Geometry and computation of 2-Weierstrass points on Kuribayashi quartic curves

Kamel Alwaleed and Fumio Sakai

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Abstract
In this paper, we study the geometry of the 2-Weierstrass points on the Kuribayashi quartic curves:

\[ C_a : x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + x^2z^2) = 0 \ (a \neq 1, \pm 2). \]

The 2-Weierstrass points on \( C_a \) are divided into flexes and sextactic points. It is known that the symmetric group \( S_4 \) acts on \( C_a \). Using the \( S_4 \)-action, we classify the 2-Weierstrass points on \( C_a \).

1. Introduction
We consider a 1-parameter family of smooth quartic curves \( C_a \subset \mathbb{P}^2(\mathbb{C}) \) (non-hyperelliptic curves of genus 3):

\[ C_a : x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + x^2z^2) = 0 \ (a \neq 1, \pm 2). \]
We call these quartic curves \( C_a \) Kuribayashi quartic curves. It is known that the Weierstrass points on a smooth quartic curve are nothing but flexes. In [8], it was shown that \( C_a \) has 12 hyperflexes (resp. 24 ordinary flexes) if \( a = 0, 3 \) (resp. otherwise). The symmetric group \( S_4 \) acts on the Kuribayashi curves \( C_a \).

Definition 1. Define the projective transformation group \( G \) to be the group generated by the three elements

\[ \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \]

The group \( G \) is isomorphic to the symmetric group \( S_4 \). Indeed, \( G \) acts on the set of four points \( O_1 = (-1 : 1 : 1), O_2 = (1 : -1 : 1), O_3 = (-1 : -1 : 1), O_4 = (1 : 1 : 1) \), as the permutations \( \sigma \to (12), \tau \to (13), \rho \to (14) \). It turns out that \( G \) also acts on \( C_a \). Thus we can regard as \( G \subset \text{Aut}(C_a) \).

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It was proved by Kuribayashi-Sekita [8] that $G_a \cong C_{a'}$ if and only if $a' = a$ except if $a = (-3 \pm 3\sqrt{-7})/2$. The group $G$ acts on Weierstrass points and on 2-Weierstrass points on $C_a$. So we can discuss the structure of $G$-orbits of the 2-Weierstrass points on $C_a$ (See [2], for Weierstrass points).

**Definition 2.** A smooth, but not a flex point $P$ on a plane curve $C$ is called a **sextactic point** if there exists an irreducible conic $D$ which meets $C$ with contact order $m \geq 6$. Such a conic $D$ (unique, if exists) is called the **sextactic conic**.

Furthermore, for a sextactic point $P$, the positive integer $m - 5$ is called the **sextactic order**. We say that $P$ is $s$-sextactic if $s = m - 5 \geq 1$.

Geometrically, a 2-Weierstrass point on a smooth quartic curve is either a flex or a sextactic point (See Section 2). The purpose of this paper is to prove the following

**Theorem.** The $G$-orbits of the 2-Weierstrass points on Kuribayashi curves $C_a$ are classified as follows. We divide the set of 2-Weierstrass points on $C_a$ into the subset of flexes and the subset of sextactic points.

<table>
<thead>
<tr>
<th>$a$</th>
<th>ordinary flexes</th>
<th>hyperflexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 3</td>
<td>2 orb. of 12 pts</td>
<td>1 orb. of 24 pts</td>
</tr>
<tr>
<td>otherwise</td>
<td>1 orb. of 24 pts</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2  $G$-orbits of sextactic points**

<table>
<thead>
<tr>
<th>$a$</th>
<th>1-sextactic pts</th>
<th>2-sextactic pts</th>
<th>3-sextactic pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 3</td>
<td>2 orb. of 12 pts</td>
<td>1 orb. of 24 pts</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3 orb. of 12 pts</td>
<td>1 orb. of 24 pts</td>
<td>1 orb. of 8 pts</td>
</tr>
<tr>
<td>$P(a)=0$</td>
<td>2 orb. of 12 pts</td>
<td>1 orb. of 24 pts</td>
<td>1 orb. of 12 pts</td>
</tr>
<tr>
<td>$Q(a)=0$</td>
<td>3 orb. of 12 pts</td>
<td>1 orb. of 24 pts</td>
<td></td>
</tr>
<tr>
<td>otherwise</td>
<td>3 orb. of 12 pts 2 orb. of 24 pts</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, we set

\[
P(a) = a^3 + 68a^2 - 91a + 98, \]
\[
Q(a) = 33a^4 - 186a^3 + 205a^2 + 364a + 196.\]

We refer to Section 4, for the location and the detailed structure of the 2-Weierstrass points. We list the numbers of the 2-Weierstrass points.
Corollary. The numbers of 2-Weierstrass points on $C_a$ with respect to their types are given in the following table.

<table>
<thead>
<tr>
<th>$a$</th>
<th>ordinary flexes</th>
<th>hyper-flexes</th>
<th>1-sextac. pts</th>
<th>2-sextac. pts</th>
<th>3-sextac. pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0</td>
<td>12</td>
<td>48</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>24</td>
<td>0</td>
<td>60</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$P(a)=0$</td>
<td>24</td>
<td>0</td>
<td>48</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>$Q(a)=0$</td>
<td>24</td>
<td>0</td>
<td>36</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>otherwise</td>
<td>24</td>
<td>0</td>
<td>84</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In Section 2, we recall basic facts on 2-Weierstrass points, Wronskian forms of quadratic differentials on a smooth quartic curve and the multiplicities of zeros of polynomials. In Section 3, we discuss the $G$-action on the Kuribayashi curves $C_a$. In Section 4, we complete the proof of Theorem. In Section 5, we numerically compute some of the 2-Weierstrass points for the cases in which $P(a) = 0$ and $Q(a) = 0$. We use the computer softwares Mathematica and Maple to perform the computations. We refer to Alwaleed [1] for detailed computations and further discussions.

2. Preliminaries

2.1 2-Weierstrass points

Let $C$ be a smooth quartic curve. The 2-Weierstrass points on $C$ are defined by the orders of quadratic differential forms $\omega \in H^0(C, (\Omega^1)^2)$. Since $(\Omega^1)^2 \cong O_C(2)$, we have $\dim H^0(C, (\Omega^1)^2) = 6$. Take a point $P \in C$. Let $\{\omega_1, \ldots, \omega_6\}$ be a basis of $H^0(C, (\Omega^1)^2)$ so that $\text{ord}_P(\omega_1) < \ldots < \text{ord}_P(\omega_6)$. Letting $n_i = \text{ord}_P(\omega_i) + 1$, the sequence $\{n_1, \ldots, n_6\}$ is called the 2-gap sequence of $P$. The quantity

$$w^{(2)}(P) = \sum_{i=1}^{6} (n_i - i)$$

is called the 2-weight of $P$. We say that $P$ is a 2-Weierstrass point if $w^{(2)}(P) > 0$. Geometrically, $P$ is a 2-Weierstrass point if and only if there is a unique conic $D$ with $I_P(C, D) = n_6 - 1 \geq 6$. We infer that either $D = 2L$ ($P$ is a flex and $L$ is the tangent line at $P$) or $D$ is an irreducible conic ($P$ is not a flex). In the latter case, the point $P$ is a sextactic point. We denote by $W_2(C)$ (resp. $W_1(C)$) the set of 2-Weierstrass points (resp. Weierstrass points) on $C$.

Lemma 1. Let $C$ be a smooth quartic curve. Then we have

(i) $W_1(C) = \{\text{flexes}\}$, $W_2(C) = \{\text{flexes}\} \cup \{\text{sextactic points}\}$. 

(ii) The possible 2-gap sequences of \( P \in W_2(C) \) are listed in the following table.

<table>
<thead>
<tr>
<th>( w^{(2)}(P) )</th>
<th>2-gap sequence</th>
<th>geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1, 2, 3, 4, 5, 7}</td>
<td>ordinary flex</td>
</tr>
<tr>
<td>5</td>
<td>{1, 2, 3, 5, 6, 9}</td>
<td>hyperflex</td>
</tr>
<tr>
<td>1</td>
<td>{1, 2, 3, 4, 5, 7}</td>
<td>1-sextactic pt</td>
</tr>
<tr>
<td>2</td>
<td>{1, 2, 3, 4, 5, 8}</td>
<td>2-sextactic pt</td>
</tr>
<tr>
<td>3</td>
<td>{1, 2, 3, 4, 5, 9}</td>
<td>3-sextactic pt</td>
</tr>
</tbody>
</table>

(iii) We have \( \sum_{P \in C} w^{(2)}(P) = 108 \).

\[ \text{Proof.} \quad \text{For (iii), we refer to [6, 9].} \]

2.2 Wronskians

In order to compute 2-Weierstrass points, one can use the Wronskian form. For a basis \( \{\omega_1, \ldots, \omega_6\} \) of the space \( H^0(C, (\Omega^1)^2) \), one can define the Wronskian form \( \Omega = W(\omega_1, \ldots, \omega_6) \in H^0(C, (\Omega^1)^{27}) \). Then the order of zeros of \( \Omega \) at \( P \) gives us the 2-weight \( w^{(2)}(P) \). Cf. [6, 9]. Letting \( f(x, y) = 0 \) be the affine equation of \( C \), we can use the basis:

\[ \{ \frac{1}{f_y^2}(dx)^2, \frac{x}{f_y^2}(dx)^2, \frac{y}{f_y^2}(dx)^2, \frac{x^2}{f_y^2}(dx)^2, \frac{xy}{f_y^2}(dx)^2, \frac{y^2}{f_y^2}(dx)^2 \} . \]

By computation, we have

**Lemma 2.**

(i) \( \Omega = W(1, x, y, x^2, xy, y^2)(dx)^{27}/f_y^{12} \),

(ii) \( W(1, x, y, x^2, xy, y^2) = 4y''\left[45y^{(4)}y^{(3)}y'' - 9y^{(5)}y''^2 - 40(y^{(3)})^3\right] \).

**Remark 1.** One can compute the term \( y^{(k)} \) by the implicit differentiation. For instance, as is well known, we have

\[ y'' = \frac{f_x f_y^2 - 2f_{xy}f_y f_x + f_y^2 f_x^2}{f_y^3} . \]

2.3 Multiplicities of zeros

**Definition 3.** Let \( p(x) \) be a polynomial. Write \( p(x) = c \prod_{i=1}^{k} (x - \alpha_i)^{m_i} \) with \( \alpha_i \neq \alpha_j \) for \( i \neq j \). We may arrange as \( m_1 \geq m_2 \geq \ldots \geq m_k \). We set

(i) \( T(p) = (m_1, \ldots, m_k) \),

(ii) \( r(p) = \max\{m_i\} \),

(iii) \( s(p) = \sum_{i=1}^{k} (m_i - 1) \).

We call \( T(p) \) the type of \( p \). We use abbreviations such as \( 1_n = 1, \ldots, 1 \) and \( 2_n = 2, \ldots, 2 \).
As for the invariants \( r(p), s(p) \), we use the following well known facts.

**Lemma 3.** Let \( p(x) \) be a polynomial.

(i) If \( V(p, p', \ldots, p^{(r-1)}) \neq \emptyset \) and \( V(p, p', \ldots, p^{(r)}) = \emptyset \), then \( r = r(p) \).

(ii) Let \( R^{(i)} \) be the \( i \)-th subresultant of \( p(x) \) and \( p'(x) \). If \( R^{(1)} = \ldots = R^{(s)} = 0, R^{(s+1)} \neq 0 \). Then we have \( s = s(p) \).

We regard a polynomial \( p(x, a) \in C[x, a] \) as a 1-parameter family of polynomials depending on the value \( a \). We sometimes write as \( p_a(x) = p(x, a) \). Consider the ideal \( I_k = (p, p', \ldots, p^{(k)}) \), where the \( p^{(i)} \) denotes the \( i \)-th differentiation by the variable \( x \). By using the Groebner basis methods (Cf. [3], Chap. 3), we can compute the ideal \( J_k = I_k \cap C[a] \) in \( C[a] \). If \( a_0 \in V(J_{r-1}) \), \( a_0 \notin V(J_r) \), then we infer that \( r(p_{a_0}) = r \). Also if \( R^{(1)}(a_0) = \ldots = R^{(s)}(a_0) = 0, R^{(s+1)}(a_0) \neq 0 \), then we conclude that \( s(p_{a_0}) = s \).

### 3. G-action on 2-Weierstrass points

We now study the \( G \)-action on the Kuribasyashi curves \( C_a \). We use the affine equation of \( C_a \):

\[
f(x, y) = x^4 + y^4 + 1 + a(x^2y^2 + x^2 + y^2) = 0.
\]

As we have seen in Section 1, a projective transformation group \( G \cong S_4 \) acts on \( C_a \). For a point \( P \) on \( C_a \), let \( G_P \) denote the stabilizer of \( P \) in \( G \). It is well known that \( G_P \) is a cyclic group. Let \( \text{Orb}(P) \) be the orbit of the point \( P \). We denote by \( X(C_a) \) the set of the points \( P \in C_a \) such that \( |G_P| > 1 \). We also write \( X_i(C_a) = \{ P \in C_a \mid |G_P| = i \} \).

**Lemma 4.** We have \( X(C_a) = X_2(C_a) \cup X_3(C_a) \).

**Proof.** Since \( G \cong S_4 \), the orders of elements in \( G \) are 1,2,3 and 4. It suffices to see that \( X_4(C_a) = \emptyset \). We show that the group \( G_P \) does not contain elements of order four for any point \( P \in C_a \). The element

\[
\varphi = \rho \tau \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]

has order four. We denote by \( \text{Fix}(\varphi, P^2) \) the set of fixed points of \( \varphi \) in \( P^2 \). We see that \( \text{Fix}(\varphi, P^2) \) consists of three points \( \{(0 : 1 : 0), (1 : 0 : \pm i)\} \), which are disjoint from \( C_a \) for \( a \neq 2 \). Since \( G \cong S_4 \), all elements of order 4 are conjugate with each other in \( G \), we conclude that \( G_P \) for \( P \in C_a \) cannot contain elements of order four. \( \square \)
In the symmetric group $S_4$, there exist 9 elements of order two, which are divided into two conjugacy classes:

\[ \{(12), (13), (14), (23), (24), (34)\}, \quad \{(12)(34), (13)(24), (14)(23)\}. \]

Using the isomorphism $G \cong S_4$ given in Section 1, we have the correspondences: $\sigma \leftrightarrow (12)$ and $\psi = (\sigma \tau \rho)^2 \leftrightarrow (13)(24)$.

For $\sigma$, we have $\text{Fix}(\sigma, P^2) = \{(-1 : 1 : 0)\} \cup \{\text{the line } x = y\}$. Thus $\text{Fix}(\sigma, P^2) \cap C_a$ consists of four points $\{(\alpha : \alpha : 1)\}$, where the $\alpha$ are the four distinct roots of the equation: $(2 + a)x^4 + 2ax^2 + 1 = 0$. Note that $\alpha \neq \pm 1$, since $a \neq -2$. We have

\[ \text{Disc}((2 + a)x^4 + 2ax^2 + 1; x) = 256(a + 1)^2(a - 2)^2(a + 2) \neq 0. \]

We have

\[ \text{Orb}((\alpha : \alpha : 1)) = \begin{cases} (\alpha : \alpha : 1), & (\alpha : -\alpha : 1), \quad (-\alpha : -\alpha : 1), \quad (\alpha : -\alpha : 1) \\ (1 : 1/\alpha : 1), & (1 : -1/\alpha : 1), \quad (-1 : 1/\alpha : 1), \quad (-1 : -1/\alpha : 1) \end{cases}. \]

We see that $G_{(\alpha : \alpha : 1)} = \{1, \sigma\}$. For the element

\[ \psi = (\sigma \tau \rho)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

we have $\text{Fix}(\psi, P^2) = \{(0 : 1 : 0)\} \cup \{\text{the line } y = 0\}$. So it follows that $\text{Fix}(\psi, P^2) \cap C_a = \{(\beta : 0 : 1)\}$, where the $\beta$ are the four distinct roots of the equation: $x^4 + ax^2 + 1 = 0$. Note that $\beta \neq 0, \pm 1$, since $a \neq -2$. We have

\[ \text{Orb}((\beta : 0 : 1)) = \begin{cases} (\beta : 0 : 1), & (1/\beta : 0 : 1), \quad (-\beta : 0 : 1), \quad (-1/\beta : 0 : 1) \\ (0 : \beta : 1), & (0 : 1/\beta : 1), \quad (0 : -\beta : 1), \quad (0 : -1/\beta : 1) \\ (\beta : 1 : 0), & (1 : \beta : 0), \quad (\beta : -1 : 0), \quad (-1 : \beta : 0) \end{cases}. \]

We see that $G_{(\beta : 0 : 1)} = \{1, \psi\}$.

Summarizing, we obtain

**Proposition 1.** We have

(i) $X_2(C_a) = \text{Orb}((\alpha_1 : \alpha_1 : 1)) \cup \text{Orb}((\alpha_2 : \alpha_2 : 1)) \cup \text{Orb}((\beta : 0 : 1))$,

where the $\alpha_1$ and the $\alpha_2$ are two distinct roots ($\alpha_1 \neq -\alpha_2$) of the equation: $u(x) = (2 + a)x^4 + 2ax^2 + 1 = 0$. The $\beta$ is a root of the equation:
\( e(x) = x^4 + ax^2 + 1 = 0. \)

(ii) \( X_2(C_a) \subseteq W_2(C_a) \),

(iii) \( X_2(C_a) \cap W_1(C_a) = \begin{cases} \text{Orb}(\beta : 0 : 1) & \text{if } a = 0, \\ \text{Orb}(i : i : 1) & \text{if } a = 3, \\ \emptyset & \text{otherwise,} \end{cases} \)

where \( \beta = (1 + i)/\sqrt{2} \).

**Proof.** (ii) We can apply Duma’s criterion in [5], Satz 6.3. (iii) Let \( H(x, y, z) \) be the Hessian of the curve \( C_a \). By computation, we obtain

\[
\text{Res}(H(x, y, 1), f(x, y); y) = \text{Const.}(a-2)^6(a+2)^4h(x)^2,
\]

where

\[
h(x) = a^2(x^{12}+1)+6a(x^{10}+x^2) - 3(a^3-a^2-3a-6)(x^8+x^4) - 2a(3a^2-2a-15)x^6.
\]

We have

\[
h(1) = -12(a-3)(a+1)^2, \quad h(0) = a^2.
\]

Thus, if \( a \neq 0,3 \), then we have \( h(0)h(1) \neq 0 \). If \( X_2(C_a) \cap W_1(C_a) \neq \emptyset \), then by (i), we must have \((1 : 1/\alpha : 1) \in W_1(C_a) \) or \((0 : \beta : 1) \in W_1(C_a) \), which is not the case.

**Proposition 2.** We have

\[
X_3(C_a) = \text{Orb}((\omega : \omega^2 : 1))
\]

\[
= \left\{ \begin{array}{l}
(\omega : \omega^2 : 1), \quad (-\omega : -\omega^2 : 1), \quad (-\omega : \omega^2 : 1), \quad (\omega : -\omega^2 : 1) \\
(\omega^2 : \omega : 1), \quad (-\omega^2 : -\omega : 1), \quad (\omega^2 : -\omega : 1), \quad (-\omega^2 : \omega : 1)
\end{array} \right\}.
\]

**Proof.** There are 8 elements of order three in \( G \cong S_4 \), which are conjugate with each other. For instance, the element

\[
\sigma \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]

has order three. We have \( \text{Fix}(\sigma \tau, \mathbb{P}^2) = \{(1 : 1 : 1), (\omega : \omega^2 : 1), (\omega^2 : \omega : 1)\} \), where the \( \omega \) is the third root of unity. Thus, we obtain

\[
\text{Fix}(\sigma \tau, \mathbb{P}^2) \cap C_a = \{(\omega : \omega^2 : 1), (\omega^2 : \omega : 1)\}.
\]

\( \square \)
Remark 2. If \((\gamma : \delta : 1) \notin X(C_a)\), then \(\text{Orb}(\gamma : \delta : 1)\) consists of the following 24 points.

\[
\{ \begin{array}{llll}
(\gamma : \delta : 1), & (\gamma : -\delta : 1), & (-\gamma : -\delta : 1), & (-\gamma : \delta : 1) \\
(\delta : \gamma : 1), & (-\delta : -\gamma : 1), & (\delta : -\gamma : 1), & (-\delta : \gamma : 1) \\
(1/\gamma : \delta/\gamma : 1), & (1/\gamma : -\delta/\gamma : 1), & (-1/\gamma : -\delta/\gamma : 1), & (-1/\gamma : \delta/\gamma : 1) \\
(\delta/\gamma : 1/\gamma : 1), & (-\delta/\gamma : -1/\gamma : 1), & (\delta/\gamma : -1/\gamma : 1), & (-\delta/\gamma : 1/\gamma : 1) \\
(1/\delta : \gamma/\delta : 1), & (1/\delta : -\gamma/\delta : 1), & (-1/\delta : -\gamma/\delta : 1), & (-1/\delta : \gamma/\delta : 1) \\
(\gamma/\delta : 1/\delta : 1), & (\gamma/\delta : -1/\delta : 1), & (-\gamma/\delta : -1/\delta : 1), & (-\gamma/\delta : 1/\delta : 1) \\
\end{array} \}
\]

4. Proof of Theorem

Using Lemma 2 and implicit differentiation, we can write the Wronskian form \(\Omega\) of quadratic differentials on \(C_a\) as:

\[
\Omega = \Phi(x, y)(dx/f_y)^{27}.
\]

We have

\[
\Phi(x, y) = \text{Const} \cdot (a + 1)(a - 2)^2(a^2 - 4)xy(x^2 - y^2)\Phi_1(x, y)\Phi_2(x, y),
\]

where \(\deg(\Phi_1(x, y)) = 8\) and \(\deg(\Phi_2(x, y)) = 18\).

We first determine the 2-Weierstrass points on \(C_0\).

Lemma 5. For the case in which \(a = 0\), we have

\[
\begin{array}{ll}
W_1(C_0) = \text{Orb}(\beta : 0 : 1), \\
W_2(C_0) \setminus W_1(C_0) = \text{Orb}(\alpha : \alpha : 1) \cup \text{Orb}((\overline{\alpha} : \overline{\alpha} : 1)) \cup \text{Orb}((\alpha : \overline{\alpha} : 1)),
\end{array}
\]

where \(\beta = (1 + i)\sqrt{2}\), \(\alpha = \sqrt{2}(1 + i)/2\). Note that \(\text{Orb}(\alpha : \overline{\alpha} : 1)\) consists of 24 points.

Proof. By Proposition 1, we obtain

\[
X_2(C_0) = \text{Orb}(\beta : 0 : 1) \cup \text{Orb}(\alpha : \alpha : 1) \cup \text{Orb}((\overline{\alpha} : \overline{\alpha} : 1)) \subset W_2(C_0).
\]

We have \(\Phi(x, y) = \text{Const} \cdot (xy)^5(x^4 - y^4)(2x^4 + y^4)(x^4 + 2y^4)\). It follows that \(\text{Orb}(\alpha : \overline{\alpha} : 1) \subset W_2(C_0)\).

In what follows, we assume that \(a \neq 0\). We compute the resultant of \(\Phi\) (or \(\Phi_i\)) and \(f\) with respect to \(y\). Set

\[
\phi(x) = \text{Res}(\Phi, f; y), \quad \phi_i(x) = \text{Res}(\Phi_i, f; y).
\]

It turns out that \(\phi_1(x)\) coincides with the polynomial \(h(x)\) up to constant, which appeared in Lemma 1 as the resultant of the Hessian with respect to \(y\). This is
a consequence of Lemma 2, (ii) and Remark 1.

**Lemma 6.** We obtain

(i) \( \phi_2(x) = (a + 2)^8(a - 2)^{14}(x^2 - 1)^4v(x)^2g(x)^2, \)

(ii) \( \phi(x) = \text{Const.}(a + 2)^{16}(a - 2)^{32}(a + 1)^4 \)
\( \times x^4(x^2 - 1)^4u(x)^2v(x)^2h(x)^2g(x)^2. \)

where \( u(x) = (2 + a)x^4 + 2ax^2 + 1, \) \( v(x) = x^4 + 2ax^2 + a + 2 \) and

\[
g(x) = 9a^2(a - 2)(a + 2)^2(x^{24} + 1) + 6a(a - 1)(a - 2)(a + 2)(15a + 14)(x^{22} + x^2)
+ (36a^6 + 294a^5 - 720a^4 - 768a^3 + 956a^2 - 112a - 784)(x^{20} + x^4)
+ (18a^6 + 276a^5 - 1920a^4 - 662a^3 + 1216a^2 - 1848a - 1568)(x^{18} + x^6)
+ (3a^7 + 450a^6 - 255a^5 - 2990a^4 - 277a^3 + 34a^2 - 4172a - 3528)(x^{16} + x^8)
+ (12a^7 + 642a^6 - 790a^5 - 3346a^4 - 2098a^3 + 980a^2 - 7224a - 5488)(x^{14} + x^{10})
+ (18a^7 + 684a^6 - 688a^5 - 5208a^4 + 1130a^3 - 2004a^2 - 8904a - 5488)x^{12}.
\]

**Remark 3.** Note that \( g(-x) = g(x) \) and \( x^{24}g(1/x) = g(x) \). Note also that \( v(x) = x^4u(1/x) \). Thus, if we write \( u(x) = (a + 2)(x^2 - \alpha_1^2)(x^2 - \alpha_2^2) \) as in Proposition 1, then we have \( v(x) = (x^2 - (1/\alpha_1)^2)(x^2 - (1/\alpha_2)^2) \).

**Lemma 7.** We have

\[
\text{Disc}(g; x) = \text{Const.}(a - 2)^{16}(a + 2)^{18}(a + 1)^{20}
\times a^{16}(a - 3)^{18}(a - 14)^8P(a)^6Q(a)^{12}\eta(a)^8,
\]

where

\[
P(a) = a^3 + 68a^2 - 91a + 98,
Q(a) = 33a^4 - 186a^3 + 205a^2 + 364a + 196,
\eta(a) = (3a^3 - 6a^2 + 9a - 14)(9a^{12} - 162a^{11} + 1683a^{10} + \cdots).
\]

**Proof.** This follows from a direct computation.

**Lemma 8.** For the case in which \( a = 3 \), we have

\[
\begin{align*}
W_1(C_3) &= \text{Orb}(i : i : 1), \\
W_2(C_3) \setminus W_1(C_3) &= \text{Orb}((i : 0 : 1)) \cup \text{Orb}((\alpha : \alpha : 1)) \cup \text{Orb}((\gamma : \delta : 1)),
\end{align*}
\]

where \( \beta = (1 + \sqrt{5})i/2, \alpha = i/\sqrt{5} \) and \( g(\gamma) = 0 \). Note that \( \text{Orb}((\gamma : \delta : 1)) \)
consists of 24 points.

**Proof.** Letting \( a = 3 \), we have \( g(x) = (x^2 + 1)^4(x^2 - 1)^2g_{12}(x) \), where \( g_{12}(x) = \)
2025 \cdot x^{12} + 6570 \cdot x^{10} + \ldots + 2025. We have Disc\((g_{12}; x) \neq 0. Now, it follows from Proposition 1 that the points

\[ X_2(C_3) = \text{Orb}(\{(i : i : 1)\}) \cup \text{Orb}(\{(\beta : 0 : 1)\}) \cup \text{Orb}(\{(i/\sqrt{5} : i\sqrt{5} : 1)\}) \]

are contained in \(W_2(C_3)\). Take a root \(\gamma\) of \(g_{12}(x)\), Then, there exists a 2-Weierstrass point \((\gamma : \delta : 1) \in C_3\). Since \(g_{12}\) has no multiple roots, we have \(w^{(2)}((\gamma : \delta : 1)) = 1. \)

**Lemma 9.** If \(a \neq 0, 3\), then there exists an ordinary flex \((\gamma_0 : \delta_0 : 1) \not\in X(C_a)\) so that \(W_1(C_a) = \text{Orb}((\gamma_0 : \delta_0 : 1))\).

**Proof.** See Proposition 1, (iii).

**Lemma 10.** For the case in which \(a = 14\), we have

\[ W_2(C_{14}) \setminus W_1(C_{14}) = \text{Orb}(\{(\omega : \omega^2 : 1)\}) \cup X_2(C_{14}) \cup \text{Orb}(\{(\gamma : \delta : 1)\}), \]

where \(g(\gamma) = 0\) and

\[ X_2(C_{14}) = \text{Orb}(\{(\beta : 0 : 1)\}) \cup \text{Orb}(\{(\alpha_1 : \alpha_1 : 1)\}) \cup \text{Orb}(\{(\alpha_2 : \alpha_2 : 1)\}), \]

where \(\beta = 2 + \sqrt{3}i\), \(\alpha_1 = (\sqrt{5} + 3)i/4\), \(\alpha_2 = (\sqrt{5} - 3)i/4\). Furthermore, we have \(w^{(2)}((\omega : \omega^2 : 1)) = 3\) and \(X_3(C_{14}) = \text{Orb}(\{(\omega : \omega^2 : 1)\}). \) Note that \(\text{Orb}(\{(\gamma : \delta : 1)\})\) consists of 24 points.

**Proof.** Letting \(a = 14\), we have \(g(x) = 200704 \cdot (x^2 + x + 1)^3(x^2 - x + 1)^3g_{12}(x)\), where \(g_{12}(x) = 27 \cdot x^{12} + \ldots + 27. We have Disc\((g_{12}; x) \neq 0. Now, it follows from Proposition 1 that \(X_2(C_{14}) \subset W_2(C_{14})\). By a direct computation, we see that \((\omega : \omega^2 : 1) \in C_{14}\) and \(w^{(2)}((\omega : \omega^2 : 1)) = 3\) (See Remark 4 below). Finally, take a root \(\gamma\) of \(g_{12}(x)\), Then there exists a 2-Weierstrass point \((\gamma : \delta : 1) \in C_3\). Since \(g_{12}\) has no multiple roots, we have \(w^{(2)}((\gamma : \delta : 1)) = 1. \)

**Lemma 11.** Suppose \(a \neq 0, 3, 14\). The type of the multiplicities of the roots of the polynomial \(g(x)\) is given in the following table:

<table>
<thead>
<tr>
<th>(a)</th>
<th>(T(g))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(a) = 0)</td>
<td>(2_6, 1_{12})</td>
</tr>
<tr>
<td>(Q(a) = 0)</td>
<td>(2_{12})</td>
</tr>
<tr>
<td>(\eta(a) = 0)</td>
<td>(2_{4}, 1_{16})</td>
</tr>
<tr>
<td>otherwise</td>
<td>(1_{24})</td>
</tr>
</tbody>
</table>

**Proof.** By using the Groebner basis method, we obtain

\[(g, g', g'') \cap \mathbb{C}[a] = ((a - 2)(a + 2)^2(a + 1)^2a^2(a - 3)^2(a - 14)).\]
Thus, we infer that $r(g) = 2$ (resp. $r(g) = 1$) if $P(a)Q(a)\eta(a) = 0$ (resp. otherwise) (See Definition 3). So if $P(a)Q(a)\eta(a) \neq 0$, then $g$ has no multiple roots. Let $R^{(k)}$ denote the $k$-th subresultant of $g$ and $g'$ with respect to $x$.

**Case** $P(a) = 0$. By computation, we have $P(a) | R^{(k)}$ for $k = 1, \ldots, 6$ but $\text{Res}(P(a), R^{(7)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 6)$. We can easily conclude that $T(g) = (2, 1, 12)$.

**Case** $Q(a) = 0$. In this case, we have $Q(a) | R^{(k)}$ for $k = 1, \ldots, 12$ but $\text{Res}(Q(a), R^{(13)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 1, 12)$. We conclude that $T(g) = (2, 1, 12)$.

**Case** $\eta(a) = 0$. In this case, we have $\eta(a) | R^{(k)}$ for $k = 1, \ldots, 4$, but we obtain $\text{Res}(\eta(a), R^{(5)}; a) \neq 0$. It follows that $(r(g), s(g)) = (2, 4)$. We conclude that $T(g) = (2, 1, 16)$.

Now we pass to coordinates change. We use the affine coordinates: $(X, Y) = (x + 2y, y)$. Write $\overline{f}(X, Y) = f(X - 2Y, Y)$ and

$$\Omega = \overline{\Phi}(X, Y)(dX/\overline{f}(X, Y))^{27}$$

so that $\overline{\Phi}(X, Y) = \Phi(X - 2Y, Y)$. Set also $\overline{\Phi}_2(X, Y) = \Phi_2(X - 2Y, Y)$. Letting $\overline{\phi}(X) = \text{Res}(\overline{\Phi}, \overline{f}; Y), \overline{\phi}_2(X) = \text{Res}(\overline{\Phi}_2, \overline{f}; Y)$, by computation, we have

**Lemma 12.**

(i) $T(\overline{\phi}_2) = (2, 1, 16)$, if $Q(a) = 0$,

(ii) $\text{Res}(\eta, \text{Disc}(\overline{\phi}; X); a) \neq 0$.

**Lemma 13.** Suppose $a \neq 3$. Let $\alpha_1, \alpha_2$ be as in Proposition 1, (i). Let $m_i$ denote the 2-weight of the 2-Weierstrass point $(\alpha_i : \alpha_i : 1) \in X_2(C_a)$. We may assume $m_1 \leq m_2$. Then we have

$$m_1, m_2) = \begin{cases} (1, 3) & \text{if } P(a) = 0, \\ (1, 1) & \text{otherwise.} \end{cases}$$

**Proof.** We recall that the point $(1 : 1/\alpha_i : 1)$ belongs to the orbit of the point $(\alpha_i : \alpha_i : 1)$. By computation, we obtain

$$g(1) = \text{Const.}(a - 3)(a + 2)(a + 1)^2P(a),$$

$$g'(1) = \text{Const.}(a - 3)(a + 2)(a + 1)^2P(a),$$

$$\text{Res}(P(a), g''(1); a) \neq 0.$$
of \((x - \alpha_2)\) in \(g(x)\) must be equal to 2. Hence we have \((m_1, m_2) = (1, 3)\). In case \(P(a) \neq 0\), the multiplicity of \((x - 1)\) in \(\phi(x)\) is equal to 4. It follows that \(m_1 + m_2 = 2\) and hence we have \((m_1, m_2) = (1, 1)\). 

**Lemma 14.** If \(Q(a) = 0\), then there exists a 2-Weierstrass point \(P \notin X(C_a)\) with \(w^{(2)}(P) = 2\).

**Proof.** Let \(\gamma\) be a root of \(g(x)\). Since \(T(g) = (2_{12})\), there exists a 2-Weierstrass point \(P = (\gamma : \delta : 1) \in C_a\). We have two possible cases:

(i) \(w^{(2)}(P) = 2\).

(ii) \(w^{(2)}(P) = 1\). In this case, we can find another 2-Weierstrass point \(\tilde{P} = (\gamma : \delta : 1) \in C_a\) with \(\delta \neq \pm \delta\).

The case (ii) does not occur. In fact, we use the affine coordinates \((X,Y)\) in Lemma 12. We see that the \(X\)-coordinate of all points in \(\text{Orb}(P)\) and \(\text{Orb}(\tilde{P})\) are different. For a proof of this fact, see Lemma 16 in Section 5. It follows that the number of the different roots of \(\phi_2\) is greater than or equal to 48. But, we infer from Lemma 12 that if \(Q(a) = 0\), then \(T(\phi_2) = (2_{24}, 1_{16})\), which is a contradiction.

**Proof of Theorem.** For the cases in which \(a = 0, 3, 14\), we refer to Lemmata 5, 8, 10.

**Case (1).** \(P(a) = 0\). By Lemma 13, we can find two 2-Weierstrass points \((\alpha_1 : \alpha_1 : 1), (\alpha_2 : \alpha_2 : 1) \in X_2(C_a)\) such that \(w^{(2)}((\alpha_1 : \alpha_1 : 1)) = 1\) and \(w^{(2)}((\alpha_2 : \alpha_2 : 1)) = 3\). Since \(T(g) = (2_6, 1_{12})\), we can also find a 2-Weierstrass point \((\gamma : \delta : 1) \notin X(C_a)\) with \(w^{(2)}((\gamma : \delta : 1)) = 1\). We have

\[
W_2(C_a) \setminus W_1(C_a) = \text{Orb}((\beta : 0 : 1)) \cup \text{Orb}((\alpha_1 : \alpha_1 : 1)) \cup \text{Orb}((\alpha_2 : \alpha_2 : 1)) \cup \text{Orb}((\gamma : \delta : 1)).
\]

**Case (2).** \(Q(a) = 0\). By Lemma 14, there is a 2-Weierstrass point \((\gamma : \delta : 1) \notin X_2(C_a)\) with \(w^{(2)}((\gamma : \delta : 1)) = 2\). In this case, we have

\[
W_2(C_a) \setminus W_1(C_a) = X_2(C_a) \cup \text{Orb}((\gamma : \delta : 1)).
\]

**Case (3).** \(P(a)Q(a) \neq 0\). We can find two 2-Weierstrass points \((\gamma_1 : \delta_1 : 1)\) and \((\gamma_2 : \delta_2 : 1)\) with \(g(\gamma_i) = 0\) and \(w^{(2)}((\gamma_i : \delta_i : 1)) = 1\) for \(i = 1, 2\) such that

\[
W_2(C_a) \setminus W_1(C_a) = X_2(C_a) \cup \text{Orb}((\gamma_1 : \delta_1 : 1)) \cup \text{Orb}((\gamma_2 : \delta_2 : 1)).
\]

The assertion follows from Lemma 11 (resp. Lemma 12, (ii)) for the case in which \(\eta(a) \neq 0\) (resp. \(\eta(a) = 0\)). We here sketch a proof for the case in which \(\eta(a) \neq 0\).

Set \(S = \{\gamma \mid g(\gamma) = 0, \text{or } \text{or } v(\gamma) \neq 0\}\). By Lemma 11, we have \(#(S) \geq 16\).
For any $\gamma_1 \in S$, there is a 2-Weierstrass point $(\gamma_1 : \delta_1 : 1) \not\in X_2(C_a)$. Choose $\gamma_2 \in S$ so that $\gamma_2 \not\in \{ \pm \gamma_1, \pm \delta_1, \pm 1/\gamma_1, \pm 1/\delta_1, \pm \gamma_1/\delta_1, \pm \delta_1/\gamma_1 \}$. Then there is a 2-Weierstrass point $(\gamma_2 : \delta_2 : 1)$. Since $g(x)$ has no multiple roots, we infer that $w^{(2)}((\gamma_i : \delta_i : 1)) = 1$ for $i = 1, 2$. □

Remark 4. A 3-sextactic point $P$ on a smooth quartic curve $C$ is a total sextactic point, i.e., the sextactic conic $D$ at $P$ meets $C$ only at $P$. Among Kuribayashi quartic curves $C_a$, there exist total sextactic points if $a = 14$ or if $P(a) = 0$. In case $a = 14$, the 2-Weierstrass points in $\text{Orb}(\omega : \omega^2 : 1)$ are all total sextactic points. We remark that they lie on bitangent lines. Namely, the total sextactic points $P_1 = (\omega : \omega^2 : 1)$ and $P_2 = (\omega^2 : \omega : 1)$ lie on a bitangent line $L : x + y + z = 0$. The sextactic conics at these points are the following:

$$D_1 : \Delta_2(x, y, z) = (x^2 + 5yz) + \omega(y^2 + 5xz) + \omega^2(z^2 + 5xy) = 0,$$
$$D_2 : \Delta_1(x, y, z) = (x^2 + 5yz) + \omega(y^2 + 5xz) + \omega^2(z^2 + 5xy) = 0.$$

Note that we can write the defining equation of $C_{14}$ as

$$\frac{5}{9}(x + y + z)^4 + \frac{4}{9}\Delta_1(x, y, z)\Delta_2(x, y, z) = 0.$$

5. Computational aspects

We now discuss the computational aspects. By a numerical method, we determine the coordinates of 2-Weierstrass points on the Kuribayashi curves $C_a$ for the cases in which $P(a) = 0$ and $Q(a) = 0$. We use the following tool.

Lemma 15. Let $C : f(x, y) = 0$ a smooth quartic curve. Take a non-flex point $P = (\alpha, \beta) \in C$. We can compute the osculating conic $D$ at $P$ (i.e., the irreducible conic having the contact order $\geq 5$ to $C$ at $P$) in the following way.

(1) Compute the defining equation $l(x, y) = y - \beta - m(x - \alpha) = 0$ of the tangent line $L$ of $C$ at $P$.

(2) Parametrize those irreducible conics passing through the point $P$ with the tangent line $L$:

$$l(x, y) + A(x - \alpha)^2 + B(x - \alpha)l(x, y) + Cl(x, y)^2 = 0 \quad (A \neq 0)$$

as

$$\begin{cases} x(t) = \alpha - t(A + Bt + Ct^2), \\ y(t) = \beta - t(t + m)/(A + Bt + Ct^2). \end{cases}$$

(3) Write
\[ f(x(t), y(t)) = \frac{s_2t^2 + s_3t^3 + s_4t^4 + s_5t^5 + s_6t^6 + s_7t^7 + s_8t^8}{(A + Bt + Ct^2)^4}, \]

where \( s_i \in \mathbb{C}[A, B, C] \) for \( i = 2, \ldots, 8 \).

(4) Determine \( A, B, C \) by solving the equations: \( s_2 = s_3 = s_4 = 0 \).

**Proof.** For the assertion (2), it suffices to parametrize the intersection points of the conic with the pencil of lines \( l(x, y) - t(x - \alpha) = 0 \).

5.1 **The case** \( P(a) = 0 \)

We consider the case in which \( P(a) = 0 \). The cubic equation \( P(a) = 0 \) has three roots:

\[ a_1 = -69.3328950\ldots, \quad a_2 = 0.6664475\ldots - (0.9845395\ldots)i, \quad a_3 = \overline{a_2}. \]

We here consider the real root \( a_1 \). The equation: \( u(x) = 0 \) has 4 distinct roots \( \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2\} \), where

\[ \alpha_1 = 0.0847732623\ldots, \quad \alpha_2 = (1.43756489\ldots)i. \]

**Proposition 3.** Suppose \( a = a_1 \). The point \( P_1 = (\alpha_1 : \alpha_1 : 1) \) is a 1-sextactic point and the point \( P_2 = (\alpha_2 : \alpha_2 : 1) \) is a 3-sextactic point.

**Proof.** **Case (i).** We first check the point \( P_1 \). Using the method in Lemma 15, we have the following approximate solutions for the equations \( s_2 = s_3 = s_4 = 0 \):

\[ A = 11.62294385\ldots, \quad B = -11.62294386\ldots, \quad C = 6.03059824\ldots. \]

In fact, we have

\[ s_2 = A^2\{-137.5830294 + 11.83719298A - (8.881\ldots \times 10^{-16})B^2 + (8.881\ldots \times 10^{-16})AC\}. \]

So if we put \( A = -137.5830294/11.83719298 = 11.6229438\ldots \), then \( s_2 \) is very close to zero. Solving the equations \( s_3 = 0, s_4 = 0 \), we can find the approximate solutions \( B, C \). In this case, we have \( s_5 = -8.367\ldots \times 10^{-11} \), but \( s_6 = 146.955\ldots \neq 0 \). Thus we infer that \( P_1 \) is a 1-sextactic point.

**Case (ii).** We now check the point \( P_2 \). In a similar manner as in Case (i), we have the following approximate solutions for the equations \( s_2 = s_3 = s_4 = 0 \):

\[ A = (2.2419441\ldots)i, \quad B = -(2.2419441\ldots)i, \quad C = (0.1079171\ldots)i. \]

In this case, we have \( s_5 = -3.637\ldots \times 10^{-12}, s_6 = 3.069\ldots \times 10^{-12} \) and \( s_7 = -6.430\ldots \times 10^{-13} \). We therefore conclude that \( P_2 \) is a 3-sextactic point. \( \square \)
5.2 The case $Q(a) = 0$

We now pass to the case in which $Q(a) = 0$. The quartic equation $Q(a) = 0$ has four roots:

$$a_1 = 3.359188060 \ldots + (1.319606687 \ldots) i, \quad \overline{a_1},$$

$$a_2 = -0.5410062419 \ldots + (0.4040965957 \ldots) i, \quad \overline{a_2}$$

We first prove the following fact which was used in Lemma 14.

**Lemma 16.** Let $a$ be a root of the equation $Q(a) = 0$. Let $\gamma$ be a root of the equation $g(x) = 0$. Let $\delta, \tilde{\delta}$ be two roots of the equation: $f(\gamma, y) = 0$ with $\tilde{\delta} \neq -\delta$. Put $P = (\gamma : \delta : 1)$ and $\tilde{P} = (\gamma : \tilde{\delta} : 1)$. Let $(X, Y)$ be the affine coordinates used in Lemma 12. Then the $X$-coordinates of the points in Orb($P$) and in Orb($\tilde{P}$) are all different.

**Proof.** We here consider the root $a_1$. We can similarly deal with the other cases. Assume $a = a_1$. Since $T(g) = (2, 12)$ (See Lemma 11), the equation $g(x) = 0$ has 12 distinct multiple roots. Let $\gamma = -1.0207 \ldots + (0.8732 \ldots) i$ be one of such roots. The equation: $f(\gamma, y) = 0$ has four roots $\{\pm \delta, \pm \tilde{\delta}\}$, where

$$\delta = 0.4070 \ldots + (0.8911 \ldots) i,$$

$$\tilde{\delta} = 0.7003 \ldots + (2.5518 \ldots) i.$$

Using Remark 2, we can list the $X$-coordinates of 48 points in Orb($P$) and in Orb($\tilde{P}$), from which follows the assertion. We omit the details. \qed

We can numerically determine which of the points $P$ and $\tilde{P}$ is a sextactic point.

**Proposition 4.** Let $P, \tilde{P}$ have the same meaning as in the proof of Lemma 16. Then, $P$ is a 2-sextactic point and $\tilde{P}$ is not a sextactic point.

**Proof.** Using the method in Lemma 15, we can find the osculating conic $D$ at $P$. The tangent line of $C_{a_1}$ at $P$ is given by $l(x, y) = y - \delta - m(x - \gamma) = 0$, where $m = -(0.10111 \ldots + (0.21625 \ldots) i$. The coefficients $A, B, C$ have the following numerical solutions for $s_2 = s_3 = s_4 = 0$:

$$A = -0.1604 \ldots - (0.2374 \ldots) i,$$

$$B = -0.9787 \ldots + (0.3595 \ldots) i,$$

$$C = -0.4780 \ldots + (0.1791 \ldots) i.$$  

In this case, we have

$$s_5 = 1.243 \cdots \times 10^{-14} - (5.329 \ldots \times 10^{-15}) i,$$

$$s_6 = 8.751 \ldots \times 10^{-9} - (1.681 \ldots \times 10^{-8}) i,$$
\[ s_7 = 5.175 \ldots - (2.620\ldots)i \neq 0. \]

Therefore, we conclude that \( P \) is a sextactic point with contact order 7. Similarly, we can compute the osculating conic \( \tilde{D} \) at \( \tilde{P} \) and we find that \( s_5 = 0.75 \ldots - (0.54\ldots)i \neq 0 \). So \( \tilde{P} \) is not a sextactic point.

**Added in proof:** It came to our attention that Egde (Edinburgh Math. Notes 35 (1945), 10–13) discussed the curves \( C_a \) and found the 12 hyperflexes (undulations) on \( C_3 \). He cited Ciani (Palermo Rendiconti, 13 (1899), 347–373) as a predecessor.

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