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Existence of crepant resolution for abelian quotient singularities by order p elements in dimension 4

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Abstract

The existence of a toric projective crepant resolution is shown for series of finite abelian noncyclic quotient singularities of dimension four which are not complete intersections.

1. Introduction

We denote the canonical line bundle of a normal $\boldsymbol{Q}$-factorial algebraic variety X by K_X. Let $\varphi : Y \to X$ be a resolution and $\{D_i\}_{i=1,2,\cdots,n}$ be the exceptional divisors of φ. Then the adjunction formula $K_Y = \varphi^* K_X + \sum_{i=1}^n a_i D_i$ holds for a rational number a_i, the discrepancy for D_i, which we denote by $\mathrm{discr}(D_i)$. If $\mathrm{discr}(D_i)$ equals to 0 for all i, then φ is called a crepant resolution.

It is known that a crepant resolution always exists for a quotient singularity by a finite subgroup of $SL(n, \boldsymbol{C})$ if n is equal to three or less [6][8][9][13][14]. But not in the case in general when n is greater than or equal to four. Since around 1990, arithmetic conditions for the existence of a crepant resolution has been shown for some series of cyclic quotient singularities [2][4][5]. It is also shown that a crepant resolution exists for c.i. singularities [1][3].

In this paper, we show the existence of a projective crepant resolution for a few series of quotient singularities by abelian noncyclic and non-c.i. type subgroups of $SL(4, \boldsymbol{C})$ by toric method. Roughly speaking, our proof of the main theorem consists of two steps. Let p be an arbitrary prime number. First, we classify the finite abelian subgroups generated by order p elements of $SL(4, \boldsymbol{C})$ into five types and focus on a class which is named (2_{13}) type (see Proposition 4.1). These actions are "simplest" in view of elemental divisors theory. We construct crepant resolution for the quotient singularity by those groups, such as $G := \langle \frac{1}{p}(1, a, 0, p-a-1), \frac{1}{p}(0, 0, 1, p-1)\rangle \subset SL(4, \boldsymbol{C})$ ($a = 1$, $\frac{p-1}{2}$, $p-2$ or $p-1$).

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2. Toric quotient singularities

In this section, we shall recall a relation between age and discrepancy by using toric language. Let us fix some notations. The main reference are [7], [10].

Throughout the section, N is $\boldsymbol{Z}^n$ and M is its dual module $\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$. We denote the canonical pairing by $\langle\ ,\ \rangle$. We write the extension of scalar $N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ as $N_{\boldsymbol{R}}$. For a finite fan (N, Δ), we denote the toric variety corresponding to it by $X(N, \Delta)$ and also written as $X(N, \sigma)$ if Δ consists of the faces of a cone σ.

Proposition 2.1. *A toric variety $X(N, \Delta)$ is nonsingular if and only if each $\sigma \in \Delta$ is generated by a part of a basis of N.* (See p.15 of [10].)

We shall use the following.

Corollary 2.1. *Let σ be an n-dimensional simplicial convex cone generated by n primitive elements $x_1, \ldots, x_n$ in N. Then $X(N, \sigma)$ is nonsingular, if and only if $\{x_1, \ldots, x_n\}$ is a basis of N.*

Let g be an element of finite order in $GL(n, \boldsymbol{C})$. Then g is diagonalizable and there exists $h \in GL(n, \boldsymbol{C})$ such that

$$hgh^{-1} = \begin{pmatrix} e^{2\pi i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i\theta_n} \end{pmatrix}$$

where $\theta_1, \theta_2, \cdots, \theta_n$ are rational numbers in $[0, 1)$. We define the *age* of g as

$$\mathrm{age}(g) := \theta_1 \ + \ \theta_2 \ + \ \cdots \ + \ \theta_n.$$

The age is independent of the choice of h. The age of g is an integer if g is in $SL(n, \boldsymbol{C})$.

We shall use the following vector notation:

$$\frac{1}{s}(t_1, t_2, \cdots, t_n) := \begin{pmatrix} e^{2\pi i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i\theta_n} \end{pmatrix}$$

where θ_i equals to $\frac{t_i}{s}$ and $t_1, t_2, \cdots, t_n$ are nonnegative integers which are less than s. The age of g equals to $\frac{1}{s}(t_1 + \cdots + t_n)$. We denote the vector $\frac{1}{s}(t_1, t_2, \cdots, t_n)$ by $v(g)$.

Let $g_1, g_2, \cdots, g_m$ be elements in $SL(n, \boldsymbol{C})$. If $g_1, g_2, \cdots, g_m$ are commutative each other, then the elements $g_1, g_2, \cdots, g_m$ are simultaneously diagonalizable.

Let $G \subset SL(n, \boldsymbol{C})$ be an abelian finite subgroup. We may assume that G is diagonalized. We consider the natural action of G on $\boldsymbol{C}^n$. In this case, the action of groups is represented as the following theorem by toric technique.

Theorem 2.1. *Let $\{e_1, e_2, \cdots, e_n\}$ be the canonical basis of N. Let N' be the module $N + \sum_{g \in G} v(g)\boldsymbol{Z}$. Then N is a submodule of N' with finite index.*

Let Δ be the finite fan which is generated by $\sigma := \langle e_1, e_2, \cdots, e_n \rangle_{\boldsymbol{R}_{\geq 0}}$ and $\psi : (N, \Delta) \to (N', \Delta)$ be the natural morphism of finite fans. Then ψ corresponds to the morphism of toric varieties denoted by $X(\psi) : X(N, \Delta) \to X(N', \Delta)$ and $X(\psi)$ is the quotient map by $N'/N \simeq G$.

It is known that any toric variety admits an equivariant resolution of singularities.

Theorem 2.2. *Let Δ' be a locally finite nonsingular subdivision of a fan Δ in N. Then the equivariant holomorphic map $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ corresponding to the natural map $(N, \Delta') \to (N, \Delta)$ is proper birational and is an equivariant resolution of singularities for $X(N, \Delta)$.*

Moreover, for a primitive vector $v \in N'$ such that $v\boldsymbol{R}_{\geq 0}$ is in Δ', there exists an element g in G such that $v(g) = v$ by the quotient map in Theorem 2.1. For an exceptional divisor D_g, the following formula holds:

$$\mathrm{discr}(D_g) = \mathrm{age}(g) - 1,$$

where $D_g = \overline{\mathrm{orb}(\boldsymbol{R}_{\geq 0} v(g))}$.

For the details of the above proposition, theorems or corollary, see [10] (especially Sections 1.4 and 1.5) and see [15] for the assertions with respect to age and discrepancy.

Next, we shall review a projectivity condition for toric morphisms.

Definition 2.1. An $\boldsymbol{R}$-valued function h on the support $|\Delta|$ is called *a Δ-linear support function* if h is $\boldsymbol{Z}$-valued on $N \cap |\Delta|$ and linear on each $\sigma \in \Delta$ where $|\Delta|$ means $\cup_{\sigma \in \Delta} \sigma$.

The set consisting of all Δ-linear support functions becomes an additive group. The group is denoted by $\mathrm{SF}(N, \Delta)$. In Definition 2.1, if Δ is a finite fan and h is $\boldsymbol{Q}$-valued on $N \cap |\Delta|$, h is also *Δ-linear support function* by taking some multiple. Let $\Delta(1)$ be the set of all 1-dimensional cones in Δ. For $\rho \in \Delta(1)$, we denote the primitive element in $N \cap \rho$ by $n(\rho)$.

Proposition 2.2. *There exists an injective homomorphism*

$$\mathrm{SF}(N, \Delta) \hookrightarrow \boldsymbol{Z}^{\Delta(1)}$$

$$h \mapsto (h(n(\rho)))_{\rho \in \Delta(1)}.$$

A support function h is determined by integers $h(n(\rho))$.

If X is nonsingular, there exists an isomorphism such that

$$\mathrm{SF}(N, \Delta) \xrightarrow{\sim} \boldsymbol{Z}^{\Delta(1)}.$$

Proposition 2.3. *A toric resolution $\phi : X(N, \tilde{\Delta}) \to X(N, \Delta)$ is complete if and only if $|\tilde{\Delta}|$ equals to $|\Delta|$.*

Definition 2.2. Suppose $|\tilde{\Delta}|$ equals to $|\Delta|$. A $\tilde{\Delta}$-linear support function h is said to be *strictly upper convex on* $\tilde{\Delta}$ if h satisfies the following conditions.
(a) $\langle l_\sigma, x\rangle \geq h(x)$ for all $\sigma \in \tilde{\Delta}$ and for all $x \in N_{\boldsymbol{R}}$,
(b) $\langle l_\sigma, x\rangle = h(x)$ if and only if $x \in \sigma$,
where l_σ is an element in M such that $\langle l_\sigma, x\rangle$ equals to $h(x)$ if x is in σ and $\langle l_\sigma, x\rangle$ equals to $\langle l_\tau, x\rangle$ for $x \in \tau$ if τ is a face of σ.

Let (N, Δ) be a finite fan and $\Delta(n)$ be the set of the n-dimensional cones in Δ. A set $\{l_\sigma;\ \sigma \in \Delta(n)\} \subset M$ is determined uniquely by $h \in \mathrm{SF}(N, \Delta)$.

Proposition 2.4. *For a complete toric resolution $\phi : X(N, \tilde{\Delta}) \to X(N, \Delta)$, the following conditions are equivalent.*
(a) ϕ is projective.
(b) There exists $h \in \mathrm{SF}(N, \tilde{\Delta})$ such that h is strictly upper convex on $\tilde{\Delta}$.

3. Complete intersection quotient singularities

In this section, we shall describe a part of the results of [11] and [12]; which give the criterion for some quotient singularities to be complete intersection. Notations and actions used here are similar to the previous sections.

Definition 3.1. *Let S be the ring $\boldsymbol{C}[X_1, X_2, \cdots, X_n]$ and I be the index set $\{1, 2, 3, \cdots, n\}$ of the variables, D be a set consisting of subsets of I and ω be a map from D to the set of the positive integers $\boldsymbol{Z}_{>0}$. The pair (D, ω) is said to be a special datum, if D and I satisfy the following conditions.*
(a) the subset $\{i\}$ is an element in D for any $i \in I$,
(b) if J and J' are elements in D, then J and J' satisfy the condition $J \subset J'$, $J' \subset J$ or $J \cap J' = \varnothing$,
(c) if J is a maximal set in D, then $\omega(J)$ equals to 1,
(d) if J and J' are elements in D and if J' contains J properly, then $\omega(J')$ divides $\omega(J)$ and $\omega(J)$ is bigger than $\omega(J')$,
(e) if J_1, J_2 and J are elements in D and if $J_i \prec J$ $(i = 1, 2)$, then $\omega(J_1)$ equals to $\omega(J_2)$, where the notation $\prec$ means that J_i is a subset of J and there exist no element in D between J_i and J.

Definition 3.2. *Let $\boldsymbol{D} = (D, \omega)$ be a special datum, we put $R_{\boldsymbol{D}}$ to be the subring $\boldsymbol{C}[X_J; J \in D]$ of S where $X_J = (\prod_{i\in J} X_i)^{\omega(J)}$.*

We denote the diagonal matrix whose (i,i) component is a (resp. (i,i) component is a and (j,j) component is b) and the other diagonal components are 1 by $(a;i)$ (resp. $(a,b;i,j)$) here.

Definition 3.3. *Let $\boldsymbol{D}=(D,\omega)$ be a special datum. A group $G_{\boldsymbol{D}}$ is the one generated by the following elements: $\{(e_\omega,\ e_\omega^{-1};i,j)|J_1,\ J_2,\ J\in D,\ i\in J_1,\ j\in J_2,\ J_1\prec J,\ J_2\prec J$ and $\omega=\omega(J_1)=\omega(J_2)\}$.*

Proposition 3.1. *If $\boldsymbol{D}=(D,\omega)$ is a special datum, then*
(1) *the ring $R_{\boldsymbol{D}}$ is a complete intersection*
(2) *$R_{\boldsymbol{D}}$ is the invariant subring under the action of the group $G_{\boldsymbol{D}}$.*

Theorem 3.1. *If G is a finite abelian subgroup of $SL(n,\boldsymbol{C})$ and if the invariant ring S^G is a complete intersection, then there is a special datum $\boldsymbol{D}$ such that $S^G=R_{\boldsymbol{D}}$ and $G=G_{\boldsymbol{D}}$.*

4. Result

We have found some infinite series of noncyclic and non-c.i. finite subgroups G of $SL(4,\boldsymbol{C})$ such that $\boldsymbol{C}^4/G$ admits a toric projective crepant resolution.

Proposition 4.1. *Let p be a prime number and G be an abelian finite subgroup of $SL(4,\boldsymbol{C})$ generated by order p elements. Then G is a vector space on the prime field of order p. The dimension of G as a vector space is at most three and G is conjugate in $SL(4,\boldsymbol{C})$ to one of the followings.*

(1) $(\dim G=1)$ *A cyclic group* $\langle\frac{1}{p}(a,b,c,d)\rangle\cong\boldsymbol{Z}/p\boldsymbol{Z}$.

(2) $(\dim G=2)$ *Noncyclic groups with two generators*
$\langle\frac{1}{p}(1,0,a,p-a-1),\frac{1}{p}(0,1,b,p-b-1)\rangle\cong(\boldsymbol{Z}/p\boldsymbol{Z})^2,\quad(2_{12}),$
$\langle\frac{1}{p}(1,a,0,p-a-1),\frac{1}{p}(0,0,1,p-1)\rangle\cong(\boldsymbol{Z}/p\boldsymbol{Z})^2\ (a\neq 0),\quad(2_{13}),$
$\langle\frac{1}{p}(0,1,0,p-1),\frac{1}{p}(0,0,1,p-1)\rangle\cong(\boldsymbol{Z}/p\boldsymbol{Z})^2,\quad(2_{23}).$

(3) $(\dim G=3)$ *Noncyclic groups with three generators*
$\langle\frac{1}{p}(1,0,0,p-1),\frac{1}{p}(0,1,0,p-1),\frac{1}{p}(0,0,1,p-1)\rangle\cong(\boldsymbol{Z}/p\boldsymbol{Z})^3.$

Proof. Let G be an abelian finite subgroup of $SL(4,\boldsymbol{C})$ generated by diagonal matrices. We define an injective homomorphism of groups $\varphi:(\boldsymbol{R}/\boldsymbol{Z})^{\oplus 4}\hookrightarrow GL(4,\boldsymbol{C})$ as

$$(x,y,z,w)\mapsto\begin{pmatrix} e^{2\pi ix} & & & 0\\ & e^{2\pi iy} & & \\ & & e^{2\pi iz} & \\ 0 & & & e^{2\pi iw}\end{pmatrix}$$

and $\psi : (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} \hookrightarrow SL(4,\boldsymbol{C})$ as

$$(x,y,z) \mapsto \begin{pmatrix} e^{2\pi ix} & & & 0 \\ & e^{2\pi iy} & & \\ & & e^{2\pi iz} & \\ 0 & & & e^{2\pi i(1-x-y-z)} \end{pmatrix}.$$

Then we have the following diagram.

$$\begin{array}{ccc} (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} & \hookrightarrow & SL(4,\boldsymbol{C}) \\ \downarrow & & \downarrow \\ (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 4} & \hookrightarrow & GL(4,\boldsymbol{C}) \end{array}$$

And we also have the following diagram where the vertical arrows are canonical quotient maps.

$$\begin{array}{ccc} G+\boldsymbol{Z}^{\oplus 3} & \hookrightarrow & \boldsymbol{R}^{\oplus 3} \\ \downarrow & & \downarrow \\ G & \hookrightarrow & (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} \end{array}$$

We write $G+\boldsymbol{Z}^{\oplus 3}$ as $\tilde{G}$. $\tilde{G}$ is a free abelian group of rank at most three. We choose an isomorphism $\rho : \tilde{G} \xrightarrow{\sim} \boldsymbol{Z}^{\oplus 3}$. Then the composition of the inclusion map $\boldsymbol{Z}^{\oplus r} \to \tilde{G}(\cong \boldsymbol{Z}^{\oplus r})$ and ρ is $\boldsymbol{Z}$-linear. We write the composition: $\boldsymbol{Z}^{\oplus 3} \hookrightarrow \tilde{G} \xrightarrow{\sim} \boldsymbol{Z}^{\oplus 3}$ as ν.

We may assume the image of ν equals to $d_1\boldsymbol{Z} \oplus d_2\boldsymbol{Z} \oplus d_3\boldsymbol{Z}$ where d_1, d_2 and d_3 are integers with $d_1|d_2|d_3$. Clearly, G is isomorphic to $\tilde{G}/\boldsymbol{Z}^{\oplus 3}$. Hence, G is isomorphic to Coker $\nu = (\boldsymbol{Z}/d_1\boldsymbol{Z}) \oplus (\boldsymbol{Z}/d_2\boldsymbol{Z}) \oplus (\boldsymbol{Z}/d_3\boldsymbol{Z})$.

By assumption, d_i equals to p or 1 where i is 1, 2 or 3. Hence an abelian finite subgroup of $SL(4,\boldsymbol{C})$ generated by order p elements becomes the type (1), (2_{12}), (2_{13}), (2_{23}) or (3) using a change of bases. □

Theorem 4.1. *There exists a projective toric crepant resolution for the following types:*

(a) *Type* (2_{12}) *where* $a=b=1$,

(b) *Type* (2_{13}) *where* $a=1$, $\frac{p-1}{2}$, $p-2$ *or* $p-1$,

(c) *Type* (2_{23}),

(d) *Type* (3).

Proof. The case (a).

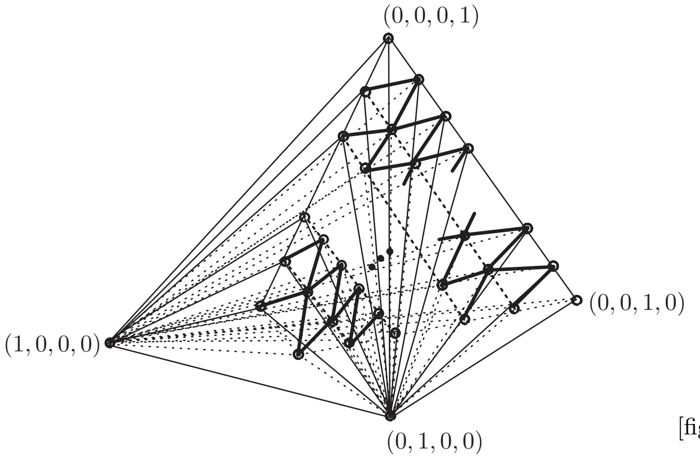
We will prove this case at the latter part of the proof of the case (b), $a=p-2$.

The case (b).

We define the hypersurface spanned by the elements $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ as H. Let G be a group of the type (2_{13}).

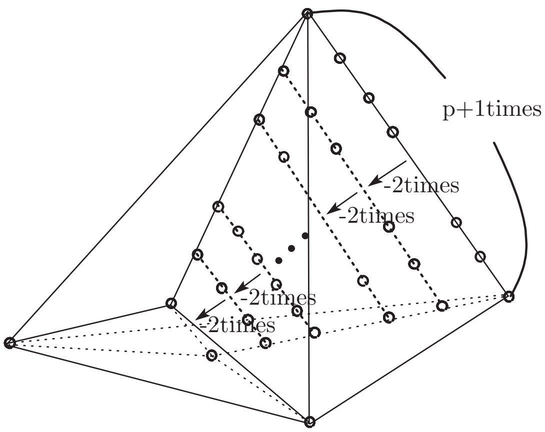
If a equals to 1 and $p \neq 2$, the quotient space $\boldsymbol{C}^4/G$ corresponds to the toric variety $X(N', \Delta)$ where the lattice set N' is $\boldsymbol{Z}^4 + \frac{1}{p}(1,1,0,p-2)\boldsymbol{Z} + \frac{1}{p}(0,0,1,p-1)\boldsymbol{Z}$ and Δ is the finite fan which consists of the faces of the cone generated by the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. We define the subset $\{\frac{1}{p}(i,i,j,p-2i-j); i = 0,1,2,\cdots,\frac{p-1}{2},\ j = 0,1,\cdots,p-2i\} \subset N'$ as P. The age of all the points in P equals to 1.

We give a resolution for the singularity $X(N', \Delta)$ by subdividing $\Delta \cap H$ as the figure [fig(213)-(1)].



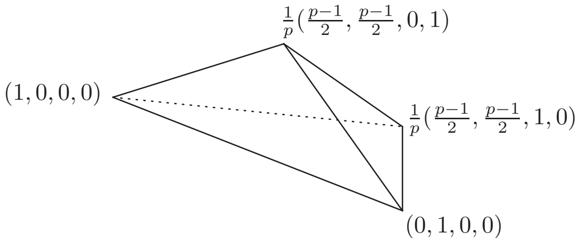
[fig(213)-(1)]

On the edge connecting $(\frac{k}{p}, \frac{k}{p}, \frac{p-2k}{p}, 0)$ and $(\frac{k}{p}, \frac{k}{p}, 0, \frac{p-2k}{p})$, there appear $(p-2k+1)$ points where the integer k satisfies the condition $0 \leq k \leq \frac{p+1}{2} - 1$. See [fig(213)-(1)-pt].

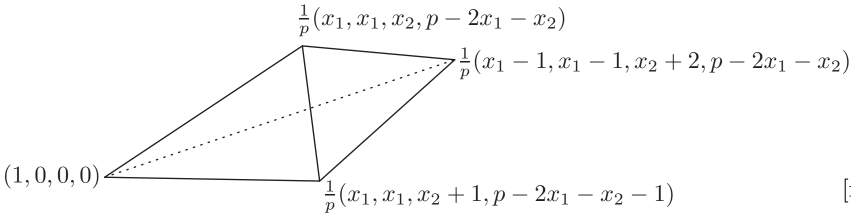


[fig(213)-(1)-pt]

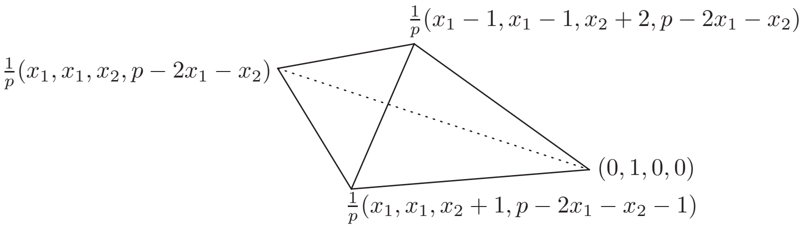
The figure [fig(213)-(1)] includes p^2 triangular pyramids of the following types: [fig1], [fig2], [fig3], [fig4], [fig5], [fig6] and [fig7].



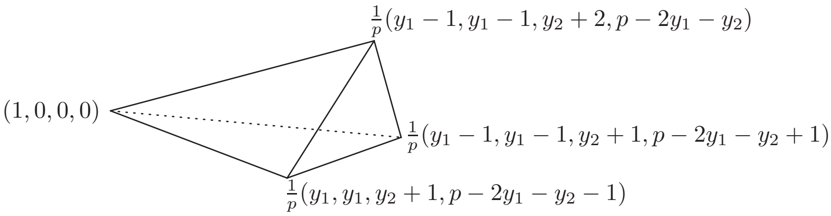
[fig1]



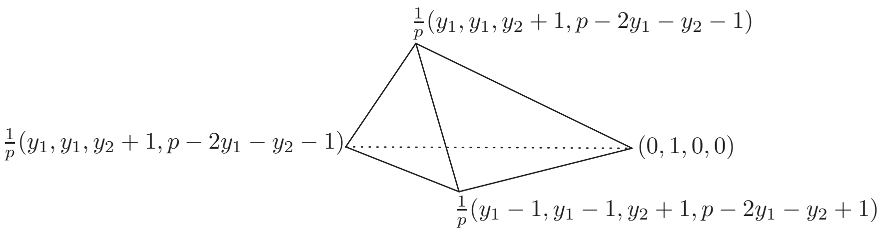
[fig2]



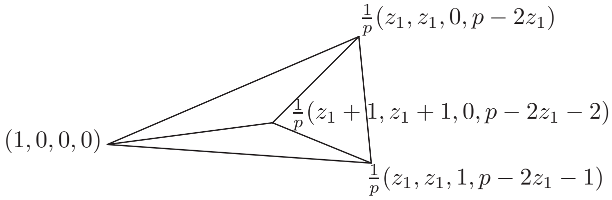
[fig3]



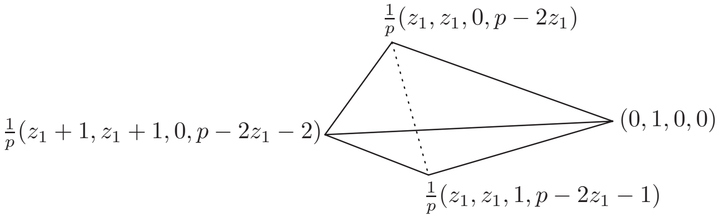
[fig4]



[fig5]



[fig6]



[fig7]

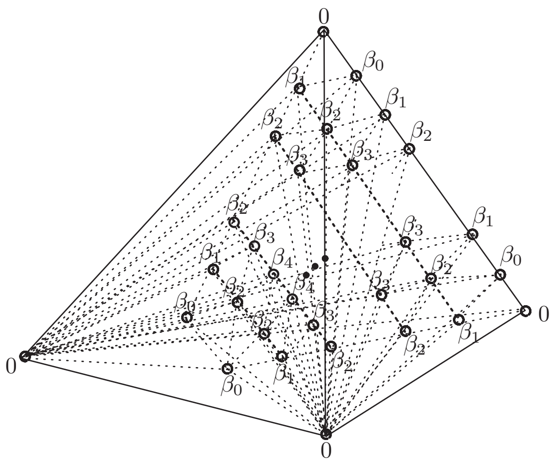
The variables satisfy the conditions $x_1, y_1, z_1 \in [1, \frac{p-1}{2}] \cap \boldsymbol{Z}$, $x_2, y_2 \in [0, \frac{p-1}{2}] \cap \boldsymbol{Z}$, $p - 2x_1 - x_2 \geq 1$ and $p - 2y_1 - y_2 \geq 0$.

All the determinants of the matrices made by the generators of the triangular pyramids equal to $\frac{1}{p^2}$ for the [fig1], [fig2], $\cdots$, [fig7]. Therefore, the cones generated by the four vertices of each triangular pyramid are nonsingular and the variety corresponding to the fan $(N', \tilde{\Delta})$ is a resolution for $X(N', \Delta)$ where $\tilde{\Delta}$ is the finite fan decomposed as the figure.

The age of every lattice point corresponding to the exceptional divisors for this resolution equals to 1. Hence, the resolution is crepant.

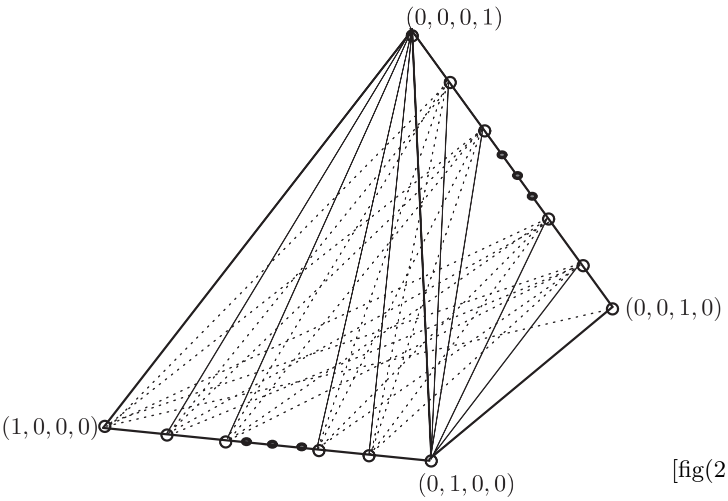
Next, we prove that this resolution is projective.

We shall define $\tilde{\Delta}$-linear support function h which is strictly upper convex on $\tilde{\Delta}$ by giving a $\boldsymbol{Q}$-value for each lattice point in $\Delta \cap H$. If h is strictly upper convex on $\tilde{\Delta} \cap H$, h is strictly upper convex on $\tilde{\Delta}$. Let h have the following $\boldsymbol{Q}$-value β_i at each lattice point in $\Delta \cap H$, then h is strictly upper convex on $\tilde{\Delta}$,



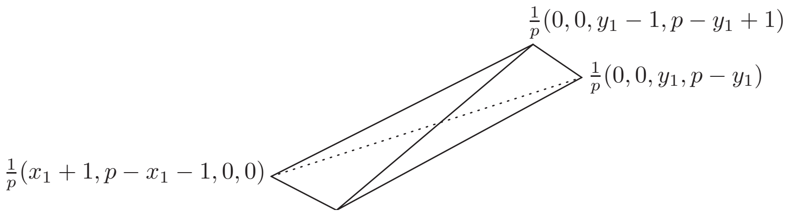
where the rational number β_i equals to $1+\sum_{k=0}^{i}\frac{1}{2^i}$ and the integer i runs from 0 to $\frac{p-3}{2}$.

From here, we shall construct a projective crepant resolution for the case a equals to $p-1$. We also treat the case $p=2$. The lattice set N' is $\boldsymbol{Z}^4+\frac{1}{p}(1,p-1,0,0)\boldsymbol{Z}+\frac{1}{p}(0,0,1,p-1)\boldsymbol{Z}$. We define a subset $\{\frac{1}{p}(i,p-i,0,0),\frac{1}{p}(0,0,j,p-j); i,j=0,2,\cdots,p\}\subset N'$ as P. The age of the elements in P is always 1. We shall give a resolution for $X(N',\Delta)$ by subdividing $\Delta\cap H$ as the figure [fig(213)-(p-1)].



[fig(213)-(p-1)]

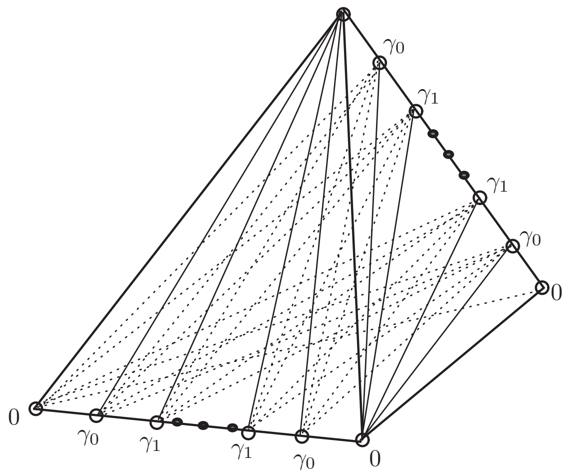
The figure [fig(213)-(p-1)] only contains the triangular pyramids as shown in [fig8].



[fig8]

By the similar way as the case that a equals to 1, the cone as shown in [fig8] is nonsingular and the variety $X(N',\tilde{\Delta})$ is a crepant resolution for $X(N',\Delta)$ where the finite fan $\tilde{\Delta}$ is decomposed as the figure [fig(213)-(p-1)].

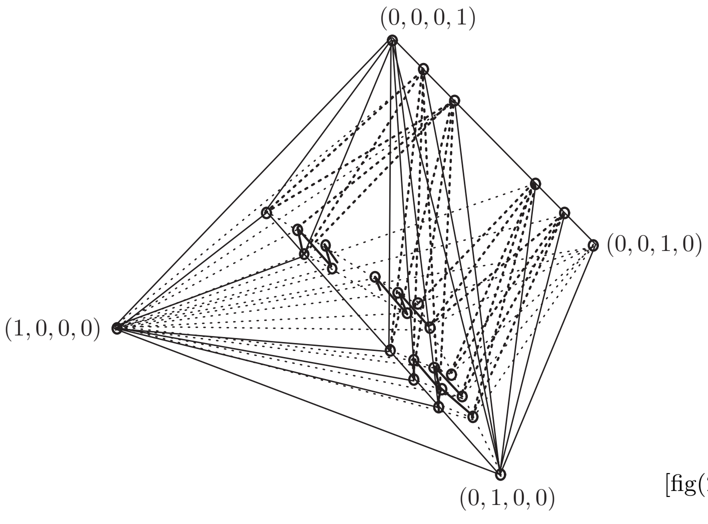
Next, we prove that this resolution is projective. If h is determined by the value at each point as follows, then the $\tilde{\Delta}$-linear support function h is strictly upper convex on $\tilde{\Delta}$ where the rational number γ_i equals to $2+\sum_{k=0}^{i}\frac{1}{2^i}$ and the integer i runs from 0 to $\frac{p-3}{2}$.



In the remaining case $a = 1$ and $p = 2$, the condition that G is the type (2_{13}) and $a = 1$, $p = 2$ is a equivalent one that G is the type (2_{13}) and $a = p-1$, $p = 2$. So, we have proved the case $a = 1,\ p-1$ of the type (2_{13}).

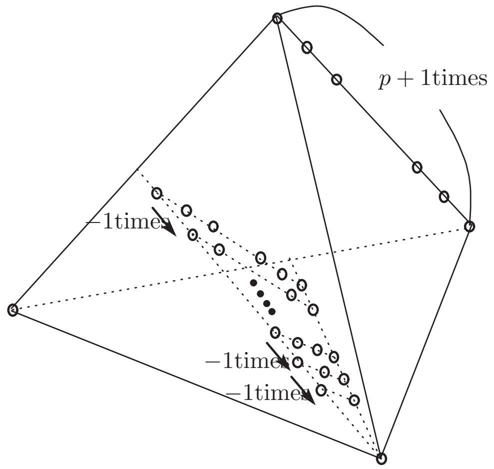
In the following, we shall prove the case $a = \frac{p-1}{2},\ p-2$. First, we consider the case $a = p-2$ where $p \neq 2$. The lattice set N' is $\boldsymbol{Z}^4 + \frac{1}{p}(1, p-2, 0, 1)\boldsymbol{Z} + \frac{1}{p}(0, 0, 1, p-1)\boldsymbol{Z}$ and Δ is as above. We define the subset $\{(1,0,0,0),\ \frac{1}{p}(0,0,i,p-i),\ \frac{1}{p}(j,k,l,j-l);\ i \in \boldsymbol{Z} \cap [0,p],\ j \in \boldsymbol{Z} \cap [0, \frac{p-1}{2}],\ k = p-2j,\ l \in \boldsymbol{Z} \cap [0,j]\} \subset N'$ as P.

The age of all the points in P equals to 1. We will give a resolution for the singularity $X(N', \Delta)$ by subdividing $\Delta \cap H$ as the figure [fig(213)-(p-2)].



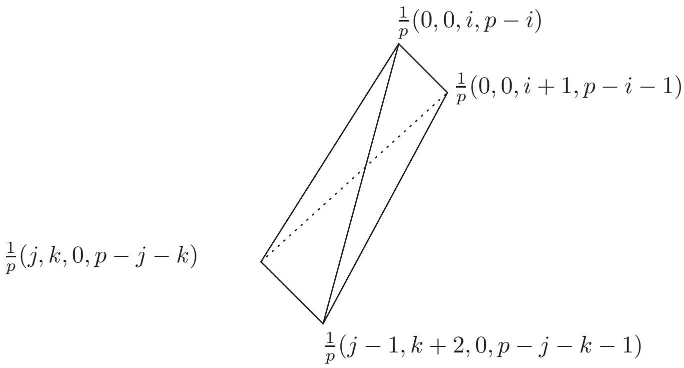
[fig(213)-(p-2)]

On the edge connecting two points $(\frac{j}{p}, \frac{p-2j}{p}, \frac{j}{p}, 0)$ and $(\frac{j}{p}, \frac{p-2j}{p}, 0, \frac{j}{p})$, there appear $j+1$ points where the integer j satisfies the condition $0 \leq j \leq \frac{p-1}{2}$. See [fig(213)-(p-2)-pt].

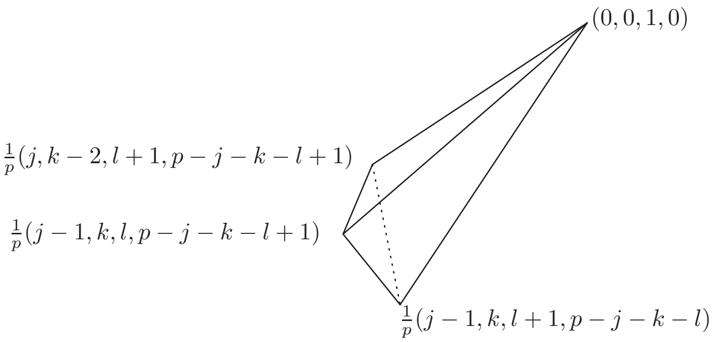


[fig(213)-(p-2)-pt]

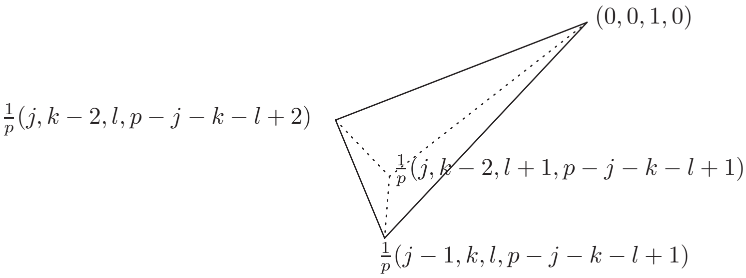
The figure [fig(213)-(p-2)] includes p^2 triangular pyramids of the following types: [fig9], [fig10], [fig11], [fig12], [fig13], [fig14] and [fig15].



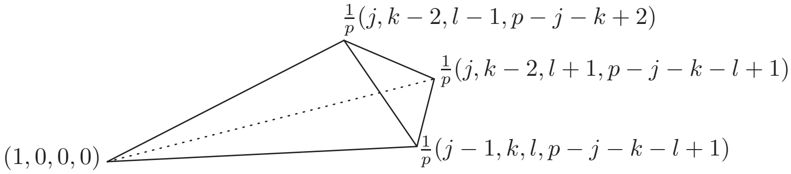
[fig9]



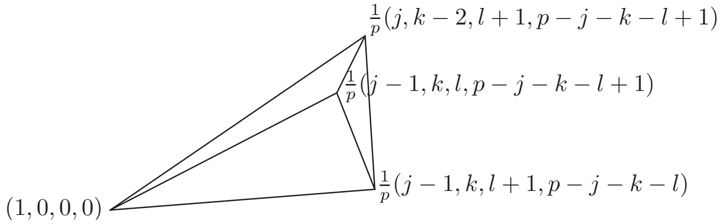
[fig10]



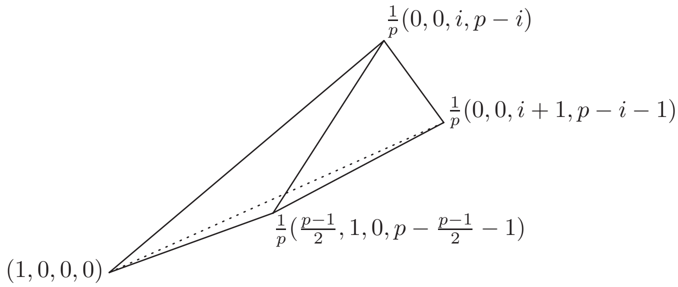
[fig11]



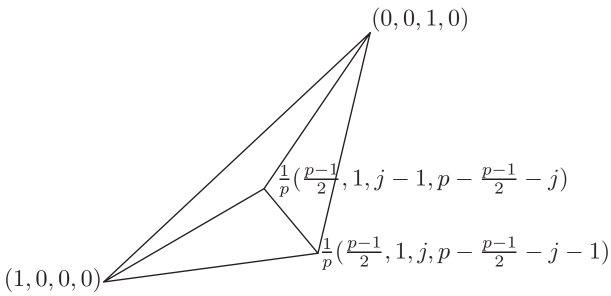
[fig12]



[fig13]



[fig14]



[fig15]

The variables satisfy the conditions $j \in [1, \frac{p-1}{2}] \cap \boldsymbol{Z}$, $k = p - 2j$, $l \in [0, p-j-k] \cap \boldsymbol{Z}$ and $i \in [0, p-1] \cap \boldsymbol{Z}$.

All the determinants of the matrices made by the generators of the triangular pyramids equal to $\frac{1}{p^2}$ for the [fig9], [fig10], $\cdots$, [fig15]. Therefore, the cone generated by the four vertices of each triangular pyramid is nonsingular and the variety corresponding to the fan $(N', \tilde{\Delta})$ is a resolution for $X(N', \Delta)$, where $\tilde{\Delta}$ is the finite fan decomposed as the figure.

The age of every lattice point corresponding to the exceptional divisors for this resolution equals to 1. Hence, the resolution is crepant.

We define h as the similar way for the case $a = 1$. Then it is confirmed that the crepant resolution is projective.

Here, we consider the case $p = 2$. The invariant ring is $\boldsymbol{C}[X_1^2, X_2, X_3^2, X_4^2, X_1X_2X_4]$. This case is clearly complete intersection type (see Section 3) and there exist projective crepant resolutions.

Finally, We get the case $a = \frac{p-1}{2}$ from the case $a = p-2$ by interchanging bases $(1,0,0,0)$ and $(0,1,0,0)$ and also the type (a) from this case by changing bases and by changing generators of the group. So we have proved the case (a) and (b).

The case (c) and (d).

The quotient singularities of this type are c.i. singularity. In the case (c) (resp. (d)), the invariant ring is $\boldsymbol{C}[X_1, X_2^p, X_3^p, X_4^p, X_2X_3X_4]$ (resp. $\boldsymbol{C}[X_1^p, X_2^p, X_3^p, X_4^p, X_1X_2X_3X_4]$). We can define a special datum $\boldsymbol{D}$ as $D := \{\{1\}, \{2\}, \{3\}, \{4\}, \{2,3,4\}\}$ (resp. $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3,4\}\}$) and a map ω as $\{1\} \to 1$, $\{i\} \to p$ $(i = 2,3,4)$ and $\{2,3,4\} \to 1$ (resp. $\{i\} \to p$ $(i = 1,2,3,4)$ and $\{1,2,3,4\} \to 1$). So there exist projective crepant resolutions. $\square$

4.1 Comment

For the group G of the type (2_{13}), if a equals to $p-1$ then $\boldsymbol{C}^4/G$ is c.i. singularity. We shall show $\boldsymbol{C}^4/G$ is non-c.i. if a equals to $1, \frac{p-1}{2}, p-2$ by showing that there are no special datum $\boldsymbol{D}$ for the group. Let I be the set $\{1,2,3,4\}$. Assume that group G is the type (2_{13}) and a equals to 1. Then the invariant ring S^G equals to $\boldsymbol{C}[{X_1}^p, {X_2}^p, {X_3}^p, {X_4}^p, {X_1}^i {X_2}^{p-i}, {X_1}^2X_3X_4, {X_2}^2X_3X_4, X_1X_2X_3X_4]$ $(1 \le i \le p-1)$. Now, Let D be the set $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$ and ω be a map from D to $\boldsymbol{Z}_{>0}$. Then the pair (D, ω) never becomes a special datum, since the images of ω for the elements $\{1,2\}$, $\{1,3,4\}$ and $\{2,3,4\}$ do not satisfy the condition that $\omega(E)$ equals to 1 or p where E is $\{1,2\}$, $\{1,3,4\}$ or $\{2,3,4\}$. Hence, in this case, we have the fact that G is non-c.i. type.

We conjecture that there exists a projective crepant resolution for type (2_{13}) for any p and a.

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