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3412, 4231 patterns produce singular points of essential sets

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Abstract

This article introduces new concepts dual bigrassmannian permutations and dual essential sets with the discussion of singular patterns (3412, 4231). We show that (dual) bigrassmannian permutations may contain only 3412 (4231). Then we define singularity of points of essential sets by singularity of certain corresponding (dual) bigrassmannian permutations (this discussion depends on the authors' recent result that given a permutation, there exists a bijection between the set of all bigrassmannian permutations maximal below it and its essential set). Our main theorem shows that if a permutation contains singular a pattern, then there exists a singular point in its (dual) essential set.

1. Introduction

This article treats three objects on the symmetric groups together: singular (3412, 4231) patterns, bigrassmannian permutations and essential sets. Each of them is quite important and deep in topics of combinatorics of permutations, Coxeter groups with Bruhat order, Schubert varieties, and so on. There are many recent developments in the literature, for exmaple, see [1, 2, 7, 12] for 3412 and 4231 patterns and singularity of Schubert varieties, [8, 10] for bigrassmannian permutations, [4, 5, 6, 11, 14] for essential sets.

The author recently proved [9] that given a permutation, there exists a bijection between the set of all bigrassmannian permutations maximal below it and its essential set. This article further investigates this bijection from two points of view, *singularity* and *duality*. Let us say that a permtation is *singular* if it contains a subsequence of four elements with the same relative order as 3412 or 4231 as in Section 2. Then Section 3 introduces *bigrassmannian* permutations and its dual. Here "dual" bigrassmannian permutations has two interpretations. First, it is certainly "order-dual" of bigrassmannian permutations meaning that $x \in S_n$ is bigrassmannian if and only if w_0x is dual bigrassmannian where w_0 is the maximum element of S_n. Second, given a bigrassmannian permutation x, there exists precisely one dual bigrassmannian permutation x^* such that the essential set of x (which consists

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of actually the only one point) is equal to the dual essential set of x^* and values of certain rank functions coincide at that point (so this may be called "essentially dual"). In Section 4 we will see these results and examples with definitions of essential sets and such rank functions. After that we come to define singularity of points of essential sets under the bijection in the author's result as mentioned above. Our main theorem in Section 5 proves that every singular pattern necessarily produces a singular point in (dual) essential sets.

Theorem. *Let $x \in S_n$. If x is singular, then there exists a singular point in its (dual) essential sets.*

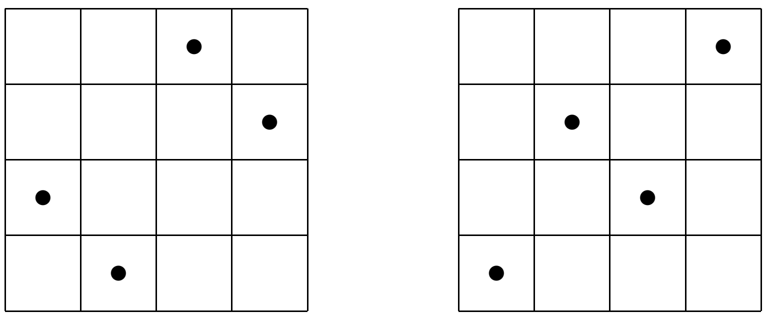
2. 3412 and 4231 patterns

We begin with a definition of two types of singular patterns.

Definition 2.1. Let $x \in S_n$ and (i_1, i_2, i_3, i_4) be a quadruple of integers. We say that it is a *3412 pattern* of x if $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ and $x(i_3) < x(i_4) < x(i_1) < x(i_2)$. Denote by $P_{3412}(x)$ the set of all 3412 patterns of x. We say that x *contains 3412* if $P_{3412}(x) \neq \varnothing$. (i_1, i_2, i_3, i_4) is a *4231 pattern* of x if $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ and $x(i_4) < x(i_2) < x(i_3) < x(i_1)$. Similarly define $P_{4231}(x)$ and say x *contains 4231* if $P_{4231}(x) \neq \varnothing$. Moreover, x is *singular* if x contains 3412 or 4231.

To express and understand permutations visibly, it is useful to place a dot (●) at positions $\{(i, x(i)) \mid 1 \leq i \leq n\}$ in an $n \times n$ matrix. Figure 1 shows such placements for singular patterns.

Example 2.2. Some permutations may contain both of singular patterns. Let $x = 5736241$. Then $P_{3412}(x) = \{(1,2,3,6), (1,2,5,6), (1,4,5,6)\}$ as 5734, 5724, 5624



3412 4231

Figure 1 Singular patterns

seen from one-line notation of x and $P_{4231}(x) = \{(1,3,6,7),(1,5,6,7),(2,3,4,7),$ $(2,3,6,7),(2,5,6,7),(4,5,6,7)\}$ as 5341, 5241, 7361, 7341, 7241, 6241.

3. bigrassmannian permutations

From now on we begin to view S_n as a poset with *Bruhat order*. For its definition, which we do not actually need here, refer to [3, Chapters 1, 2]. In this order, *bigrassmannian permutations* play an important role. In particular, the most important property is that bigrassmannian permutations are equivalently join-irreducible elements in this order (Remark 3.2). We will also introduce dual bigrassmannian permutations.

Definition 3.1. For $x \in S_n$, define the set of *left* and *right descents* as

$$D_L(x) = \{1 \le i \le n-1 \mid x^{-1}(i) > x^{-1}(i+1)\},$$
$$D_R(x) = \{1 \le i \le n-1 \mid x(i) > x(i+1)\}.$$

We say that x is *bigrassmannian* if $\#D_L(x) = \#D_R(x) = 1$ and x is *dual bigrassmannian* if $\#D_L(x) = \#D_R(x) = n-2$. From now on by $\le$ we mean the Bruhat order on S_n. Then denote by $B(S_n)$ and $B^*(S_n)$ the set of all (dual) bigrassmannian permutations, respectively. Define $B(x) = \{w \in B(S_n) \mid w \le x\}$ and $B^*(x) = \{w \in B^*(S_n) \mid w \ge x\}$.

Remark 3.2. Each $x \in S_n$ can be written as $x = \vee \operatorname{Max} B(x) = \wedge \operatorname{Min} B^*(x)$, the join of maximal elements of $B(x)$ and the meet of minimal elements of $B^*(x)$ since (dual) bigrassmannian permutations are equivalently join-irreducible (meet-irreducilbe) elements in S_n. For details, see [9, Sections 1, 2]. In Section 4, we will particularly discuss the bijection between elements of $\operatorname{Max} B(x)$ ($\operatorname{Min} B^*(x)$) and (dual) essential sets. These decompositions are in fact unique so that they encode many properties of x. Our main theorem will show that existence of singular patterns of x guarantees existence of singular permutations in $\operatorname{Max} B(x)$ or $\operatorname{Min} B^*(x)$.

Following [13, Section 8], we next introduce a convenient parameter of (dual) bigrassmannian permutations with three positive integers.

Definition 3.3. Let $A_n = \{(a,b,c) \in \mathbf{N}^3 \mid 1 \le b \le a \le n-1 \text{ and } b+1 \le c \le n-a+b\}$. For each $(a,b,c) \in A_n$, define $J_{abc} \in B(S_n)$ and $M_{abc} \in B^*(S_n)$ by

$$J_{abc}(i) = \begin{cases} i & \text{if } 1 \le i \le b-1, \\ i+c-b & \text{if } b \le i \le a, \\ i-a+b-1 & \text{if } a+1 \le i \le a-b+c, \\ i & \text{if } a-b+c+1 \le i \le n \end{cases}$$

and

$$M_{abc}(i) = \begin{cases} n-i+1 & \text{if } 1 \le i \le a-b, \\ c+a-b-i & \text{if } a-b+1 \le i \le a, \\ n+b+1-i & \text{if } a+1 \le i \le n+b-c+1, \\ n-i+1 & \text{if } n+b-c+2 \le i \le n. \end{cases}$$

Example 3.4. For $n = 9$, we see $J_{536} = 126783459$ and $M_{536} = 985437621$.

Proposition 3.5.

(1) *J_{abc} contains 3412 if and only if $a-b \ge 1$ and $c-b \ge 2$. Moreover, any J_{abc} does not contain 4231.*

(2) *M_{abc} contains 4231 if and only if $a-b \ge 1$ and $c-b \ge 2$. Moreover, any M_{abc} does not contain 3412.*

Proof. Notice that one-line notation of J_{abc} consists of (at most four and at least two) consecutive increasing sequences. At most four means the first or fourth sequence(s) may be empty if $b-1 = 0$ or $n-a+b-c = 0$. At least two means the middle two sequences are not empty. More explicitly,

$$J_{abc} = \underbrace{\boxed{1\ 2 \cdots}}_{b-1}\underbrace{\boxed{c\ c+1 \cdots}}_{a-b+1}\underbrace{\boxed{b\ b+1 \cdots}}_{c-b}\underbrace{\boxed{a-b+c+1\ a-b+c+2 \cdots .}}_{n-a+b-c}$$

It is easy to see that if J_{abc} contains 3412, then the two middle sequences must have length greater than 1, i.e., $a-b+1 \ge 2$ and $c-b \ge 2$ (and vice versa). Obviously any J_{abc} cannot contain 4231.

In the same way, M_{abc} consists of (at most four and at least two) consecutive decreasing sequences as

$$M_{abc} = \underbrace{\boxed{n\ n-1 \cdots}}_{a-b}\underbrace{\boxed{c{-}1\ c{-}2 \cdots}}_{b}\underbrace{\boxed{n{-}a{+}b\ n{-}a{+}b{-}1 \cdots}}_{n-a+b-c+1}\underbrace{\boxed{c{-}b{-}1\ c{-}b{-}2 \cdots .}}_{c-b-1}$$

If M_{abc} contains 4231, then the first and fourth sequences cannot be empty, i.e., $a-b \ge 1$ and $c-b-1 \ge 1$ (and vice versa). Obviously any M_{abc} cannot contain 3412. $\square$

Remark 3.6.

(1) Consequently, J_{abc} is singular if and only if so is M_{abc} (with different types of singular patterns).

(2) Later in Definition 4.5, we will talk about singularity of $J_{a,b,c+1}$, not J_{abc}. Thus actually singular condition for $J_{a,b,c+1}$ is $a-b \ge 1$ and $(c+1)-b \ge 2$, i.e., $c-b \ge 1$.

4. singular points in (dual) essential sets

Fulton [6] introduced the *essential set* for permutations. This section introduces new concepts and objects: dual essential set, b- and b*-rank and singularity for each point in (dual) essential sets.

Definition 4.1. Let $x \in S_n$.

(1) The *essential set* of x in $\{1, 2, \dots, n\}^2$ is

$$\mathrm{Ess}(x) = \{(a, c) \mid a < x^{-1}(c), c < x(a), x(a+1) \leq c, x^{-1}(c+1) \leq a\}.$$

To each $(a, c) \in \mathrm{Ess}(x)$, we associate a set $B = B(a, c)$ and a positive integer (call it b-*rank*) $b = b(a, c)$ defined by

$$B = \{a' \mid 1 \leq a' \leq a \text{ and } 1 \leq x(a') \leq c+1\}$$

and $b = \#B$.

(2) The *dual essential set* of x is

$$\mathrm{Ess}^*(x) = \{(a, c) \mid a \geq x^{-1}(c), c \geq x(a), x(a+1) > c, x^{-1}(c+1) > a\}.$$

Also to each $(a, c) \in \mathrm{Ess}^*(x)$, we associate a set $B^* = B^*(a, c)$ and a positive integer (call it b*-*rank*) $b^* = b^*(a, c)$ defined by

$$B^* = \{a' \mid 1 \leq a' \leq a \text{ and } 1 \leq x(a') \leq c\}$$

and $b^* = \#B^*$ (this happens to be the usual rank function r_x in context of essential sets, though).

We should have written $B(a, c, x)$ and $b(a, c, x)$, but we drop x from notation whenever no confusion arises.

The following two propositions show that there exists a bijection between each element of $\mathrm{Max}\, B(x)$ ($\mathrm{Min}\, B^*(x)$) and each point of $\mathrm{Ess}(x)$ ($\mathrm{Ess}^*(x)$).

Proposition 4.2. [9, Lemma 4.2] *For* $1 \leq a, c \leq n-1$*, we have* $(a, c) \in \mathrm{Ess}(x) \iff J_{a,b,c+1} \in \mathrm{Max}\, B(x)$ *where* $b = b(a, c)$.

Proposition 4.3. $(a, c) \in \mathrm{Ess}^*(x) \iff M_{a,b,c+1} \in \mathrm{Min}\, B^*(x)$ *where* $b = b^*(a, c)$.

Proof. The idea of this proof is quite similar to that of [9, Lemma 4.2]. For each $M_{abc} \in B^*(S_n)$, define

$$\begin{aligned} C_1^*(M_{abc}) &= M_{a,b+1,c+1} && \text{if } a - b \geq 1, \\ C_2^*(M_{abc}) &= M_{a-1,b,c} && \text{if } a - b \geq 1, \\ C_3^*(M_{abc}) &= M_{a+1,b+1,c} && \text{if } c - b \geq 2, \\ C_4^*(M_{abc}) &= M_{a,b,c-1} && \text{if } c - b \geq 2. \end{aligned}$$

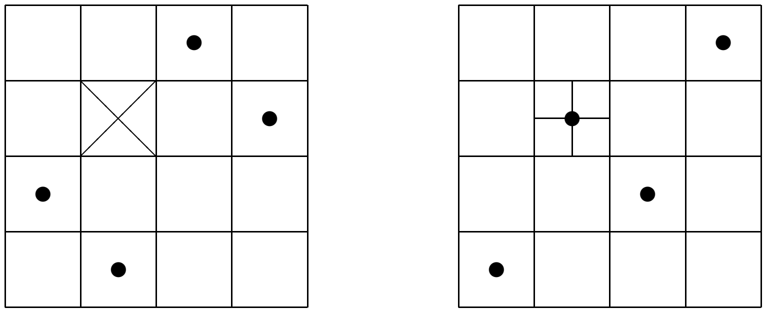


Figure 2 Ess(3412) = $\{(2,2)\}$ = Ess*(4231) (singular)

Then (some of) these four elements cover M_{abc} in $B^*(S_n)$ and furthermore any other element does not cover M_{abc} in $B^*(S_n)$ as [9, Proposition 2.7]. It is the fact [13, Section 8] that M_{abc} satisfies the property that if $x \in S_n$ and $\{x_{ab} \mid 1 \leq b \leq a\}$ is the increasing arrangement (i.e., $x_{a1} < x_{a2} < \cdots < x_{aa}$) of $\{x(1), x(2), \ldots, x(a)\}$, then we have $x_{ab} \leq c-1 \iff M_{abc} \in B^*(x)$.

Now let $y = M_{a,b,c+1}$. Due to the above facts, to show that $y \in \operatorname{Min} B^*(x)$, it is enough to verify that

$$\begin{array}{lll}
\text{(a)}\ x \not\leq C_1^*(y) = M_{a,b+1,c+2} & \text{or equivalently} & x_{a,b+1} \leq c+2 \\
\text{(b)}\ x \not\leq C_2^*(y) = M_{a-1,b,c+1} & \text{or equivalently} & x_{a-1,b} \leq c+1 \\
\text{(c)}\ x \not\leq C_3^*(y) = M_{a+1,b+1,c+1} & \text{or equivalently} & x_{a+1,b+1} \leq c+1 \\
\text{(d1)}\ x \not\leq C_4^*(y) = M_{abc} & \text{or equivalently} & x_{ab} \geq c \\
\text{(d2)}\ y \in B^*(x) & \text{or equivalently} & x_{ab} \leq c
\end{array}$$

where $b = b^*(a,c)$. Then check that (a) $\iff x^{-1}(c+1) > a$, (b) $\iff x(a) \leq c$, (c) $\iff x(a+1) > c$ and (d1)&(d2) $\iff x^{-1}(c) \leq a$. $\square$

Example 4.4. As a consequence of these propositions, we obtain $\mathrm{Ess}(J_{a,b,c+1}) = \{(a,c)\} = \mathrm{Ess}^*(M_{a,b,c+1})$. In fact, this implies further that b- and b*-rank coincide at this point, i.e, $b(a,c,J_{a,b,c+1}) = b = b^*(a,c,M_{a,b,c+1})$. Figure 2 shows such an example that 3412 ($= J_{213}$) and 4231 ($= M_{213}$) share the point $(2,2)$ (crossed boxes $\boxed{\times}$ ($\boxed{+}$) indicate points of (dual) essential sets). Observe that order duals of these, 1324 and 2143 also share $(2,2)$ in their (dual) essential sets, but these permutations are not singular anymore (Figure 3).

Note that $b^*(a,c)$ is a weakly increasing function with respect to a and c. Let us write the matrix whose (a,c) entry is $b^*(a,c)$ of x as seen in Figure 4. Then at a point $(a,c) \in \mathrm{Ess}(x)$ in this matrix, b^* was invariant from north and west while b^* will do increase to south and east (the essential set is the set of such points in the matrix). Dually, at a point $(a,c) \in \mathrm{Ess}^*(x)$, b^* will be invariant to south and east

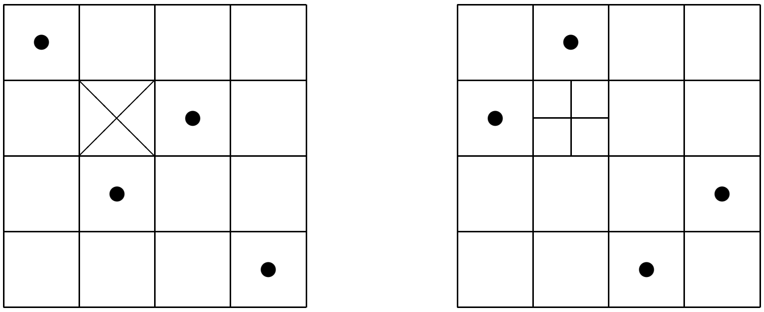


Figure 3 Ess(1324) = $\{(2,2)\}$ = Ess*(2143) (nonsingular)

0	0	1	1
0	**0**	1	2
1	1	2	3
1	2	3	4

Ess(3412)

0	0	0	1
0	**1**	1	2
0	1	2	3
1	2	3	4

Ess*(4231)

Figure 4 local behavior of b*-rank at points of (dual) essential sets

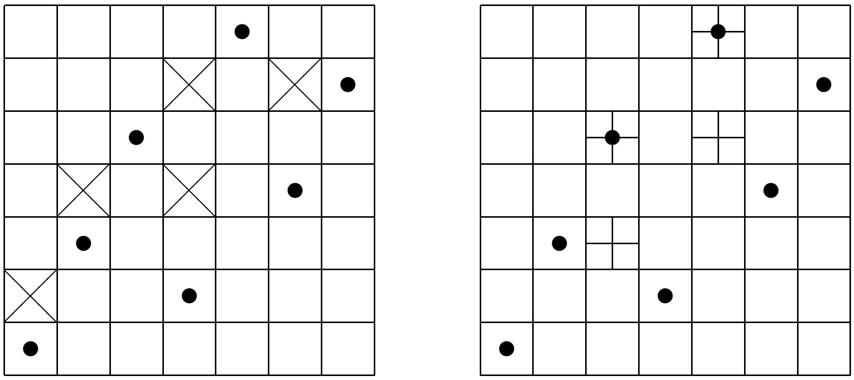


Figure 5 Ess(5736241) (taken from [5]) and Ess*(5736241)

while b^* did increase from north and west. In general, dual essential sets are the set of such points.

We have seen in Sections 2 and 3 that (dual) bigrassmannian permutations may be singular with the pattern 3412 (4231). Under bijections of Propositions 4.2 and 4.3, it is natural to define singularity for points of (dual) essential sets.

Definition 4.5. Let us say that $(a, c) \in \mathrm{Ess}(x)$ is *singular* if $a - b \geq 1$ and $c - b \geq 1$ (so that the corresponding $J_{a,b,c+1} \in \mathrm{Max}\, B(x)$ is singular). Similarly $(a, c) \in \mathrm{Ess}^*(x)$ is *singular* if $a - b^* \geq 1$ and $c - b^* \geq 1$.

Example 4.6. Figure 5 gives another example of essential sets for $x = 5736241$. Tables 1 and 2 show b- and b*-rank of points.

Table 1 Ess(5736241)

(a, c)	$(2, 4)$	$(2, 6)$	$(4, 2)$	$(4, 4)$	$(6, 1)$
b	1	2	1	2	1
$a - b$	1	0	3	2	5
$c - b$	3	4	1	2	0
singularity	$\surd$	—	$\surd$	$\surd$	—

Table 2 Ess*(5736241)

(a, c)	$(1, 5)$	$(3, 3)$	$(3, 5)$	$(5, 3)$
b^*	1	1	2	2
$a - b^*$	0	2	1	3
$c - b^*$	4	2	3	1
singularity	—	$\surd$	$\surd$	$\surd$

5. Main theorem

Theorem. *Let $x \in S_n$. If x is singular, then there exists a singular point in* $\mathrm{Ess}(x) \cup \mathrm{Ess}^*(x)$. *More precisely, if x contains 3412, then there exists a singular point in* $\mathrm{Ess}(x)$. *If x contains 4231, then there exists a singular point in* $\mathrm{Ess}^*(x)$.

Theorem follows from Propositions 3.5, 4.2, 4.3 and the following two lemmas.

Lemma 1. *Let $(i_1, i_2, i_3, i_4) \in P_{3412}(x)$. Define*

$$
\begin{aligned}
C &= \{c' \mid (x^{-1}(c'), i_2, i_3, i_4) \in P_{3412}(x)\},\\
c &= \min C - 1,\\
A &= \{i \mid i_2 \leq i \text{ and } x(i+1) \leq c\},\\
a &= \min A.
\end{aligned}
$$

Then (a, c) is a singular point of $\mathrm{Ess}(x)$.

For convenience, say $x(i_0) = \min C$, with $(i_0, i_2, i_3, i_4) \in P_{3412}(x)$ so that

$x(i_0) = c+1$. Note that C, A are nonempty since $x(i_0) \in C$ and $i_3 - 1 \in A$ (because $i_2 \leq i_3 - 1$ and $x((i_3-1)+1) = x(i_3) < x(i_4) < x(i_0) = c+1$, i.e., $x(i_3) \leq c$).

Proof. It is enough to show that (i) $a < x^{-1}(c)$ (ii) $c < x(a)$ (iii) $x(a+1) \leq c$ (iv) $x^{-1}(c+1) \leq a$ (v) $a - b \geq 1$ and (vi) $c - b \geq 1$ where $b = b(a,c)$. (iii), (ii) follow from minimality of a (i.e., $a - 1 \notin A$ and $a \in A$) as $x(a+1) \leq c$ and $x(a) = x((a-1)+1) > c$. (iv) is clear since $x^{-1}(c+1) = i_0 < i_2 \leq a$. We next prove (i). Suppose, toward a contradiction, that $a \geq x^{-1}(c)$. Then $c = x(j)$ for some $j \leq a$. First, if $i_2 + 1 \leq j \leq a$, then $i_2 \leq j - 1 \leq a - 1$ and hence $x(j) = x((j-1)+1) > c$ by minimality of a, but $c = x(j)$, a contradiction. Second, if $j = i_2$, then $x(j) = c < x(i_0) < x(i_2)$, a contradiction. So $j \neq i_2$. Third, assume $j < i_2$. Note that $x(i_4) \leq c = x(j) < x(i_0) < x(i_2)$ ($x(i_4) \leq c$ follows from $(i_0, i_2, i_3, i_4) \in P_{3412}(x)$ and hence $x(i_4) < x(i_0) = c+1$. The equality is acutually impossible since $j < i_2 < i_3 < i_4$). Then $j < i_2$ and $x(i_4) \leq c = x(j) < x(i_0) < x(i_2)$ together mean $(j, i_2, i_3, i_4) \in P_{3412}(x)$, a contradiction for minimality of $x(i_0)$. Thus we proved (i).

It remains to show (v), (vi). Recall that

$$b = b(a,c) = \#\{a' \mid 1 \leq a' \leq a \text{ and } 1 \leq x(a') \leq c+1\}.$$

Here we remark that $b \leq \min\{a, c+1\}$. Note that $1 \leq i_2 \leq a$ but $x(i_2) > x(i_0) = c+1$. As a result, the number of a' such that $1 \leq a' \leq a$ and $1 \leq x(a) \leq c+1$ is at most $a-1$ (i.e., we excluded the case $a' = i_2$). Thus $b \leq a-1$. Note also that $x(i_3) < x(i_4) \leq c+1$ but $a < i_3 < i_4$. Thus $b \leq c+1-2$ (i.e., we excluded the case both $x(a') = x(i_3)$ and $x(a') = x(i_4)$). Conclude that $a - b \geq 1$ and $c - b \geq 1$. $\square$

Lemma 2. *Let $(i_1, i_2, i_3, i_4) \in P_{4231}(x)$. Define*

$$\begin{aligned} C &= \{c' \mid x^{-1}(c') \leq i_2 \textit{ and } c' < x(i_3)\}, \\ c &= \max C, \\ A &= \{i \mid i_2 \leq i \textit{ and } x(i+1) > c\}, \\ a &= \min A. \end{aligned}$$

Then (a,c) is a singular point of $\mathrm{Ess}^*(x)$.

Again, C, A are nonempty since $x(i_2) \in C$ and $i_3 - 1 \in A$ (because $c < x(i_3)$).

Proof. It is enough to show that (i) $a \geq x^{-1}(c)$ (ii) $c \geq x(a)$ (iii) $x(a+1) > c$ (iv) $x^{-1}(c+1) > a$ (v) $a - b^* \geq 1$ and (vi) $c - b^* \geq 1$ where $b^* = b^*(a,c)$. (iii), (ii) follow from minimality of a. (i) is clear since $x^{-1}(c) \leq i_2 \leq a$. We next prove (iv). Suppose, toward a contradiction, that $x^{-1}(c+1) \leq a$, say $x(j) = c+1$ for $j \leq a$.

First, if $i_2+1 \le j \le a$ then $i_2 \le j-1 \le a-1$ and hence $x(j) = x((j-1)+1) \le c$ by minimality of a, but $c+1 = x(j)$, a contradiction. Second, assume $j \le i_2$. Note that $x(j) = c+1 \le x(i_3)$ since $c < x(i_3)$. But here the equality $x(j) = x(i_3)$ is impossible since we are assuming that $j \le i_2$ $(< i_3)$. Thus $x(j) = c+1 < x(i_3), j \le i_2$, which is a contradiction for maximality of c. Thus we proved (iv).

It remains to show that (v), (vi). Recall that

$$b^* = b^*(a, c) = \#\{a' \mid 1 \le a' \le a \text{ and } 1 \le x(a') \le c\}.$$

Here we remark that $b^* \le \min\{a, c\}$. Note that $i_1 < i_2 \le a$ but $x(i_1) > x(i_3) > c$. Thus $b^* \le a-1$ (i.e., we excluded the case $a' = i_1$). Note also that $x(i_4) < x(i_2) \le \max C = c$ but $i_4 > i_3 - 1 \ge a$. Then we have $b^* \le c-1$ (i.e., we excluded the case $x(a') = x(i_4)$). Conclude that $a - b^* \ge 1$ and $c - b^* \ge 1$. $\square$

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