Saitama Math. J.
Vol. 27 (2010), 35–80

The gonality conjecture for curves on toric surfaces with two $\mathbb{P}^1$-fibrations

Ryo Kawaguchi

(*Received March 25, 2010; Revised December 3, 2010*)

Abstract

The gonality is an important invariant for the study of linear systems on a curve. Although it is not so easy to determine the gonality of a given curve, the gonality conjecture posed by Green and Lazarsfeld predicts that the gonality could be read off from the minimal resolution of any one line bundle of sufficiently large degree. In this paper, we consider this conjecture for a curve embedded in a toric surface which has two $\mathbb{P}^1$-fibrations by toric morphisms, and prove it affirmatively under several conditions.

1. Introduction

As mentioned in Abstract, the aim of this paper is to show the gonality conjecture (Conjecture 1.2) under certain conditions. First of all, in this section, we would like to roughly review the background and preliminary results for gonality and the gonality conjecture. Our main theorem will be stated at the end of this section.

In this paper, a *curve* will always mean a nonsingular irreducible complex projective curve unless otherwise stated. We denote by g^1_k a 1-dimensional linear system of degree k on a curve. For a curve C, the gonality is defined to be the minimal degree of surjective morphisms from C to $\mathbb{P}^1$:

$$\mathrm{gon}(C) = \min\{\deg f \mid f : C \to \mathbb{P}^1 \text{ surjective morphism}\} = \min\{k \mid C \text{ has } g^1_k\}.$$

It is known that the gonality of a nonsingular plane curve of degree $d\,(\geq 2)$ is equal to $d-1$ ([11]). Coppens and Kato generalized this result to the case of singular plane curves in [2]. They computed the gonality of the normalization of a plane curve with double points under several numerical conditions on its degree and the number of singular points. Recently, more general cases were investigated by Ohkouchi and Sakai ([13]). Besides, in [9], Martens determined the gonality of a nonsingular curve lying on a Hirzebruch surface.

On the other hand, there exists a close relation between the theory of syzygies and geometric properties of a projective variety. In particular, the *gonality*

2000 Mathematics Subject Classification. Primary 14H51; Secondary 14M25.
Key words and phrases. The gonality conjecture, toric surfaces

conjecture (Conjecture 1.2) suggested by Green and Lazarsfeld in [7] predicted the interaction between the gonality of a curve and the syzygies defined by a line bundle on the curve. For a projective variety X, a line bundle L on X and a coherent sheaf $\mathcal{F}$ on X, we denote by $K_{p,q}(X,\mathcal{F},L)$ the Koszul cohomology, which is introduced by Green in [5] as the cohomology of the Koszul complex

$$\bigwedge^{p+1} H^0(X,L) \otimes H^0(X,\mathcal{F}\otimes(q-1)L) \to \bigwedge^{p} H^0(X,L) \otimes H^0(X,\mathcal{F}\otimes qL) \\ \to \bigwedge^{p-1} H^0(X,L) \otimes H^0(X,\mathcal{F}\otimes(q+1)L).$$

For simplicity, if $\mathcal{F} = \mathcal{O}_X$, we suppress it and write $K_{p,q}(X,L)$. It is known that there exists the following relation between the Koszul cohomology and the syzygies:

Theorem 1.1 ([5])**.** *Let X and L be as above. Denote by S the symmetric algebra of $H^0(X,L)$, and consider the minimal free resolution of a graded S-module as*

$$\cdots \to \bigoplus_{q\geq q_2} S(-q)\otimes M_{2,q} \to \bigoplus_{q\geq q_1} S(-q)\otimes M_{1,q} \to \bigoplus_{q\geq q_0} S(-q)\otimes M_{0,q} \\ \to \bigoplus_{q=0}^{\infty} H^0(X,qL) \to 0.$$

Then $K_{p,q}(X,L)$ is isomorphic to $M_{p,p+q}$ as a complex vector space.

Conjecture 1.2 ([7])**.** *Let C be a curve of genus g and L a line bundle on C of sufficiently large degree compared to $2g$. Then $K_{p,1}(C,L) = 0$ for any integer $p \geq h^0(C,L) - \mathrm{gon}(C)$.*

This conjecture means that we can read off the gonality of a curve from the minimal resolution of any one line bundle of sufficiently large degree. For the cases where $\mathrm{gon}(C) = 1, 2$, Green has shown this conjecture affirmatively in [5]. The case where $\mathrm{gon}(C) = 3$ has been done by Ehbauer ([4]). As for curves on a Hirzebruch surface Σ_a, Aprodu showed it, and computed their gonality independently of Martens' result ([1]). In his proof, the following Theorem 1.4 played an essential role.

Definition 1.3. Let L be a line bundle on a curve C, and q a non-negative integer. We say that L satisfies the property (M_q) if $K_{p,1}(C,L) = 0$ for any integer $p \geq h^0(C,L) - q - 1$.

Theorem 1.4 ([1])**.** *Let C be an irrational curve which has a g^1_k. If there is a nonspecial and globally generated line bundle L on C with the property (M_{k-1}), then C is k-gonal, and the gonality conjecture is true for C.*

Recall that a toric surface has a finite number of $\mathbb{P}^1$-fibrations by toric morphisms, and Hirzebruch surfaces are the simplest examples of toric surfaces which have one or two such $\mathbb{P}^1$-fibrations. In the wake of Aprodu's result, we proved the gonality conjecture for curves lying on a toric surface with a unique $\mathbb{P}^1$-fibration ([8]). This is an extension of Aprodu's result for Σ_a with $a \geq 1$. Therefore, it is a valid question to ask whether we can generalize Aprodu's result for the case $a = 0$, that is, whether the gonality conjecture is true for the cruves on a toric surface with two $\mathbb{P}^1$-fibrations. In this paper, we shall show the following theorem:

Theorem 1.5 (Main theorem). *Let S be a compact nonsingular toric surface which has two $\mathbb{P}^1$-fibrations f and f' by toric morphisms, and C an irrational curve on S. If $2 \leq \deg f|_C < \deg f'|_C$ and $|C - F|$ is base point free (where F denotes the fiber of f), then the gonality conjecture is true for C and $\mathrm{gon}(C) = \deg f|_C$.*

In Section 2, we introduce toric surfaces which are the main stage of our study, and collect basic facts about them. In Section 3, we will see the main idea to prove the main theorem. Although a key point of the proof is the existence of a certain divisor called an auxiliary divisor, we do not mention its concrete construction at all. Instead, we spend Section 4 and 5 to construct an auxiliary divisor. In these two sections, we will treat essentially different cases. We close this section with some basics of Koszul cohomology, which are essential for our study.

Theorem 1.6 ([1]). *Let X be a nonsingular projective variety, L a line bundle on X and $Y \in |L|$ an irreducible divisor on X. If the irregularity of X is zero, then $K_{p,1}(X, L) \simeq K_{p,1}(Y, L|_Y)$ for any integer p.*

Theorem 1.7 ([5]). *Let L be a line bundle on a curve C and put $m = \dim\varphi_{|L|}(C)$. Then L satisfies (M_{m-1}).*

Theorem 1.8 ([1]). *Let C be an irrational curve, L a nonspecial and globally generated line bundle on C, and q a non-negative integer such that L satisfies (M_q). Then, for any effective divisor D on C, also $L + D$ satisfies (M_q).*

2. Summary of toric surfaces

In this section, we briefly review the theory of toric surfaces. It has the close connection with the geometry of convex polytopes in the real affine space. Many basic properties of toric varieties and divisors on them can be interpreted in terms of the elementary convex geometry.

We will henceforth assume that a surface is always compact and nonsingular.

At first, it is a basic fact that a toric surface is obtained from the projective plane or a Hirzebruch surface by a finite succession of blowing-ups with T-fixed points (i.e. points which are invariant with respect to the action on a toric surface by the algebraic torus T) as centers. A composite of such blowing-ups is called a *toric morphism*. For a toric surface (except for the projective plane), a $\mathbb{P}^1$-fibration obtained by composing a toric morphism and a ruling of the Hirzebruch surface is called a *toric ruling*. Naturally, a toric surface has a finite number of toric rulings.

In the present paper, we consider a toric surface S with two toric rulings. In this case, the fan Δ_S associated to S has two lines passing through the origin. For each half-line (which is called a *cone*) in Δ_S, a lattice point on it is called a *primitive element* if it is closest to the origin. We denote by D_1 the T-invariant divisor whose associated cone has a primitive element $(0,1)$ and number other cones in Δ_S clockwise. In particular, we set D_{d_0} if its associated cone has a primitive element $(0,-1)$ (see Fig. 1). We denote by $\sigma(D_i)$ the cone associated

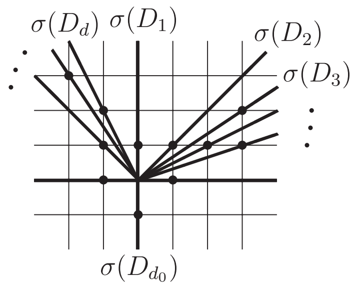

Figure 1

to T-invariant divisor D_i and by (x_i, y_i) the primitive element of $\sigma(D_i)$. We set $D_0 = D_d$ and $D_{d+1} = D_1$ formally. Then the self-intersection numbers of T-invariant divisors are computed by the following formula.

Theorem 2.1 ([12]). *For any integer i with $1 \leq i \leq d$, two equalities $x_i D_i^2 = -x_{i-1} - x_{i+1}$ and $y_i D_i^2 = -y_{i-1} - y_{i+1}$ hold.*

The Picard group of S is generated (not freely) by the classes of $D_1, \dots, D_d$. For instance, the canonical divisor of S has the relation $K_S \sim -\sum_{i=1}^{d} D_i$. Besides, the fibers of two toric rulings can be written as

$$F_1 \sim \sum_{i=2}^{d_0-1} x_i D_i \sim -\sum_{i=d_0+1}^{d} x_i D_i$$

$$F_2 \sim D_{d_0} \sim \sum_{i=1}^{d_0-2} y_i D_i + \sum_{i=d_0+2}^{d} y_i D_i.$$

We denote by f_1 (resp. f_2) the toric ruling of S whose fiber is F_1 (resp. F_2).

Next we collect several basic properties of divisors on a toric surface. In the remaining part of this section, let D be a divisor on S. In the case where the complete linear system $|D|$ is base point free, we have the following two results.

Theorem 2.2 ([12]). *If $|D|$ is base point free, then $h^i(S, D) = 0$ for any positive integer i.*

Theorem 2.3 ([10]). *The complete linear system $|D|$ is base point free if and only if D has a non-negative intersection number with every T-invariant divisor on S.*

Let $\square_D$ be a lattice polytope associated to $D = \sum_{i=1}^{d} n_i D_i$, that is, we define

$$\square_D = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w \leq n_i \text{ for } 1 \leq i \leq d\}.$$

Although this definition depends on the description of the linear equivalence class of D, differences of the description cause only parallel translations of $\square_D$. Hence the shape of the lattice polytope is determined uniquely. We write the lines which form the boundaries of $\square_D$ as

$$l_i(D) = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w = n_i\}.$$

We can read off the dimension of cohomology group of D from the number of lattice points contained in $\square_D$:

Theorem 2.4 ([12]). *The equation $h^0(S, D) = \sharp(\square_D \cap \mathbb{Z}^2)$ holds. In particular, for a curve C on S, its genus is equal to the number of the lattice points contained in the interior of $\square_C$.*

Proposition 2.5. *Let C be a curve on S and assume that $C.F_1, C.F_2 \geq 2$. If either $|C - F_1|$ or $|C - F_2|$ is base point free, then C is irrational.*

Proof. We prove the case where $\mathrm{Bs}|C - F_1| = \emptyset$. By Theorem 2.4, it is sufficient to verify that there exists at least one lattice point in the interior of $\square_C$. We first note that our assumption and Theorem 2.3 imply that $(C - F_1).D_1 = C.D_1 - 1 \geq 0$. Considering the construction of $\square_C$, in the case where $C.D_1 \geq 2$, we see that the lattice point $l_1(C) \cap l_2(C) + (-1, -1)$ is contained in the interior of $\square_C$. In the case where $C.D_1 = 1$, we put $a = \min\{i \geq 2 \mid C.D_i \geq 1\}$ and $b = \max\{i \leq d \mid C.D_i \geq 1\}$. The inequality

$$C.F_2 = 1 + \sum_{i=2}^{d_0-2} y_i C.D_i + \sum_{i=d_0+2}^{d} y_i C.D_i \geq 2$$

implies that either $a \leq d_0 - 2$ or $b \geq d_0 + 2$ holds. We see that the lattice point

$l_1(C) \cap l_2(C) + (0, -1)$ (resp. $l_1(C) \cap l_d(C) + (0, -1)$) is contained in the interior of $\square_C$ if $a \leq d_0 - 2$ (resp. $b \geq d_0 + 2$). $\square$

3. Outline of the proof

In order to prove Theorem 1.5, we need to introduce a certain divisor defined with respect to a given curve, which is called an auxiliary divisor. Its construction is, however, very complicated. Hence, in this section, we admit the existence of auxiliary divisors satisfying Assertion 3.1–3.3 below, and will prove Theorem 1.5. We postpone their precise construction to Section 4 (the case where $\deg f_1|_C \leq \deg f_2|_C$) and Section 5 (the case where $\deg f_1|_C \geq \deg f_2|_C$).

Let S be a toric surface as in Section 2, and C a curve on S. We denote by F and F' the fibers of two toric rulings of S, and put $k = C.F$ and $k' = C.F'$.

Assertion 3.1. *Assume that $k' \geq k \geq 2$ and $|C - F|$ is base point free. Then there exists an effective divisor I_C satisfying the following properties* (i)–(vi) *:*

(i) $I_C.F = 1$,

(ii) $H^1(S, -I_C) = 0$,

(iii) *The complete linear systems $|C - I_C|$ and $|C - I_C - F|$ are base point free,*

(iv) $(C - I_C)^2 \geq 1$,

(v) $(C - I_C)|_C$ *is nonspecial,*

(vi) $K_{p,1}(S, C - I_C) = 0$ *holds for any integer $p \geq h^0(S, C - I_C) - 2$.*

We call I_C the auxiliary divisor of C. By the properties (iii) and (iv), we can take a nonsingular irreducible curve $C_1 \in |C - I_C|$. If $C_j.F' \geq C_j.F \geq 2$, we can take inductively

$$\begin{aligned} I_j &: \text{the auxiliary divisor of } C_j, \\ C_{j+1} &: \text{a nonsingular irreducible curve in } |C_j - I_j| \end{aligned}$$

for a positive integer j. We put $C_0 = C$ and $I_0 = I_C$. In this section, we admit the following two assertions.

Assertion 3.2. *Assume that $k' > k \geq 2$ and $|C - F|$ is base point free. There exists a positive integer $m_0 \leq k$ such that $C_j.F' = C_j.F$ for $j = 1, \ldots, m_0 - 1$, and moreover, the inequality $C_{m_0}.F' > C_{m_0}.F$ holds if $m_0 \leq k - 1$.*

Assertion 3.3. *Assume that $k' > k \geq 2$ and $|C - F|$ is base point free, and let m_0 be a positive integer in Assertion 3.2. Then the inequality*

$$h^0(S, C_j) - h^0(S, C_j - I_{j-1}) \geq C_j.F + 2$$

holds for any integer j with $1 \leq j \leq \min\{m_0, k-1\}$. Besides, if $k' = k = 2$, then this inequality holds for $j = 1$.

We have the following lemma, which can be shown by the same argument as in the proof of Lemma 2.3 in [8].

Lemma 3.4. *The isomorphism $K_{p,1}(S, C - I_C) \simeq K_{p,1}(C, (C - I_C)|_C)$ holds for any integer $p \geq h^0(S, C - 2I_C) + 1$.*

Theorem 1.5 follows immediately from the following proposition and Theorem 1.4.

Proposition 3.5. *Assume that C is irrational, $k' > k \geq 2$ and $|C - F|$ is base point free. If Assertion 3.1–3.3 hold, then $\mathcal{O}_C(C)$ satisfies the property (M_{k-1}).*

Proof. We first show the case where $k = 2$. By Assertion 3.1 and Lemma 3.4, we see that $K_{p,1}(C, C_1|_C) = 0$ for any integer $p \geq \max\{h^0(S, C_1) - 2, h^0(S, C_1 - I_0) + 1\}$. Hence, by combining this fact with Assertion 3.3, we see that $K_{p,1}(C, C_1|_C) = 0$ for any integer $p \geq h^0(S, C_1) - 2$. The short exact sequence of sheaves $0 \to \mathcal{O}_S(-I_0) \to \mathcal{O}_S(C_1) \to \mathcal{O}_C(C_1) \to 0$ induces the cohomology long exact sequence

$$0 \to H^0(S, -I_0) \to H^0(S, C_1) \to H^0(C, C_1|_C) \to H^1(S, -I_0) \to \cdots .$$

Since $H^0(S, -I_0) = H^1(S, -I_0) = 0$, we have $h^0(S, C_1) = h^0(C, C_1|_C)$. In sum, we can conclude that $K_{p,1}(C, C_1|_C) = 0$ for any integer $p \geq h^0(C, C_1|_C) - 2$, that is, $\mathcal{O}_C(C_1)$ satisfies (M_1). Note that $\mathcal{O}_C(C_1)$ is nonspecial and globally generated by Assertion 3.1. Hence Theorem 1.8 implies that also $\mathcal{O}_C(C)$ satisfies (M_1). Here we note that in fact this lemma holds in the case where $k' = k = 2$ also. Indeed, for such case, we can develop the same argument as above by using the latter part of Assertion 3.3.

We next consider the case where $k \geq 3$ under the assumption that our proposition is valid for a curve C' on S if $C'.F \leq k - 1$, $C'.F' > C'.F \geq 2$ and $\mathrm{Bs}|C' - F| = \emptyset$. For C, let m_0 be a positive integer as in Assertion 3.2. If $m_0 \leq k - 2$, then $2 \leq C_{m_0}.F \leq k - 1$. Since $|C_{m_0} - F|$ is base point free by Assertion 3.1, we see that $\mathcal{O}_{C_{m_0}}(C_{m_0})$ satisfies $(M_{C_{m_0}.F-1})$ by the hypothesis of our induction. Consider the case where $m_0 \geq k-1$. We have $C_{k-2}.F' = C_{k-2}.F = 2$ by the property of m_0. Since $|C_{k-2} - F|$ is base point free by Assertion 3.1, we see that $\mathcal{O}_{C_{k-2}}(C_{k-2})$ satisfies (M_1). In sum, in any case, we can conclude that there is an integer $m \leq k-2$ such that $\mathcal{O}_{C_m}(C_m)$ satisfies $(M_{C_m.F-1})$. Note that $C_0, \ldots, C_m$ are irrational by Proposition 2.5 and (iii) in Assertion 3.1.

By Theorem 1.6, we have $K_{p,1}(C_m, C_m|_{C_m}) \simeq K_{p,1}(S, C_m)$ for any integer p. On the other hand, the short exact sequence of sheaves $0 \to \mathcal{O}_S \to \mathcal{O}_S(C_m) \to \mathcal{O}_{C_m}(C_m) \to 0$ induces the cohomology long exact sequence

(1) $0 \to H^0(S, \mathcal{O}_S) \to H^0(S, C_m) \to H^0(C_m, C_m|_{C_m}) \to H^1(S, \mathcal{O}_S) \to \cdots.$

Since $H^0(S, \mathcal{O}_S) = \mathbb{C}$ and $H^1(S, \mathcal{O}_S) = 0$, we obtain $h^0(C_m, C_m|_{C_m}) = h^0(S, C_m) - 1$. Hence the vanishing $K_{p,1}(S, C_m) = 0$ holds for any integer $p \geq h^0(S, C_m) - C_m.F - 1$. Then by the same argument as in the case where $k = 2$, one can show that $\mathcal{O}_{C_{m-1}}(C_{m-1})$ satisfies $(M_{C_{m-1}.F-1})$. We can inductively verify that $\mathcal{O}_{C_j}(C_j)$ satisfies $(M_{C_j.F-1})$ for integers $0 \leq j \leq m-2$. The case where $j = 0$ is the statement of our proposition. $\square$

4. The case where $\deg f_1|_C \leq \deg f_2|_C$

Let S be a toric surface defined in Section 2, and C a curve on S. In this section, we consider the case where $\deg f_1|_C \leq \deg f_2|_C$, that is, $C.F_1 \leq C.F_2$. We put $F = F_1$ and $F' = F_2$. The aim of this section is to construct a divisor which satisfies Assertion 3.1–3.3.

4.1 Division of the primitive element

First of all, we remark a basic fact about primitive elements of cones.

Fact 4.1. *Let i be an integer with $2 \leq i \leq d_0 - 1$, and define $c = \min\{c' \in \mathbb{Z} \mid c'x_i - y_i \geq 0\}$. Then there is an integer r with $2 \leq r \leq i$ such that $(x_r, y_r) = (1, c)$. In particular, if $x_i \geq 2$, then $r \leq i-1$ and there exists a pair of integers (s, t) with $r \leq s < i < t \leq d_0 - 1$ such that $(x_i, y_i) = (x_s, y_s) + (x_t, y_t)$ and $x_t y_s - y_t x_s = 1$.*

Remark 4.2. Let i be an integer with $2 \leq i \leq d_0 - 1$. If $x_i \geq 2$, then $d_0 \geq 5$, $3 \leq i \leq d_0 - 2$, and moreover, the equality $c = \min\{c' \in \mathbb{Z} \mid c'x_s - y_s \geq 0\}$ holds in Fact 4.1.

The following lemma gives us more detailed properties of the above division.

Lemma 4.3. *For an integer i with $3 \leq i \leq d_0 - 2$ such that $x_i \geq 2$, the pair (s, t) in Fact 4.1 is uniquely determined. If $x_s \geq 2$, then we can divide (x_s, y_s) into the sum of primitive elements as*

$$(x_s, y_s) = (x_u, y_u) + (x_v, y_v) \qquad (r \leq u < s < v \leq d_0 - 1,\ x_v y_u - y_v x_u = 1)$$

by Fact 4.1. Then the inequality $v \geq t$ holds.

Proof. Assume that $r \leq s < i < t \leq d_0 - 1$, $r \leq s' < i < t' \leq d_0 - 1$,

$$(x_i, y_i) = (x_s, y_s) + (x_t, y_t) = (x_{s'}, y_{s'}) + (x_{t'}, y_{t'})$$

and $x_t y_s - y_t x_s = x_{t'} y_{s'} - y_{t'} x_{s'} = 1$. We put $x = x_{t'} - x_t$ and $y = y_{t'} - y_t$. Since $x_t y_i - y_t x_i = x_{t'} y_i - y_{t'} x_i = 1$, we have $x_i y - y_i x = 0$. Hence there exists an integer n such that $x = n x_i$ and $y = n y_i$. We thus have $x_i = x_s + x_t = x_s + x_{t'} - n x_i$, which implies $n \geq 0$. Similarly, the equation $x_i = x_{s'} + x_{t'} = x_{s'} + nx_i + x_{t'}$ implies $n \leq 0$. Hence we have $n = 0$, that is, $(x_{s'}, y_{s'}) = (x_s, y_s)$ and $(x_{t'}, y_{t'}) = (x_t, y_t)$.

Next we will show the latter part of the lemma. We put $x' = x_t - x_v$ and $y' = y_t - y_v$. Since $x'y_s - y'x_s = 0$, there exists an integer m such that $x' = mx_s$ and $y' = my_s$. Then the equation $x_s = x_u + x_v = x_u + x_t - mx_s$ implies $m \geq 0$. Hence we have $x_v y_t - y_v x_t = x_v y' - y_v x' = m \geq 0$, which means $v \geq t$. □

4.2 The auxiliary divisor

Let S, C, F and F' be as above, and put $k = C.F$ and $k' = C.F'$. If $k \geq 1$, we define

$$\begin{aligned}
\lambda &= \max\{i \leq d_0 - 1 \mid C.D_i \geq 1\},\\
\mu &= \min\{i \geq d_0 + 1 \mid C.D_i \geq 1\},\\
c_0 &= \min\{c \in \mathbb{Z} \mid cx_\lambda - y_\lambda \geq 0\},\\
e_0 &= -\max\{e \in \mathbb{Z} \mid ex_\mu - y_\mu \geq 0\}.
\end{aligned}$$

If $x_\lambda \geq 2$, then by Fact 4.1 and Lemma 4.3, we can divide primitive elements repeatedly as

$$\begin{aligned}
(x_\lambda, y_\lambda) &= (x_{s_1}, y_{s_1}) + n_{\alpha_1}(x_{\alpha_1}, y_{\alpha_1}),\ x_{\alpha_1} y_{s_1} - y_{\alpha_1} x_{s_1}\\
&= 1\,(s_1 < \lambda < \alpha_1 < d_0 - 1)\,,\\
(x_{s_1}, y_{s_1}) &= (x_{s_2}, y_{s_2}) + n_{\alpha_2}(x_{\alpha_2}, y_{\alpha_2}),\ x_{\alpha_2} y_{s_2} - y_{\alpha_2} x_{s_2}\\
&= 1\,(s_2 < s_1,\ \alpha_1 < \alpha_2 < d_0 - 1)\,,\\
&\ \ \vdots\\
(x_{s_{a_0-1}}, y_{s_{a_0-1}}) &= (1, c_0) + n_{\alpha_{a_0}}(x_{\alpha_{a_0}}, y_{\alpha_{a_0}}),\ x_{\alpha_{a_0}} c_0 - y_{\alpha_{a_0}}\\
&= 1\,(\alpha_{a_0-1} < \alpha_{a_0} \leq d_0 - 1)\,.
\end{aligned}$$

On the other hand, in the case where $x_\lambda = 1$, it is obvious that the equation $(x_\lambda, y_\lambda) = (1, c_0)$ holds. Consequently, we obtain the unique division

$$(x_\lambda, y_\lambda) = (1, c_0) + n_{\alpha_1}(x_{\alpha_1}, y_{\alpha_1}) + \cdots + n_{\alpha_{a_0}}(x_{\alpha_{a_0}}, y_{\alpha_{a_0}}), \tag{2}$$

where $\lambda < \alpha_1 < \cdots < \alpha_{a_0} \leq d_0 - 1$. Considering the definition of the division and Fact 4.1, one can obtain the equality

$$x_{\alpha_a}\left(c_0 + \sum_{j=a+1}^{a_0} n_{\alpha_j} y_{\alpha_j}\right) - y_{\alpha_a}\left(1 + \sum_{j=a+1}^{a_0} n_{\alpha_j} x_{\alpha_j}\right) = 1 \tag{3}$$

for each integer $1 \leq a \leq a_0$.

Similarly, we can divide

$$(x_\mu, y_\mu) = (-1, e_0) + n_{\beta_1}(x_{\beta_1}, y_{\beta_1}) + \cdots + n_{\beta_{b_0}}(x_{\beta_{b_0}}, y_{\beta_{b_0}}), \tag{4}$$

where $d_0+1 \le \beta_1 < \cdots < \beta_{b_0} < \mu$. The equality

$$\Big(-1+\sum_{j=1}^{b-1} m_{\beta_j} x_{\beta_j}\Big) y_{\beta_b} - \Big(e_0+\sum_{j=1}^{b-1} m_{\beta_j} y_{\beta_j}\Big) x_{\beta_b} = 1 \tag{5}$$

holds for each integer $1 \le b \le b_0$.

Definition 4.4. Let C be a curve on S such that $k \ge 1$, and set $n_i = 0$ for integers $\lambda+1 \le i \le d_0$ except for $i = \alpha_1, \ldots, \alpha_{a_0}$ and $m_i = 0$ for integers $d_0+1 \le i \le \mu-1$ except for $i = \beta_1, \ldots, \beta_{b_0}$. We define the *auxiliary divisor* $I_C = \sum_{i=1}^{d} p_i D_i$ of C as follows:

$$p_i = \begin{cases} 0 & (1 \le i \le \lambda,\ \mu \le i \le d), \\ x_i\Big(c_0+\sum_{j=i+1}^{d_0-1} n_j y_j\Big)+y_i\Big(-1-\sum_{j=i+1}^{d_0-1} n_j x_j\Big) & (\lambda+1 \le i \le d_0), \\ x_i\Big(-e_0-\sum_{j=d_0+1}^{i-1} m_j y_j\Big)+y_i\Big(-1+\sum_{j=d_0+1}^{i-1} m_j x_j\Big) & (d_0+1 \le i \le \mu-1). \end{cases}$$

We note that (3), (5) and an easy computation imply that $p_i = 1$ for $i = \alpha_1, \ldots, \alpha_{a_0}, d_0, \beta_1, \ldots, \beta_{b_0}$. Besides, it is obvious that $I_C.F = 1$ by definition. The rest of this subsection is devoted to verify that I_C satisfies Assertion 3.1.

Proposition 4.5. *The divisor I_C is effective.*

Proof. Let i_1 be an integer with $\lambda+1 \le i_1 \le d_0-1$. Since $x_\lambda y_i - y_\lambda x_i < 0$ for any integer i with $\lambda+1 \le i \le d_0-1$, we have

$$\begin{aligned} & x_\lambda\Big(c_0+\sum_{i=i_1+1}^{d_0-1} n_i y_i\Big)-y_\lambda\Big(1+\sum_{i=i_1+1}^{d_0-1} n_i x_i\Big) \\ &= x_\lambda\Big(c_0+\sum_{i=\lambda+1}^{d_0-1} n_i y_i\Big)-y_\lambda\Big(1+\sum_{i=\lambda+1}^{d_0-1} n_i x_i\Big)-\sum_{i=\lambda+1}^{i_1} n_i(x_\lambda y_i - y_\lambda x_i) \ge x_\lambda y_\lambda - y_\lambda x_\lambda = 0. \end{aligned}$$

Hence we have the inequalities

$$\frac{y_{i_1}}{x_{i_1}} < \frac{y_\lambda}{x_\lambda} \le \frac{c_0+\sum_{i=i_1+1}^{d_0-1} n_i y_i}{1+\sum_{i=i_1+1}^{d_0-1} n_i x_i},$$

which implies that $p_{i_1} > 0$. Let i_2 be an integer with $d_0+1 \le i_2 \le \mu-1$. Then the inequality

$$x_\mu\Big(e_0+\sum_{i=d_0+1}^{i_2-1} m_i y_i\Big)+y_\mu\Big(1-\sum_{i=d_0+1}^{i_2-1} m_i x_i\Big) = x_\mu y_\mu - y_\mu x_\mu - \sum_{i=i_2}^{\mu-1} m_i(x_\mu y_i - y_\mu x_i) \le 0$$

implies that

$$\frac{y_{i_2}}{x_{i_2}} > \frac{y_\mu}{x_\mu} \geq \frac{-e_0 - \sum_{i=d_0+1}^{i_2-1} m_i y_i}{1 - \sum_{i=d_0+1}^{i_2-1} m_i x_i}.$$

We thus obtain $p_{i_2} > 0$. □

Proposition 4.6. *The complete linear system $|C - I_C|$ is base point free.*

Proof. We put $H = C - I_C$. By Theorem 2.3, it is sufficient to verify that $H.D_i \geq 0$ for each integer $1 \leq i \leq d$. For an integer i with $1 \leq i \leq \lambda - 1$ or $\mu + 1 \leq i \leq d$, since $I_C.D_i = 0$, we have $H.D_i = C.D_i \geq 0$. For λ, we have

$$\begin{aligned} I_C.D_\lambda = p_{\lambda+1} &= x_{\lambda+1}\Bigg(c_0 + \sum_{i=\lambda+2}^{d_0-1} n_i y_i\Bigg) + y_{\lambda+1}\Bigg(-1 - \sum_{i=\lambda+2}^{d_0-1} n_i x_i\Bigg) \\ &= x_{\lambda+1}\Bigg(c_0 + \sum_{i=\lambda+1}^{d_0-1} n_i y_i\Bigg) + y_{\lambda+1}\Bigg(-1 - \sum_{i=\lambda+1}^{d_0-1} n_i x_i\Bigg) = x_{\lambda+1}y_\lambda - y_{\lambda+1}x_\lambda = 1 \end{aligned}$$

and $H.D_\lambda = C.D_\lambda - 1 \geq 0$.

For the case where $i = d_0$, since $(c_0 - 1)x_{\lambda_0} < y_{\lambda_0}$, $(e_0 - 1)x_{\mu_0} < -y_{\mu_0}$ and $D_{d_0}^2 = 0$, easy computations give

$$\begin{aligned} C.D_{d_0} &= C.\Bigg(\sum_{i=1}^{d_0-2} y_i D_i + \sum_{i=d_0+2}^{d} y_i D_i\Bigg) \\ &\geq C.(D_1 + y_{\lambda_0} D_{\lambda_0} + y_{\mu_0} D_{\mu_0}) \geq C.D_1 + c_0 + e_0, \end{aligned} \tag{6}$$

$$I_C.D_{d_0} = \begin{cases} c_0 + e_0 & (\lambda + 2 \leq d_0 \leq \mu - 2), \\ c_0 & (\lambda + 2 \leq d_0 = \mu - 1), \\ e_0 & (\lambda + 1 = d_0 \leq \mu - 2), \\ 0 & (\lambda + 1 = d_0 = \mu - 1). \end{cases}$$

Hence we have $H.D_{d_0} \geq 0$.

We next consider the case where $i = \lambda + 1$. Since the case where $i = d_0$ was already checked, we assume that $\lambda + 1 < d_0$. By Theorem 2.1, we have

$$\begin{aligned} I_C.D_{\lambda+1} &= p_{\lambda+1} D_{\lambda+1}^2 + p_{\lambda+2} \\ &= -(x_\lambda + x_{\lambda+2})\Bigg(c_0 + \sum_{i=\lambda+2}^{d_0-1} n_i y_i\Bigg) + (y_\lambda + y_{\lambda+2})\Bigg(1 + \sum_{i=\lambda+2}^{d_0-1} n_i x_i\Bigg) \\ &\quad + x_{\lambda+2}\Bigg(c_0 + \sum_{i=\lambda+3}^{d_0-1} n_i y_i\Bigg) - y_{\lambda+2}\Bigg(1 + \sum_{i=\lambda+3}^{d_0-1} n_i x_i\Bigg) \\ &= -x_\lambda\Bigg(c_0 + \sum_{i=\lambda+1}^{d_0-1} n_i y_i\Bigg) + y_\lambda\Bigg(1 + \sum_{i=\lambda+1}^{d_0-1} n_i x_i\Bigg) \end{aligned} \tag{7}$$

$$
\begin{aligned}
&\quad + n_{\lambda+1}(x_\lambda y_{\lambda+1} - y_\lambda x_{\lambda+1}) \\
&= -n_{\lambda+1}.
\end{aligned}
$$

Namely, we have $H.D_{\lambda+1} \geq 0$. For an integer i with $\lambda + 2 \leq i \leq d_0 - 1$, we have $I_C.D_i = -n_i$ by a computation similar to (7).

Let us consider the case where $i = \mu$. If $\mu \geq d_0 + 2$, then we have

$$
\begin{aligned}
I_C.D_\mu &= p_{\mu-1} = x_{\mu-1}\Big(- e_0 - \sum_{j=d_0+1}^{\mu-2} m_j y_j \Big) + y_{\mu-1}\Big(- 1 + \sum_{j=d_0+1}^{\mu-2} m_j x_j \Big) \\
&= -x_{\mu-1} y_\mu + y_{\mu-1} x_\mu = 1.
\end{aligned}
$$

If $\mu = d_0 + 1$, then we have $I_C.D_\mu = p_{d_0} = 1$. Hence we have $H.D_\mu = C.D_\mu - 1 \geq 0$.

Consider the case where $i = d_0 + 1$ under the assumption that $d_0 + 1 \neq \mu$. If $\mu \geq d_0 + 3$, then

$$
\begin{aligned}
&I_C.D_{d_0+1} \\
&\quad = 1 - e_0 x_{d_0+1} D_{d_0+1}^2 + x_{d_0+2}(-e_0 - m_{d_0+1} y_{d_0+1}) + y_{d_0+2}(-1 + m_{d_0+1} x_{d_0+1}) \\
&\quad = 1 + e_0 x_{d_0} - m_{d_0+1} - y_{d_0+2} = -m_{d_0+1}.
\end{aligned}
$$

If $\mu = d_0 + 2$, then we have $e_0 = 1$ and $m_{d_0+1} = -x_{d_0+2} - 1$ by definition. We thus have $I_C.D_{d_0+1} = p_{d_0} + p_{d_0+1} D_{d_0+1}^2 = 1 - x_{d_0+1} D_{d_0+1}^2 = -m_{d_0+1}$. Hence, in any case, the inequality $H.D_{d_0+1} \geq 0$ holds. Let i be an integer with $d_0 + 2 \leq i \leq \mu - 2$. Then we have $I_C.D_i = -m_i$ and $H.D_i \geq 0$ by computing similarly to (7).

Lastly, let us check the case where $i = \mu - 1$. Since the cases of d_0 and $d_0 + 1$ were already checked, we assume that $\mu - 1 \geq d_0 + 2$. Then we have $I_C.D_{\mu-1} = -m_{\mu-1}$ by computing. Namely, we have $H.D_{\mu-1} \geq 0$. □

We list the intersection numbers of I_C and the T-invariant divisors for the later use.

$$
I_C.D_i = \begin{cases} -n_i & (i = \alpha_1, \ldots, \alpha_{a_0}), \\ 1 & (i = \lambda, \mu), \\ -m_i & (i = \beta_1, \ldots, \beta_{b_0}), \\ c_0 + e_0 & (i = d_0), \\ 0 & (\text{otherwise}). \end{cases} \tag{8}
$$

Proposition 4.7. *If $|C - F|$ is base point free, then also $|C - I_C - F|$ is base point free.*

Proof. Since

$$
(C - I_C - F).D_i = \begin{cases} (C - I_C).D_i - 1 & (i = 1, d_0), \\ (C - I_C).D_i \geq 0 & (\text{otherwise}), \end{cases}
$$

it is sufficient to verify that $(C-I_C).D_1$ and $(C-I_C).D_{d_0}$ are positive. Since $(C-F).D_1 = C.D_1 - 1 \geq 0$ and $I_C.D_1 = 0$, we deduce that $(C-I_C).D_1$ is positive. Besides, by an easy computation, we have

$$(C-I_C).D_{d_0} = (C-I_C).\bigg(\sum_{i=1}^{d_0-1} y_i D_i + \sum_{i=d_0+1}^{d} y_i D_i\bigg) \geq (C-I_C).D_1 \geq 1. \qquad \square$$

We next check the property (ii) in Assertion 3.1.

Proposition 4.8. *The vanishing $H^1(S,-I_C)=0$ holds.*

Proof. By Riemann-Roch theorem and Theorem 2.4, we have

$$\begin{aligned} h^1(S,-I_C) &= h^0(S,-I_C) + h^0(S,K_S+I_C) - \frac{1}{2}I_C.(K_S+I_C) - 1 \\ &= -\frac{1}{2}I_C.(K_S+I_C) - 1. \end{aligned}$$

Recall the remark after Definition 4.4 and (8). Then

$$\begin{aligned} &I_C.(K_S+I_C) \\ &= I_C.\bigg(-D_\lambda + \sum_{j=1}^{a_0}(p_{\alpha_j}-1)D_{\alpha_j} + \sum_{j=1}^{b_0}(p_{\beta_j}-1)D_{\beta_j} - D_\mu + (p_{d_0}-1)D_{d_0}\bigg) \\ &= I_C.(-D_\lambda - D_\mu) = -2. \end{aligned} \qquad \square$$

Proposition 4.9. *Assume that $k \geq 2$ and $|C-F|$ is base point free. Then the divisor $C-I_C$ satisfies the properties* (iv)–(vi) *in Assertion 3.1.*

Proof. (iv) We put $H = C - I_C$ and write $H \sim \sum_{i=2}^{d-1} h_i D_i$. As we saw in the Proposition 4.6, $|H|$ is base point free. By computing (using Theorem 2.1 and 2.3), we see that h_i is non-negative for each integer $2 \leq i \leq d-1$. We have $h_{d_0} = H.F = k-1 \geq 1$. Hence by (6) and (8), we have

$$H^2 \geq (C-I_C).D_{d_0} \geq C.D_{d_0} - c_0 - e_0 \geq C.D_1 = (C-F).D_1 + F.D_1 \geq 1.$$

(v) The short exact sequence of sheaves $0 \to \mathcal{O}_S(-I_C) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0$ induces the cohomology long exact sequence

$$\cdots \to H^1(S,H) \to H^1(C,H|_C) \to H^2(S,-I_C) \to \cdots.$$

Since $|H|$ is base point free, we have $H^1(S,H)=0$ by Theorem 2.2. On the other hand, since $p_1 = 0$ and $p_{d_0} = 1$, we have $H^2(S,-I_C) = H^0(S,K_S+I_C) = 0$ by Theorem 2.4. We thus obtain $H^1(C,H|_C)=0$.
(vi) We can take a nonsingular irreducible curve $C_1 \in |H|$ by (iii), (iv) and Bertini's theorem. Then Theorem 1.7 shows that $K_{p,1}(C_1, C_1|_{C_1}) = 0$ for any

integer $p \geq h^0(C_1, C_1|_{C_1}) - 1$. Considering the cohomology long exact sequence similar to (1), we obtain $h^0(C_1, C_1|_{C_1}) = h^0(S, C_1) - 1$. Thus we see that $K_{p,1}(C_1, C_1|_{C_1}) = 0$ for any integer $p \geq h^0(S, C_1) - 2$. On the other hand, by Theorem 1.6, $K_{p,1}(C_1, C_1|_{C_1}) \simeq K_{p,1}(S, C_1)$ holds for any integer p. $\square$

The following lemma is necessary to prove Assertion 3.3.

Lemma 4.10. *Under the same assumption as in Proposition 4.9, we can take a nonsingular irreducible curve $C_1 \in |C - I_C|$. The divisor $(C - 2I_C)|_{C_1}$ is nonspecial.*

This follows from the inequality

$$\deg(C_1 - I_C)|_{C_1} - 2g_1 = C_1.(-I_C - K_S) - 2 \geq C_1.D_1 - 2 \geq -1,$$

where g_1 denotes the genus of C_1. Here we used the equalities

$$\begin{cases} p_i = 0 & (1 \leq i \leq \lambda, \mu \leq i \leq d), \\ p_i = 1 & (i = \alpha_1, \cdots, \alpha_{a_0}, d_0, \beta_1, \ldots, \beta_{b_0}), \\ C_1.D_i = 0 & (\text{otherwise}), \end{cases}$$

which are obtained by (8) and the mention after Definition 4.4.

4.3 Proof of Assertion 3.2 and 3.3

In order to prove Assertion 3.2 and 3.3 for I_C in Definition 4.4, in this subsection, we consider the operation to take auxiliary divisors repeatedly.

Let D be a divisor on S such that $D.D_1 \geq 1$. For an integer y with $0 \leq y < \sum_{i=1}^{d_0-1} y_i D.D_i$, we define

$$\begin{aligned} i(D, y) &= \max\left\{ j \geq 1 \;\middle|\; y < \sum_{i=j}^{d_0-1} y_i D.D_i \right\}, \\ x(D, y) &= \sum_{i=i(D,y)+1}^{d_0-1} x_i D.D_i + \frac{x_{i(D,y)}}{y_{i(D,y)}} \left(y - \sum_{i=i(D,y)+1}^{d_0-1} y_i D.D_i \right). \end{aligned}$$

Let C be a curve on S such as at the beginning of this section, and k, k', λ and μ the integers as in the previous subsection.

Remark 4.11. By definition, we have $x(C, 0) = 0$ if $y_\nu \geq 1$. Besides, $x(C, y) = x_\lambda y / y_\lambda$ if $i(C, y) = \lambda$.

Lemma 4.12. *Assume that $k \geq 1$ and $C.D_1 \geq 1$. Let y be an integer with $c_0 \leq y < \sum_{i=2}^{d_0-1} y_i C.D_i$, and define $p = \min\{n \in \mathbb{Z} \mid n > x(C - I_C, y - c_0)\}$. Then*

$$x(C - I_C, y - c_0) + 1 \leq x(C, y) < p + 1.$$

PROOF. (i) We first consider the case where $y \geq y_\lambda$. By definition, $i(C,y) \leq \lambda$ and

$$\sum_{i=i(C,y)+1}^{d_0-1} y_i C.D_i \leq y < \sum_{i=i(C,y)}^{d_0-1} y_i C.D_i. \tag{9}$$

By computing, we have

$$\begin{aligned}\sum_{i=i(C,y)}^{d_0-1} y_i(C-I_C).D_i &= \sum_{i=i(C,y)}^{d_0-1} y_i C.D_i - y_\lambda I_C.D_\lambda - \sum_{j=1}^{a_0} y_{\alpha_j} I_C.D_{\alpha_j} \\ &= \sum_{i=i(C,y)}^{d_0-1} y_i C.D_i - y_\lambda + \sum_{j=1}^{a_0} n_{\alpha_j} y_{\alpha_j} = \sum_{i=i(C,y)}^{d_0-1} y_i C.D_i - c_0,\end{aligned}$$

$$\sum_{i=i(C,y)+1}^{d_0-1} y_i(C-I_C).D_i = \begin{cases} y_\lambda - c_0 & (i(C,y)=\lambda), \\ \sum_{i=i(C,y)+1}^{d_0-1} y_i C.D_i - c_0 & (i(C,y) \leq \lambda-1). \end{cases}$$

Hence, by (9) and the assumption $y \geq y_\lambda$, we have

$$\sum_{i=i(C,y)+1}^{d_0-1} y_i(C-I_C).D_i \leq y - c_0 < \sum_{i=i(C,y)}^{d_0-1} y_i(C-I_C).D_i,$$

which means that $i(C-I_C, y-c_0) = i(C,y)$. If $i(C,y)=\lambda$, then

$$\begin{aligned}x(C-I_C, y-c_0) &= -\sum_{i=\lambda+1}^{d_0-1} x_i I_C.D_i + \frac{x_\lambda}{y_\lambda}\left(y - c_0 + \sum_{i=\lambda+1}^{d_0-1} y_i I_C.D_i\right) \\ &= x_\lambda - 1 + \frac{x_\lambda}{y_\lambda} y - x_\lambda = x(C,y) - 1.\end{aligned}$$

If $i(C,y) \leq \lambda - 1$, then

$$\begin{aligned}x(C-I_C, y-c_0) &= \sum_{i=i(C,y)+1}^{d_0-1} x_i C.D_i - x_\lambda + \sum_{j=1}^{a_0} n_{\alpha_j} x_{\alpha_j} \\ &\quad + \frac{x_{i(C,y)}}{y_{i(C,y)}}\left(y - c_0 - \sum_{i=i(C,y)+1}^{d_0-1} y_i C.D_i + y_\lambda - \sum_{j=1}^{a_0} n_{\alpha_j} y_{\alpha_j}\right) \\ &= \sum_{i=i(C,y)+1}^{d_0-1} x_i C.D_i - 1 \\ &\quad + \frac{x_{i(C,y)}}{y_{i(C,y)}}\left(y - \sum_{i=i(C,y)+1}^{d_0-1} y_i C.D_i\right) = x(C,y) - 1.\end{aligned}$$

We obtain $x(C-I_C, y-c_0) = x(C,y)-1$ in both cases. Therefore, the claim is trivial.
(ii) We next consider the case where $y < y_\lambda$. Note that $y_\lambda \geq 1$ and $i(C,y) = \lambda$ in this case. Since

$$y - c_0 < y_\lambda - c_0 = \sum_{j=1}^{a_0} n_{\alpha_j} y_{\alpha_j} = -\sum_{i=\alpha_1}^{d_0-1} y_i I_C.D_i = \sum_{i=\alpha_1}^{d_0-1} y_i (C-I_C).D_i,$$

there exists an integer a with $1 \leq a \leq a_0$ such that $i(C-I_C, y-c_0) = \alpha_a$. Then

$$\begin{aligned}
x(C-I_C, y-c_0) &= \sum_{j=a+1}^{a_0} n_{\alpha_j} x_{\alpha_j} + \frac{x_{\alpha_a}}{y_{\alpha_a}}\left(y - c_0 - \sum_{j=a+1}^{a_0} n_{\alpha_j} y_{\alpha_j}\right) \\
&= x_\lambda - 1 - \sum_{j=1}^{a-1} n_{\alpha_j} x_{\alpha_j} + \frac{x_{\alpha_a}}{y_{\alpha_a}}\left(y - y_\lambda + \sum_{j=1}^{a-1} n_{\alpha_j} y_{\alpha_j}\right) \\
&< x_\lambda - 1 - \frac{x_\lambda}{y_\lambda}\sum_{j=1}^{a-1} n_{\alpha_j} y_{\alpha_j} + \frac{x_\lambda}{y_\lambda}\left(y - y_\lambda + \sum_{j=1}^{a-1} n_{\alpha_j} y_{\alpha_j}\right) \\
&= \frac{x_\lambda}{y_\lambda} y - 1 = x(C,y) - 1.
\end{aligned}$$

We next show the inequality $x(C,y) < p+1$. Recall the notation s_i which appeared at the beginning of Subsection 4.2. Then $y_{s_a} \leq y < y_{s_{a-1}}$ and

$$\begin{aligned}
x(C-I_C, y-c_0) &= \sum_{j=a+1}^{a_0} n_{\alpha_j} x_{\alpha_j} + \frac{x_{\alpha_a}}{y_{\alpha_a}}\left(y - c_0 - \sum_{j=a+1}^{a_0} n_{\alpha_j} y_{\alpha_j}\right) \\
&= x_{s_a} - 1 + \frac{x_{\alpha_a}}{y_{\alpha_a}}(y - y_{s_a}).
\end{aligned}$$

Suppose that $p+1 = x(C,y) = x_\lambda y / y_\lambda$. Then we have $y = 0$ by the inequality $y < y_\lambda$ and the fact that x_λ and y_λ are relatively prime. It follows that p is negative, a contradiction. Suppose that $p+1 < x_\lambda y / y_\lambda$. Since $x_\lambda / y_\lambda < x_{\alpha_a}/y_{\alpha_a}$, we have

$$(p+1) y_{\alpha_a} - y x_{\alpha_a} \leq -1 = -x_{\alpha_a}\left(c_0 + \sum_{i=\alpha_a+1}^{d_0-1} n_i y_i\right) + y_{\alpha_a}\left(1 + \sum_{i=\alpha_a+1}^{d_0-1} n_i x_i\right)$$

$$\begin{aligned}
p &\leq \sum_{i=\alpha_a+1}^{d_0-1} n_i x_i + \frac{x_{\alpha_a}}{y_{\alpha_a}}\left(y - c_0 + \sum_{i=\alpha_a+1}^{d_0-1} n_i y_i\right) \\
&= \sum_{i=\alpha_a+1}^{d_0-1} x_i (C-I_C).D_i + \frac{x_{\alpha_a}}{y_{\alpha_a}}\left(y - c_0 + \sum_{i=\alpha_a+1}^{d_0-1} y_i (C-I_C).D_i\right) \\
&= x(C-I_C, y-c_0).
\end{aligned}$$

This contradicts the definition of p. Hence we can conclude that $p+1 > x(C,y)$. □

We define $i'(C,y')$ and $x'(C,y')$ in a similar way to $i(C,y)$ and $x(C,y)$, respectively. Concretely, in the case where $C.D_1 \geq 1$, for an integer y' with $0 \leq y' < \sum_{i=d_0+1}^{d+1} y_i C.D_i$, we define

$$i'(C,y') = \min\left\{ j \leq d+1 \;\middle|\; y' < \sum_{i=d_0+1}^{j} y_i C.D_i \right\},$$
$$x'(C,y') = \sum_{i=d_0+1}^{i'(C,y')-1} x_i C.D_i + \frac{x_{i'(C,y')}}{y_{i'(C,y')}}\left(y' - \sum_{i=d_0+1}^{i'(C,y')-1} y_i C.D_i \right).$$

Then one can obtain the following lemma by an argument similar to that in the proof of Lemma 4.12.

Lemma 4.13. *Assume that $k \geq 1$ and $C.D_1 \geq 1$. Let y' be an integer with $v_0 \leq y' < \sum_{i=d_0+2}^{d+1} y_i C.D_i$, and define $p' = \max\{n \in \mathbb{Z} \mid n < x'(C - I_C, y' - e_0)\}$. Then*

$$p' - 1 < x'(C,y') \leq x'(C - I_C, y' - e_0) - 1.$$

Let us consider the situation that we take auxiliary divisors repeatedly. From the convex geometrical view point, this means that we reduce the size of the lattice polytope step by step. We put $C_0 = C$, and define inductively

$$\begin{aligned} I_{j-1} &: \text{the auxiliary divisor of } C_{j-1}, \\ C_j &: \text{a nonsingular irreducible curve in } |C_{j-1} - I_{j-1}|, \\ \lambda_j &= \max\{i \leq d_0 - 1 \mid C_j.D_i \geq 1\}, \\ \mu_j &= \min\{i \geq d_0 + 1 \mid C_j.D_i \geq 1\}, \\ c_j &= \min\{c \in \mathbb{Z} \mid c x_{\lambda_j} - y_{\lambda_j} \geq 0\}, \\ e_j &= -\max\{e \in \mathbb{Z} \mid e x_{\mu_j} - y_{\mu_j} \geq 0\} \end{aligned}$$

for an integer $1 \leq j \leq k$.

Lemma 4.14. *Assume that $k \geq 1$ and $C.D_1 \geq 1$. Let y and j be integers with $y \leq \sum_{i=2}^{d_0-1} y_i C_0.D_i$ and $1 \leq j \leq k$. If $y \geq c_0 + \cdots + c_{j-1}$, then $x(C_j, y - c_0 - \cdots - c_{j-1}) + j \leq x(C_0, y)$ and there is no integer in the half-open interval $\big(x(C_j, y - c_0 - \cdots - c_{j-1}) + j, x(C_0,y)\big]$.*

Proof. The inequality $x(C_j, y - c_0 - \cdots - c_{j-1}) + j \leq x(C_0, y)$ follows immediately from Lemma 4.12. Let j' be an integer with $0 \leq j' \leq j-1$. We have

$$\sum_{i=2}^{d_0-1} y_i C_{j'}.D_i = \sum_{i=2}^{d_0-1} y_i C_0.D_i - c_0 - \cdots - c_{j'-1}$$

by computing, where we define $c_0+\cdots+c_{j'-1}=0$ in the case where $j'=0$. We thus have $c_{j'} \leq y-c_0-\cdots-c_{j'-1} \leq \sum_{i=2}^{d_0-1} y_i C_{j'}.D_i$. Then by Lemma 4.12, there is no integer in the half-open interval $\big(x(C_{j'+1}, y-c_0-\cdots-c_{j'})+j'+1, x(C_{j'}, y-c_0-\cdots-c_{j'-1})+j'\big]$. Hence the statement of the lemma is obvious. □

A similar lemma holds for the opposite side of the lattice polytope.

Lemma 4.15. *Assume that $k \geq 1$ and $C.D_1 \geq 1$. Let y' and j be integers with $y' \leq \sum_{i=d_0+2}^{d+1} y_i C_0.D_i$ and $1 \leq j \leq k$. If $y' \geq e_0+\cdots+e_{j-1}$, then $x'(C_j, y'-e_0-\cdots-e_{j-1})-j \geq x'(C_0, y')$ and there is no integer in the half-open interval $\big[x'(C_0, y'), x'(C_j, y'-e_0-\cdots-e_{j-1})-j\big)$.*

Lemma 4.16. *The equality $c_0+\cdots+c_{x_{\lambda_0}-1}=y_{\lambda_0}$ holds.*

Proof. Suppose $c_0+\cdots+c_{x_{\lambda_0}-1} \leq y_{\lambda_0}-1$. Then we have

$$\begin{aligned} x(C_0, c_0+\cdots+c_{x_{\lambda_0}-1}) &= \frac{x_{\lambda_0}}{y_{\lambda_0}}(c_0+\cdots+c_{x_{\lambda_0}-1}) \\ &\leq \frac{x_{\lambda_0}}{y_{\lambda_0}}(y_{\lambda_0}-1) < x_{\lambda_0} \\ &\leq x(C_{x_{\lambda_0}}, 0)+x_{\lambda_0}, \end{aligned}$$

which contradicts Lemma 4.14. Suppose $c_0+\cdots+c_{x_{\lambda_0}-1} \geq y_{\lambda_0}+1$ and put $s=\max\{i \leq x_{\lambda_0}-1 \mid c_i \geq 1\}$. Then we have

$$\begin{aligned} x(C_s, c_s-1)+s &= \frac{x_{\lambda_s}}{y_{\lambda_s}}(c_s-1)+s < s+1, \\ x(C_0, c_0+\cdots+c_s-1) &= x(C_0, c_0+\cdots+c_{x_{\lambda_0}-1}-1) \geq x(C_0, y_{\lambda_0}) \\ &= x_{\lambda_0} \geq s+1. \end{aligned}$$

They contradict Lemma 4.14. Hence we can conclude that $c_0+\cdots+c_{x_{\lambda_0}-1} = y_{\lambda_0}$. □

Lemma 4.17. *If $c_0 \geq 1$ and $x_{\lambda_0} \geq 2$, then $c_0-1 \leq y_{\lambda_j}/x_{\lambda_j} \leq y_{\lambda_0}/x_{\lambda_0}$ for integers $1 \leq j \leq x_{\lambda_0}-1$, especially the equality $y_{\lambda_j}/x_{\lambda_j} = c_0-1$ holds for $j=x_{\lambda_0}-1$.*

Proof. Let j be an integer with $1 \leq j \leq x_{\lambda_0 - 1}-1$. We first show the inequality $y_{\lambda_j}/x_{\lambda_j} \geq c_0-1$.
(i) We consider the case where $x_{\lambda_j} \geq 2$. Since x_j and y_j are relatively prime, we obtain $c_j x_{\lambda_j} > y_{\lambda_j}$ in this case. Hence

$$x(C_j, y_{\lambda_j}-c_j)+j = \frac{x_{\lambda_j}}{y_{\lambda_j}}(y_{\lambda_j}-c_j)+j = x_{\lambda_j} - \frac{x_{\lambda_j}}{y_{\lambda_j}}c_j + j < x_{\lambda_j}-1+j.$$

On the other hand, we have

$$\begin{aligned} x(C_0, c_0+\cdots+c_{j-1}+y_{\lambda_j}-c_j) &= \frac{x_{\lambda_0}}{y_{\lambda_0}}(c_0+\cdots+c_{j-1}+y_{\lambda_j}) - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_j \\ &= x(C_0, c_0+\cdots+c_{j-1}+y_{\lambda_j}) - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_j \\ &\geq x(C_j, y_{\lambda_j}) + j - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_j = x_{\lambda_j} + j - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_j. \end{aligned}$$

Since $(x(C_j, y_{\lambda_j}-c_j)+j, x(C_0, c_0+\cdots+c_{j-1}+y_{\lambda_j}-c_j)] \cap \mathbb{Z} = \emptyset$ by Lemma 4.14, we have $c_0 \leq c_j$ and $y_{\lambda_j}/x_{\lambda_j} > c_j - 1 \geq c_0 - 1$.

(ii) We consider the case where $x_{\lambda_j} = 1$. Note that $y_{\lambda_j} = c_j$. We define $s = \max\{i \leq j-1 \mid c_i \geq 1\}$. Then we have

$$x(C_s, c_s - 1) + s = \frac{x_{\lambda_s}}{y_{\lambda_s}}(c_s - 1) + s < s + 1 \leq j.$$

On the other hand, we have

$$\begin{aligned} x(C_0, c_0+\cdots+c_s-1) &= x(C_0, c_0+\cdots+c_j-(c_j+1)) \\ &\geq x(C_j, c_j) + j - \frac{x_{\lambda_0}}{y_{\lambda_0}}(c_j+1) \geq j+1 - \frac{x_{\lambda_0}}{y_{\lambda_0}}(c_j+1). \end{aligned}$$

Then by Lemma 4.14, we have $c_0 - 1 \leq c_j = y_{\lambda_j}/x_{\lambda_j}$.

We next show the inequality $y_{\lambda_j}/x_{\lambda_j} \leq y_{\lambda_0}/x_{\lambda_0}$. If $y_{\lambda_j}/x_{\lambda_j} = 0$, then the inequality is obviously holds. Hence we consider the case where $y_{\lambda_j}/x_{\lambda_j} > 0$. Suppose $y_{\lambda_j}/x_{\lambda_j} > y_{\lambda_0}/x_{\lambda_0}$. Then we have

$$\begin{aligned} x\left(C_0, c_0+\cdots+c_{j-1}+\frac{y_{\lambda_0}}{x_{\lambda_0}}\right) &= x(C_0, c_0+\cdots+c_{j-1}) + 1 \\ &\geq x(C_j, 0) + j + 1 = j+1, \\ x\left(C_j, \frac{y_{\lambda_0}}{x_{\lambda_0}}\right) + j &= \frac{x_{\lambda_j}}{y_{\lambda_j}} \cdot \frac{y_{\lambda_0}}{x_{\lambda_0}} + j < j+1. \end{aligned}$$

They contradict Lemma 4.14.

Lastly, we show $y_{\lambda_a}/x_{\lambda_a} = c_0 - 1$, where we put $a = x_{\lambda_0} - 1$ for simplicity. Because of the above arguments, it is sufficient to verify that $y_{\lambda_a}/x_{\lambda_a} \leq c_0 - 1$. Since this is obvious in the case where $y_{\lambda_a}/x_{\lambda_a} = 0$, we assume $y_{\lambda_a}/x_{\lambda_a} > 0$. Then we have $x(C_a, 0) + a = a = x_{\lambda_0} - 1$. On the other hand, we have

$$\begin{aligned} x(C_0, c_0+\cdots+c_{a-1}) &= x(C_0, c_0+\cdots+c_a) - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_a \\ &= x(C_0, y_{\lambda_0}) - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_a \\ &= x_{\lambda_0} - \frac{x_{\lambda_0}}{y_{\lambda_0}}c_a. \end{aligned}$$

Note that $c_0 x_{\lambda_0} > y_{\lambda_0}$ since $x_{\lambda_0} \geq 2$. Then by Lemma 4.14, we have $x_{\lambda_0} - x_{\lambda_0} c_a / y_{\lambda_0} \geq x_{\lambda_0} - 1$, which implies that $c_0 > y_{\lambda_0}/x_{\lambda_0} \geq c_a \geq y_{\lambda_a}/x_{\lambda_a}$. □

Assertion 3.2 follows immediately from the following proposition. Needless to say, the integer m_0 in the following proposition coincides with that in Assertion 3.2.

Proposition 4.18. *Assume that $k' > k \geq 2$ and $|C - F|$ is base point free. Then there exists a positive integer $m_0 \leq k$ such that $C_j.F' = C_j.F$ for $j = 1, \dots, m_0 - 1$ and $C_{m_0}.F' > C_{m_0}.F$.*

PROOF. (i) We first consider the case where $c_0 \geq 2$. Note that $x_i y_{\lambda_0} \leq y_i x_{\lambda_0}$ for integers $2 \leq i \leq d_0 - 1$ if $C.D_i \geq 1$. Then we have

$$\begin{aligned}
C_1.F &= k - 1 = \sum_{i=2}^{\lambda_0} x_i C.D_i - 1 \\
&= \sum_{i=2}^{\lambda_0 - 1} x_i C.D_i + x_{\lambda_0}(C.D_{\lambda_0} - 1) + x_{\lambda_0} - 1, \\
C_1.F' &= k' - c_0 - e_0 \\
&= \sum_{i=1}^{\lambda_0} y_i C.D_i + \sum_{i=\mu_0}^{d} y_i C.D_i - c_0 - e_0 \\
&\geq C.D_1 + \frac{y_{\lambda_0}}{x_{\lambda_0}} \sum_{i=2}^{\lambda_0 - 1} x_i C.D_i + y_{\lambda_0} C.D_{\lambda_0} + y_{\mu_0} - c_0 - e_0 \\
&> 1 + (c_0 - 1) \sum_{i=2}^{\lambda_0 - 1} x_i C.D_i + (c_0 - 1) x_{\lambda_0} C.D_{\lambda_0} - c_0 \\
&= (c_0 - 1) \sum_{i=2}^{\lambda_0 - 1} x_i C.D_i + (c_0 - 1) x_{\lambda_0}(C.D_{\lambda_0} - 1) + (c_0 - 1)(x_{\lambda_0} - 1).
\end{aligned}$$

Hence we have $C_1.F' > C_1.F$. Namely, the lemma is valid for $m_0 = 1$ in this case.
(ii) Similarly, in the case where $e_0 \geq 2$, one can show that the lemma is valid for $m_0 = 1$.

From now on, we assume that $c_0 \leq 1$ and $e_0 \leq 1$.
(iii) Assume that $c_0 + e_0 \leq 1$ or $k' \geq k + 2$. Then, since

$$C_1.F' - C_1.F = k' - I_0.D_{d_0} - k + 1 = k' - k - c_0 - e_0 + 1 \geq 1,$$

the lemma is valid for $m_0 = 1$.
(iv) For the remaining case where $c_0 = e_0 = 1$ and $k' = k + 1$, it is sufficient to verify the following claim:
Claim A: Two inequalities $2 \leq x_{\lambda_0} \leq k$, $-k \leq x_{\mu_0} \leq -2$ hold and there

exists an integer m_0 with $2 \leq m_0 \leq \min\{x_{\lambda_0}, -x_{\mu_0}\}$ such that $I_j.D_{d_0} = \begin{cases} 1 & (1 \leq j \leq m_0 - 2), \\ 0 & (j = m_0 - 1). \end{cases}$

First we mention the inequalities $k = \sum_{i=2}^{d_0-1} x_i C.D_i \geq x_{\lambda_0}$ and $k = -\sum_{i=d_0+1}^{d} x_i C.D_i \geq -x_{\mu_0}$. By the definition of λ_0, $x_i y_{\lambda_0} \leq y_i x_{\lambda_0}$ holds if $2 \leq i \leq d_0 - 1$ and $C.D_i \geq 1$. Similarly, $x_i y_{\mu_0} \geq y_i x_{\mu_0}$ holds if $d_0 + 1 \leq i \leq d$ and $C.D_i \geq 1$. Hence we obtain

$$\begin{aligned} k' &= C.D_1 + \sum_{i=2}^{d_0-1} y_i C.D_i + \sum_{i=d_0+1}^{d} y_i C.D_i \\ &\geq 1 + \frac{y_{\lambda_0}}{x_{\lambda_0}} \sum_{i=2}^{d_0-1} x_i C.D_i + \frac{y_{\mu_0}}{x_{\mu_0}} \sum_{i=d_0+1}^{d} x_i C.D_i \\ &= 1 + \left(\frac{y_{\lambda_0}}{x_{\lambda_0}} - \frac{y_{\mu_0}}{x_{\mu_0}} \right) k. \end{aligned}$$

It follows that

$$\frac{y_{\lambda_0}}{x_{\lambda_0}} - \frac{y_{\mu_0}}{x_{\mu_0}} \leq 1. \tag{10}$$

Since $c_0 = e_0 = 1$, we have $y_{\lambda_0}/x_{\lambda_0} < 1$ and $y_{\mu_0}/x_{\mu_0} > -1$, which mean that $x_{\lambda_0} \geq 2$ and $x_{\mu_0} \leq -2$.

From now on, we assume that $x_{\lambda_0} \leq -x_{\mu_0}$. The case where $x_{\lambda_0} \geq -x_{\mu_0}$ can be shown in a similar way. Let j be an integer with $1 \leq j \leq x_{\lambda_0} - 1$ such that $I_1.D_{d_0} = \cdots = I_{j-1}.D_{d_0} = 1$. We verify that $I_j.D_{d_0} \leq 1$, especially $I_j.D_{d_0} = 0$ if $j = x_{\lambda_0} - 1$. By Lemma 4.14 and 4.15, we have

$$\begin{aligned} x(C_0, c_0 + \cdots + c_{j-1}) &= \frac{x_{\lambda_0}}{y_{\lambda_0}}(c_0 + \cdots + c_{j-1}) < x(C_j, 0) + j + 1, \\ x'(C_0, e_0 + \cdots + e_{j-1}) &= \frac{x_{\mu_0}}{y_{\mu_0}}(e_0 + \cdots + e_{j-1}) > x'(C_j, 0) - j - 1. \end{aligned} \tag{11}$$

On the other hand, it follows from Lemma 4.17 that $c_j, e_j \leq 1$. If $c_j = e_j = 1$, then we have $x(C_j, 0) = x'(C_j, 0) = 0$ and

$$\begin{aligned} \frac{y_{\lambda_0}}{x_{\lambda_0}} - \frac{y_{\mu_0}}{x_{\mu_0}} &> \frac{c_0 + \cdots + c_{j-1} + e_0 + \cdots + e_{j-1}}{j+1} \\ &= \frac{c_0 + e_0 + I_1.D_{d_0} + \cdots + I_{j-1}.D_{d_0}}{j+1} = 1, \end{aligned}$$

a contradiction. Hence we obtain $I_j.D_{d_0} = c_j + e_j \leq 1$.

In the case where $j = x_{\lambda_0} - 1$, we have $c_j = 0$ by Lemma 4.17 and

$$\frac{c_0 + \cdots + c_{j-1}}{j+1} = \frac{y_{\lambda_0}}{x_{\lambda_0}}$$

by Lemma 4.16. If $e_j = 1$, then we have $x'(C_j, 0) = 0$. Hence, by noting (11), we obtain a contradiction

$$\frac{y_{\lambda_0}}{x_{\lambda_0}} - \frac{y_{\mu_0}}{x_{\mu_0}} > \frac{c_0 + \cdots + c_{j-1} + e_0 + \cdots + e_{j-1}}{j+1} = 1.$$

Therefore, we can conclude $e_j = 0$, that is, $I_j.D_{d_0} = 0$ in this case. □

Let us show one more lemma needed in the proof of Assertion 3.3.

Lemma 4.19. *Assume that $k' > k \geq 2$ and $|C - F|$ is base point free, and let m_0 be a positive integer in Proposition 4.18. If $c_0 = e_0 = 1$, then either of the inequalities $x_{\lambda_{j-1}} \geq 2$ or $x_{\mu_{j-1}} \leq -2$ holds for each integer $1 \leq j \leq m_0 - 1$.*

Proof. Note that, in this case, the integer m_0 is equal to that in Claim A in the proof of Proposition 4.18. The case where $j = 1$ is contained in Claim A. Hence let j be an integer with $2 \leq j \leq m_0 - 1$, and suppose $x_{\lambda_{j-1}} = -x_{\mu_{j-1}} = 1$. Since $I_{j-1}.D_{d_0} = c_{j-1} + e_{j-1} = 1$, either c_{j-1} or e_{j-1} is equal to one. If $c_{j-1} = 1$, then we have $y_{\lambda_0}/x_{\lambda_0} \geq y_{\lambda_{j-1}}/x_{\lambda_{j-1}} = 1$ by Lemma 4.17. Hence (10) implies that $y_{\mu_0}/x_{\mu_0} = 0$. This contradicts the fact that $e_0 = 1$. We obtain a similar contradiction in the case where $e_{j-1} = 1$. Therefore, two equalities $x_{\lambda_{j-1}} = 1$ and $x_{\mu_{j-1}} = -1$ do not occur at the same time. □

Finally, we prove Assertion 3.3.

Proposition 4.20. *Assume that $k' > k \geq 2$ and $|C - F|$ is base point free, and let m_0 be a positive integer in Proposition 4.18. Then the inequality in Assertion 3.3 holds for each integer $1 \leq j \leq m_0$. Besides, if $k' = k = 2$, then this inequality holds for $j = 1$.*

Proof. Considering the cohomology long exact sequence similar to (1), we obtain $h^0(S, C_j) = h^0(C_j, C_j|_{C_j}) + 1$. On the other hand, the short exact sequence $0 \to \mathcal{O}_S(-I_{j-1}) \to \mathcal{O}_S(C_j - I_{j-1}) \to \mathcal{O}_{C_j}(C_j - I_{j-1}) \to 0$ and Proposition 4.8 induce the cohomology exact sequence

$$0 \to H^0(S, C_j - I_{j-1}) \to H^0(C_j, (C_j - I_{j-1})|_{C_j}) \to 0. \tag{12}$$

Hence it is sufficient to verify the inequality

$$h^0(C_j, C_j|_{C_j}) - h^0(C_j, (C_j - I_{j-1})|_{C_j}) \geq C_j.F + 1. \tag{13}$$

The divisor $(C_j - I_{j-1})|_{C_j}$ is nonspecial by Lemma 4.10. By Riemann-Roch theorem, we have

$$\deg C_j|_{C_j} - 2g_j = -C_j.K_S - 2 \geq C_j.D_1 - 2 \geq -1, \tag{14}$$

where g_j denotes the genus of C_j. Hence also $C_j|_{C_j}$ is nonspecial. Thus (13) is equivalent to the inequality

$$C_j.I_{j-1} \geq C_j.F + 1. \tag{15}$$

In the case where $c_0 \neq 1$ or $e_0 \neq 1$ or $k' \geq k+2$, m_0 is equal to one by (i)–(iii) in the proof of Proposition 4.18. Since $C_1.I_0 \geq C_1.D_{d_0} = C_1.F' \geq C_1.F + 1$ by the definition of m_0, (15) holds in this case.

Let us consider the case where $c_0 = 1$, $e_0 = 1$ and $k' = k+1$. In the case where $1 \leq j \leq m_0 - 1$, we have $x_{\lambda_{j-1}} \geq 2$ or $x_{\mu_{j-1}} \leq -2$ by Lemma 4.19. Hence if we divide $(x_{\lambda_{j-1}}, y_{\lambda_{j-1}})$ and $(x_{\mu_{j-1}}, y_{\mu_{j-1}})$ as (2) and (4), then either α_{a_0} or β_{b_0} exists. If α_{a_0} exists, then by (8), we have

$$\begin{aligned} C_j.I_{j-1} \geq C_j.(D_{\alpha_{a_0}} + D_{d_0}) &= -I_{j-1}.D_{\alpha_{a_0}} + C_0.D_{d_0} - (I_0 + \cdots + I_{j-1}).D_{d_0} \\ &= n_{\alpha_{a_0}} + C_0.F' - j - 1 \geq C_0.F - j + n_{\alpha_{a_0}} \\ &= C_j.F + n_{\alpha_{a_0}}. \end{aligned}$$

A similar computation is carried out in the case where β_{b_0} exists to verify the inequality $C_j.I_{j-1} \geq C_j.F + m_{\beta_{b_0}}$. In the case where $j = m_0$, we have $C_{m_0}.I_{m_0-1} \geq C_{m_0}.D_{d_0} = C_{m_0}.F' \geq C_{m_0}.F + 1$.

Lastly, let us show (15) for $j = 1$ under the assumption that $k' = k = 2$. Namely, we will show the inequality $C_1.I_0 \geq 2$. If $x_{\lambda_0} \geq 2$ or $x_{\mu_0} \leq -2$, then this can be proved by the same argument as above. Assume $x_{\lambda_0} = 1$ and $x_{\mu_0} = -1$. Since $k = \sum_{i=2}^{d_0-1} x_i C.D_i \geq 2$, $C.D_{\lambda_0} \geq 2$ or there exists an integer i with $2 \leq i \leq d_0 - 1$ such that $i \neq \lambda_0$ and $C.D_i \geq 1$. We thus have

$$1 = k' - 1 \geq \sum_{i=2}^{d_0-1} y_i C.D_i \geq \frac{y_{\lambda_0}}{x_{\lambda_0}} \sum_{i=2}^{d_0-1} x_i C.D_i > y_{\lambda_0}.$$

Hence y_{λ_0} must be zero. Similarly, one can obtain $y_{\mu_0} = 0$. We thus have $I_0.D_{d_0} = c_0 + e_0 = 0$ and $C_1.I_0 \geq C_1.D_{d_0} = C.D_{d_0} = k' = 2$. □

5. The case where $\deg f_1|_C \geq \deg f_2|_C$

In this section, we consider the case where $\deg f_1|_C \geq \deg f_2|_C$, that is, $C.F_1 \geq C.F_2$. We put $F = F_2$, $F' = F_1$, $k = C.F$ and $k' = C.F'$. In order to simplify the argument, we renumber the T-invariant divisors. Concretely, we denote by D_1 the T-invariant divisor whose associated cone has the primitive element $(-1, 0)$, and number other cones in Δ_S clockwise. In particular, we set D_{d_0} if its primitive element is $(0, 1)$ (see Fig. 2). Note that, in this case, the fiber F and F' are written as

$$F \sim D_d \sim \sum_{i=2}^{d-2} y_i D_i$$

$$F' \sim -\sum_{i=1}^{d_0-1} x_i D_i \sim \sum_{i=d_0+1}^{d-1} x_i D_i.$$

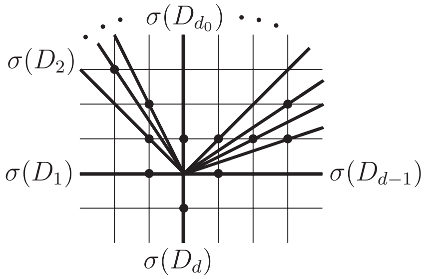

Figure 2

If $k' \geq k \geq 2$, we define

$$\begin{aligned} \nu &= \max\{i \leq d_0 \mid C.D_i \geq 1\}, \\ \xi &= \min\{i \geq d_0 \mid C.D_i \geq 1\}, \\ c_0 &= -\max\{c \in \mathbb{Z} \mid x_\nu - c y_\nu \geq 0\}, \\ e_0 &= \min\{e \in \mathbb{Z} \mid x_\xi - e y_\xi \leq 0\}, \\ f_0 &= \min\{c_0, e_0\}. \end{aligned}$$

If $\nu = 1$ or $\xi = d-1$, then C becomes a curve considered in Section 4. We thus assume that $\nu \geq 2$ and $\xi \leq d-2$ in this section. Note that the five statements $c_0 = 0$, $e_0 = 0$, $f_0 = 0$, $\nu = d_0$ and $\xi = d_0$ are equivalent. By Fact 4.1 and the argument in Subsection 4.1, we obtain the unique divisions

$$\begin{aligned} (16)\quad (x_\nu, y_\nu) &= (-c_0, 1) + n_{\gamma_1}(x_{\gamma_1}, y_{\gamma_1}) + \cdots + n_{\gamma_{s_0}}(x_{\gamma_{s_0}}, y_{\gamma_{s_0}}) \\ &\qquad\qquad (\nu < \gamma_1 < \cdots < \gamma_{s_0} \leq d_0), \\ (x_\xi, y_\xi) &= (e_0, 1) + m_{\delta_1}(x_{\delta_1}, y_{\delta_1}) + \cdots + m_{\delta_{t_0}}(x_{\delta_{t_0}}, y_{\delta_{t_0}}) \\ &\qquad\qquad (d_0 \leq \delta_1 < \cdots < \delta_{t_0} < \xi), \end{aligned}$$

where n_{γ_i} and m_{δ_j} are positive integers. We note that

$$(17)\qquad x_{\gamma_s}\left(1 + \sum_{j=s+1}^{s_0} n_{\gamma_j} y_{\gamma_j}\right) - y_{\gamma_s}\left(-c_0 + \sum_{j=s+1}^{s_0} n_{\gamma_j} x_{\gamma_j}\right) = 1$$

for each integer $1 \leq s \leq s_0$ and

$$(18)\qquad \left(e_0 + \sum_{j=1}^{t-1} m_{\delta_j} x_{\delta_j}\right) y_{\delta_t} - \left(1 + \sum_{j=1}^{t-1} m_{\delta_j} y_{\delta_j}\right) x_{\delta_t} = 1$$

for each integer $1 \le t \le t_0$. There exists an integer ι such that $(x_\iota, y_\iota) = (e_0 - c_0, 1)$. Obviously, we see that $\iota = d_0$ (resp. $\nu + 1 \le \iota \le \xi - 1$) if $f_0 = 0$ (resp. $f_0 \ge 1$).

5.1 The auxiliary divisor

In this subsection, we keep the above notation. Let C be a curve on S such that $k \le k'$, $\nu \ge 2$ and $\xi \le d - 2$.

Definition 5.1. Let C be a curve on S such that $k \ge 2$, and set $n_i = 0$ for integers $\nu+1 \le i \le d_0$ except for $i = \gamma_1, \ldots, \gamma_{s_0}$ and $m_i = 0$ for integers $d_0 \le i \le \xi-1$ except for $i = \delta_1, \ldots, \delta_{t_0}$. We define the *auxiliary divisor* $I_C = \sum_{i=1}^{d} q_i D_i + F'$ of C as follows:

(i) The case where $\iota \le d_0$ (that is, $c_0 \ge e_0$).

$$q_i = \begin{cases} x_i\left(1 + \sum_{j=i+1}^{d_0} n_j y_j\right) + y_i\left(c_0 - \sum_{j=i+1}^{d_0} n_j x_j\right) & (\nu + 1 \le i \le \iota - 1), \\ e_0 y_i & (\iota \le i \le d_0), \\ x_i\left(-1 - \sum_{j=d_0}^{i-1} m_j y_j\right) + y_i\left(e_0 + \sum_{j=d_0}^{i-1} m_j x_j\right) & (d_0 + 1 \le i \le \xi - 1), \\ 0 & (\text{otherwise}). \end{cases}$$

(ii) The case where $\iota \ge d_0 + 1$ (that is, $c_0 < e_0$).

$$q_i = \begin{cases} x_i\left(1 + \sum_{j=i+1}^{d_0} n_j y_j\right) + y_i\left(c_0 - \sum_{j=i+1}^{d_0} n_j x_j\right) & (\nu + 1 \le i \le d_0), \\ c_0 y_i & (d_0 + 1 \le i \le \iota), \\ x_i\left(-1 - \sum_{j=d_0}^{i-1} m_j y_j\right) + y_i\left(e_0 + \sum_{j=d_0}^{i-1} m_j x_j\right) & (\iota + 1 \le i \le \xi - 1), \\ 0 & (\text{otherwise}). \end{cases}$$

It is obvious that $I_C.F = 1$ by definition. We remark that $I_C = F'$ if $f_0 = 0$. Besides, by (17) and (18), we have $q_i = 1$ for $i = \gamma_1, \ldots, \gamma_{i_0}, \delta_1, \ldots, \delta_{j_0}$.

Lemma 5.2. *If $c_0 \ge e_0 \ge 1$, then $\iota \ge \gamma_{s_0}$.*

PROOF. If $\iota < \gamma_{s_0}$, then we have $x_{\gamma_{s_0}}/y_{\gamma_{s_0}} > x_\iota/y_\iota = e_0 - c_0$. By (17), the inequality $y_{\gamma_{s_0}} e_0 < x_{\gamma_{s_0}} + y_{\gamma_{s_0}} c_0 = 1$ holds. This implies that $y_{\gamma_{s_0}} = 0$, a contradiction. □

Corollary 5.3. *If $c_0 \ge e_0 \ge 1$, then $n_i = 0$ for any integer $\iota + 1 \le i \le d_0$.*

Lemma 5.4. *Assume that $f_0 \ge 1$. If $\iota = \nu + 1$ (resp. $\iota = \xi - 1$), then $e_0 = 1$ (resp. $c_0 = 1$).*

PROOF. We note that c_0 and e_0 are positive by assumption. Assume that

$\iota = \nu + 1$. Then, since $x_\iota y_\nu - y_\iota x_\nu = 1$, we have

$$e_0 = c_0 + \frac{x_\nu}{y_\nu} + \frac{1}{y_\nu} < c_0 + (-c_0 + 1) + \frac{1}{y_\nu} = 1 + \frac{1}{y_\nu} \le 2.$$

Similarly, if $\iota = \xi - 1$, then the equality $x_\xi y_\iota - y_\xi x_\iota = 1$ implies that

$$c_0 = e_0 - \frac{x_\xi}{y_\xi} + \frac{1}{y_\xi} < e_0 - (e_0 - 1) + \frac{1}{y_\xi} = 1 + \frac{1}{y_\xi} \le 2. \qquad \square$$

There is no essential difference between the cases where $c_0 \ge e_0$ and $c_0 \le e_0$. Hence we will prove all the remaining propositions and lemmas in this subsection only for the former case.

Proposition 5.5. *The divisor I_C is effective.*

This lemma can be shown by a computation similar to that in the proof of Proposition 4.5. The following Proposition 5.6 and 5.8 correspond to the property (iii) in Assertion 3.1.

Proposition 5.6. *Assume that $k \ge 2$. The complete linear system $|C - I_C|$ is base point free.*

Proof. If $f_0 = 0$, then $C.D_{d_0} \ge 1$ and $I_C = F'$. Hence our lemma is clear in this case. We thus consider the case where $f_0 \ge 1$, that is, c_0 and e_0 are positive and $\xi \ge d_0 + 1$. We will compute the intersection number $I_C.D_i$ only for $i = d_0, \iota$. In the other cases, we can compute similarly to the proof of Proposition 4.6 to verify that $(C - I_C).D_i$ is non-negative.

In the case where $i = d_0$, we divide the situation into six cases as follows:

$$\begin{aligned}
&\text{(i)} && \nu + 1 = \iota = d_0 = \xi - 1,\\
&\text{(ii)} && \nu + 2 \le \iota = d_0 = \xi - 1,\\
&\text{(iii)} && \iota + 1 \le d_0 = \xi - 1,\\
&\text{(iv)} && \nu + 1 = \iota = d_0 \le \xi - 2,\\
&\text{(v)} && \nu + 2 \le \iota = d_0 \le \xi - 2,\\
&\text{(vi)} && \iota + 1 \le d_0 \le \xi - 2.
\end{aligned}$$

Lemma 5.4 and the assumption $c_0 \ge e_0$ implies that $e_0 = 1$ in the case (iv), and $c_0 = e_0 = 1$ in the cases (i) and (ii). By definition, $n_{d_0} = y_{d_0-1} - 1$ if $\nu = d_0 - 1$, and $m_{d_0} = y_{d_0+1} - 1$ if $\xi = d_0 + 1$.

(i) Since $I_C = D_{d_0} + F'$, we have $I_C.D_{d_0} = -y_{d_0-1} - y_{d_0+1} + 1 = -n_{d_0} - m_{d_0} - 1$.

(ii) Since $I_C = (-1 - n_{d_0} + y_{d_0-1})D_{d_0-1} + D_{d_0} + F'$, we have

$$I_C.D_{d_0} = -1 - n_{d_0} + y_{d_0-1} - y_{d_0-1} - y_{d_0+1} + 1 = -n_{d_0} - m_{d_0} - 1.$$

(iii) In this case, we have $e_0 = 1$ by definition. Besides, $\gamma_{s_0} \leq d_0 - 1$ by Lemma 5.2, which implies that $n_{d_0} = 0$. Hence $I_C.D_{d_0} = (y_{d_0-1}D_{d_0-1} + D_{d_0} + F').D_{d_0} = -y_{d_0+1} + 1 = -n_{d_0} - m_{d_0}$.
(iv) Since $I_C = D_{d_0} + (-1 - m_{d_0} + y_{d_0+1})D_{d_0+1} + F'$, we have

$$I_C.D_{d_0} = -y_{d_0-1} - y_{d_0+1} - 1 - m_{d_0} + y_{d_0+1} + 1 = -y_{d_0-1} - m_{d_0} = -n_{d_0} - m_{d_0} - 1.$$

(v) Note that $c_0 = e_0$. Then, since $q_{d_0-1} = -1 - n_{d_0} + y_{d_0-1}$, $q_{d_0} = e_0$ and $q_{d_0+1} = -1 - m_{d_0} + e_0 y_{d_0+1}$, we have

$$\begin{aligned} I_C.D_{d_0} &= -1 - n_{d_0} + y_{d_0-1} - e_0(y_{d_0-1} + y_{d_0+1}) - 1 - m_{d_0} + e_0 y_{d_0+1} + 1 \\ &= -n_{d_0} - m_{d_0} - 1. \end{aligned}$$

(vi) In this case, we have $n_{d_0} = 0$ by the same argument as in (iii). Hence

$$I_C.D_{d_0} = e_0 y_{d_0-1} - e_0(y_{d_0-1} + y_{d_0+1}) - 1 - m_{d_0} + e_0 y_{d_0+1} + 1 = -n_{d_0} - m_{d_0}.$$

We next compute $I_C.D_\iota$ under the assumption $\iota \leq d_0 - 1$. If $\iota \geq \nu + 2$, then we have

$$\begin{aligned} I_C.D_\iota &= x_{\iota-1}(1 + n_\iota y_\iota) + y_{\iota-1}(c_0 - n_\iota x_\iota) - e_0(y_{\iota-1} + y_{\iota+1}) + e_0 y_{\iota+1} \\ &= x_{\iota-1} + n_\iota(x_{\iota-1}x_\iota - y_{\iota-1}x_\iota) + (c_0 - e_0)y_{\iota-1} \\ &= y_\iota x_{\iota-1} - x_\iota y_{\iota-1} - n_\iota = -n_\iota - 1. \end{aligned}$$

In the case where $\iota = \nu + 1$, since $e_0 = 1$ by Lemma 5.4, we have $I_C.D_\iota = -(y_{\iota-1} + y_{\iota+1}) + y_{\iota+1} = -y_\nu$. If $y_\nu = 1$, then we have $n_\iota = 0$, that is, $I_C.D_\iota = -n_\iota - 1$. If $y_\nu \geq 2$, then since $\gamma_{s_0} \leq \nu + 1$ by Lemma 5.2, we have $\gamma_{s_0} = \nu + 1$ and $s_0 = 1$. Hence $I_C.D_\iota = -1 - n_{\gamma_1} y_{\gamma_1} = -n_\iota - 1$. □

List 5.7. We list the intersection numbers of I_C and the T-invariant divisors on S.
(A) In the case where $f_0 = 0$,

$$I_C.D_i = \begin{cases} 1 & (i = d_0, d), \\ 0 & (\text{otherwise}). \end{cases}$$

(B) In the case where $f_0 \geq 1$ and $\iota = d_0$,

$$I_C.D_i = \begin{cases} -n_{d_0} - m_{d_0} - 1 & (i = d_0), \\ -n_i & (i \neq d_0, i = \gamma_1, \ldots, \gamma_{s_0}), \\ 1 & (i = \nu, \xi, d), \\ -m_i & (i \neq d_0, i = \delta_1, \ldots, \delta_{t_0}), \\ 0 & (\text{otherwise}). \end{cases}$$

(C) In the case where $f_0 \geq 1$ and $\iota \leq d_0 - 1$,

$$I_C.D_i = \begin{cases} -n_{d_0} - m_{d_0} & (i = d_0), \\ -n_i - 1 & (i = \iota), \\ -n_i & (i \neq \iota, i = \gamma_1, \dots, \gamma_{s_0}), \\ 1 & (i = \nu, \xi, d), \\ -m_i & (i \neq d_0, i = \delta_1, \dots, \delta_{t_0}), \\ 0 & (\text{otherwise}). \end{cases}$$

Proposition 5.8. *If $|C - F|$ is base point free, then also $|C - I_C - F|$ is base point free.*

We can easily show the above proposition by using List 5.7. We next check the property (ii) in Assertion 3.1.

Proposition 5.9. *The vanishing $H^1(S, -I_C) = 0$ holds.*

Proof. We write the linear equivalence class of F' as $F' \sim -\sum_{i=1}^{d_0-1} x_i D_i$. Clearly the coefficients of D_1 and D_{d-1} in I_C are one and zero, respectively. It follows from Theorem 2.4 that $H^0(S, K_S + I_C) = 0$. Hence, by the same computation as that in Proposition 4.8, it is enough to show $I_C.(K_S + I_C) = -2$.

In the case (A) in List 5.7, since $I_C = F'$, we have $I_C.(K_S + I_C) = F'.K_S = F'.(-D_{d_0} - D_d) = -2$. Recall that $q_i = 1$ for $i = \gamma_1, \dots, \gamma_{s_0}, \delta_1, \dots, \delta_{t_0}$. In the case (B), we have

$$\begin{aligned} I_C.(K_S + I_C) &= I_C.((q_\nu - 1)D_\nu + (q_{d_0} - 1)D_{d_0} + (q_\xi - 1)D_\xi + (q_d - 1)D_d + F') \\ &= -1 + (q_{d_0} - 1)(-n_{d_0} - m_{d_0} - 1) - 1 - 1 + q_{d_0} \\ &= (e_0 - 1)(-n_{d_0} - m_{d_0}) - 2. \end{aligned}$$

This value is clearly equal to minus two if $e_0 = 1$. In the case where $e_0 \geq 2$, since $n_{d_0} = m_{d_0} = 0$ by definition, we have $I_C.(K_S + I_C) = -2$. Lastly, we consider the case (C). By computing, we have $I_C.(K_S + I_C) = (e_0 - 1)(-n_{d_0} - m_{d_0} - n_\iota) - 2$. This value is equal to minus two if $e_0 = 1$. Consider the case where $e_0 \geq 2$. Note that $m_{d_0} = 0$ in this case. Let us show that $n_\iota = n_{d_0} = 0$. This is obvious if $y_\nu = 1$ by definition. We assume $y_\nu \geq 2$, and prove the inequality $\iota > \gamma_{s_0}$. The inequality $\iota \geq \gamma_{s_0}$ follows from Lemma 5.2. Suppose that $x_{\gamma_{s_0}} \geq -c_0 + 2$. Then by (17), we have the inequality $1 = x_{\gamma_{s_0}} + c_0 y_{\gamma_{s_0}} \geq -c_0 + 2 + c_0 y_{\gamma_{s_0}}$, which implies a contradiction $c_0(y_{\gamma_{s_0}} - 1) \leq -1$. Hence we have $x_\iota = e_0 - c_0 \geq 2 - c_0 > x_{\gamma_{s_0}}$. It follows that $n_\iota = n_{d_0} = 0$ and $I_C.(K_S + I_C) = -2$. □

Proposition 5.10. *Assume that $k \geq 2$ and $|C - F|$ is base point free. Then the divisor $C - I_C$ satisfies the properties* (iv)–(vi) *in Assertion 3.1.*

Proof. (iv) We put $H = C - I_C$. Since $|H|$ has no base points, we can write $H \sim \sum_{i=2}^{d-1} h_i D_i$ with non-negative integers h_i. Then we have $h_{d-1} = H.D_d = C.D_d - I_C.D_d = C.F - 1 \geq 1$. By assumption, we have $(C - F).D_{d-1} =$

$C.D_{d-1} - 1 \geq 0$. It follows that $H^2 \geq H.D_{d-1} = C.D_{d-1} - I_C.D_{d-1} \geq 1$. Recall that $H^2(S, -I_C) = 0$ as mentioned at the beginning of the proof of Proposition 5.9. Then (v) and (vi) can be shown by the same argument as that in the proof of Proposition 4.9. □

We next aim to show the lemma similar to Lemma 4.10. Although it is ideal that $(C - 2I_C)|_{C_1}$ is nonspecial for a curve $C_1 \in |C - I_C|$ in any case, in fact, we need a certain condition (see Lemma 5.12).

Lemma 5.11. *Assume that $k \geq 2$, $f_0 \geq 1$ and $|C - F|$ is base point free. Besides, we assume that C satisfies neither of the following conditions* (a) *nor* (b) *:*
(a) $y_\nu = 1$, $C.D_\nu = 1$ *and* $C.D_i = 0$ *for any integer* $2 \leq i \leq \nu - 1$.
(b) $y_\xi = 1$, $C.D_\xi = 1$ *and* $C.D_i = 0$ *for any integer* $\xi \leq i \leq d - 2$.
Then there exists an effective divisor E on S such that $|C - 2I_C - E|$ is base point free, $h^0(S, C - 2I_C - E) = h^0(S, C - 2I_C)$ and $E.(E + K_S) \leq 2E.(C - 2I_C)$.

Proof. We first aim to prove the existence of an effective divisor E_1 satisfying the following properties (i)–(iii) :
(i) $(C - 2I_C - E_1).D_i \geq 0$ for $1 \leq i \leq d_0$ and $i = d$,
(ii) $h^0(S, C - 2I_C - E_1) = h^0(S, C - 2I_C)$,
(iii) $E_1.(E_1 + K_S) \leq 2E_1.(C - 2I_C)$.
If $C.D_\nu \geq 2$, the zero divisor satisfies (i)–(iii). We thus assume that $C.D_\nu = 1$, and define

$$\nu' = \max\{i \leq \nu - 1 \mid C.D_i \geq 1\},$$

$$\gamma' = \min\{i \geq \nu + 1 \mid (C - 2I_C).D_i \geq 1\} = \begin{cases} \gamma_1 & (y_\nu \geq 2), \\ \iota & (y_\nu = 1). \end{cases}$$

By the information of the intersection numbers

$$(C - 2I_C).D_i \begin{cases} \geq 1 & (i = \nu'), \\ = -1 & (i = \nu), \\ \geq 1 & (i = \gamma'), \\ = 0 & (\nu' + 1 \leq i \leq \nu - 1, \nu + 1 \leq i \leq \gamma' - 1), \end{cases}$$

we can see the partial shape of $\square_{C-2I_C}$ around the line $l_\nu(C - 2I_C)$ (see Fig. 3 (1)).

Let us verify the following two claims (see Fig. 3 (2)) :
Claim 1 : The intersection point $P_0 = l_{\nu'-1}(C - 2I_C) \cap l_{\nu'}(C - 2I_C)$ lies on the half-line $L_1 = \{R - a(y_{\gamma'}, -x_{\gamma'}) \mid a \geq 0\}$.
Claim 2 : The intersection point $Q_0 = l_{\gamma'}(C - 2I_C) \cap l_{\gamma'+1}(C - 2I_C)$ lies on the half-line $L_2 = \{R + a(y_{\nu'}, -x_{\nu'}) \mid a \geq 0\}$.
If we put $P = (p_1, p_2)$, then the X-coordinate of R is greater than or equal to $p_1 - y_\nu$ (see Fig. 4). Since the X-coordinate of P_0 is less than or equal to $p_1 - y_{\nu'}$,

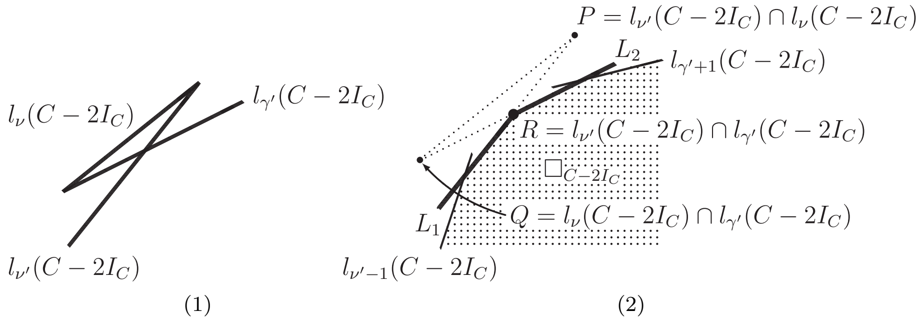

Figure 3

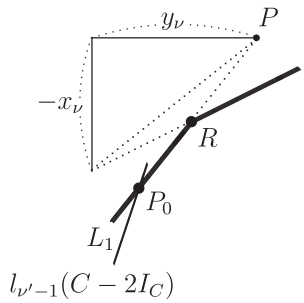

Figure 4

Claim 1 is true if $y_\nu < y_{\nu'}$. Let us consider the case where $y_\nu > y_{\nu'}$. It is sufficient to show the inequality

$$x_{\gamma'}(y_\nu - y_{\nu'}) \leq y_{\gamma'}(x_\nu - x_{\nu'}). \tag{19}$$

In the case where $y_\nu = 1$, we have $x_\nu = -c_0$ and $\gamma' = \iota$. On the other hand, $\nu' \geq 2$ since C does not satisfies the condition (a). We thus have

$$y_{\gamma'}(x_\nu - x_{\nu'}) - x_{\gamma'}(y_\nu - y_{\nu'}) = e_0(y_{\nu'} - 1) - x_{\nu'} - c_0 y_{\nu'} > 0.$$

Assume that $y_\nu \geq 2$. Since $\gamma' = \gamma_1$ and $x_{\gamma_1} y_\nu - y_{\gamma_1} x_\nu = 1$ by (17), the inequality (19) is obvious. We next show Claim 2. Since

$$Q_0 = Q + (C - 2I_C).D_{\gamma'}(y_{\gamma'}, -x_{\gamma'}) = Q - 2I_C.D_{\gamma'}(y_{\gamma'}, -x_{\gamma'}),$$

it is sufficient for Claim 2 to verify the inequality $-2y_{\gamma'} I_C.D_{\gamma'} \geq y_\nu$. This is obvious if $y_\nu = 1$. We thus consider the case where $y_\nu \geq 2$. Note that $\gamma' = \gamma_1$ in this case. If $s_0 = 1$, then since $(x_\nu, y_\nu) = (-c_0, 1) + n_{\gamma_1}(x_{\gamma_1}, y_{\gamma_1})$, we have

$$y_\nu = 1 + n_{\gamma_1} y_{\gamma_1} \leq 2n_{\gamma_1} y_{\gamma_1} \leq -2y_{\gamma_1} I_C.D_{\gamma_1}.$$

If $s_0 \geq 2$, then we have $x_{\gamma_2}(y_\nu - n_{\gamma_1} y_{\gamma_1}) - y_{\gamma_2}(x_\nu - n_{\gamma_1} x_{\gamma_1}) = 1$ by (17). Hence we can write $(x_{\gamma_1}, y_{\gamma_1}) = \varepsilon(x_{\gamma_2}, y_{\gamma_2}) + \zeta(x_\nu - n_{\gamma_1} x_{\gamma_1}, y_\nu - n_{\gamma_1} y_{\gamma_1})$ with integers ε and ζ. Since $x_{\gamma_2} y_{\gamma_1} - y_{\gamma_2} x_{\gamma_1} = x_{\gamma_1} y_\nu - y_{\gamma_1} x_\nu = 1$, we have $\varepsilon = \zeta = 1$. It follows that $(n_{\gamma_1} + 1) y_{\gamma_1} > y_\nu$. Since the inequality $I_C.D_{\gamma_1} \leq -n_{\gamma_1}$ holds by List 5.7, we have $-2y_{\gamma_1} I_C.D_{\gamma_1} > y_\nu$. Hence Claim 2 is true. Besides, we remark that

$$-I_C.D_{\gamma'} \geq y_\nu \tag{20}$$

if $\gamma' = \iota$. This is obvious in the case where $y_\nu = 1$. If $y_\nu \geq 2$, then we have $\gamma_1 = \iota$, that is, $y_{\gamma_1} = 1$. Hence the above two inequalities $I_C.D_{\gamma_1} \leq -n_{\gamma_1}$ and $(n_{\gamma_1} + 1) y_{\gamma_1} > y_\nu$ implies that $-I_C.D_{\gamma'} \geq y_\nu$.

We denote by $Q_{\gamma'} = (z_{\gamma'}, w_{\gamma'})$ the lattice point in $l_{\gamma'}(C - 2I_C) \cap \square_{C-2I_C}$ which is closest to L_1. Note that the line $l_i(C - 2I_C)$ passes through the point P (resp. Q) for any integer $\nu' + 1 \leq i \leq \nu - 1$ (resp. $\nu + 1 \leq i \leq \gamma' - 1$). Thus we can inductively define positive integers τ_i for $\nu' + 1 \leq i \leq \gamma' - 1$ as follows (see Fig. 5) :

τ_i : the positive integer such that the line $l_i(C - 2I_C - \tau_i D_i)$ passes through Q_{i+1},
$Q_i = (z_i, w_i)$: the lattice point in $l_i(C - 2I_C - \tau_i D_i) \cap \square_{C-2I_C}$ which is closest to L_1.

By definition, we have

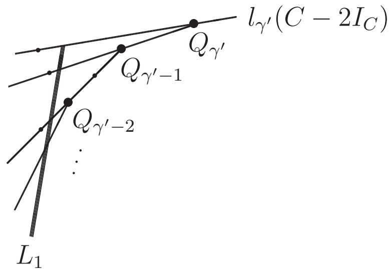

Figure 5

$$\tau_i = \begin{cases} x_i(z_0 - z_i + y_\nu) + y_i(w_0 - w_i - x_\nu) & (\nu' + 1 \leq i \leq \nu - 1), \\ x_i(z_0 - z_i) + y_i(w_0 - w_i) & (\nu \leq i \leq \gamma' - 1), \end{cases} \tag{21}$$

where we set $Q = (z_0, w_0)$. We put $E_1 = \sum_{i=\nu'+1}^{\gamma'-1} \tau_i D_i$. By definition, it is obvious that E_1 satisfies the property (i). Besides, for an integer i with $\nu' + 1 \leq i \leq \gamma' - 1$, the lattice points contained in the domain surrounded by $l_{i-1}(C - 2I_C - E_1)$, $l_i(C - 2I_C - E_1)$ and L_1 must lie on the line $l_i(C - 2I_C - E_1)$. This means that E_1 satisfies the property (ii). For the later use, we note that

(22) $$\tau_{\gamma'-1} \leq -I_C.D_{\gamma'}$$

holds if $\gamma' = \iota$. Indeed, if we suppose that $\gamma' = \iota$ and $\tau_{\gamma'-1} > -I_C.D_{\gamma'}$, then the point $Q' = Q + y_\nu(y_{\gamma'}, -x_{\gamma'})$ lies on L_2 since the X-coordinate of R (resp. Q') is less (resp. greater) than or equal to that of P. Namely, Q' lies on $l_{\gamma'}(C - 2I_C) \cap \Box_{C-2I_C}$. On the other hand, by (20), we deduce that Q' is nearer to L_1 than $Q_{\gamma'} = Q + \tau_{\gamma'-1}(y_{\gamma'}, -x_{\gamma'})$, which contradicts the definition of $Q_{\gamma'}$.

Let us show that E_1 satisfies (iii). By computing, we have $(C - 2I_C).E_1 = \tau_\nu(C - 2I_C).D_\nu = -\tau_\nu$. Hence it is sufficient to verify $E_1.(E_1 + K_S) \leq -2\tau_\nu$. We first consider the case where $y_\nu \geq 2$. Since $\gamma' = \gamma_1$ in this case, we have $x_{\gamma'}y_\nu - y_{\gamma'}x_\nu = 1$ by (17). Hence the lattice point $P - (y_{\nu'}, -x_{\nu'})$ lies on the half-line L_1. We thus see that $Q_{\nu'+1} = P - (y_{\nu'}, -x_{\nu'})$ by the property (ii), which implies that

$$\begin{aligned}\tau_{\nu'+1} &= x_{\nu'+1}(z_0 - z_{\nu'+1} + y_\nu) + y_{\nu'+1}(w_0 - w_{\nu'+1} - x_\nu)\\ &= x_{\nu'+1}y_{\nu'} + y_{\nu'+1}(-x_{\nu'}) = 1.\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}Q_{\gamma'} &= Q + s(y_{\gamma'}, -x_{\gamma'}),\\ Q_{\gamma'-1} &= Q_{\gamma'} + t(y_{\gamma'-1}, -x_{\gamma'-1}),\\ Q_\nu &= Q_{\gamma'} + (a, b)\end{aligned}$$

with integers s, t, a, and b. We note that $ax_\nu + by_\nu \geq 0$. Then, by (21), we have

$$\begin{aligned}\tau_{\gamma'-1} &= x_{\gamma'-1}(z_0 - z_{\gamma'-1}) + y_{\gamma'-1}(w_0 - w_{\gamma'-1})\\ &= x_{\gamma'-1}(-sy_{\gamma'} - ty_{\gamma'-1}) + y_{\gamma'-1}(sx_{\gamma'} + tx_{\gamma'-1}) = s,\\ \tau_\nu &= x_\nu(z_0 - z_\nu) + y_\nu(w_0 - w_\nu) = x_\nu(-sy_{\gamma'} - a) + y_\nu(sx_{\gamma'} - b)\\ &= \tau_{\gamma'-1} - (ax_\nu + by_\nu) \leq \tau_{\gamma'-1}.\end{aligned}$$

In sum, we obtain

$$\begin{aligned}E_1.(E_1 + K_S) &= E_1.\Bigg(-D_{\nu'} + \sum_{i=\nu'+1}^{\gamma_1-1}(\tau_i - 1)D_i - D_{\gamma_1}\Bigg)\\ &\leq E_1.(-D_{\nu'} + (\tau_\nu - 1)D_\nu - D_{\gamma_1})\\ &= -\tau_{\nu'+1} + (\tau_\nu - 1)E_1.D_\nu - \tau_{\gamma_1-1}\\ &\leq -\tau_{\nu'+1} - \tau_\nu + 1 - \tau_{\gamma_1-1} \leq -2\tau_\nu.\end{aligned}$$

Let us consider the case where $y_\nu = 1$. We note that $x_\nu = -c_0$ and $\gamma' = \iota$ in this case. If $(C - 2I_C - E_1).D_i = 0$ for integers $\nu' + 1 \leq i \leq \gamma' - 1$, then we have $Q_{\gamma'} = Q_{\nu'-1}$. Namely, R must be a lattice point. Then, since the X-coordinate of R is greater than (resp. less than or equal to) that of Q (resp.

P), we have $\nu' = 1$, which contradicts the condition (a). Hence we can take an integer $\kappa = \max\{n \mid \nu'+1 \le n \le \gamma'-1, (C-2I_C-E_1).D_n \ge 1\}$. If $\kappa > \nu$, then by computing, we have

$$\begin{aligned} \tau_\kappa &= x_\kappa(-y_{\gamma'}) + y_\kappa x_{\gamma'} = -x_\kappa + (e_0 - c_0)y_\kappa, \\ \tau_\nu &\le x_\nu(-y_{\gamma'} + y_\kappa) + y_\nu(x_{\gamma'} - x_\kappa) = -c_0 y_\kappa + e_0 - x_\kappa \le \tau_\kappa. \end{aligned}$$

We thus have

$$\begin{aligned} (23) \qquad E_1.(E_1+K_S) &= E_1.\Big(-D_{\nu'} + \sum_{i=\nu'+1}^{\gamma'-1}(\tau_i - 1)D_i - D_{\gamma'}\Big) \\ &\le -E_1.D_{\nu'} + (\tau_\kappa - 1)E_1.D_\kappa + (\tau_\nu - 1)E_1.D_\nu - E_1.D_{\gamma'} \\ &\le -\tau_{\nu'+1} - \tau_\kappa - \tau_\nu + 2 - \tau_{\gamma'-1} \le -\tau_\kappa - \tau_\nu \le -2\tau_\nu. \end{aligned}$$

In the case where $\kappa = \nu$, we have $E_1.D_\nu \le -2$ and

$$E_1.(E_1+K_S) \le -E_1.D_{\nu'} + (\tau_\nu - 1)E_1.D_\nu - E_1.D_{\gamma'} \le -\tau_{\nu'+1} - 2\tau_\nu + 2 - \tau_{\gamma'-1} \le -2\tau_\nu.$$

In the case where $\kappa < \nu$, we have

$$\begin{aligned} \tau_\kappa &= x_\kappa(y_\nu - y_{\gamma'}) + y_\kappa(-x_\nu + x_{\gamma'}) = e_0 y_\kappa, \\ \tau_\nu &\le x_\nu(-y_{\gamma'}) + y_\nu x_{\gamma'} = e_0 \le \tau_\kappa. \end{aligned}$$

Hence the inequality (23) holds in this case also. Therefore, E_1 satisfies the property (iii).

We define

$$\begin{aligned} \delta' &= \max\{i \le \xi - 1 \mid (C-2I_C).D_i \ge 1\} = \begin{cases} \delta_{t_0} & (y_\xi \ge 2), \\ \iota & (y_\xi = 1), \end{cases} \\ \xi' &= \min\{i \ge \xi + 1 \mid C.D_i \ge 1\}. \end{aligned}$$

Then, by a similar way to that in the case of E_1, we can construct the effective divisor $E_2 = \sum_{i=\delta'+1}^{\xi'-1} \omega_i D_i$ satisfying the following properties (i)$'$–(iii)$'$:

(i)$'$ $(C - 2I_C - E_2).D_i \ge 0$ for $d_0 \le i \le d$,
(ii)$'$ $h^0(S, C - 2I_C - E_2) = h^0(S, C - 2I_C)$,
(iii)$'$ $E_2.(E_2 + K_S) \le 2E_2.(C - 2I_C)$.

Note that $E_2 = 0$ in the case where $C.D_\xi \ge 2$. In sum, we obtain the effective divisors

$$E_1 = \begin{cases} 0 & (C.D_\nu \ge 2), \\ \displaystyle\sum_{i=\nu'+1}^{\gamma'-1} \tau_i D_i & (C.D_\nu = 1), \end{cases} \qquad \gamma' = \begin{cases} \gamma_1 & (y_\nu \ge 2), \\ \iota & (y_\nu = 1), \end{cases}$$

$$E_2 = \begin{cases} 0 & (C.D_\xi \geq 2), \\ \sum_{i=\delta'+1}^{\xi'-1} \omega_i D_i & (C.D_\xi = 1), \end{cases} \qquad \delta' = \begin{cases} \delta_{t_0} & (y_\xi \geq 2), \\ \iota & (y_\xi = 1) \end{cases}$$

which satisfy the properties (i)–(iii) and (i)′–(iii)′.

We put $E = E_1 + E_2$. Let us check that $(C - 2I_C - E).D_i \geq 0$ holds for any integer $1 \leq i \leq d$. This is obvious if either E_1 or E_2 is equal to zero. Hence we consider the case where $C.D_\nu = C.D_\xi = 1$. By noting $\gamma' \leq \iota \leq \delta'$, $(C - 2I_C - E).D_i$ is non-negative if $i \neq \iota$. Since $(C - 2I_C - E).D_\iota$ is non-negative if $\gamma' < \iota$ or $\delta' > \iota$, we consider the case where $\gamma' = \delta' = \iota$. Then we have

$$(C - 2I_C - E).D_\iota \geq -2I_C.D_\iota - \tau_{\gamma'-1} - \omega_{\delta'+1}.$$

Similarly to (22), one can show the inequality $\omega_{\delta'+1} \leq -I_C.D_{\delta'}$ in the case where $\delta' = \iota$. We thus have $(C - 2I_C - E).D_\iota \geq 0$. In sum, we can conclude that $\mathrm{Bs}|C - 2I_C - E| = \emptyset$. Since

$$\begin{aligned} \square_{C-2I_C} \setminus \square_{C-2I_C-E} &= \square_{C-2I_C} \setminus (\square_{C-2I_C-E_1} \cap \square_{C-2I_C-E_1}) \\ &= (\square_{C-2I_C} \setminus \square_{C-2I_C-E_1}) \cup (\square_{C-2I_C} \setminus \square_{C-2I_C-E_2}), \end{aligned}$$

the properties (ii) and (ii)′ imply that $(\square_{C-2I_C} \setminus \square_{C-2I_C-E}) \cap \mathbb{Z} = \emptyset$. Namely, we obtain $h^0(S, C - 2I_C - E) = h^0(S, C - 2I_C)$. Lastly, we have

$$\begin{aligned} &E.(E + K_S) - 2E.(C - 2I_C) \\ &\quad = E_1.(E_1 + K_S) - 2E_1.(C - 2I_C) + E_2.(E_2 + K_S) - 2E_2.(C - 2I_C) + 2E_1.E_2 \\ &\quad \leq 2E_1.E_2 = 0 \end{aligned}$$

by the properties (iii) and (iii)′. □

Lemma 5.12. *Assume that $k \geq 2$ and $|C - F|$ is base point free. By Proposition 5.10, we can take a nonsingular irreducible curve $C_1 \in |C - I_C|$. If $f_0 \geq 1$ and C satisfies neither of the conditions* (a) *nor* (b) *in Lemma 5.11, then the divisor $(C_1 - I_C)|_{C_1}$ is nonspecial.*

Proof. Since $|C - F_1|$ has no base points, we have $(C - F_1).D_1 = C.D_1 - 1 \geq 0$. Hence, by Riemann-Roch theorem and List 5.7, we have

$$\begin{aligned} &\deg C_1|_{C_1} - 2g_1 \\ &\quad = -C_1.K_S - 2 \geq C_1.D_1 - 2 = C.D_1 - 2 \geq -1, \end{aligned}$$

where g_1 denotes the genus of C_1. Hence also $C_1|_{C_1}$ is nonspecial.

Assume that C satisfies neither of (a) nor (b). Since $H^1(S, -I_C) = H^2(S, -I_C) = 0$, the cohomology exact sequence similar to (12) implies that $h^1(C_1, (C - 2I_C)|_{C_1}) = h^1(S, C - 2I_C)$. We take the effective divisor E as in

Lemma 5.11. Then we have

$$\begin{aligned} & h^1(S, C-2I_C) \\ & = h^0(S, C-2I_C) + \frac{1}{2}(C-2I_C).(K_S - C + 2I_C) - 1 \\ & = h^0(S, C-2I_C-E) + \frac{1}{2}(C-2I_C-E).(K_S - C + 2I_C + E) - 1 \\ & \quad + \frac{1}{2}E.(E+K_S) - E.(C-2I_C) \\ & = h^1(S, C-2I_C-E) + \frac{1}{2}E.(E+K_S) - E.(C-2I_C) \leq 0. \qquad \square \end{aligned}$$

We need Lemma 5.12 in the proof of Assertion 3.3. Hence we must show Assertion 3.3 independently for the cases excluded in Lemma 5.12.

Lemma 5.13. *Assume that $k \geq 2$, $f_0 \geq 1$ and $|C-F|$ is base point free. If either of the conditions* (a) *or* (b) *in Lemma 5.11 holds, then $h^0(S, C-I_C) - h^0(S, C-2I_C) \geq (C-I_C).F + 2$.*

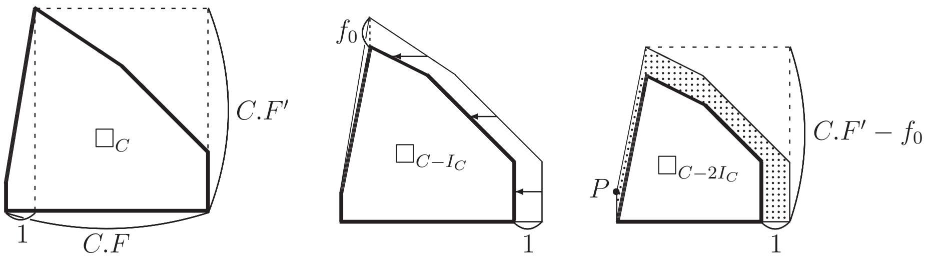

Figure 6

Proof. We prove the case of (a). By Theorem 2.4, the statement of the lemma is equivalent to the inequality $\sharp(\square_{C-I_C} \cap \mathbb{Z}^2) - \sharp(\square_{C-2I_C} \cap \mathbb{Z}^2) \geq C.F + 1$. Comparing the two lattice polytope $\square_C$ and $\square_{C-I_C}$, by the definition of I_C, we see that the horizontal distance between $l_i(C)$ and $l_i(C-I_C)$ (similarly, $l_i(C-I_C)$ and $l_i(C-2I_C)$) is at least one for each integer $d_0+1 \leq i \leq d-1$ (see Fig. 6). Indeed, if we write the linear equivalence class of F' as $F' \sim \sum_{d_0+1}^{d-1} x_i D_i$, then the coefficient of D_i in the linear equivalence class of I_C is at least x_i for each integer $d_0+1 \leq i \leq d-1$. By noting the existence of the point P, we see that there exist at least $C.F' - f_0 + 2$ lattice points in $\square_{C-I_C} \setminus \square_{C-2I_C}$. Hence the lemma is clear if $f_0 = 1$. Let us consider the case where $f_0 \geq 2$. In this case, we have

$$C.F = \sum_{i=2}^{d-2} y_i C.D_i = 1 + \sum_{i=d_0+1}^{d-2} y_i C.D_i,$$

$$\begin{aligned} C.F' &= \sum_{i=d_0+1}^{d-1} x_i C.D_i \geq C.D_{d-1} + \frac{x_\xi}{y_\xi} \sum_{i=d_0+1}^{d-2} y_i C.D_i > 1 + (e_0 - 1)(C.F - 1) \\ &\geq 1 + (f_0 - 1)(C.F - 1) = C.F + (f_0 - 2)(C.F - 1) \geq C.F + f_0 - 2. \end{aligned}$$

Hence the lemma is true in this case also. □

Lemma 5.14. *Assume that $k \geq 2$ and $f_0 = 0$. Then $h^0(S, C - I_C) - h^0(S, C - 2I_C) \geq (C - I_C).F + 2$.*

PROOF. Since $I_C = F'$ in this case, the vertical distance between $l_i(C)$ and $l_i(C - I_C)$ (similarly, $l_i(C - I_C)$ and $l_i(C - 2I_C)$) is just one for each integer $d_0 + 1 \leq i \leq d - 1$. Hence, by a similar argument to that in the proof of Lemma 5.13, we have $\sharp(\Box_{C-I_C} \cap \mathbb{Z}^2) - \sharp(\Box_{C-2I_C} \cap \mathbb{Z}^2) \geq C.F + 1$. □

5.2 Proof of Assertion 3.2 and 3.3

In this subsection, we will prove Assertion 3.2 and 3.3. Let D be a divisor on S such that $D.D_1 \geq 1$. For an integer x with $0 \leq x \leq D.F'$, we define

$$i(D, x) = \max \left\{ j \geq 1 \;\middle|\; x < -\sum_{i=j}^{d_0-1} x_i D.D_i \right\},$$

$$y(D, x) = \sum_{i=i(D,x)+1}^{d_0-1} y_i D.D_i - \frac{y_{i(D,x)}}{x_{i(D,x)}} \left(x + \sum_{i=i(D,x)+1}^{d_0-1} x_i D.D_i \right).$$

Let C be a curve on S such as at the beginning of this section and I_C the auxiliary divisor of C. In this subsection, we write the linear equivalence class of F' as $F' \sim \sum_{d_0+1}^{d-1} x_i D_i$. Then I_C is written as

$$I_C = \sum_{i=0}^{d_0} q_i D_i + \sum_{d_0+1}^{d-1} (q_i + x_i) D_i,$$

where we formally set $q_0 = q_d$ and $D_0 = D_d$.

Remark 5.15. By definition, we have $y(C, 0) = 0$. Besides, $y(C, x) = -y_\nu x / x_\nu$ if $i(C, x) = \nu$.

Lemma 5.16. *Assume that $k \geq 2$ and $C.D_1 \geq 1$. Let x be an integer with $f_0 \leq x \leq C.F'$, and define $q = \min\{n \in \mathbb{Z} \mid n > y(C - I_C, x - f_0)\}$. Then*

$$\begin{aligned} y(C - I_C, x - f_0) + n_{d_0} &\leq y(C, x) < q + n_{d_0} && (c_0 > e_0), \\ y(C - I_C, x - f_0) + n_{d_0} + 1 &\leq y(C, x) < q + n_{d_0} + 1 && (c_0 < e_0). \end{aligned}$$

PROOF. We prove only the former case. The latter case can be proved by a similar procedure. Since $c_0 > e_0$, we have $\nu, \iota \leq d_0 - 1$. Let us compute $\sum_{i=i(C,x)}^{d_0-1} x_i I_C.D_i$. Recall (C) in List 5.7. If $\iota = \gamma_{s_0}$, then by noting $x_{\gamma_{s_0}} = e_0 - c_0$, we have

$$\begin{aligned}\sum_{i=i(C,x)}^{d_0-1} x_i I_C.D_i &= x_\nu - n_{\gamma_1}x_{\gamma_1} - \cdots - n_{\gamma_{s_0-1}}x_{\gamma_{s_0-1}} - (n_{\gamma_{s_0}}+1)x_{\gamma_{s_0}} \\ &= -c_0 - x_{\gamma_{s_0}} = -e_0.\end{aligned}$$

If $\iota \geq \gamma_{s_0}+1$, then by noting $n_\iota = 0$, we have

$$\sum_{i=i(C,x)}^{d_0-1} x_i I_C.D_i = x_\nu - n_{\gamma_1}x_{\gamma_1} - \cdots - n_{\gamma_{s_0}}x_{\gamma_{s_0}} - x_\iota = -c_0 - x_\iota = -e_0.$$

We thus obtain

$$\sum_{i=i(C,x)+1}^{d_0-1} x_i I_C.D_i = \begin{cases} -e_0 - x_\nu & (i(C,x)=\nu), \\ -e_0 & (i(C,x)\leq \nu-1). \end{cases} \tag{24}$$

Similarly, we can obtain

$$\sum_{i=i(C,x)+1}^{d_0-1} y_i I_C.D_i = \begin{cases} n_{d_0} - y_\nu & (i(C,x)=\nu), \\ n_{d_0} & (i(C,x)\leq \nu-1). \end{cases}$$

(i) Consider the case where $x \geq -x_\nu$. If $i(C,x)=\nu$, then we have

$$\begin{aligned} y(C-I_C, x-e_0) &= -\sum_{i=\nu+1}^{d_0-1} y_i I_C.D_i + \frac{y_\nu}{x_\nu}\left(x - e_0 - \sum_{i=\nu+1}^{d_0-1} x_i I_C.D_i\right) \\ &= y_\nu - n_{d_0} - \frac{y_\nu}{x_\nu}x - y_\nu = y(C,x) - n_{d_0}. \end{aligned}$$

On the other hand, if $i(C,x)\leq \nu-1$, then we have

$$\begin{aligned} -\sum_{i=i(C,x)+1}^{d_0-1} x_i C.D_i &\leq x < -\sum_{i=i(C,x)}^{d_0-1} x_i C.D_i \\ -\sum_{i=i(C,x)+1}^{d_0-1} x_i (C-I_C).D_i &\leq x - e_0 < -\sum_{i=i(C,x)}^{d_0-1} x_i (C-I_C).D_i \end{aligned}$$

by (24). This means that $i(C-I_C, x-e_0) = i(C,x)$. Hence

$$\begin{aligned} & y(C-I_C, x-e_0) \\ &= \sum_{i=i(C,x)+1}^{d_0-1} y_i (C-I_C).D_i - \frac{y_{i(C,x)}}{x_{i(C,x)}}\left(x - e_0 + \sum_{i=i(C,x)+1}^{d_0-1} x_i (C-I_C).D_i\right) \\ &= \sum_{i=i(C,x)+1}^{d_0-1} y_i C.D_i - n_{d_0} - \frac{y_{i(C,x)}}{x_{i(C,x)}}\left(x + \sum_{i=i(C,x)+1}^{d_0-1} x_i C.D_i\right) = y(C,x) - n_{d_0}. \end{aligned}$$

Consequently, our lemma is true in any case.
(ii) We next consider the case where $x < -x_\nu$. Note that $i(C,x) = \nu$ and $y(C,x) = -y_\nu x/x_\nu$ in this case. Since $C.D_i = 0$ for $i(C,x)+1 \le i \le d_0 - 1$, we have

$$x - e_0 < -x_\nu - e_0 = \sum_{i=i(C,x)+1}^{d_0-1} x_i I_C.D_i = -\sum_{i=i(C,x)+1}^{d_0-1} x_i(C - I_C).D_i$$

by (24). Hence there exists an integer s with $1 \le s \le s_0$ such that $i(C - I_C, x - e_0) = \gamma_s$. Then

$$\begin{aligned}
y(C - I_C, x - e_0) &= \sum_{i=\gamma_s+1}^{d_0-1} y_i(C - I_C).D_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\Big(x - e_0 + \sum_{i=\gamma_s+1}^{d_0-1} x_i(C - I_C).D_i\Big) \\
&= \sum_{i=\gamma_s}^{d_0-1} y_i(C - I_C).D_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\Big(x - e_0 + \sum_{i=\gamma_s}^{d_0-1} x_i(C - I_C).D_i\Big) \\
&\le \sum_{i=\gamma_s}^{d_0-1} y_i(C - I_C).D_i - \frac{y_\nu}{x_\nu}\Big(x - e_0 + \sum_{i=\gamma_s}^{d_0-1} x_i(C - I_C).D_i\Big) \\
&= -\sum_{i=\gamma_s}^{d_0-1} y_i I_C.D_i - \frac{y_\nu}{x_\nu}\Big(x - e_0 - \sum_{i=\gamma_s}^{d_0-1} x_i I_C.D_i\Big) \\
&= -n_{d_0} + y_\nu + \sum_{i=\nu+1}^{\gamma_s-1} y_i I_C.D_i - \frac{y_\nu}{x_\nu}\Big(x + x_\nu + \sum_{i=\nu+1}^{\gamma_s-1} x_i I_C.D_i\Big) \\
&= -n_{d_0} - \frac{y_\nu}{x_\nu}x + \sum_{i=\nu+1}^{\gamma_s-1}\Big(y_i - \frac{y_\nu}{x_\nu}x_i\Big) I_C.D_i \\
&\le -n_{d_0} - \frac{y_\nu}{x_\nu}x = y(C,x) - n_{d_0}.
\end{aligned}$$

We next show the inequality $y(C,x) < q + n_{d_0}$. Suppose that $q + n_{d_0} = y(C,x) = -y_\nu x/x_\nu$. Note that $q + n_{d_0}$ is the integer. Then we have $x = 0$ by the inequality $x < -x_\nu$ and the fact that x_ν and y_ν are relatively prime. This implies that q is nonpositive, a contradiction. Suppose that $q + n_{d_0} < -y_\nu x/x_\nu$. Since $y_\nu/x_\nu > y_{\gamma_s}/x_{\gamma_s}$, we have

$$-(q + n_{d_0})x_{\gamma_s} - xy_{\gamma_s} \le -1 = -x_{\gamma_s}\Big(1 + \sum_{i=\gamma_s+1}^{d_0} n_i y_i\Big) + y_{\gamma_s}\Big(-c_0 + \sum_{i=\gamma_s+1}^{d_0} n_i x_i\Big)$$

$$
\begin{aligned}
q &\le 1+\sum_{i=\gamma_s+1}^{d_0} n_i y_i - n_{d_0} - \frac{y_{\gamma_s}}{x_{\gamma_s}}\left(x-c_0+\sum_{i=\gamma_s+1}^{d_0} n_i x_i\right)\\
&= 1+\sum_{i=\gamma_s+1}^{d_0-1} n_i y_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\left(x-c_0+\sum_{i=\gamma_s+1}^{d_0-1} n_i x_i\right)\cdots(*).
\end{aligned}
$$

If $\iota \ge \gamma_s+1$, then by List 5.7, we have

$$
\begin{aligned}
(*) &= -\sum_{i=\gamma_s+1}^{d_0-1} y_i I_C.D_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\left(x-e_0-\sum_{i=\gamma_s+1}^{d_0-1} x_i I_C.D_i\right)\\
&= \sum_{i=\gamma_s+1}^{d_0-1} y_i (C-I_C).D_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\left(x-e_0+\sum_{i=\gamma_s+1}^{d_0-1} x_i (C-I_C).D_i\right)\\
&= x(C-I_C, x-e_0).
\end{aligned}
$$

If $\iota = \gamma_s$, then since $(C-I_C).D_i = 0$ for $\gamma_s+1 \le i \le d_0-1$, we have

$$
\begin{aligned}
(*) &= 1-\frac{y_{\gamma_s}}{x_{\gamma_s}}(x-c_0) = -\frac{y_{\gamma_s}}{x_{\gamma_s}}(x-e_0)\\
&= \sum_{i=\gamma_s+1}^{d_0-1} y_i (C-I_C).D_i - \frac{y_{\gamma_s}}{x_{\gamma_s}}\left(x-e_0+\sum_{i=\gamma_s+1}^{d_0-1} x_i (C-I_C).D_i\right)\\
&= x(C-I_C, x-e_0).
\end{aligned}
$$

Obviously, they contradict the definition of q. Consequently, we can conclude that $y(C,x) < q + n_{d_0}$. $\square$

We define $i'(D,x')$ and $y'(D,x')$ in a way similar to $i(D,x)$ and $y(D,x)$, respectively. Concretely, in the case where $D.D_{d-1} \ge 1$, for an integer x' with $0 \le x' \le D.F'$, we define

$$
\begin{aligned}
i'(D,x') &= \min\left\{ j \le d-1 \;\middle|\; x' < \sum_{i=d_0+1}^{j} x_i D.D_i \right\},\\
y'(D,x') &= \sum_{i=d_0+1}^{i'(D,x')-1} y_i D.D_i + \frac{y_{i'(D,x')}}{x_{i'(D,x')}}\left(x' - \sum_{i=d_0+1}^{i'(D,x')-1} x_i D.D_i\right).
\end{aligned}
$$

Then we can obtain the following lemma by an argument similar to that in the proof of Lemma 5.16.

Lemma 5.17. *Assume that $k \ge 2$ and $C.D_{d-1} \ge 1$. Let x' be an integer with $f_0 \le x' \le C.F'$, and define $q' = \min\{n \in \mathbb{Z} \mid n > y'(C-I_C, x'-f_0)\}$. Then*

$$
\begin{aligned}
y'(C-I_C, x'-f_0) + m_{d_0} + 1 \le y'(C,x') &< q' + m_{d_0} + 1 \quad (c_0 > e_0),\\
y'(C-I_C, x'-f_0) + m_{d_0} \le y'(C,x') &< q' + m_{d_0} \quad (c_0 < e_0).
\end{aligned}
$$

Similar to Section 4, we next consider the operation to take auxiliary divisors repeatedly. We put $C_0 = C$ and $I_0 = I_C$, and take a nonsingular irreducible curve $C_1 \in |C_0 - I_0|$. For an integer $n \geq 2$, if $C_j.F \geq 2$, $C_j.F' \geq 2$ and

$$\begin{aligned}\nu_j &= \max\{i \leq d_0 \mid C_j.D_i \geq 1\} \geq 2,\\ \xi_j &= \min\{i \geq d_0 \mid C_j.D_i \geq 1\} \leq d-2\end{aligned}$$

for any integer j with $1 \leq j \leq n-1$, then we can take the auxiliary divisor I_{n-1} of C_{n-1} and a nonsingular irreducible curve $C_n \in |C_{n-1} - I_{n-1}|$ inductively. Note that $n \leq C.F - 1$ by the condition $C_{n-1}.F \geq 2$. We define

$$\begin{aligned}c_j &= -\max\{c \in \mathbb{Z} \mid x_{\nu_j} - c y_{\nu_j} \geq 0\},\\ e_j &= \min\{e \in \mathbb{Z} \mid x_{\xi_j} - e y_{\xi_j} \leq 0\},\\ f_j &= \min\{c_j, e_j\},\end{aligned}$$

$$\alpha_j : \text{the cardinality of the set} \quad \{l \mid 0 \leq l \leq j, c_l < e_l\}$$

for an integer j with $0 \leq j \leq n-1$. We divide each (x_{ν_j}, y_{ν_j}) and (x_{ξ_j}, y_{ξ_j}) in ways similar to (16) as

$$(x_{\nu_j}, y_{\nu_j}) = (-c_j, 1) + \sum_{i=\nu_j+1}^{d_0} n_i^j (x_i, y_i),$$

$$(x_{\xi_j}, y_{\xi_j}) = (e_j, 1) + \sum_{i=d_0}^{\xi_j - 1} m_i^j (x_i, y_i).$$

In the following Lemma 5.18–5.20, we assume that we can take nonsingular irreducible curves $C_1, \ldots, C_n$ in the above way.

Lemma 5.18. *Assume that $C.D_1 \geq 1$. Let x be an integer with $x \leq -\sum_{i=1}^{d_0-1} x_i C_0.D_i$. If $x \geq f_0 + \cdots + f_{n-1}$ and $C_j.D_{d_0} = 0$ for $j = 0, \ldots, n$, then $y(C_n, x - f_0 - \cdots - f_{n-1}) + \alpha_{n-1} \leq y(C_0, x)$ and there is no integer in the half-open interval $\big(y(C_n, x - f_0 - \cdots - f_{n-1}) + \alpha_{n-1}, y(C_0, x)\big]$.*

Proof. Let j be an integer with $0 \leq j \leq n-1$. By assumption, we have $c_j \neq e_j$ and $n_{d_0}^j = m_{d_0}^j = 0$. Besides, by computing, we have

$$\sum_{i=1}^{d_0-1} x_i C_j.D_i = \sum_{i=1}^{d_0-1} x_i C_0.D_i + f_0 + \cdots + f_{j-1},$$

where we define $f_0 + \cdots + f_{j-1} = 0$ in the case where $j = 0$. We thus have $f_j \leq x - f_0 - \cdots - f_{j-1} \leq -\sum_{i=1}^{d_0-1} x_i C_j.D_i$. In the case where $c_j > e_j$, Lemma 5.16 implies that $y(C_{j+1}, x - f_0 - \cdots - f_j) \leq y(C_j, x - f_0 - \cdots - f_{j-1})$ and there exists no integer in the half-open interval $\big(y(C_{j+1}, x - f_0 - \cdots - f_j), y(C_j, x - f_0 - \cdots - f_{j-1})\big]$.

Similarly, if $c_j < e_j$, then there exists no integer in the half-open interval $\big(y(C_{j+1}, x - f_0 - \cdots - f_j) + 1, y(C_j, x - f_0 - \cdots - f_{j-1})\big]$. Therefore, our lemma is clear. □

A similar lemma holds for the opposite side of the lattice polytope.

Lemma 5.19. *Assume that $C.D_{d-1} \geq 1$. Let x' be an integer with $x' \leq \sum_{i=d_0+1}^{d-1} x_i C_0.D_i$. If $x' \geq f_0 + \cdots + f_{n-1}$ and $C_j.D_{d_0} = 0$ for $j = 0, \ldots, n$, then $y'(C_n, x' - f_0 - \cdots - f_{n-1}) + n - \alpha_{n-1} \leq y'(C_0, x')$ and there is no integer in the half-open interval $\big(y'(C_n, x' - f_0 - \cdots - f_{n-1}) + n - \alpha_{n-1}, y'(C_0, x')\big]$.*

Lemma 5.20. *Assume that $k' - k = f_0 - 1 \geq 1$ and $|C - F|$ is base point free. If $n \leq k - 2$ and $f_1 = \cdots = f_{n-1} = 1$, then $\nu_n \geq 2$ and $\xi_n \leq d - 2$.*

Proof. An easy computation shows that $C_{n-1}.F' = C.F' - f_0 - \cdots - f_{n-2} = C.F - n + 1 = C_{n-1}.F$. Suppose that $\nu_n = 1$. Since $C_n.D_i = 0$ for any integer $2 \leq i \leq d_0$, by List 5.7, C_{n-1} must satisfy the properties $c_{n-1} < e_{n-1}$, $y_{\nu_{n-1}} = 1$, $C_{n-1}.D_{\nu_{n-1}} = 1$ and $C_{n-1}.D_i = 0$ for any integer $2 \leq i \leq \nu_{n-1} - 1$. Since $e_{n-1} \geq 2$, by the same computation as that at the last of the proof of Lemma 5.13, we have $C_{n-1}.F' > C_{n-1}.F$, a contradiction. One can induce the same contradiction under the assumption that $\xi_n = d - 1$. □

Let us consider Assertion 3.2 with respect to a special case:

Lemma 5.21. *Assume $k' = k+1 \geq 3$, $f_0 = 2$ and $|C - F|$ is base point free. Then $x_{\nu_0} \leq -2$, $x_{\xi_0} \geq 2$ and there exists an integer m with $2 \leq m \leq \min\{-x_{\nu_0}, x_{\xi_0}\}$ such that $f_1 = \cdots = f_{m-2} = 1$ and $f_{m-1} = 0$.*

Proof. Since $f_0 = 2$, we have $C.D_{d_0} = 0$, $x_{\nu_0} < -y_{\nu_0}$ and $x_{\xi_0} > y_{\xi_0}$. It follows that $x_{\nu_0} \leq -2$ and $x_{\xi_0} \geq 2$. As we saw in the proof of Proposition 5.10 and Lemma 5.12, $C.D_1$ and $C.D_{d-1}$ are positive. By the inequality

$$
\begin{aligned}
(25)\quad k &= \sum_{i=2}^{d-2} y_i C.D_i \\
&= \sum_{i=2}^{d_0-1} y_i C.D_i + \sum_{i=d_0+1}^{d-2} y_i C.D_i \leq \frac{y_{\nu_0}}{x_{\nu_0}} \sum_{i=2}^{d_0-1} x_i C.D_i + \frac{y_{\xi_0}}{x_{\xi_0}} \sum_{i=d_0+1}^{d-2} x_i C.D_i \\
&= \frac{y_{\nu_0}}{x_{\nu_0}}(-C.F' + C.D_1) + \frac{y_{\xi_0}}{x_{\xi_0}}(C.F' - C.D_{d-1}) \leq \left(-\frac{y_{\nu_0}}{x_{\nu_0}} + \frac{y_{\xi_0}}{x_{\xi_0}}\right)(k' - 1),
\end{aligned}
$$

we obtain

$$
(26)\qquad -\frac{y_{\nu_0}}{x_{\nu_0}} + \frac{y_{\xi_0}}{x_{\xi_0}} \geq 1.
$$

Consider the case where $\min\{-x_{\nu_0}, x_{\xi_0}\} = 2$. If $x_{\nu_0} = -2$, we have $y_{\nu_0} = 1$, that is, $c_0 = 2$ by the assumption $f_0 = 2$. Hence, by (26), we have

$1/2 \le y_{\xi_0}/x_{\xi_0} < 1/(e_0-1)$, that is, $e_0 = 2$. Similarly, one can show $c_0 = e_0 = 2$ in the case where $x_{\xi_0} = 2$. Therefore, by (B) in List 5.7, we have $C_1.D_{d_0} \ge 1$ in this case. Hence the lemma is valid for $m = 2$.

We assume that $m_0 = \min\{-x_{\nu_0}, x_{\xi_0}\} \ge 3$. Let us consider the following operation:

$$\text{We can take } C_1. \to \begin{cases} f_1 = 0. \to \text{We can finish our proof.} \\ f_1 = 1. \to \text{We can take } C_2 \text{ by Lemma 5.20 if } k \ge 3. \swarrow \\ f_1 \ge 2. \end{cases}$$

$$\to \begin{cases} f_2 = 0. \to \text{We can finish our proof.} \\ f_2 = 1. \to \text{We can take } C_2 \text{ by Lemma 5.20 if } k \ge 4. \swarrow \\ f_2 \ge 2. \end{cases}$$

$$\to \cdots .$$

Then, since $k = k' - 1 \ge -x_1 C.D_1 - x_{\nu_0} C.D_{\nu_0} - 1 \ge m_0$, it is sufficient to check that

(i) $f_j \le 1$ if $f_1 = \cdots = f_{j-1} = 1$ for an integer j with $1 \le j \le m_0 - 1$,
(ii) especially $f_{m_0-1} = 0$ if $f_1 = \cdots = f_{m_0-2} = 1$.

(i) Let j be an integer with $1 \le j \le m_0 - 1$ such that $f_1 = \cdots = f_{j-1} = 1$. By Lemma 5.20, we can take the auxiliary divisor I_{j-1} of C_{j-1} and a nonsingular irreducible curve $C_j \in |C_{j-1} - I_{j-1}|$. Suppose that $f_j \ge 2$. Since $x_{\nu_j} \le -2$ and $x_{\xi_j} \ge 2$, we have $y(C_j, 1) = -y_{\nu_j}/x_{\nu_j}$ and $y'(C_j, 1) = y_{\xi_j}/x_{\xi_j}$. Hence, by Lemma 5.18 and 5.19,

$$\begin{aligned} y(C_0, j+2) &= -\frac{y_{\nu_0}}{x_{\nu_0}}(j+2) < \min\{l \in \mathbb{Z} \mid l > y(C_j,1) + \alpha_{j-1}\} \\ &= \min\left\{ l \in \mathbb{Z} \,\middle|\, l > -\frac{y_{\nu_j}}{x_{\nu_j}} + \alpha_{j-1} \right\} = \alpha_{j-1} + 1, \\ y'(C_0, j+2) &= \frac{y_{\xi_0}}{x_{\xi_0}}(j+2) < \min\{l \in \mathbb{Z} \mid l > y'(C_j,1) + j - \alpha_{j-1}\} \\ &= \min\left\{ l \in \mathbb{Z} \,\middle|\, l > \frac{y_{\xi_j}}{x_{\xi_j}} + j - \alpha_{j-1} \right\} = j - \alpha_{j-1} + 1. \end{aligned}$$

These inequalities imply that $-y_{\nu_0}/x_{\nu_0} + y_{\xi_0}/x_{\xi_0} < 1$, a contradiction. We thus obtain $f_j \le 1$.
(ii) Considering $y(C_0, m_0-1)$, $y(C_{m_0-2}, 0)$, $y'(C_0, m_0-1)$ and $y'(C_{m_0-2}, 0)$, Lemma 5.18 and 5.19 implies that

$$(27) \qquad \left(\alpha_{m_0-3}, -\frac{y_{\nu_0}}{x_{\nu_0}}(m_0-1)\right] \cap \mathbb{Z} = \emptyset,$$

$$\left(m_0 - 2 - \alpha_{m_0-3}, \frac{y_{\xi_0}}{x_{\xi_0}}(m_0-1)\right] \cap \mathbb{Z} = \emptyset.$$

Note that $x_{\nu_0} < -y_{\nu_0}$ and $x_{\xi_0} > y_{\xi_0}$ by the assumption $f_0 = 2$. In the case where $m_0 = -x_{\nu_0}$, we obtain $\alpha_{m_0-3} = y_{\nu_0} - 1$ by the first equality of (27). Since

$$(y(C_{m_0-2}, 1) + \alpha_{m_0-3}, y(C_0, m_0)] \cap \mathbb{Z} = \left(-\frac{y_{\nu_{m_0-2}}}{x_{\nu_{m_0-2}}} + y_{\nu_0} - 1, y_{\nu_0}\right] \cap \mathbb{Z} = \emptyset,$$

we have $y_{\nu_{m_0-2}}/x_{\nu_{m_0-2}} = -1$, that is, $c_{m_0-2} = 1$. On the other hand, since

$$\begin{aligned}&(y'(C_{m_0-2}, 1) + m_0 - 2 - \alpha_{m_0-3}, y'(C_0, m_0)] \cap \mathbb{Z}\\ &= \left(\frac{y_{\xi_{m_0-2}}}{x_{\xi_{m_0-2}}} - x_{\nu_0} - y_{\nu_0} - 1, -\frac{y_{\xi_0}}{x_{\xi_0}} x_{\nu_0}\right] \cap \mathbb{Z} = \emptyset,\end{aligned}$$

we have $y_{\xi_{m_0-2}}/x_{\xi_{m_0-2}} \geq 1$. Indeed, if $y_{\xi_{m_0-2}}/x_{\xi_{m_0-2}} < 1$, then we have $-y_{\xi_0}x_{\nu_0}/x_{\xi_0} < -x_{\nu_0} - y_{\nu_0}$, which contradicts (26). Hence we obtain $e_{m_0-2} = 1$. Therefore, we can conclude that $C_{m_0-1}.D_{d_0} \geq 1$ by (B) in List 5.7, which means that $f_{m_0-1} = 0$. In the case where $m_0 = x_{\xi_0}$, we obtain $\alpha_{m_0-3} = x_{\xi_0} - y_{\xi_0} - 1$ by the second equality of (27). Since

$$(y(C_{m_0-2}, 1) + \alpha_{m_0-3}, y(C_0, m_0)] \cap \mathbb{Z} = \left(-\frac{y_{\nu_{m_0-2}}}{x_{\nu_{m_0-2}}} + x_{\xi_0} - y_{\xi_0} - 1, -\frac{y_{\nu_0}}{x_{\nu_0}} x_{\xi_0}\right] \cap \mathbb{Z} = \emptyset,$$

we have $y_{\nu_{m_0-2}}/x_{\nu_{m_0-2}} \geq -1$. Indeed, if $y_{\nu_{m_0-2}}/x_{\nu_{m_0-2}} < -1$, then we have $-y_{\nu_0}x_{\xi_0}/x_{\nu_0} < x_{\xi_0} - y_{\xi_0}$, which contradicts (26). We thus have $c_{m_0-2} = 1$. On the other hand, since

$$(y'(C_{m_0-2}, 1) + m_0 - 2 - \alpha_{m_0-3}, y'(C_0, m_0)] \cap \mathbb{Z} = \left(\frac{y_{\xi_{m_0-2}}}{x_{\xi_{m_0-2}}} + y_{\xi_0} - 1, y_{\xi_0}\right] \cap \mathbb{Z} = \emptyset,$$

we have $y_{\xi_{m_0-2}}/x_{\xi_{m_0-2}} = 1$, that is, $e_{m_0-2} = 1$. Consequently, we have $f_{m_0-1} = 0$ in this case also. □

In keeping with the Lemma 5.21, let us show Assertion 3.2.

Proposition 5.22. *For the curve C, Assertion 3.2 is true.*

Proof. First, if $k = 2$ and $C_1.F' > C_1.F$ (resp. $C_1.F' = C_1.F$), then the statement is obviously true for $m_0 = 1$ (resp. $m_0 = 2$). Hence we assume that $k \geq 3$.

Consider the case where $f_0 \leq 1$. Since

$$C_1.F' - C_1.F = C.F' - I_0.F' - C.F + I_0.F = k' - f_0 - k + 1 \geq 2 - f_0 \geq 1,$$

the lemma is valid for $m_0 = 1$ in this case.

Let us consider the case where $f_0 = 2$. If $k' \geq k + 2$, then we have $C_1.F' \geq C_1.F + 1$, and can finish the proof. On the other hand, in the case where $k' = k + 1$, the lemma is valid for $m_0 = m$, where m is the integer in

Lemma 5.21. Indeed, this is clear in the case where $C_{m-1}.F = 1$. If $C_{m-1}.F \geq 2$, then we can take the auxiliary divisor I_{m-1} of C_{m-1} and a nonsingular irreducible curve $C_m \in |C_{m-1} - I_{m-1}|$ by Lemma 5.20. Since $f_{m-1} = 0$, we obtain $C_m.F' \geq C_m.F + 1$.

From now on, we assume that $f_0 \geq 3$. Since $x_{\nu_0} < -2y_{\nu_0}$ and $x_{\xi_0} > 2y_{\xi_0}$, we have $k' \geq k + 2$ by the same computation as (25). We put $a = \sum_{i=2}^{d_0-1} y_i C.D_i$ and $b = \sum_{i=d_0+1}^{d-2} y_i C.D_i$.

(i) Consider the case where $f_0 \geq 3$ and $k' = k + 2$. Then we have

$$a \leq \frac{y_{\nu_0}}{x_{\nu_0}} \sum_{i=2}^{d_0-1} x_i C.D_i < -\frac{1}{c_0 - 1} \sum_{i=2}^{d_0-1} x_i C.D_i = \frac{k' - C.D_1}{c_0 - 1} = \frac{k + 2 - C.D_1}{c_0 - 1},$$
$$b \leq \frac{y_{\xi_0}}{x_{\xi_0}} \sum_{i=d_0+1}^{d-2} x_i C.D_i < \frac{1}{e_0 - 1} \sum_{i=2}^{d_0-1} x_i C.D_i = \frac{k' - C.D_{d-1}}{e_0 - 1} = \frac{k + 2 - C.D_{d-1}}{e_0 - 1}.$$

Here we note that $C.D_1 = 1$. Indeed, if $C.D_1 \geq 2$ and k is even (resp. odd), then we have $a < k/2$ and $b \leq k/2$ (resp. $a \leq (k-1)/2$ and $b < (k+1)/2$), which contradicts the equality $k = a + b$. Similarly, one can obtain $C.D_{d-1} = 1$. If $c_0 \geq 4$, then we have

$$a \begin{cases} \leq 1 & (k = 3, 4), \\ < \dfrac{k+1}{3} & (k \geq 5), \end{cases}$$

which contradicts that $k = a + b$ and $b \leq (k+1)/2$. We thus have $c_0 = 3$. Similarly, one can obtain $e_0 = 3$. In sum, we have $f_1 = 0$ by (B) in List 5.7, and moreover

$$C_1.F' = C.F' - I_0.F' = C.F + 2 - f_0 = C.F - 1 = C_1.F.$$

Since $C_1.F = k - 1 \geq 2$, we can take the auxiliary divisor I_1 of C_1 and a nonsingular irreducible curve $C_2 \in |C_1 - I_1|$ by Lemma 5.20. Then we obtain $C_2.F' > C_2.F$ by $f_1 = 0$. Therefore, the lemma is valid for $m_0 = 2$ in this case.

(ii) Consider the case where $f_0 \geq 3$ and $k' \geq k + 3$. If $f_0 = 3$, then we have

$$C_1.F' = C.F' - 3 \geq C.F = C_1.F + 1.$$

Assume that $f_0 \geq 4$ and $a \geq b$. Then we have

$$\begin{aligned} C_1.F &= C.F - 1 = a + b - 1 \leq 2a - 1 \\ &= 2 \sum_{i=2}^{\nu_0 - 1} y_i C.D_i + 2y_{\nu_0}(C.D_{\nu_0} - 1) + 2y_{\nu_0} - 1, \\ C_1.F' &= C.F' - f_0 \end{aligned}$$

$$= -\sum_{i=1}^{d_0-1} x_i C.D_i - f_0 \geq C.D_1 - \frac{x_{\nu_0}}{y_{\nu_0}} a - f_0 > 1 + (f_0 - 1)a - f_0$$
$$= (f_0 - 1)\sum_{i=2}^{\nu_0-1} y_i C.D_i + (f_0 - 1)y_{\nu_0}(C.D_{\nu_0} - 1) + (f_0 - 1)(y_{\nu_0} - 1).$$

We thus obtain $C_1.F' > C_1.F$ if $y_{\nu_0} \geq 2$. If $y_{\nu_0} = 1$, then by noting $k \geq 3$, we have the inequality $\sum_{i=2}^{\nu_0} y_i C.D_i \geq 2$. Hence, in this case, either $\sum_{i=2}^{\nu_0-1} y_i C.D_i \geq 1$ or $C.D_{\nu_0} \geq 2$ holds, which implies that $C_1.F' > C_1.F$. Similarly, one can obtain the same inequality in the case where $f_0 \geq 4$ and $a \leq b$. Therefore, the lemma is valid for $m_0 = 1$ in this case. □

Proposition 5.23. *For the curve C, Assertion 3.3 is true.*

Proof. Let j be an integer with $1 \leq j \leq \min\{m_0, k-1\}$. If $f_{j-1} = 0$ (resp. $f_{j-1} \geq 1$ and C_{j-1} satisfies (a) or (b) in Lemma 5.11), then the statement follows from Lemma 5.14 (resp. Lemma 5.13).

Consider the case where $f_{j-1} \geq 1$ and C_{j-1} satisfies neither (a) nor (b). Note that $H^1(S, -I_{j-1}) = 0$ by Proposition 5.9 and $(C_j - I_{j-1})|_{C_j}$ is nonspecial by Lemma 5.12. Besides, a similar computation to (14) shows that also $C_j|_{C_j}$ is nonspecial. Hence we see that the statement of our lemma is equivalent to the inequality $C_j.I_{j-1} \geq C_j.F + 1$ by the same argument as that in the proof of Proposition 4.20. Let ι_{j-1} be an integer such that $(x_{\iota_{j-1}}, y_{\iota_{j-1}}) = (e_{j-1} - c_{j-1}, 1)$. For the auxiliary divisor $I_{j-1} = \sum_{i=1}^{d} q_i D_i + F'$, we have $q_{\iota_{j-1}} = f_{j-1} \geq 1$. Since $C_j.D_{\iota_{j-1}} \geq 1$ by List 5.7, we have

$$C_j.I_{j-1} \geq C_j.(q_{\iota_{j-1}} D_{\iota_{j-1}} + F') \geq C_j.F' + 1 \geq C_j.F + 1.$$

The above argument all goes through for $j = 1$ if $C.F' = C.F = 2$. □

References

[1] M. Aprodu, On the vanishing of higher syzygies of curves, Math. Z. **241** (2002), 1–15.

[2] M. Coppens and T. Kato, The gonality of smooth curves with plane models, Manuscripta Math. **70** (1990), 5–25.

[3] M. Coppens and T. Kato, Correction to: "The gonality of smooth curves with plane models", Manuscripta Math. **71** (1991), 337–338.

[4] S. Ehbauer, Syzygies of points in projective space and applications, Zero-dimensional schemes, Proceedings to the International conference (Ravello, 1992), de Gruyter, Berlin (1994), 145–170.

[5] M. Green, Koszul cohomology and the geometry of projective varieties, J. Diff. Geom. **19** (1984), 125–171.

[6] M. Green and R. Lazarsfeld, The nonvanishing of certain Koszul cohomology groups, J. Diff. Geom. **19** (1984), 168–170. (Appendix to [5])

[7] M. Green and R. Lazarsfeld, On the projective normality of complete linear series on

an algebraic curve, Invent. Math. **83** (1986), 73–90.
[8] R. Kawaguchi, The gonality conjecture for curves on certain toric surfaces, Osaka J. Math. **45** (2008), 113-126.
[9] G. Martens, The gonality of curves on a Hirzebruch surface, Arch. Math. **64** (1996), 349–452.
[10] M. Mustaţă, Vanishing Theorems on toric varieties, Tohoku Math. J. **54** (2002), 451–470.
[11] M. Namba, Geometry of projective algebraic curves, Monographs and Textbooks in Pure and Appl. Math. **88**, Marcel Dekker, New York, 1984.
[12] T. Oda, Convex bodies and algebraic geometry, Springer-Verlag, Berlin, 1988.
[13] M. Ohkouchi and F. Sakai, The gonality of singular plane curves, Tokyo J. Math. **27** (2004), 137–147.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560–0043, Japan
e-mail: kawaguchir@cr.math.sci.osaka-u.ac.jp