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Expressions of the heat kernels on spheres by elementary functions and their recurrence relations

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Abstract

We show that, near the diagonal set of $S^n \times S^n$ and ignoring the terms decaying exponentially when $t \to 0$, the derivatives of heat kernel on the sphere S^n can be described explicitly by using elementary functions and the kernels on different dimensional spheres have certain recurrence relations. Consequently, known various results concerning its asymptotics may be obtained by investigating only the elementary functions.

1. Introduction and the main result

Let us take the Laplacian $\Delta = d^*d$ acting on functions on the standard n-sphere (S^n, g) with curvature 1 where d is the exterior differential and d^* is its formal adjoint. We consider the fundamental solution or the heat kernel $K_n(t, r(P,Q))$ associated to the heat equation with initial condition

$$(\partial/\partial t + \Delta)f = 0, \quad f|_{t=0} = f_0. \tag{1.1}$$

Note that, due to homogeneity of sphere, the kernel depends only on t and the distance $r(P,Q)$ of two points P, Q on S^n. In the paper we will show that, if $r(P,Q)$ is small, then the behavior of the derivatives of $K_n(t, r(P,Q))$ when $t \to 0$ can be described explicitly by using some elementary functions and there exist interesting recurrence relations among the kernels on different dimensional spheres. As a result, known various results of study on their behavior may be obtained by investigating only the elementary functions.

Let us take $0 < \delta < \pi/2$ and consider the C^∞-function on $(0,\infty) \times [0,\delta)$

$$k_n(t,r) = \frac{e^{m^2 t}}{(2\pi)^m (4\pi t)^{1/2}} \left(\frac{-1}{\sin r} \frac{\partial}{\partial r} \right)^m e^{-r^2/4t} \quad (n = 2m+1), \tag{1.2}$$

$$k_n(t,r) = \frac{e^{(2m+1)^2 t/4}}{2(2\pi)^{m+3/2} t^{3/2}} \left(\frac{-1}{\sin r} \frac{\partial}{\partial r} \right)^m \int_r^\delta ds \, \frac{s e^{-s^2/4t}}{(\cos r - \cos s)^{1/2}} \quad (n = 2m+2) \tag{1.3}$$

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$$= \frac{e^{(2m+1)^2t/4}}{(2\pi)^{m+3/2}t^{3/2}} \int_r^\delta ds\,(\cos r - \cos s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} s e^{-s^2/4t} + \mathcal{O}_\infty(e^{-1/t}).$$

Here $\mathcal{O}_\infty(e^{-1/t})$ is a term, for any ℓ, whose ℓ times derivative by r can be estimated as $\mathcal{O}(e^{-\varepsilon_\ell/t})$ with some $\varepsilon_\ell > 0$, and certainly the second equality in (1.3) is obtained by integration by parts. Note that the change of $\delta > 0$ brings only an $\mathcal{O}_\infty(e^{-1/t})$-term. Then we have

Theorem 1.1. *Take small $\delta > 0$. Then, for $0 < t \le 1$ and $r = r(P,Q) < \delta$, we have*

$$K_n(t,r) = k_n(t,r) + \mathcal{O}_\infty(e^{-1/t}), \tag{1.4}$$

$$K_{n+2}(t,r) = \frac{e^{nt}}{2\pi}\Big(\frac{-1}{\sin r}\frac{\partial}{\partial r}\Big)K_n(t,r) + \mathcal{O}_\infty(e^{-1/t}), \tag{1.5}$$

$$K_{n+1}(t,r) = \int_r^\delta ds\,\frac{2^{1/2}e^{-(2n+1)t/4}K_{n+2}(t,s)\sin s}{(\cos r - \cos s)^{1/2}} + \mathcal{O}_\infty(e^{-1/t}). \tag{1.6}$$

In the normal coordinates x at a point, the distance $r(P,Q)$ is given by

$$r(P,Q) = \arccos\Big(\cos|x(P)|\cos|x(Q)| + \langle x(P), x(Q)\rangle \frac{\sin|x(P)|}{|x(P)|}\frac{\sin|x(Q)|}{|x(Q)|}\Big), \tag{1.7}$$

where $\langle x(P), x(Q)\rangle$ is the standard inner product of the coordinate space. Thus the behavior of derivatives of $K_n(t, r(P,Q))$ may be explicitly described using it and the function $k_n(t,r)$ which can be expressed roughly as (see (2.1), (2.2))

$$k_n(t,r) = \frac{e^{-r^2/4t+(n-1)^2t/4}}{(4\pi t)^{n/2}} \sum_{j=0}^{m-1} t^j\, k_{n,j}(r^2) \quad (n = 2m+1), \tag{1.8}$$

$$k_n(t,r) = \frac{e^{-r^2/4t+(n-1)^2t/4}}{(4\pi t)^{n/2}} \Big\{\sum_{j=0}^{N} t^j k_{n,j}(r^2) + \mathcal{O}_\infty(t^N)\Big\} \quad (n = 2m+2) \tag{1.9}$$

with $k_{n,j}(u) \in C^\infty(-\delta^2, \delta^2)$ $(j = 0, 1, \dots)$ and, in particular,

$$k_{n,0}(r^2) \equiv \lim_{t\to 0} (4\pi t)^{n/2} e^{r^2/4t-(n-1)^2t/4}\, k_n(t,r) = \Big(\frac{r}{\sin r}\Big)^{(n-1)/2}. \tag{1.10}$$

For example, the coefficients of the asymptotic expansion of the heat trace $\displaystyle\int_{S^n} dg(P)K_n(t, r(P,P)) = \mathrm{vol}(S^n)K_n(t,0)$ were calculated by Cahn-Wolf ([2]) using the theories of Lie group, symmetric space and representation, by Polterovich ([13]) using his formula in [12], certain combinatorial identities (due to some property of hypergeometric function) and the table of eigenvalues and their multiplicities of Laplacian, and by the others ([14], etc.). Their methods need much acquaintance with such many areas, but in our method we have only to calculate $k_{n,j}(0)$, that is, to expand $\left(\frac{-1}{\sin r}\frac{\partial}{\partial r}\right)^m e^{-r^2/4t}|_{r=0}$ at $t=0$ and $(1-\cos s)^{1/2}$ at $s=0$, etc., in a suitable sense (see (2.1)).

In [10] the author proposed a simple method of investigating the heat kernels (acting on differential forms) on arbitrary Riemannian manifolds using only basic knowledge of calculus: compare with the method in [5] using the Gilkey's invariant theory and also with the stochastic one in [4], [11], etc. In this paper, in the case of sphere we introduce the method quite simpler than in [10], that is, we need thus only some study of elementary function $k_n(t,r)$ or of the trigonometric functions consisting it. The recurrence relations (1.5) and (1.6), which may clarify the structure of heat kernels from a new viewpoint, come from also such an elementary study (see (2.3)). Then how did he find out $k_n(t,r)$? We will discuss it in §3. In short, inspired by a certain duality between sphere (of curvature 1) and hyperbolic space (of curvature -1) and motivated by the explicit description ([3], [6]) of the heat kernel on the latter, naturally we expected it is $k_n(t,r)$ that is an appropriate parametrix for the former.

2. The proof of Theorem 1.1.

We will mainly discuss the proof of the formula (1.4) in the section. Before it let us lightly ascertain (1.5), (1.6), (1.8)–(1.10), all of which will be obvious. First, as for (1.8)–(1.10): If $n=2m+1$ they will be all obvious. If $n=2m+2$, referring to the second expression at (1.3) we have

$$
\begin{aligned}
&(\cos r-\cos s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} se^{-s^2/4t}\\
&=2^{-1/2}(s^2-r^2)^{1/2}\Big(\frac{\sin\frac{s+r}{2}}{\frac{s+r}{2}}\Big)^{1/2}\Big(\frac{\sin\frac{s-r}{2}}{\frac{s-r}{2}}\Big)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} se^{-s^2/4t}\\
&=\sum_{k=1}^{m+1} t^{-k}se^{-s^2/4t}(s^2-r^2)^{1/2}V_k(s^2,r^2)\quad (V_k(u,v)\in C^\infty((-\delta^2,\delta^2)^2)),
\end{aligned}
$$

$$
\begin{aligned}
(2.1)\quad &\int_r^\delta ds\,(\cos r-\cos s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} se^{-s^2/4t}\\
&=e^{-r^2/4t}\sum_{k=1}^{m+1} t^{-k}\int_r^\delta ds\,se^{-(s^2-r^2)/4t}(s^2-r^2)^{1/2}V_k(s^2,r^2)\\
&=e^{-r^2/4t}4t^{3/2}\sum_{k=1}^{m+1} t^{-k}\int_0^{(\delta^2-r^2)/4t} du\,e^{-u}u^{1/2}V_k(4tu+r^2,r^2),
\end{aligned}
$$

which yield (1.9). We have then

$$
(2.2)\quad \frac{(4\pi t)^{m+1}e^{r^2/4t}}{(2\pi)^{m+3/2}t^{3/2}}\int_r^\delta ds\,(\cos r-\cos s)^{1/2}\Big(\frac{s}{2t\sin s}\Big)^{m+1} se^{-s^2/4t}
$$

$$= \frac{2}{\pi^{1/2}} \int_0^{\frac{\delta^2-r^2}{4t}} du\, e^{-u} u^{1/2} \Big(\frac{\sin\frac{s+r}{2}}{\frac{s+r}{2}} \frac{\sin\frac{s-r}{2}}{\frac{s-r}{2}} \Big)^{1/2} \Big(\frac{s}{\sin s} \Big)^{m+1} \Big|_{s=(4tu+r^2)^{1/2}} + \mathcal{O}(t)$$
$$\to \frac{2}{\pi^{1/2}} \int_0^{\infty} du\, e^{-u} u^{1/2} \Big(\frac{\sin\frac{s+r}{2}}{\frac{s+r}{2}} \frac{\sin\frac{s-r}{2}}{\frac{s-r}{2}} \Big)^{1/2} \Big(\frac{s}{\sin s} \Big)^{m+1} \Big|_{s=r} \quad (t \to 0)$$
$$= \frac{2}{\pi^{1/2}} \int_0^{\infty} du\, e^{-u} u^{1/2} \Big(\frac{\sin r}{r} \Big)^{1/2} \Big(\frac{r}{\sin r} \Big)^{m+1} = \Big(\frac{r}{\sin r} \Big)^{m+1/2}.$$

Thus we obtain (1.10). Next, if (1.4) holds, then the formula (1.5) will be obvious and, setting $c(r,s) = \cos r - \cos s$, we have

$$(2.3) \quad \int_r^\delta ds \frac{2^{1/2} e^{-(4m+1)t/4} k_{2m+2}(t,s) \sin s}{c(r,s)^{1/2}}$$
$$= \frac{t^{-3/2} e^{m^2 t}}{2^{m+2} \pi^{m+3/2}} \int_r^\delta ds \int_s^\delta du \, \frac{\sin s}{c(r,s)^{1/2}} \frac{1}{c(s,u)^{1/2}} \Big(\frac{\partial}{\partial u} \frac{-1}{\sin u} \Big)^m u e^{-u^2/4t} + \mathcal{O}_\infty(e^{-1/t})$$
$$= \frac{t^{-3/2} e^{m^2 t}}{2^{m+2} \pi^{m+3/2}} \int_r^\delta du \int_r^u ds \, \frac{\sin s}{c(r,s)^{1/2}} \frac{1}{c(s,u)^{1/2}} \Big(\frac{\partial}{\partial u} \frac{-1}{\sin u} \Big)^m u e^{-u^2/4t} + \mathcal{O}_\infty(e^{-1/t})$$
$$= \frac{t^{-3/2} e^{m^2 t}}{2^{m+2} \pi^{m+3/2}} \int_r^\delta du \Big(\frac{\partial}{\partial u} \frac{-1}{\sin u} \Big)^m u e^{-u^2/4t} \int_r^u ds \, \frac{\sin s}{c(r,s)^{1/2} c(s,u)^{1/2}} + \mathcal{O}_\infty(e^{-1/t})$$
$$= \frac{t^{-3/2} e^{m^2 t}}{2^{m+2} \pi^{m+1/2}} \int_r^\delta du \Big(\frac{\partial}{\partial u} \frac{-1}{\sin u} \Big)^m u e^{-u^2/4t} + \mathcal{O}_\infty(e^{-1/t})$$
$$= \frac{t^{-3/2} e^{m^2 t}}{2^{m+2} \pi^{m+1/2}} \frac{1}{\sin r} \Big(\frac{\partial}{\partial r} \frac{-1}{\sin r} \Big)^{m-1} r e^{-r^2/4t} + \mathcal{O}_\infty(e^{-1/t})$$
$$= \frac{t^{-1/2} e^{m^2 t}}{2^{m+1} \pi^{m+1/2}} \Big(\frac{-1}{\sin r} \frac{\partial}{\partial r} \Big)^m e^{-r^2/4t} + \mathcal{O}_\infty(e^{-1/t}),$$

which implies (1.6).

We will now prove (1.4), which says that $k_n(t, r(P,Q))$ approximates $K_n(t, r(P,Q))$ quite well. Namely we have only to show that $k_n(t, r(P,Q))$ is such a good parametrix of $K_n(t, r(P,Q))$.

Proposition 2.1. *If we take small $\delta > 0$, then, for $0 < t \le 1$ and $r = r(P,Q) < \delta$, we have*

$$(2.4) \qquad \Big(\frac{\partial}{\partial r} \Big)^\ell k_n(t,r) = \mathcal{O}(t^{-n/2-\ell/2} e^{-r^2/5t}) \quad (\forall \ell),$$

$$(2.5) \qquad (\partial/\partial t + \Delta_P) k_n(t,r) = \mathcal{O}_\infty(e^{-1/t}).$$

And, for a C^∞-function $f(P)$ with $\operatorname{supp} f \subset \{P \in S^n \mid r(P,Q) < \delta\}$, *we have*

$$(2.6) \qquad \lim_{t \to 0} \int_{S^n} dg(Q)\, f(P) k_n(t, r(P,Q)) = f(Q).$$

If the proposition holds then we can easily show (1.4) by using the standard Duhamel's principle as follows: We take small $\delta_1 > 0$ and a cut-off function ϕ on $[0, \infty)\,(\ni a)$ satisfying $\phi(a) = 1$ if $a \leq \delta_1^2/4$ and $\phi(a) = 0$ if $a \geq \delta_1^2$. Then, for $r(P, Q) < \delta_1/3$, we have

$$
\begin{aligned}
&K_n(t, P, Q) - k_n(t, P, Q)\,(\equiv K_n(t, r(P, Q)) - k_n(t, r(P, Q)))\\
&= K_n(t, P, Q)\,\phi(r(P, Q)^2) - k_n(t, P, Q)\,\phi(r(P, Q)^2)\\
&= -\int_0^t d\tau\,\frac{\partial}{\partial\tau}\int dg(P')\,K_n(t-\tau, P, P')\,k_n(\tau, P', Q)\phi(r(P, P')^2)\phi(r(P', Q)^2)\\
&= -\int_0^t d\tau\int dg(P')\Big\{\Big(\Delta_{P'}K_n(t-\tau, P, P')\Big)\,k_n(\tau, P', Q)\phi(r(P, P')^2)\phi(r(P', Q)^2)\\
&\quad - K_n(t-\tau, P, P')\,\Big(\Delta_{P'}k_n(\tau, P', Q)\Big)\phi(r(P, P')^2)\phi(r(P', Q)^2)\Big\} + \mathcal{O}_\infty(e^{-1/t})\\
&= -\int_0^t d\tau\int dg(P')\Big\{\Big(\Delta_{P'}K_n(t-\tau, P, P')\Big)\,k_n(\tau, P', Q)\phi(r(P, P')^2)\phi(r(P', Q)^2)\\
&\quad - K_n(t-\tau, P, P')\,\Delta_{P'}\Big(k_n(\tau, P', Q)\phi(r(P, P')^2)\phi(r(P', Q)^2)\Big)\\
&\quad - 2K_n(t-\tau, P, P')\,g\Big(d_{P'}(\phi(r(P, P')^2)\phi(r(P', Q)^2)), d_{P'}k_n(\tau, P', Q)\Big)\\
&\quad + K_n(t-\tau, P, P')\Delta_{P'}\Big(\phi(r(P, P')^2)\phi(r(P', Q)^2)\Big)k_n(\tau, P', Q)\Big\} + \mathcal{O}_\infty(e^{-1/t})\\
&= \int_0^t d\tau\int_{r(P', Q)>\delta_1/6}^{r(P, P')>\delta_1/6} dg(P')\\
&\quad \Big\{2K_n(t-\tau, P, P')\,g\Big(d_{P'}(\phi(r(P, P')^2)\phi(r(P', Q)^2)), d_{P'}k_n(\tau, P', Q)\Big)\\
&\quad - K_n(t-\tau, P, P')\Delta_{P'}\Big(\phi(r(P, P')^2)\phi(r(P', Q)^2)\Big)k_n(\tau, P', Q)\Big\} + \mathcal{O}_\infty(e^{-1/t})\\
&= \mathcal{O}_\infty(e^{-1/t}).
\end{aligned}
$$

Thus (1.4) was proved.

Proof of Proposition 2.1. Obviously (1.8) and (1.9) imply (2.4). If we take normal coordinates x at Q then the volume element near Q can be written as $dg = \left(\frac{\sin r}{r}\right)^{n-1} dx$. Hence, using (1.8)–(1.10) we know (2.6) holds, that is, we have

$$
\begin{aligned}
(2.7)\quad &\int_{S^n} dg(P)\,f(P)k_n(t, r(P, Q))\\
&= \int_{\mathbb{R}^n} dx\,\Big(\frac{\sin r}{r}\Big)^{n-1} f(x)\frac{e^{-r^2/4t+(n-1)^2t/4}}{(4\pi t)^{n/2}}\Big\{\Big(\frac{r}{\sin r}\Big)^{(n-1)/2} + \mathcal{O}(t)\Big\}\\
&\to \Big(\frac{\sin r}{r}\Big)^{n-1} f(x)\,e^{(n-1)^2t/4}\Big\{\Big(\frac{r}{\sin r}\Big)^{(n-1)/2} + \mathcal{O}(t)\Big\}\Big|_{(t,x)=(0,0)} = f(0).
\end{aligned}
$$

In order to show (2.5) remark that, near the point Q, the Laplacian acting on the functions depending only on $r = r(P, Q)$ can be expressed as

$$\Delta = -\frac{\partial}{\partial r}\frac{\partial}{\partial r} - (n-1)\cot r \frac{\partial}{\partial r}. \tag{2.8}$$

This comes from, for example, [7, Chapter II Proposition 5.26] and the fact that the volume of $\{P \in S^n \mid r(P,Q) = r\}$ is equal to $\mathrm{vol}(S^{n-1})(\sin r)^{n-1}$. As for (2.5) with $n = 2m+1$: It is enough to prove

$$(\partial/\partial t + \Delta_P)k_n(t,r) = 0. \tag{2.9}$$

Put $\left[\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell}, \sin^2 r\right] = \left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell} \circ \sin^2 r - \sin^2 r \circ \left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell}$, etc. Then inductively we know

$$\begin{aligned}
&\left[\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell}, \sin^2 r\right] = -\ell(\ell-1)\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell-2} + 2\ell\cos r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell-1},\\
&\left[\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell}, \cos r\right] = -\ell\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{\ell-1}.
\end{aligned} \tag{2.10}$$

These together with (2.8) imply that the left hand side of (2.9) is equal to

$$\begin{aligned}
&\frac{(-1)^m}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m\left(-\frac{1}{2t} + m^2 + \frac{r^2}{4t^2}\right)e^{m^2t - r^2/4t}\\
&\quad - \frac{(-1)^m}{(2\pi)^m}\frac{\sin^2 r}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+2}e^{m^2t-r^2/4t}\\
&\quad - \frac{(-1)^m}{(2\pi)^m}\frac{(2m+1)\cos r}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+1}e^{m^2t-r^2/4t}\\
&= \frac{(-1)^m}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m\Big\{\sin^2 r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^2 + \cos r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)\\
&\quad + m^2\Big\}e^{m^2t-r^2/4t} - \frac{(-1)^m}{(2\pi)^m}\frac{\sin^2 r}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+2}e^{m^2t-r^2/4t}\\
&\quad - \frac{(-1)^m}{(2\pi)^m}\frac{(2m+1)\cos r}{(4\pi t)^{1/2}}\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+1}e^{m^2t-r^2/4t}\\
&= \frac{(-1)^m}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\Big\{\left[\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m, \sin^2 r\right]\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^2\\
&\quad + \left[\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m, \cos r\right]\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right) + m^2\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m\\
&\quad - 2m\cos r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+1}\Big\}e^{m^2t-r^2/4t}\\
&= \frac{(-1)^m}{(2\pi)^m}\frac{1}{(4\pi t)^{1/2}}\Big\{-m(m-1)\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m + 2m\cos r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+1}\\
&\quad - m\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m + m^2\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^m - 2m\cos r\left(\frac{1}{\sin r}\frac{\partial}{\partial r}\right)^{m+1}\Big\}e^{m^2t-r^2/4t}\\
&= 0.
\end{aligned}$$

Thus we obtain (2.9). As for (2.5) with $n = 2m+2$: Set

$$b(s,t) = \frac{st^{-3/2}e^{-s^2/4t+(2m+1)^2t/4}}{(2\pi)^{m+3/2}}, \quad c(r,s) = \cos r - \cos s.$$

Then we have

$$\text{(2.11)} \quad \frac{\partial}{\partial t}k_n(t,r) = \int_r^\delta ds\, c(r,s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}\frac{\partial b(s,t)}{\partial t} + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, c(r,s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}\Big\{-\frac{3}{2t} + \frac{(2m+1)^2}{4} + \frac{s^2}{4t^2}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, c(r,s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}\Big\{\frac{(2m+1)^2}{4} + \frac{\partial}{\partial s}\frac{\partial}{\partial s}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t}).$$

And, using the transformations

$$\begin{aligned} k_n(t,r) &= \int_r^\delta ds\, \frac{2}{3}c(r,s)^{3/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}b(s,t) + \mathcal{O}_\infty(e^{-1/t}) \\ &= \int_r^\delta ds\, \frac{4}{15}c(r,s)^{5/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}b(s,t) + \mathcal{O}_\infty(e^{-1/t}) \end{aligned}$$

and the same formulas as (2.10), we have

$$\text{(2.12)} \quad \Delta k_n(t,r) = \int_r^\delta ds\, \Big\{-\sin^2 r\Big(\frac{-1}{\sin r}\frac{\partial}{\partial r}\Big)^2\frac{4}{15}c(r,s)^{5/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}$$

$$+ (2m+2)\cos r\Big(\frac{-1}{\sin r}\frac{\partial}{\partial r}\Big)\frac{2}{3}c(r,s)^{3/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, \Big\{-c(r,s)^{1/2}\sin^2 r\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}$$

$$+ (2m+2)c(r,s)^{1/2}\cos r\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, \Big\{(c(r,s)^{5/2} + 2c(r,s)^{3/2}\cos s - c(r,s)^{1/2}\sin^2 s)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}$$

$$+ (2m+2)(c(r,s)^{3/2} + c(r,s)^{1/2}\cos s)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, c(r,s)^{1/2}\Big\{\frac{12m+15}{4}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}$$

$$+ (2m+5)\cos s\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2} - \sin^2 s\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}\Big\}b(s,t) + \mathcal{O}_\infty(e^{-1/t})$$

$$= \int_r^\delta ds\, c(r,s)^{1/2}\Big\{\frac{12m+15}{4}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} + (2m+5)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}\cos s$$

$$-\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+3}\sin^2 s - (2m+5)(m+2)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}$$

$$
\begin{aligned}
&+(m+3)(m+2)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}-2(m+3)\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+2}\cos s\Big\}b(s,t)\\
&+\mathcal{O}_\infty(e^{-1/t})\\
=&\int_r^\delta ds\, c(r,s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}\Big\{-\frac{(2m+1)^2}{4}-\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)\cos s\\
&-\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^2\sin^2 s\Big\}b(s,t)+\mathcal{O}_\infty(e^{-1/t})\\
=&\int_r^\delta ds\, c(r,s)^{1/2}\Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1}\Big\{-\frac{(2m+1)^2}{4}-\frac{\partial}{\partial s}\frac{\partial}{\partial s}\Big\}b(s,t)+\mathcal{O}_\infty(e^{-1/t}).
\end{aligned}
$$

(2.11) and (2.12) imply (2.5) with $n = 2m+2$. ∎

3. Hyperbolic space and the reason for having expected $k_n(t,r)$ to be an appropriate parametrix

In the section we present some facts having supported the expectation that $k_n(t,r)$ is an appropriate parametrix. Let us consider the hyperbolic space

$$
H^n = \{\xi \in \mathbb{R}^n \mid |\xi| < 2\} \text{ with } g_H = \frac{\sum d\xi_i \otimes d\xi_i}{(1-|\xi|^2/4)^2}. \tag{3.1}
$$

(Notice that, by using a stereographic projection, the metric on S^n near a point Q may be expressed as $g_{S^n} = \dfrac{\sum d\xi_i \otimes d\xi_i}{(1+|\xi|^2/4)^2}$.) As well as (2.8), near a point $Q \in H^n$, the Laplacian acting on the functions depending only on $r = r(P,Q)$ can be expressed as

$$
\Delta^H = -\frac{\partial}{\partial r}\frac{\partial}{\partial r} - (n-1)\coth r\frac{\partial}{\partial r} \tag{3.2}
$$

because the volume of $\{P \in H^n \mid r(P,Q) = r\}$ is equal to $\mathrm{vol}(S^{n-1})(\sinh r)^{n-1}$. And, according to [3] and [6], the heat kernel $K_n^H(t, r(P,Q))$ on the hyperbolic space is expressed explicitly as

$$
K_n^H(t,r) = \begin{cases} \dfrac{e^{-m^2 t}}{(2\pi)^m(4\pi t)^{1/2}}\Big(\dfrac{-1}{\sinh r}\dfrac{\partial}{\partial r}\Big)^m e^{-r^2/4t} & (n = 2m+1), \\[2ex] \dfrac{e^{-(2m+1)^2 t/4}}{2(2\pi)^{m+3/2}t^{3/2}}\Big(\dfrac{-1}{\sinh r}\dfrac{\partial}{\partial r}\Big)^m \displaystyle\int_r^\infty ds\, \frac{s e^{-s^2/4t}}{(\cosh s - \cosh r)^{1/2}} & (n = 2m+2). \end{cases} \tag{3.3}
$$

Observing the correspondence between (2.8) and (3.2), it will be natural to have expected that it is (1.4) that corresponds to (3.3).

As for the expression (3.3), in [6] it was introduced by considering the property of the wave kernel $\cos(t\sqrt{\Delta^H - (n-1)^2/4})$ and the Fourier transform

$$(3.4)\quad e^{-t(\Delta^H-(n-1)^2/4)} = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2/4t}\cos\Big(s\sqrt{\Delta^H-(n-1)^2/4}\Big),$$

and in [3] the authors introduced two kinds of recurrence relations ([3, Theorem 2.1(ii)])

$$(3.5)\qquad K^H_{n+2}(t,r) = \frac{e^{-nt}}{2\pi}\Big(\frac{-1}{\sinh r}\frac{\partial}{\partial r}\Big)K^H_n(t,r),$$

$$(3.6)\qquad K^H_{n+1}(t,r) = \int_r^{\infty} ds\,\frac{2^{1/2}e^{(2n+1)t/4}K^H_{n+2}(t,s)\sinh s}{(\cosh s-\cosh r)^{1/2}}$$

by using the Selberg transform and then obtained the formula (3.3) by using the relations and the obvious formula $K_1^H(t,r) = (4\pi t)^{-1/2}e^{-r^2/4t}$ (see (3.2) with $n = 1$). Here it will be also natural to expect the heat kernels on spheres have the relations (1.5), (1.6). The formula for $K_2^H(t,r)$ was originally introduced in [8] by using the Legendre function theory, etc.: Put $z = \cosh r$ then (3.2) with $n = 2$ is transformed into $(1-z^2)(\partial/\partial z)^2 - 2z\,\partial/\partial z$ which is the differential part of the Legendre differential equation. Using the formula (3.2) and by an elementary calculation, we can easily show the identity

$$(3.7)\qquad (\partial/\partial t+\Delta^H_{n+2})\frac{e^{-nt}}{2\pi}\Big(\frac{-1}{\sinh r}\frac{\partial}{\partial r}\Big) = \frac{e^{-nt}}{2\pi}\Big(\frac{-1}{\sinh r}\frac{\partial}{\partial r}\Big)(\partial/\partial t+\Delta^H_n)$$

which will be almost equivalent to (3.5). The identity and the formulas for $K_n^H(t,r)$ $(n = 1, 2)$ also imply the general formula (3.3).

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