatin

$$\mathrm{Res}(H(x, y, 1), f(x, y); y) = \mathrm{Const.}(a-2)^6(a+2)^4 h(x)^2,$$

where

$$h(x) = a^2(x^{12}+1)+6a(x^{10}+x^2)-3(a^3-a^2-3a-6)(x^8+x^4)-2a(3a^2-2a-15)x^6.$$

We have

$$h(1) = -12(a-3)(a+1)^2, \quad h(0) = a^2.$$

Thus, if $a \neq 0, 3$, then we have $h(0)h(1) \neq 0$. If $X_2(C_a) \cap W_1(C_a) \neq \emptyset$, then by (i), we must have $(1 : 1/\alpha : 1) \in W_1(C_a)$ or $(0 : \beta : 1) \in W_1(C_a)$, which is not the case. □

Proposition 2. *We have*

$$\begin{aligned} X_3(C_a) &= \mathrm{Orb}((\omega : \omega^2 : 1)) \\ &= \left\{ \begin{array}{llll} (\omega : \omega^2 : 1), & (-\omega : -\omega^2 : 1), & (-\omega : \omega^2 : 1), & (\omega : -\omega^2 : 1) \\ (\omega^2 : \omega : 1), & (-\omega^2 : -\omega : 1), & (\omega^2 : -\omega : 1), & (-\omega^2 : \omega : 1) \end{array} \right\}. \end{aligned}$$

Proof. There are 8 elements of order three in $G \cong S_4$, which are conjugate with each other. For instance, the element

$$\sigma\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

has order three. We have $\mathrm{Fix}(\sigma\tau, \mathbf{P}^2) = \{(1 : 1 : 1), (\omega : \omega^2 : 1), (\omega^2 : \omega : 1)\}$, where the ω is the third root of unity. Thus, we obtain

$$\mathrm{Fix}(\sigma\tau, \mathbf{P}^2) \cap C_a = \{(\omega : \omega^2 : 1), (\omega^2 : \omega : 1)\}.$$

□

Remark 2. If $(\gamma:\delta:1)\notin X(C_a)$, then $\mathrm{Orb}(\gamma:\delta:1)$ consists of the following 24 points.

$$\left\{\begin{array}{llll}(\gamma:\delta:1), & (\gamma:-\delta:1), & (-\gamma:-\delta:1), & (-\gamma:\delta:1)\\ (\delta:\gamma:1), & (\delta:-\gamma:1), & (-\delta:-\gamma:1), & (-\delta:\gamma:1)\\ (1/\gamma:\delta/\gamma:1), & (1/\gamma:-\delta/\gamma:1), & (-1/\gamma:-\delta/\gamma:1), & (-1/\gamma:\delta/\gamma:1)\\ (\delta/\gamma:1/\gamma:1), & (\delta/\gamma:-1/\gamma:1), & (-\delta/\gamma:-1/\gamma:1), & (-\delta/\gamma:1/\gamma:1)\\ (1/\delta:\gamma/\delta:1), & (1/\delta:-\gamma/\delta:1), & (-1/\delta:-\gamma/\delta:1), & (-1/\delta:\gamma/\delta:1)\\ (\gamma/\delta:1/\delta:1), & (\gamma/\delta:-1/\delta:1), & (-\gamma/\delta:-1/\delta:1), & (-\gamma/\delta:1/\delta:1)\end{array}\right\}.$$

4. Proof of Theorem

Using Lemma 2 and implicit differentiation, we can write the Wronskian form Ω of quadratic differentials on C_a as:

$$\Omega=\Phi(x,y)\big(dx/f_y\big)^{27}.$$

We have

$$\Phi(x,y)=\mathrm{Const}\cdot(a+1)(a-2)^2(a^2-4)xy(x^2-y^2)\Phi_1(x,y)\Phi_2(x,y),$$

where $\deg(\Phi_1(x,y))=8$ and $\deg(\Phi_2(x,y))=18$.

We first determine the 2-Weierstrass points on C_0.

Lemma 5. *For the case in which $a=0$, we have*

$$\left\{\begin{array}{rcl}W_1(C_0) & = & \mathrm{Orb}((\beta:0:1)),\\ W_2(C_0)\setminus W_1(C_0) & = & \mathrm{Orb}((\alpha:\alpha:1))\cup\mathrm{Orb}((\overline{\alpha}:\overline{\alpha}:1))\cup\mathrm{Orb}((\alpha:\overline{\alpha}:1)),\end{array}\right.$$

where $\beta=(1+i)/\sqrt{2}$, $\alpha=\sqrt[4]{2}(1+i)/2$. Note that $\mathrm{Orb}((\alpha:\overline{\alpha}:1))$ *consists of 24 points.*

Proof. By Proposition 1, we obtain

$$X_2(C_0)=\mathrm{Orb}((\beta:0:1))\cup\mathrm{Orb}((\alpha:\alpha:1))\cup\mathrm{Orb}((\overline{\alpha}:\overline{\alpha}:1))\subset W_2(C_0).$$

We have $\Phi(x,y)=\mathrm{Const}\cdot(xy)^5(x^4-y^4)(2x^4+y^4)(x^4+2y^4)$. It follows that $\mathrm{Orb}(\alpha:\overline{\alpha}:1)\subset W_2(C_0)$. □

In what follows, we assume that $a\neq 0$. We compute the resultant of Φ (or Φ_i) and f with respect to y. Set

$$\phi(x)=\mathrm{Res}(\Phi,f;y),\quad \phi_i(x)=\mathrm{Res}(\Phi_i,f;y).$$

It turns out that $\phi_1(x)$ coincides with the polynomial $h(x)$ up to constant, which appeared in Lemma 1 as the resultant of the Hessian with respect to y. This is a consequence of Lemma 2, (ii) and Remark 1.

Lemma 6. *We obtain*

$$\begin{aligned}\text{(i)}\ \phi_2(x) &= (a+2)^8(a-2)^{14}(x^2-1)^4 v(x)^2 g(x)^2,\\ \text{(ii)}\ \phi(x) &= \text{Const.}(a+2)^{16}(a-2)^{32}(a+1)^4\\ &\quad \times x^4(x^2-1)^4 u(x)^2 v(x)^2 h(x)^2 g(x)^2.\end{aligned}$$

where $u(x) = (2+a)x^4 + 2ax^2 + 1$, $v(x) = x^4 + 2ax^2 + a + 2$ *and*

$$\begin{aligned}g(x) &= 9a^2(a-2)(a+2)^2\,(x^{24}+1)\\ &\quad + 6a(a-1)(a-2)(a+2)(15a+14)\,(x^{22}+x^2)\\ &\quad + (36\,a^6+294\,a^5-720\,a^4-768\,a^3+956\,a^2-112\,a-784)(x^{20}+x^4)\\ &\quad + (186\,a^6+276\,a^5-1920\,a^4-662\,a^3+1216\,a^2-1848\,a-1568)\,(x^{18}+x^6)\\ &\quad + (3\,a^7+450\,a^6-255\,a^5-2990\,a^4-277\,a^3+34\,a^2-4172\,a-3528)\,(x^{16}+x^8)\\ &\quad + (12\,a^7+642\,a^6-790\,a^5-3346\,a^4-2098\,a^3+980\,a^2-7224\,a-5488)\,(x^{14}+x^{10})\\ &\quad + (18\,a^7+684\,a^6-688\,a^5-5208\,a^4+1130\,a^3-2004\,a^2-8904\,a-5488)\,x^{12}.\end{aligned}$$

Remark 3. Note that $g(-x) = g(x)$ and $x^{24}g(1/x) = g(x)$. Note also that $v(x) = x^4u(1/x)$. Thus, if we write $u(x) = (a+2)(x^2-\alpha_1^2)(x^2-\alpha_2^2)$ as in Proposition 1, then we have $v(x) = (x^2-(1/\alpha_1)^2)(x^2-(1/\alpha_2)^2)$.

Lemma 7. *We have*

$$\begin{aligned}\text{Disc}(g;x) &= \text{Const.}\,(a-2)^{16}(a+2)^{18}(a+1)^{20}\\ &\quad \times a^{16}(a-3)^{18}(a-14)^8 P(a)^6 Q(a)^{12}\eta(a)^8,\end{aligned}$$

where

$$\begin{aligned}P(a) &= a^3+68a^2-91a+98,\\ Q(a) &= 33a^4-186a^3+205a^2+364a+196,\\ \eta(a) &= (3a^3-6a^2+9a-14)(9a^{12}-162a^{11}+1683a^{10}+\cdots).\end{aligned}$$

Proof. This follows from a direct computation. □

Lemma 8. *For the case in which* $a = 3$, *we have*

$$\begin{cases} W_1(C_3) = \text{Orb}((i:i:1)),\\ W_2(C_3)\setminus W_1(C_3) = \text{Orb}((\beta:0:1))\cup\text{Orb}((\alpha:\alpha:1))\cup\text{Orb}((\gamma:\delta:1)),\end{cases}$$

where $\beta = (1+\sqrt{5})i/2$, $\alpha = i/\sqrt{5}$ *and* $g(\gamma) = 0$. *Note that* $\text{Orb}((\gamma:\delta:1))$ *consists of* 24 *points.*

Proof. Letting $a = 3$, we have $g(x) = (x^2+1)^4(x^2-1)^2 g_{12}(x)$, where $g_{12}(x) =$

$2025\,x^{12} + 6570\,x^{10} + \ldots + 2025$. We have $\mathrm{Disc}(g_{12}; x) \neq 0$. Now, it follows from Proposition 1 that the points

$$X_2(C_3) = \mathrm{Orb}((i : i : 1)) \cup \mathrm{Orb}((\beta : 0 : 1)) \cup \mathrm{Orb}((i/\sqrt{5} : i/\sqrt{5} : 1))$$

are contained in $W_2(C_3)$. Take a root γ of $g_{12}(x)$, Then, there exists a 2-Weierstrass point $(\gamma : \delta : 1) \in C_3$. Since g_{12} has no multiple roots, we have $w^{(2)}((\gamma : \delta : 1)) = 1$. □

Lemma 9. *If $a \neq 0, 3$, then there exists an ordinary flex $(\gamma_0 : \delta_0 : 1) \notin X(C_a)$ so that $W_1(C_a) = \mathrm{Orb}((\gamma_0 : \delta_0 : 1))$.*

Proof. See Proposition 1, (iii). □

Lemma 10. *For the case in which $a = 14$, we have*

$$W_2(C_{14}) \setminus W_1(C_{14}) = \mathrm{Orb}((\omega : \omega^2 : 1)) \cup X_2(C_{14}) \cup \mathrm{Orb}((\gamma : \delta : 1)),$$

where $g(\gamma) = 0$ and

$$X_2(C_{14}) = \mathrm{Orb}(((\beta : 0 : 1)) \cup \mathrm{Orb}((\alpha_1 : \alpha_1 : 1)) \cup \mathrm{Orb}((\alpha_2 : \alpha_2 : 1)),$$

where $\beta = 2 + \sqrt{3}i$, $\alpha_1 = (\sqrt{5} + 3)i/4$, $\alpha_2 = (\sqrt{5} - 3)i/4$. Furthermore, we have $w^{(2)}((\omega : \omega^2 : 1)) = 3$ and $X_3(C_{14}) = \mathrm{Orb}((\omega : \omega^2 : 1))$. Note that $\mathrm{Orb}((\gamma : \delta : 1))$ consists of 24 points.

Proof. Letting $a = 14$, we have $g(x) = 200704\,(x^2 + x + 1)^3(x^2 - x + 1)^3 g_{12}(x)$, where $g_{12}(x) = 27\,x^{12} + \ldots + 27$. We have $\mathrm{Disc}(g_{12}; x) \neq 0$. Now, it follows from Proposition 1 that $X_2(C_{14}) \subset W_2(C_{14})$. By a direct computation, we see that $(\omega : \omega^2 : 1) \in C_{14}$ and $w^{(2)}(\omega : \omega^2 : 1) = 3$ (See Remark 4 below). Finally, take a root γ of $g_{12}(x)$, Then there exists a 2-Weierstrass point $(\gamma : \delta : 1) \in C_3$. Since g_{12} has no multiple roots, we have $w^{(2)}((\gamma : \delta : 1)) = 1$. □

Lemma 11. *Suppose $a \neq 0, 3, 14$. The type of the multiplicities of the roots of the polynomial $g(x)$ is given in the following table:*

a	$T(g)$
$P(a) = 0$	$(2_6, 1_{12})$
$Q(a) = 0$	(2_{12})
$\eta(a) = 0$	$(2_4, 1_{16})$
otherwise	(1_{24})

Proof. By using the Groebner basis method, we obtain

$$(g, g', g'') \cap \mathbf{C}[a] = ((a-2)(a+2)^2(a+1)^2a^2(a-3)^2(a-14)).$$

Thus, we infer that $r(g) = 2$ (resp. $r(g) = 1$) if $P(a)Q(a)\eta(a) = 0$ (resp. otherwise) (See Definition 3). So if $P(a)Q(a)\eta(a) \neq 0$, then g has no multiple roots. Let $R^{(k)}$ denote the k-th subresultant of g and g' with respect to x.

Case $P(a) = 0$. By computation, we have $P(a) \,|\, R^{(k)}$ for $k = 1, \dots, 6$ but $\mathrm{Res}(P(a), R^{(7)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 6)$. We can easily conclude that $T(g) = (2_6, 1_{12})$.

Case $Q(a) = 0$. In this case, we have $Q(a) \,|\, R^{(k)}$ for $k = 1, \dots, 12$ but $\mathrm{Res}(Q(a), R^{(13)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 12)$. We conclude that $T(g) = (2_{12})$.

Case $\eta(a) = 0$. In this case, we have $\eta(a) \,|\, R^{(k)}$ for $k = 1, \dots, 4$, but we obtain $\mathrm{Res}(\eta(a), R^{(5)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 4)$. We conclude that $T(g) = (2_4, 1_{16})$. □

Now we pass to coordinates change. We use the affine coordinates: $(X, Y) = (x + 2y, y)$. Write $\overline{f}(X, Y) = f(X - 2Y, Y)$ and

$$\Omega = \overline{\Phi}(X, Y)\big(dX/\overline{f}_Y\big)^{27}$$

so that $\overline{\Phi}(X, Y) = \Phi(X - 2Y, Y)$. Set also $\overline{\Phi_2}(X, Y) = \Phi_2(X - 2Y, Y)$. Letting $\overline{\phi}(X) = \mathrm{Res}(\overline{\Phi}, \overline{f}; Y)$, $\overline{\phi_2}(X) = \mathrm{Res}(\overline{\Phi_2}, \overline{f}; Y)$, by computation, we have

Lemma 12.

(i) $T(\overline{\phi_2}) = (2_{24}, 1_{16})$, *if* $Q(a) = 0$,

(ii) $\mathrm{Res}(\eta, \mathrm{Disc}(\overline{\phi}; X); a) \neq 0$.

Lemma 13. *Suppose $a \neq 3$. Let α_1, α_2 be as in Proposition 1, (i). Let m_i denote the 2-weight of the 2-Weierstrass point $(\alpha_i : \alpha_i : 1) \in X_2(C_a)$. We may assume $m_1 \leq m_2$. Then we have*

$$(m_1, m_2) = \begin{cases} (1,3) & \text{if } P(a) = 0, \\ (1,1) & \text{otherwise.} \end{cases}$$

Proof. We recall that the point $(1 : 1/\alpha_i : 1)$ belongs to the orbit of the point $(\alpha_i : \alpha_i : 1)$. By computation, we obtain

$$\begin{aligned} g(1) &= \mathrm{Const.}(a-3)(a+2)(a+1)^2 P(a), \\ g'(1) &= \mathrm{Const.}(a-3)(a+2)(a+1)^2 P(a), \\ \mathrm{Res}(P(a), g''(1); a) &\neq 0. \end{aligned}$$

In case $P(a) = 0$, we infer that the multiplicity of $(x - 1)$ in $\phi(x)$ is equal to 8. Hence $m_1 + m_2 = 4$. It follows that (m_1, m_2) is either $(2, 2)$ or $(1, 3)$. By computation, we also have $r(ug) = 3$. Since $T(g) = (2_6, 1_{12})$, the multiplicity of $(x - \alpha_2)$ in $g(x)$ must be equal to 2. Hence we have $(m_1, m_2) = (1, 3)$. In case $P(a) \neq 0$, the multiplicity of $(x - 1)$ in $\phi(x)$ is equal to 4. It follows that $m_1 + m_2 = 2$ and hence we have $(m_1, m_2) = (1, 1)$. □

Lemma 14. *If $Q(a) = 0$, then there exists a 2-Weierstrass point $P \notin X(C_a)$ with $w^{(2)}(P) = 2$.*

Proof. Let γ be a root of $g(x)$. Since $T(g) = (2_{12})$, there exists a 2-Weierstrass point $P = (\gamma : \delta : 1) \in C_a$. We have two possible cases:

(i) $w^{(2)}(P) = 2$.

(ii) $w^{(2)}(P) = 1$. In this case, we can find another 2-Weierstrass point $\tilde{P} = (\gamma : \tilde{\delta} : 1) \in C_a$ with $\tilde{\delta} \neq \pm\delta$.

The case (ii) does not occur. In fact, we use the affine coordinates (X, Y) in Lemma 12. We see that the X-coordinate of all points in $\mathrm{Orb}(P)$ and $\mathrm{Orb}(\tilde{P})$ are different. For a proof of this fact, see Lemma 16 in Section 5. It follows that the number of the different roots of $\overline{\phi_2}$ is greater than or equal to 48. But, we infer from Lemma 12 that if $Q(a) = 0$, then $T(\overline{\phi_2}) = (2_{24}, 1_{16})$, which is a contradiction. □

Proof of Thoeorem. For the cases in which $a = 0, 3, 14$, we refer to Lemmata 5, 8, 10.

Case (1). $P(a) = 0$. By Lemma 13, we can find two 2-Weierstrass points $(\alpha_1 : \alpha_1 : 1), (\alpha_2 : \alpha_2 : 1) \in X_2(C_a)$ such that $w^{(2)}((\alpha_1 : \alpha_1 : 1)) = 1$ and $w^{(2)}((\alpha_2 : \alpha_2 : 1)) = 3$. Since $T(g) = (2_6, 1_{12})$, we can also find a 2-Weierstrass point $(\gamma : \delta : 1) \notin X(C_a)$ with $w^{(2)}((\gamma : \delta : 1)) = 1$. We have

$$W_2(C_a) \setminus W_1(C_a) = \mathrm{Orb}((\beta : 0 : 1)) \cup \mathrm{Orb}((\alpha_1 : \alpha_1 : 1)) \cup \mathrm{Orb}((\alpha_2 : \alpha_2 : 1)) \cup \mathrm{Orb}((\gamma : \delta : 1)).$$

Case (2). $Q(a) = 0$. By Lemma 14, there is a 2-Weierstrass point $(\gamma : \delta : 1) \notin X_2(C_a)$ with $w^{(2)}((\gamma : \delta : 1)) = 2$. In this case, we have

$$W_2(C_a) \setminus W_1(C_a) = X_2(C_a) \cup \mathrm{Orb}((\gamma : \delta : 1)).$$

Case (3). $P(a)Q(a) \neq 0$. We can find two 2-Weierstrass points $(\gamma_1 : \delta_1 : 1)$ and $(\gamma_2 : \delta_2 : 1)$ with $g(\gamma_i) = 0$ and $w^{(2)}((\gamma_i : \delta_i : 1)) = 1$ for $i = 1, 2$ such that

$$W_2(C_a) \setminus W_1(C_a) = X_2(C_a) \cup \mathrm{Orb}((\gamma_1 : \delta_1 : 1)) \cup \mathrm{Orb}((\gamma_2 : \delta_2 : 1)).$$

The assertion follows from Lemma 11 (resp. Lemma 12, (ii)) for the case in which $\eta(a) \neq 0$ (resp. $\eta(a) = 0$). We here sketch a proof for the case in which $\eta(a) \neq 0$. Set $S = \{\gamma \,|\, g(\gamma) = 0, u(\gamma)v(\gamma) \neq 0\}$. By Lemma 11, we have $\#(S) \geq 16$.

For any $\gamma_1 \in S$, there is a 2-Weierstrass point $(\gamma_1 : \delta_1 : 1) \notin X_2(C_a)$. Choose $\gamma_2 \in S$ so that $\gamma_2 \notin \{\pm\gamma_1, \pm\delta_1, \pm 1/\gamma_1, \pm 1/\delta_1, \pm\gamma_1/\delta_1, \pm\delta_1/\gamma_1\}$. Then there is a 2-Weierstrass point $(\gamma_2 : \delta_2 : 1)$. Since $g(x)$ has no multiple roots, we infer that $w^{(2)}((\gamma_i : \delta_i : 1)) = 1$ for $i = 1, 2$. □

Remark 4. A 3-sextactic point P on a smooth quartic curve C is a *total sextactic* point, i.e., the sextactic conic D at P meets C only at P. Among Kuribayashi quartic curves C_a, there exist total sextactic points if $a = 14$ or if $P(a) = 0$. In case $a = 14$, the 2-Weierstrass points in $\mathrm{Orb}(\omega : \omega^2 : 1)$ are all total sextactic points. We remark that they lie on bitangent lines. Namely, the total sextactic points $P_1 = (\omega : \omega^2 : 1)$ and $P_2 = (\omega^2 : \omega : 1)$ lie on a bitangent line $L : x + y + z = 0$. The sextactic conics at these points are the following:

$$D_1 : \Delta_2(x, y, z) = (x^2 + 5yz) + \omega^2(y^2 + 5xz) + \omega(z^2 + 5xy) = 0,$$
$$D_2 : \Delta_1(x, y, z) = (x^2 + 5yz) + \omega(y^2 + 5xz) + \omega^2(z^2 + 5xy) = 0.$$

Note that we can write the defining equation of C_{14} as

$$\frac{5}{9}(x + y + z)^4 + \frac{4}{9}\Delta_1(x, y, z)\Delta_2(x, y, z) = 0.$$

5. Computational aspects

We now discuss the computational aspects. By a numerical method, we determine the coordinates of 2-Weierstrass points on the Kuribayashi curves C_a for the cases in which $P(a) = 0$ and $Q(a) = 0$. We use the following tool.

Lemma 15. *Let $C : f(x, y) = 0$ a smooth quartic curve. Take a non-flex point $P = (\alpha, \beta) \in C$. We can compute the osculating conic D at P (i.e., the irreducible conic having the contact order ≥ 5 to C at P) in the following way.*

(1) *Compute the defining equation $l(x, y) = y - \beta - m(x - \alpha) = 0$ of the tangent line L of C at P.*

(2) *Parametrize those irreducible conics passing through the point P with the tangent line L:*

$$l(x, y) + A(x - \alpha)^2 + B(x - \alpha)\, l(x, y) + C\, l(x, y)^2 = 0 \quad (A \neq 0)$$

as

$$\begin{cases} x(t) = \alpha - t/(A + Bt + Ct^2), \\ y(t) = \beta - t(t + m)/(A + Bt + Ct^2). \end{cases}$$

(3) *Write*

$$f(x(t), y(t)) = \frac{s_2t^2 + s_3t^3 + s_4t^4 + s_5t^5 + s_6t^6 + s_7t^7 + s_8t^8}{(A + Bt + Ct^2)^4},$$

where $s_i \in \mathbf{C}[A, B, C]$ *for* $i = 2, \dots, 8$.

(4) *Determine* A, B, C *by solving the equations:* $s_2 = s_3 = s_4 = 0$.

Proof. For the assertion (2), it suffices to parametrize the intersection points of the conic with the pencil of lines $l(x, y) - t(x - \alpha) = 0$. $\square$

5.1 The case $P(a) = 0$

We consider the case in which $P(a) = 0$. The cubic equation $P(a) = 0$ has three roots:

$$a_1 = -69.3328950\dots, \quad a_2 = 0.6664475\dots - (0.9845395\dots)i, \quad a_3 = \overline{a_2}.$$

We here consider the real root a_1. The equation: $u(x) = 0$ has 4 distinct roots $\{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2\}$, where

$$\alpha_1 = 0.0847732623\dots, \quad \alpha_2 = (1.43756489\dots)i.$$

Proposition 3. *Suppose* $a = a_1$. *The point* $P_1 = (\alpha_1 : \alpha_1 : 1)$ *is a* 1*-sextactic point and the point* $P_2 = (\alpha_2 : \alpha_2 : 1)$ *is a* 3*-sextactic point.*

Proof. **Case (i)**. We first check the point P_1. Using the method in Lemma 15, we have the following approximate solutions for the equations $s_2 = s_3 = s_4 = 0$:

$$A = 11.62294385\dots, \quad B = -11.62294386\dots, \quad C = 6.03059824\dots.$$

In fact, we have

$$s_2 = A^2\{-137.5830294 + 11.83719298A - (8.881\dots \times 10^{-16})B^2 + (8.881\dots \times 10^{-16})AC\}.$$

So if we put $A = -137.5830294/11.83719298 = 11.6229438\dots$, then s_2 is very close to zero. Solving the equations $s_3 = 0, s_4 = 0$, we can find the approximate solutions B, C. In this case, we have $s_5 = -8.367\dots \times 10^{-11}$, but $s_6 = 146.955\dots \neq 0$. Thus we infer that P_1 is a 1-sextactic point.

Case (ii). We now check the point P_2. In a similar manner as in Case (i), we have the following approximate solutions for the equations $s_2 = s_3 = s_4 = 0$:

$$A = (2.2419441\dots)i, \quad B = -(2.2419441\dots)i, \quad C = (0.1079171\dots)i.$$

In this case, we have $s_5 = -3.637\dots \times 10^{-12}$, $s_6 = 3.069\dots \times 10^{-12}$ and $s_7 = -6.430\dots \times 10^{-13}$. We therefore conclude that P_2 is a 3-sextactic point. $\square$

5.2 The case $Q(a) = 0$

We now pass to the case in which $Q(a) = 0$. The quartic equation $Q(a) = 0$ has four roots:

$$\begin{aligned} a_1 &= 3.359188060\ldots + (1.319606687\ldots)i, \quad \overline{a_1}, \\ a_2 &= -0.5410062419\ldots + (0.4040965957\ldots)i, \quad \overline{a_2} \end{aligned}$$

We first prove the following fact which was used in Lemma 14.

Lemma 16. *Let a be a root of the equation $Q(a) = 0$. Let γ be a root of the equation $g(x) = 0$. Let $\delta, \tilde{\delta}$ be two roots of the equation: $f(\gamma, y) = 0$ with $\tilde{\delta} \neq -\delta$. Put $P = (\gamma : \delta : 1)$ and $\tilde{P} = (\gamma : \tilde{\delta} : 1)$. Let (X, Y) be the affine coordinates used in Lemma 12. Then the X-coordinates of the points in $\mathrm{Orb}(P)$ and in $\mathrm{Orb}(\tilde{P})$ are all different.*

Proof. We here consider the root a_1. We can similarly deal with the other cases. Assume $a = a_1$. Since $T(g) = (2_{12})$ (See Lemma 11), the equation $g(x) = 0$ has 12 distinct multiple roots. Let $\gamma = -1.0207\ldots + (0.8732\ldots)i$ be one of such roots. The equation: $f(\gamma, y) = 0$ has four roots $\{\pm\delta, \pm\tilde{\delta}\}$, where

$$\begin{aligned} \delta &= 0.4070\ldots + (0.8911\ldots)i, \\ \tilde{\delta} &= 0.7003\ldots + (2.5518\ldots)i. \end{aligned}$$

Using Remark 2, we can list the X-coordinates of 48 points in $\mathrm{Orb}(P)$ and in $\mathrm{Orb}(\tilde{P})$, from which follows the assertion. We omit the details. □

We can numerically determine which of the points P and $\tilde{P}$ is a sextactic point.

Proposition 4. *Let $P, \tilde{P}$ have the same meaning as in the proof of Lemma 16. Then, P is a 2-sextactic point and $\tilde{P}$ is not a sextactic point.*

Proof. Using the method in Lemma 15, we can find the osculating conic D at P. The tangent line of C_{a_1} at P is given by $l(x, y) = y - \delta - m(x - \gamma) = 0$, where $m = -(0.10111\ldots + (0.21625\ldots)i$. The coefficients A, B, C have the following numerical solutions for $s_2 = s_3 = s_4 = 0$:

$$\begin{aligned} A &= -0.1604\ldots - (0.2374\ldots)i, \\ B &= -0.9787\ldots + (0.3595\ldots)i, \\ C &= -0.4780\ldots + (0.1791\ldots)i. \end{aligned}$$

In this case, we have

$$\begin{aligned} s_5 &= 1.243\cdots \times 10^{-14} - (5.329\ldots \times 10^{-15})i, \\ s_6 &= 8.751\ldots \times 10^{-9} - (1.681\ldots \times 10^{-8})i, \end{aligned}$$

$$s_7 = 5.175\ldots - (2.620\ldots)i \neq 0.$$

Therefore, we conclude that P is a sextactic point with contact order 7. Similarly, we can compute the osculating conic $\tilde{D}$ at $\tilde{P}$ and we find that $s_5 = 0.75\cdots - (0.54\ldots)i \neq 0$. So $\tilde{P}$ is not a sextactic point. □

Added in proof: It came to our attention that Egde (Edinburgh Math. Notes **35** (1945), 10–13) discussed the curves C_a and found the 12 hyperflexes (undulations) on C_3. He cited Ciani (Palermo Rendiconti, **13** (1899), 347–373) as a predecessor.

References

[1] Alwaleed, K.: *Geometry of* 2*-Weierstrass points on certain plane curves*, Thesis, Saitama Univ., In preparation.

[2] Babu, H. and Venkataraman, P.: *Group action on genus* 3 *curves and their Weierstrass points*, Computational aspects of algebraic curves, pp.264–272, Lect. Notes Ser. Comput. **13**, World Sci. Publ., 2005.

[3] Cox, D., Little, J. and O'Shea, D.: *Ideals, Varieties and Algorithms*, Springer-Verlag, New York, 1992.

[4] Del Centina, A.: *Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s*, Ann. Univ. Ferrara, **54** (2008), 37–59.

[5] Duma, A.: *Holomorphe Differentiale höherer Ordnung auf kompakten Riemannschen Flächen*, Schriftenreihe der Univ. Münster, 2. Serie, Heft **14**, 1978.

[6] Farkas, H.M. and Kra, I.: *Riemann Surfaces*, Springer-Verlag, 1992.

[7] Kawasaki, M.: *Weierstrass points and gap sequences on certain cyclic branched coverings of the projective line*, Thesis, Saitama Univ., 2007.

[8] Kuribayashi,A. and Sekita, E.: *On a family of Riemann surfaces I*, Bull. Fac. Sci. Eng. Chuo Univ. **22** (1979), 107–129.

[9] Miranda, R.: *Algebraic Curves and Riemann Surfaces*, A.M.S. 1995.

[10] Vermeulen, A.: *Weierstrass points of weight two on curves of genus three*, Thesis, Univ. of Amsterdam, 1983.

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