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The 3-Weierstrass points on genus two curves with extra involutions

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Abstract

We classify the 3-Weierstrass points on the genus two curves

$$C_{a,b} : y^2 = x^6 + ax^4 + bx^2 + 1,$$

where $a, b \in \mathbf{C}$ are two parameters. We describe the classification in terms of the invariants $u = ab$ and $v = a^3 + b^3$ (cf. [7]).

1. Introduction

We consider a 2-parameters family of genus two curves $C_{a,b}$ defined by

$$C_{a,b} : y^2 = x^6 + ax^4 + bx^2 + 1, \quad (\Delta(a, b) \neq 0),$$

where $\Delta(a, b) = -64(27 - 18ab + 4a^3 + 4b^3 - a^2b^2)^2$ is the discriminant of the polynomial $x^6 + ax^4 + bx^2 + 1$. It is clear that the curves $C_{a,b}$ admit an extra involution $(x, y) \mapsto (-x, y)$ which differs from the hyperelliptic involution.

It is well known that, for a hyperelliptic curve C of a genus $g \geq 2$ the set of ordinary Weierstrass points on C, denoted by $W_1(C)$, are nothing but its set of the $(2g+2)$-ramification points over $\mathbb{P}^1$. Furthermore, $W_1(C)$ is contained in the set of the q-Weierstrass points on C, denoted by $W_q(C)$, for every $q \geq 2$. For a point $P \in C$, let $w^{(q)}(P)$ denote the *q-weight* of P.

For a genus two curve C, we have the equality $W_1(C) = W_2(C)$. For a point $P \in W_3(C)$, there occur three cases: $w^{(3)}(P) = 1, 2$ and 3. We have $w^{(3)}(P) = 3$ if and only if $P \in W_1(C)$. We can divide $W_3(C)$ as

$$W_3(C) = W_3(C)_1 \cup W_3(C)_2 \cup W_1(C) \quad \text{(disjoint union)},$$

where $W_3(C)_1$ (resp. $W_3(C)_2$) is the set of the 3-Weierstrass points P with $w^{(3)}(P) = 1$ (resp. $w^{(3)}(P) = 2$). We denote by N_1 (resp. N_2) the number of points in $W_3(C)_1$ (resp. $W_3(C)_2$). We have the formula: $N_1 + 2N_2 = 32$.

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Shaska and Völklein [7] found a *generic moduli* for the curves $C_{a,b}$. They introduced the invariants $u = ab$ and $v = a^3 + b^3$ of a D_6 action, where D_{2n} denotes the dihedral group of order $2n$ (see Subsection 2.2). We have $\Delta(a,b) = -64\,\delta(u,v)^2$, where $\delta(u,v) = 27 + 4v - 18u - u^2$. Set $\Lambda : \delta(u,v) = 0$. Then the complement $\mathbf{C}^2 \setminus \Lambda$ generically classifies the curves $C_{a,b}$. The curves $C_{a,b}$ with the same invariants (u,v) are isomorphic, hence they have the same pair (N_1, N_2).

We define the following curves in the (u,v)-plane:

$$\begin{aligned} S : s(u,v) &= -1125 + 4v + 110u - u^2 = 0, \\ T : t(u,v) &= v^2 - 4u^3 = 0, \\ M : m(u,v) &= 4v - u(u+16) = 0, \\ G : g(u,v) &= 20796875 - 13942500u - 571350u^2 - 98324u^3 - 3645u^4 \\ &\quad +3429000v - 235440uv + 1512u^2v + 52272v^2 = 0. \end{aligned}$$

The curves S and T were introduced in [7], where it was shown that if $(u,v) \in S$ (resp. T), then there exists an automorphism of order three (resp. four) on $C_{a,b}$. We will encounter the curves M and G in Section 3.

The purpose of this paper is to prove the following:

Theorem. *We classify the* 3*-Weierstrass points on* $C_{a,b}$ *as follows:*

N_1	N_2	(u,v)	Geometry	$\mathrm{Aut}(C_{a,b})$
0	16	$(25,-250)$	$S \cap G \cap T$	$GL_2(3)$
12	10	A	$M \cap S \cap G$	D_{12}
16	8	$B_\pm$	$G \cap T$	D_8
		Q	the node of G	V_4
20	6	$E_\pm$	$M \cap G$	V_4
24	4	$(0,0)$	$M \cap T$	$\mathbb{Z}_3 \rtimes D_8$
		$(16,128)$	$M \cap T$	D_8
		general points on S		D_{12}
		general points on G		V_4
28	2	general points on M		V_4
32	0	otherwise		V_4

Here, we used the following notations:

$$\begin{aligned} N_1 &= \#(W_3(C)_1), \quad & N_2 &= \#(W_3(C)_2), \\ A &= \left(\frac{125}{14}, \frac{43625}{784}\right), \quad & Q &= \left(-\frac{25}{2}, -\frac{11125}{176}\right), \end{aligned}$$

$$B_\pm = \left(\frac{1025}{729} \pm \frac{5200}{729}\sqrt{-2},\ -\frac{698750}{19683} \pm \frac{758000}{19683}\sqrt{-2}\right),$$

$$E_\pm = \left(-\frac{647}{256} \pm \frac{3519}{3328}\sqrt{-39},\ -\frac{33079811}{1703936} \pm \frac{4930119}{1703936}\sqrt{-39}\right).$$

In Section 2, we recall some basic facts on q-Weierstrass points and Wronskian forms of q-differentials on genus two curves. In Section 3, we compute the 3-Weierstrass points on $C_{a,b}$ and prove Theorem. In Section 4, we prove that every 3-Weierstrass point of 3-weight 2 on a genus two curve is a q-Weierstrass point of q-weight 2 ($q \geq 4$) except if $q \equiv 2 \pmod 3$. We used the computer softwares Mathematica and Maple to perform the computations.

2. Preliminaries

Let C be a non-singular projective curve of genus two. Let $H^0(C, (\Omega^1)^q)$ be the $\mathbf{C}$-vector space of holomorphic q-differentials on C. By Riemann-Roch Theorem, we have $\dim H^0(C, (\Omega^1)^q) = (2q-1)$, for $q \geq 2$.

Definition 1. For a point P on C, take a basis $\{\psi_1, \ldots, \psi_{2q-1}\}$ of $H^0(C, (\Omega^1)^q)$ so that $\mathrm{ord}_P(\psi_1) < \cdots < \mathrm{ord}_P(\psi_{2q-1})$. Letting $n_i = \mathrm{ord}_P(\psi_i) + 1$, the sequence $G^{(q)}(P) = \{n_1, n_2, \ldots, n_{2q-1}\}$ is called the *q-gap sequence* of P. The non-negative integer

$$w^{(q)}(P) = \sum_{i=1}^{2q-1} (n_i - i)$$

is called the *q-weight* of P. We say that a point $P \in C$ is a *q-Weierstrass point* if $w^{(q)}(P) > 0$. We denote by $W_q(C)$ the set of all q-Weierstrass points on C.

Definition 2. Now we define the Wronskian form:

$$\Omega_q = W(\psi_1, \ldots, \psi_{2q-1}) \in H^0(C, (\Omega^1)^{(2q-1)^2}).$$

Since every q-differential ψ_i can be written in a form $\psi_i = f_i(z)(dz)^q$, $i = 1, \ldots, 2q-1$, where f_i is a holomorphic function and z is a local coordinate, then, we define

$$\Omega_q = W(\psi_1, \ldots, \psi_{2q-1}) = W(f_1, \ldots, f_{2q-1})(dz)^{(2q-1)^2},$$

where:

$$W(f_1, \ldots, f_{2q-1}) = \begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_{2q-1}(z) \\ f_1'(z) & f_2'(z) & \cdots & f_{2q-1}'(z) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(2q-2)}(z) & f_2^{(2q-2)}(z) & \cdots & f_{2q-1}^{(2q-2)}(z) \end{vmatrix}$$

Hence, we obtain

$$\mathrm{div}(\Omega_q) = \mathrm{div}(W(f_1, \ldots, f_{2q-1})) + (2q-1)^2 \mathrm{div}(dz).$$

By using the Wronskian method, we can prove the following:

Lemma 1. *We have the formula:*

$$\sum_{P \in C} w^{(q)}(P) = 2(2q-1)^2, \quad \textit{for } q \geq 2.$$

Lemma 2 (Duma [3])**.** *For every $P \in C$, we have $w^{(q)}(P) \leq 3$. The equality occurs if and only if $P \in W_1(C)$.*

Lemma 3. *The possible 3-gap sequences of $P \in W_3(C)$ are listed as follows:*

$G^{(3)}(P)$	$w^{(3)}(P)$
$\{1,2,3,4,6\}$	1
$\{1,2,3,4,7\}$	2
$\{1,2,3,5,7\}$	3

Proof. It suffices to see that $\{1,2,3,5,6\}$ is not a 3-gap sequence. Let $P \in C$ be a 3-Weierstrass point with $w^{(3)}(P) = 2$. Since $P \notin W_1(C)$, there exists an element $\psi \in H^0(C, \Omega^1)$ such that $\mathrm{ord}_P(\psi) = 1$. We have $\mathrm{ord}_P(\psi^3) = 3$, hence $4 \in G^{(3)}(P)$. □

2.1 The 3-Weierstrass points on $C_{a,b}$

Here, we will restrict our attention to the curves $C_{a,b}$. Write $f(x,y) = y^2 - (x^6 + ax^4 + bx^2 + 1)$. Clearly, $x^6 + ax^4 + bx^2 + 1 = (x^2 - \alpha_1^2)(x^2 - \alpha_2^2)(x^2 - \alpha_3^2)$, where $\alpha_1, \alpha_2, \alpha_3$ are nonzero distinct complex numbers. The set of branch points is given by $\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. The corresponding set of the ramification points on $C_{a,b}$ is given by

$$\{R_1^{\pm} = (\pm\alpha_1, 0), R_2^{\pm} = (\pm\alpha_2, 0), R_3^{\pm} = (\pm\alpha_3, 0)\}.$$

There are two points over the point $\infty \in \mathbb{P}^1$, in the nonsingular model of $C_{a,b}$, we call these two points P_1^{∞} and P_2^{∞}. We will denote by P_1^0 and P_2^0 the two points over the point $0 \in \mathbb{P}^1$.

In order to compute the 3-Weierstrass points on $C_{a,b}$, we use the following basis of holomorphic 3-differentials:

$$\psi_0 = (1/y^3)(dx)^3, \quad \psi_1 = (x/y^3)(dx)^3, \quad \psi_2 = (x^2/y^3)(dx)^3,$$
$$\psi_3 = (x^3/y^3)(dx)^3, \quad \psi_4 = (1/y^2)(dx)^3.$$

The divisors of these holomorphic 3-differentials are given as follows:

$$\begin{aligned} &\mathrm{div}(\psi_0) = 3(P_1^{\infty} + P_2^{\infty}), \quad \mathrm{div}(\psi_1) = (P_1^0 + P_2^0) + 2(P_1^{\infty} + P_2^{\infty}), \\ &\mathrm{div}(\psi_2) = 2(P_1^0 + P_2^0) + (P_1^{\infty} + P_2^{\infty}), \quad \mathrm{div}(\psi_3) = 3(P_1^0 + P_2^0), \\ &\mathrm{div}(\psi_4) = \textstyle\sum_{i=1}^{3} \left(R_i^+ + R_i^-\right). \end{aligned}$$

We can compute the Wronskian form Ω_3 (see Definition 2) by using the 4-th differentiation $y^{(4)}$ of y by x as follows:

Lemma 4. *1)* $W(1, x, x^2, x^3, y) = 12y^{(4)}$.

2) $\Omega_3(x, a, b) = W(1, x, x^2, x^3, y)(dx)^{25}/f_y^{15} \in H^0(C_{a,b}, (\Omega^1)^{25})$.

2.2 Parameter spaces of the curves $C_{a,b}$

The parameter space of the family $C_{a,b}$, is given by $\mathbf{C}^2 \setminus \widetilde{\Lambda}$ where $\widetilde{\Lambda}$ is a quartic curve defined by

$$\widetilde{\Lambda} : \widetilde{\delta}(a, b) = 27 - 18ab + 4a^3 + 4b^3 - a^2b^2 = 0 \quad (\Delta(a, b) = -64\, \widetilde{\delta}(a, b)^2).$$

The dihedral group $D_6 = \langle\sigma_1, \sigma_2\rangle$ acts on the (a, b)-plane in the following way:

$$\sigma_1 : (a, b) \to (\omega a, \omega^2 b), \qquad \sigma_2 : (a, b) \to (b, a),$$

where $\omega = \exp(2\pi i/3)$. The invariant ring under the action of D_6 is generated by $u = ab$ and $v = a^3 + b^3$, so that

$$\mathbf{C}[a, b]^{D_6} = \mathbf{C}[u, v].$$

Note that $\widetilde{\delta}(a, b) = \delta(u, v)$ and we defined the curve $\Lambda : \delta(u, v) = 0$. We then have the isomorphism $\left(\mathbf{C}^2 \setminus \widetilde{\Lambda}\right)/D_6 \cong \mathbf{C}^2 \setminus \Lambda$. In particular, the points

$$(a, b),\ (\omega a, \omega^2 b),\ (\omega^2 a, \omega b),\ (b, a),\ (\omega b, \omega^2 a),\ (\omega^2 b, \omega a)$$

correspond to the same point $(u, v) = (ab, a^3 + b^3)$ in $\mathbf{C}^2 \setminus \Lambda$.

Lemma 5. *In $\mathbf{C}^2 \setminus \Lambda$, the intersections of the curves S, T, M and G defined in Introduction are given as follows (see also Introduction, for A, $B_\pm$, $E_\pm$):*

$$\begin{aligned}
S \cap T &= \{(25, -250),\ (225, 6750),\ (9, 54)\},\\
S \cap M &= \{A\},\\
M \cap T &= \{(0, 0),\ (16, 128)\},\\
G \cap S &= \{(25, -250),\ A\},\\
G \cap M &= \{E_\pm,\ A\},\\
G \cap T &= \{B_\pm,\ (25, -250),\ (1, -2),\ (\frac{121}{25}, \frac{2662}{125})\}.
\end{aligned}$$

Remark 1. The curve G has a node at the point Q and a tacnode on the line at infinity. Hence, G is a rational curve.

Shaska and Völklein in [7] determined the automorphism group of $C_{a,b}$ in terms of u and v as follows:

Lemma 6. *1)* $\mathrm{Aut}(C_{a,b}) \cong GL_2(3)$ *if and only if* $(u, v) = (25, -250)$,

2) $\mathrm{Aut}(C_{a,b}) \cong \mathbb{Z}_3 \rtimes D_8$ *if and only if* $(u, v) = (0, 0)$ *or* $(225, 6750)$, *3)* $\mathrm{Aut}(C_{a,b}) \cong D_{12}$ *if and only if* $(u,v) \in S$ *and* $u \neq 9, 70+30\sqrt{5}, 25,$

4) $\mathrm{Aut}(C_{a,b}) \cong D_8$ *if and only if* $(u,v) \in T$ *and* $u \neq 1, 9, 0, 25, 225,$

5) $\mathrm{Aut}(C_{a,b}) \cong V_4$ *if and only if* $(u,v) \notin (S \cup T) \setminus \Lambda.$

3. Proof of Theorem

Using Lemma 4, we can write the Wronskian form Ω_3 on $C_{a,b}$ as:

$$\Omega_3 = \Phi(x,a,b)(dx)^{25}/f_y^{22},$$

where the polynomial

$$\begin{aligned}\Phi(x,a,b) = (4a-b^2) + (60-16ab+4b^3)x^2 + (70b-56a^2+14ab^2)x^4 \\ -(196b-84a^2)x^{10} + (70a-56b^2+14ba^2)x^{12} \\ +(60-16ab+4a^3)x^{14} + (4b-a^2)x^{16}\end{aligned}$$

is of degree 14 or 16 according as the coefficient $4b-a^2$ of x^{16} is zero or not. We infer that

$$\mathrm{div}(\Omega_3) = 3\left(\sum_{i=1}^{3}\left(R_i^+ + R_i^-\right)\right) + 16(P_1^\infty + P_2^\infty) + \mathrm{div}(\Phi)_0 - \mathrm{div}(\Phi)_\infty.$$

In view of Lemma 2, we see that $\Phi(x,a,b)$ has no multiple roots whose multiplicities are greater than 2.

3.1 Case(I): $4b-a^2=0$

Suppose $4b-a^2=0$. In this case, we have

$$\begin{aligned}\Phi(x,a) = \Phi(x,a,a^2/4) = \mathrm{Const.}\Big\{a(64-a^3) + (960-64a^3+a^6)x^2 \\ +(-616a^2+14a^5)x^4 + (-3136a+84a^4)x^6 + (-4560+240a^3)x^8 \\ +(560a^2)x^{10} + (1120a)x^{12} + 960x^{14}\Big\}.\end{aligned}$$

The pole divisor of $\Phi(x,a)$ is given by $14(P_1^\infty + P_2^\infty)$, so we get

$$\mathrm{div}(\Omega_3) = 3\left(\sum_{i=1}^{3}\left(R_i^+ + R_i^-\right)\right) + 2(P_1^\infty + P_2^\infty) + \mathrm{div}(\Phi)_0,$$

which implies that $w^{(3)}(P_1^\infty) = w^{(3)}(P_2^\infty) = 2$. By computing the discriminant of $\Phi(x,a)$, we have

$$\mathrm{Disc}(\Phi) = \mathrm{Const}\,\Delta(a)^{18} a\eta(a)U(a)^2V(a)^4,$$

where $\Delta(a) = \Delta(a,a^2/4) \neq 0$ and

$$\begin{aligned}\eta(a) &= 64 - a^3,\\ U(a) &= 332750 + 8411a^3 + 416a^6,\\ V(a) &= -250 + 7a^3.\end{aligned}$$

We see that Φ has a multiple root if and only if $a\eta(a)U(a)V(a) = 0$. To determine the number of the multiple roots of $\Phi = \Phi(x, a)$, we compute the subresultants $R^{(i)}$ of the two polynomials Φ and Φ_x (the differentiation by x). Since Φ has at most double roots, if $R^{(1)}(a) = \cdots = R^{(s)}(a) = 0$, $R^{(s+1)}(a) \neq 0$, then Φ has s double roots at a. We obtain

$$\begin{aligned}R^{(1)}(\Phi, \Phi_x; x) &= \text{Const.}\,\Delta(a)^{18} a\,\eta(a)\,U(a)^2 V(a)^4,\\ R^{(2)}(\Phi, \Phi_x; x) &= \text{Const.}\,\Delta(a)^{15}\,U(a)V(a)^3\,\zeta_{12}(a),\\ R^{(3)}(\Phi, \Phi_x; x) &= \text{Const.}\,\Delta(a)^{12} V(a)^2 \zeta_9(a)\,\zeta_{12}(a),\\ R^{(4)}(\Phi, \Phi_x; x) &= \text{Const.}\,\Delta(a)^{10} a\,V(a)\zeta_9(a)\,\chi_{12}(a),\\ R^{(5)}(\Phi, \Phi_x; x) &= \text{Const.}\,\Delta(a)^{8} a^3\,\zeta_6(a)\,\chi_{12}(a),\end{aligned}$$

where $\zeta_n(a)$ and $\chi_n(a)$ are polynomials of degree n of a.

We consider the curve $\widetilde{M} : (4b - a^2)(4a - b^2) = 0$. Note that

$$(4b - a^2)(b^2 - 4a) = m(u, v) = 4v - u(u + 16).$$

In Introduction, we defined the curve $M : m(u, v) = 0$.

Case $a\eta(a) = 0$. Since $\mathrm{Res}(a\eta(a), U(a)V(a)\zeta_{12}(a); a) \neq 0$, we conclude that Φ has one multiple root if $a\eta(a) = 0$.

(i) The point $(a, b) = (0, 0)$ corresponds to the point $(0, 0) \in (M \cap T) \setminus \Lambda$. We have

$$\Phi(x, 0, 0) = 15x^2(4 - 19x^6 + 4x^{12}).$$

Thus, we have $W_3(C_{0,0})_2 = \{P_1^\infty, P_2^\infty, P_1^0, P_2^0\}$.

(ii) For the case in which $\eta(a) = 0$, we have three points

$$(4, 4), \quad (4\omega, 4\omega^2), \quad (4\omega^2, 4\omega),$$

which correspond to the point $(16, 128) \in (M \cap T) \setminus \Lambda$. We have

$$\Phi(x, 4, 4) = 5x^2(12 + 56x^2 + 112x^4 + 135x^6 + 112x^8 + 50x^{10} + 12x^{12}).$$

Thus, $W_3(C_{a,b})_2 = \{P_1^\infty, P_2^\infty, P_1^0, P_2^0\}$.

Case $U(a) = 0$. Since $\mathrm{Res}(U(a), V(a)\zeta_9(a)\zeta_{12}(a); a) \neq 0$, we infer that Φ has two multiple roots if $U(a) = 0$. There occur three points over the point E_+ and three points over the point E_-. It follows that $N_2 = 6$.

Case $V(a) = 0$. Since $\mathrm{Res}(V(a), a\zeta_6(a)\chi_{12}(a); a) \neq 0$, we infer that Φ has four multiple roots if $V(a) = 0$. There occur three points over the point A. It follows that $N_2 = 10$.

Otherwise, Φ has no multiple roots, thus, $W_3(C_{a,b})_2 = \{P_1^\infty, P_2^\infty\}$.
We summarize the results for the case in which $4b - a^2 = 0$.

	(u,v)	N_1	N_2	$\mathrm{Aut}(C_{a,b})$
$V(a) = 0$	A	12	10	D_{12}
$U(a) = 0$	$E_\pm$	20	6	V_4
$\eta(a) = 0$	$(16, 128)$	24	4	D_8
$a = 0$	$(0, 0)$	24	4	$\mathbb{Z}_3 \rtimes D_8$
otherwise		28	2	V_4

By Lemma 5, we see that $A \in S \cap G$, $E_\pm \in G$ and $(0,0), (16,128) \in M$.

3.2 Case(II): $4b - a^2 \neq 0$

We now turn to the case in which $4b - a^2 \neq 0$. In this case, we have

$$\mathrm{div}(\Omega_3) = 3\left(\sum_{i=1}^{3}\left(R_i^+ + R_i^-\right)\right) + \mathrm{div}(\Phi)_0.$$

By computing the discriminant of $\Phi = \Phi(x, a, b)$, we find that

$$\mathrm{Disc}(\Phi) = \mathrm{Const}\,\Delta(a,b)^{18}(4b - a^2)^2(4a - b^2)F_4^2(a,b)F_8^2(a,b),$$

where

$$\begin{aligned}
F_4(a,b) &= -1125 + 110ab + 4a^3 + 4b^3 - a^2b^2,\\
F_8(a,b) &= 20796875 - 13942500ab + 3429000(a^3 + b^3) - 571350a^2b^2\\
&\quad -235440ab(a^3 + b^3) + 6220a^3b^3 + 52272(a^6 + b^6)\\
&\quad +1512a^2b^2(a^3 + b^3) - 3645a^4b^4.
\end{aligned}$$

We consider the following curves in the (a,b)-plane:

$$\widetilde{S} : F_4(a,b) = 0 \text{ and } \widetilde{G} : F_8(a,b) = 0.$$

We can write as

$$\begin{aligned}
F_4(a,b) &= s(u,v) = -1125 + 4v + 110u - u^2,\\
F_8(a,b) &= g(u,v) = 20796875 - 13942500u - 571350u^2 - 98324u^3 - 3645u^4\\
&\qquad +3429000v - 235440uv + 1512u^2v + 52272v^2.
\end{aligned}$$

We have already defined the following curves in the (u,v)-plane:

$$S : s(u,v) = 0 \text{ and } G : g(u,v) = 0.$$

For the case in which $(4a - b^2)F_4(a,b)F_8(a,b) \neq 0$, we see that Φ has no multiple roots. Hence, $W_3(C_{a,b}) = W_3(C_{a,b})_1 \cup W_1(C_{a,b})$ (disjoint union).

We now determine the number of the multiple roots for the case in which $(4a - b^2)F_4(a,b)F_8(a,b) = 0$. We compute the subresultants of the two polynomials Φ and Φ_x:

$$\begin{aligned} R^{(1)}(\Phi, \Phi_x; x) &= \text{Const. } \Delta(a,b)^{18}(4b - a^2)^2(4a - b^2)F_4^2(a,b)F_8^2(a,b), \\ R^{(2)}(\Phi, \Phi_x; x) &= \text{Const. } \Delta(a,b)^{15}(4b - a^2)^2 F_4(a,b)F_8(a,b)\gamma_{21}(a,b), \\ R^{(3)}(\Phi, \Phi_x; x) &= \text{Const. } \Delta(a,b)^{12}(4b - a^2)^2\gamma_{21}(a,b)\gamma_{18}(a,b), \end{aligned}$$

where $\gamma_n(a,b)$ are polynomials of degree n of a and b.

3.2.1 On the conic M

Suppose $4a - b^2 = 0$. Since $C_{a,b} \cong C_{b,a}$ via the birational map $(x,y) \to (1/x, y/x^3)$, we have only to refer Subsection 3.1.

3.2.2 On the conic S

We now consider the points on $\widetilde{S}$. We have

$$\operatorname{Res}(F_4, \gamma_{21}\gamma_{18}; a) = \text{Const. } (-27 + b^3)^4(125 + b^3)^{12}(-15625 + 784b^3)^2 V(b)^3.$$

So the curve $\widetilde{S}$ has intersection points with the curve $\widetilde{\Gamma} : \gamma_{21}(a,b)\gamma_{18}(a,b) = 0$ if and only if $\operatorname{Res}(F_4, \gamma_{21}\gamma_{18}; a) = 0$. It suffices to consider the points in $(\widetilde{S} \cap \widetilde{\Gamma}) \setminus (\widetilde{\Lambda} \cup \widetilde{M})$. We have three points

$$(-5,-5), \quad (-5\omega, -5\omega^2), \quad (-5\omega^2, -5\omega),$$

which correspond to the point $(25, -250) \in S$. Note that the above three points also belong to $\widetilde{G}$ (see Lemma 5). We have

$$\Phi(x, -5, -5) = -5(1 + 10x^2 + x^4)^2(3 - 2x^2 + 3x^4)^2.$$

Hence, Φ has eight multiple roots so that $N_2 = 16$.

Finally, for the general points in $\widetilde{S} \setminus \widetilde{\Lambda}$, Φ has two multiple roots. It follows that $N_2 = 4$. Thus, we obtain the following table:

(u,v)	N_1	N_2	$\mathrm{Aut}(C_{a,b})$
$(25, -250)$	0	16	$GL_2(3)$
general points on S	24	4	D_{12}

3.2.3 On the quartic G

We consider the points on $\widetilde{G}$. We have

$$\begin{aligned}
&\mathrm{Res}(F_8, \gamma_{21}\gamma_{18}; a)\\
&= \text{Const. } (1+b^3)^{20}(125+b^3)^{12}(-250+11b^3)^4(1375+16b^3)^4(-15625+784b^3)^2\\
&\times (275-130b+27b^2)^4(75625+35750b+9475b^2+3510b^3+729b^4)^4V(b)^3U(b).
\end{aligned}$$

In $(\widetilde{G}\cap\widetilde{\Gamma})\setminus(\widetilde{\Lambda}\cup\widetilde{M}\cup\widetilde{S})$, we first find three points

$$(\mu_+,\mu_+),\quad (\omega\mu_+,\omega^2\mu_+),\quad (\omega^2\mu_+,\omega\mu_+)$$

over the point $B_+\in G$ and three points

$$(\mu_-,\mu_-),\quad (\omega\mu_-,\omega^2\mu_-),\quad (\omega^2\mu_-,\omega\mu_-)$$

over the point $B_-\in G$, where $\mu_\pm = 5(13\pm 8\sqrt{-2})/27$. We have

$$\Phi(x,\mu_\pm,\mu_\pm) = \text{Const. } \xi_8^\pm(x)\,(\pm 3 + (\pm 10 + 8\sqrt{-2})x^2 \pm 3x^4)^2$$

By computing the discriminant of the polynomials $\xi_8^\pm(x)$, we can check that $\xi_8^\pm(x)$ have no multiple roots. It follows that $N_2 = 8$. We can then find six points $(5(2/11)^{1/3}, (-5/2)(11/2)^{1/3}), \ldots$ over the point $Q\in G$. We have

$$\begin{aligned}
&\Phi\Big(x, 5(2/11)^{1/3}, (-5/2)(11/2)^{1/3}\Big)\\
&\qquad = \text{Const. } \Theta_8(x)\,((2^{2/3})(11^{1/3}) + 4(2^{1/3})(11^{2/3})x^2 - 4x^4)^2.
\end{aligned}$$

By computing the discriminant of the polynomial $\Theta_8(x)$, we can check that $\Theta_8(x)$ has no multiple roots. Therefore, we see that $N_2 = 8$.

Finally, for the general points in $\widetilde{G}\setminus(\widetilde{\Lambda}\cup\widetilde{S})$, Φ has two multiple roots. Therefore, we see that $N_2 = 4$. We obtain the following table:

(u,v)	N_1	N_2	$\mathrm{Aut}(C_{a,b})$
$B_\pm$	16	8	D_8
Q	16	8	V_4
general points on G	24	4	V_4

Remark 2. The quartic $\widetilde{S}$ has nodes at the 3 points over $(25,-250)$. The octic $\widetilde{G}$ has two tacnodes on the line at infinity and 15 nodes (the 3 points over $(25,-250)$, the 3 points each over $B_\pm$, the 6 points over the node Q of G). As a consequence, we see that $\widetilde{S}$ is an elliptic curve and $\widetilde{G}$ is a genus two curve.

4. Higher order Weierstrass points

Let C be a genus two curve. For $q \geq 4$, in case $w^{(q)}(P) = 2$, there exist two types of q-gap sequences:

type I: $G^{(q)}(P) = \{1, 2, ..., 2q-3, 2q-2, 2q+1\}$,

type II: $G^{(q)}(P) = \{1, 2, ..., 2q-3, 2q-1, 2q\}$.

Let us denote by $W_q(C)_{2,I}$ (resp. $W_q(C)_{2,II}$) the set of q-Weierstrass points of q-weight 2 of type I (resp. type II) on the curve C.

Lemma 7 (see also Theorem 6.13 in [1])**.** *Let C be a genus two curve. If $P \in W_3(C)_2$, then we have*

$$w^{(q)}(P) = \begin{cases} 2 & \textit{(type I)} \quad \textit{if } q \equiv 0 \pmod 3, \\ 2 & \textit{(type II)} \quad \textit{if } q \equiv 1 \pmod 3, \\ 0 & \qquad\qquad\quad \textit{if } q \equiv 2 \pmod 3. \end{cases}$$

Proof. We give the proof only for the case in which $q \equiv 0 \pmod 3$. Assume that $P \in W_3(C)_2$. In view of Lemma 3, there exists an element $\psi \in H^0(C, (\Omega^1)^3)$ such that $\mathrm{ord}_P(\psi) = 6$. Write $q = 3m$. Clearly, $\psi^m \in H^0(C, (\Omega^1)^q)$ and $\mathrm{ord}_P(\psi^m) = 6m = 2q$. Thus, we have $2q+1 \in G^{(q)}(P)$. We infer that $G^{(q)}(P) = \{1, 2, , 3, \ldots, 2q-3, 2q-2, 2q+1\}$. □

Remark 3. We can also prove that every 3-Weierstrass point of 3-weight 1 on C is not a 4-Weierstrass point.

Lemma 8. *Let C be a genus two curve. Let σ be an automorphism of C of order three. Then, σ has four fixed points and every fixed point P is a q-Weierstrass point ($q \geq 3$) except if $q \equiv 2 \pmod 3$. We have*

$$w^{(q)}(P) = \begin{cases} 2 & \textit{(type I)} \quad \textit{if } q \equiv 0 \pmod 3, \\ 2 & \textit{(type II)} \quad \textit{if } q \equiv 1 \pmod 3. \end{cases}$$

Proof. Let $\nu(\sigma)$ denote the number of the fixed points of σ. Let $\overline{g}$ be the genus of the curve $\overline{C} = C/\langle\sigma\rangle$. By using the Riemann-Hurwitz formula, we have

$$1 = 3(\overline{g} - 1) + \nu(\sigma).$$

It follows that $0 \leq \overline{g} < 2$. If $\overline{g} = 1$, then $\nu(\sigma) = 1$, which contradicts Theorem V.2.11 in [5]. So we must have $\overline{g} = 0$ and $\nu(\sigma) = 4$. By Satz 6.4 in [3], every fixed point P of σ is a q-Weierstrass point ($q \geq 3$) except if $q \equiv 2 \pmod 3$. The assertion then follows from Lemma 7. □

Example 1. We consider the following 1-parameter family of genus two curves:

$$C_t : y^2 = x^6 + tx^3 + 1, \quad (t \neq \pm 2).$$

The curve C_t has an extra involution $(x, y) \longmapsto (1/x, y/x^3)$. Clearly, $\rho : (x, y) \mapsto (\omega x, y)$ acts on C_t. The two automorphisms ρ and ρ^2 are elements of order three in $\mathrm{Aut}(C_t)$. The fixed points of both ρ and ρ^2 are given by

$$\mathrm{Fix}(\rho) = \mathrm{Fix}(\rho^2) = \{P_1^0, P_2^0, P_1^\infty, P_2^\infty\} \subset W_3(C_t)_2.$$

We remark that the curve C_t is isomorphic to the curve $C_{a,b}$ with

$$a = 3\left(\frac{10-t}{2-t}\right)\left(\frac{2-t}{2+t}\right)^{2/3}, \ b = 3\left(\frac{10+t}{2+t}\right)\left(\frac{2+t}{2-t}\right)^{2/3}.$$

Hence, $u = 9\left(100 - t^2\right) / \left(4 - t^2\right)$ and $v = 54\left(2000 + 360t^2 + t^4\right) / \left(4 - t^2\right)^2$. We can easily check that $(u, v) \in S$.

Remark 4. It turns out that the genus two curve $\widetilde{G}$ is birationally equivalent to the curve: $y^2 = x^6 + 322x^3 + 1$. We used the package software Weierstrassform (Maple).

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