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On the residues of spectral zeta functions on spheres

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Abstract

We introduce a series of functions whose residues can be calculated easily. In terms of their residues one can describe the residues of the spectral zeta functions on spheres. The formula thus gotten gives us an efficient method for calculating the residues of the zeta functions.

1. Introduction

Let us consider the n-dimensional unit sphere S^n with the canonical metric and consider the Laplacian Δ_n which acts on functions on S^n. Let

$$0 = \lambda_{n,0} < \lambda_{n,1} \leq \lambda_{n,2} \leq \dots \ (\to \infty)$$

be the sequence of its eigenvalues, which are repeated according to their multiplicities. The spectral zeta function $\zeta_n(s)$ is defined by

$$\zeta_n(s) = \sum_{j=1}^{\infty} \lambda_{n,j}^{-s} \qquad (s \in \mathbb{C}).$$

This series converges absolutely on the half plane $\mathrm{Re}(s) > n/2$ and has a meromorphic continuation to $\mathbb{C}$ with only simple poles at

$$\begin{aligned} s &= \frac{n}{2}, \ \frac{n}{2} - 1, \ \frac{n}{2} - 2, \dots \qquad \text{(if } n \text{ is odd)}, \\ s &= \frac{n}{2}, \ \frac{n}{2} - 1, \ \frac{n}{2} - 2, \dots, 1 \quad \text{(if } n \text{ is even)}, \end{aligned}$$

whose residues were studied closely by many researchers (e.g. Minakshisundaram-Pleijel [3], Carletti-Monti Bragadin [1]).

In this paper, we introduce a series of functions $Z_1(s,r), Z_2(s,r), \dots$ on $\mathbb{C} \times [0,\pi)\,(\ni (s,r))$ inductively, which are all meromorphic with respect to the variable s, and whose residues can be calculated easily. We found an interesting formula which expresses the residues of $\zeta_n(s)$ in terms of the residues of the functions. The formula gives us an efficient method for calculating the residues of

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$\zeta_n(s)$. In section 2, following the statement of the main theorem, we will report some results of calculation using the method.

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2. Main results and some calculations of residues

Let us define the functions $Z_1(s,r), Z_2(s,r), \dots$ ($(s,r) \in \mathbb{C} \times [0,\pi)$) inductively as follows:

$$Z_1(s,r) = \frac{1}{\Gamma(s)} \frac{2^{-1/2}}{(2\pi)^{1/2}} \frac{1}{s-1/2}, \tag{2.1}$$

$$\begin{aligned} Z_2(s,r) = {} & \frac{1}{\Gamma(s)} \frac{2^{-1}}{(2\pi)^{3/2}} \\ & \times \int_0^1 t^{s-1-3/2} e^{r^2/4t} \left(\int_r^\delta \frac{\sigma e^{-\sigma^2/4t}}{(\cos r - \cos\sigma)^{1/2}}\, d\sigma \right) dt, \end{aligned} \tag{2.2}$$

$$\begin{aligned} Z_{n+2}(s,r) = {} & \frac{1}{2\pi} \frac{-1}{\sin r} \frac{\partial}{\partial r} Z_n(s,r) \\ & + \frac{1}{4\pi} \frac{r}{\sin r} \frac{1}{s-1} Z_n(s-1,r). \end{aligned} \tag{2.3}$$

Then, certainly, as a function of s, every $Z_n(s,r)$ has simple poles at

$$\begin{aligned} s &= \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots, \frac{1}{2} \quad (\text{if } n \text{ is odd}), \\ s &= \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots, 1 \quad (\text{if } n \text{ is even}), \end{aligned}$$

and we obtain the following:

Theorem 2.1

(1) *The residues of $\zeta_n(s)$ are expressed by using the residues of $Z_n(s,0)$ as*

$$\begin{aligned} & \operatorname*{Res}_{s=n/2-j} \zeta_n(s) \\ & = \mathrm{vol}(S^n) \sum_{l=0}^{N} \frac{1}{(j-l)!} \left(\frac{(n-1)^2}{4} \right)^{j-l} \frac{\Gamma(n/2-l)}{\Gamma(n/2-j)} \operatorname*{Res}_{s=n/2-l} Z_n(s,0), \end{aligned} \tag{2.4}$$

where

$$N = \begin{cases} \max\{\frac{n-1}{2}, j\} & (n \text{ is odd}) \\ j & (n \text{ is even}). \end{cases}$$

Table 1 Residues of $\zeta_n(s)$ at $\frac{n}{2} - j$ $(j = 0, 1, \dots)$

$n \backslash j$	0	1	2	3	...
1	1	0	0	0	...
3	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{16}$	$\frac{1}{32}$	...
5	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{1}{6}$	$-\frac{1}{12}$	...
7	$\frac{1}{720}$	$\frac{11}{288}$	$\frac{1379}{5760}$	$\frac{527}{1280}$	...

(2) *$Z_{2m}(s,r)$ and $Z_{2m+1}(s,r)$ are related to each other as*

$$\begin{aligned} &Z_{2m}(s,r) \\ &= \frac{1}{\Gamma(s)} 2^{1/2} \sum_{j=0}^{m} \Gamma(m + \frac{1}{2} - j) \int_0^1 t^{s-1-m-1/2+j} \\ &\quad \times \int_r^\delta \operatorname*{Res}_{s=m+1/2-j} Z_{2m+1}(s,\sigma) \cdot \frac{\sin\sigma}{(\cos r - \cos\sigma)^{1/2}} e^{-(\sigma^2 - r^2)/4t} \, d\sigma \, dt \\ &\quad + \rho(s,r), \end{aligned}$$

where $\rho(s,r)$ is holomorphic on $\mathbb{C}\,(\ni s)$ and smooth at $r = 0$.

Using the theorem, we can calculate all the residues of $\zeta_n(s)$ easily in principle. As an example, in Table 1, we enumerate some of its residues when n is odd, which are obtained by using the formula (2.4) as follows:

First, let us calculate some of the residues of $Z_n(s,r)$. The function $Z_1(s,r)$, which does not depend on r, has only one pole at $s = 1/2$ with residue $1/2\pi$. By the formula (2.3), we have

$$\begin{aligned} \operatorname*{Res}_{s=(n+2)/2-j} Z_{n+2}(s,r) = &\frac{1}{2\pi} \frac{-1}{\sin r} \frac{\partial}{\partial r} \operatorname*{Res}_{s=n/2-(j-1)} Z_n(s,r) \\ &+ \frac{1}{4\pi} \frac{r}{\sin r} \frac{1}{n/2 - j} \operatorname*{Res}_{s=n/2-j} Z_n(s-1,r). \end{aligned}$$

Hence, the residues of $Z_3(s,r)$ can be calculated as

$$\operatorname*{Res}_{s=3/2} Z_3(s,r) = \left(\frac{1}{2\pi}\right)^2 \frac{r}{\sin r}, \qquad \operatorname*{Res}_{s=3/2-1} Z_3(s,r) = 0.$$

We obtain thus

$$\operatorname*{Res}_{s=3/2-0} Z_3(s,0) = \left(\frac{1}{2\pi}\right)^2, \qquad \operatorname*{Res}_{s=3/2-1} Z_3(s,0) = 0.$$

We enumerate some of the residues of $Z_n(s,0)$ in Table 2, which together with the formula (2.4) yields Table 1.

Table 2 Residues of $Z_n(s,0)$

n \ poles	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
1	$\frac{1}{2\pi}$			
3	0	$\left(\frac{1}{2\pi}\right)^2$		
5	0	$\left(\frac{1}{2\pi}\right)^3\left(-\frac{1}{3}\right)$	$\left(\frac{1}{2\pi}\right)^3\frac{1}{3}$	
7	0	$\left(\frac{1}{2\pi}\right)^4\frac{4}{15}$	$\left(\frac{1}{2\pi}\right)^4\left(-\frac{1}{3}\right)$	$\left(\frac{1}{2\pi}\right)^4\frac{1}{15}$

3. Preparation for the proof of Theorem 2.1

First, let us gather some known results, which yield us to focus on a certain function denoted $\widetilde{\zeta}_n(s,r)$ on $\mathbb{C}\times[0,\pi)$ (refer to (3.10)) for the proof of Theorem 2.1.

Let $K_n(t)=K_n(t,x,y)$ denote the fundamental solution or the heat kernel of the heat equation with the initial condition on S^n:

$$\left(\frac{\partial}{\partial t}+\Delta_n\right)f(t,x)=0,\quad f(0,x)=f_0(x).$$

By the homogeneity of sphere, the kernel depends only on t and the geodesic distance $r=r(x,y)$, so that we denote it simply by $K_n(t,r)$. It is well known that its trace

$$\operatorname{Tr}(K_n(t))=\int_{S^n}K_n(t,0)\,dg(x)=\operatorname{vol}(S^n)\,K_n(t,0) \tag{3.1}$$

(dg is the volume element of S^n) can be described as

$$\operatorname{Tr}(K_n(t))=\sum_{j=0}^{\infty}e^{-t\lambda_{n,j}}.$$

By the Mellin transform of $\sum_{j=1}^{\infty}e^{-t\lambda_{n,j}}=\operatorname{Tr}(K_n(t))-1$, we obtain

$$\zeta_n(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\operatorname{Tr}(K_n(t))-1\right)dt. \tag{3.2}$$

Proposition 3.1 (e.g. Gilkey [2])

(1) *The trace can be expanded asymptotically as*

$$\operatorname{Tr}(K_n(t))\sim\sum_{j=0}^{\infty}t^{-n/2+j}A_j\quad(t\downarrow 0).$$

(2) *Let us take $N\in\mathbb{N}$ and decompose* (3.2) *into three terms:*

$$\begin{aligned}\zeta_n(s) = &\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\sum_{j=0}^{N} t^{-n/2+j}A_j\,dt\\ &+\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\mathrm{Tr}(K_n(t))-1-\sum_{j=0}^{N} t^{-n/2+j}A_j\right)dt\\ &+\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\mathrm{Tr}(K_n(t))-1\right)dt.\end{aligned}$$

Then, the latter two terms are holomorphic on $\{s\in\mathbb{C}|\,\mathrm{Re}(s)>n/2-N-1\}$*, and, hence, we have*

$$\zeta_n(s)=\frac{1}{\Gamma(s)}\sum_{j=0}^{N}\frac{A_j}{s-n/2+j}+R(s),$$

where $R(s)$ *is holomorphic on* $\{s\in\mathbb{C}|\,\mathrm{Re}(s)>n/2-N-1\}$.

Proposition 3.2 (Nagase [5]) *Let us define* $k_n(t,r)$ *as*

$$k_n(t,r)=\frac{2^{-1/2}}{(2\pi)^{n/2}}\,t^{-1/2}e^{(n-1)^2t/4}\left(\frac{-1}{\sin r}\frac{\partial}{\partial r}\right)^m e^{-r^2/4t}\quad(n=2m+1),\tag{3.3}$$

$$\begin{aligned}k_n(t,r)=&\frac{2^{-1}}{(2\pi)^{(n+1)/2}}\,t^{-3/2}e^{(n-1)^2t/4}\\ &\times\left(\frac{-1}{\sin r}\frac{\partial}{\partial r}\right)^m\int_r^\delta\frac{\sigma e^{-\sigma^2/4t}}{(\cos r-\cos\sigma)^{1/2}}\,d\sigma\quad(n=2m+2).\end{aligned}\tag{3.4}$$

Then $K_n(t,r)$ *can be described as*

$$K_n(t,r)=k_n(t,r)+O_\infty(e^{-1/t}),\tag{3.5}$$

where $O_\infty(e^{-1/t})$ *is a function with respect to* t *and* r *whose derivative of arbitrary order with respect to* r *is estimated as* $O(e^{-\varepsilon/t})$ *with some* $\varepsilon>0$ *when* t *tends to* 0.

Corollary 3.3 (Nagase [5]) *We have*

$$k_{n+2}(t,r)=\frac{e^{nt}}{2\pi}\left(\frac{-1}{\sin r}\frac{\partial}{\partial r}\right)k_n(t,r)\tag{3.6}$$

and

$$k_{n+1}(t,r)=2^{1/2}e^{-(2n+1)t/4}\int_r^\delta\frac{k_{n+2}(t,\sigma)\sin\sigma}{(\cos r-\cos\sigma)^{1/2}}\,d\sigma+O_\infty(e^{-1/t}).\tag{3.7}$$

Moreover, according to Nagase [5], $k_n(s,r)$ has the following expansion:

$$k_n(t,r) = e^{-r^2/4t+(n-1)^2t/4}\, t^{-n/2} \sum_{j=0}^{m} t^j A_{m,j}(r) \quad (n = 2m+1), \tag{3.8}$$

$$\begin{aligned} k_n(t,r) &= e^{-r^2/4t+(n-1)^2t/4} \\ &\quad\times \left(t^{-n/2} \sum_{j=0}^{N} t^j B_{m,j}(r) + O_\infty(t^{-n/2+N+1}) \right) \quad (n = 2m+2), \end{aligned} \tag{3.9}$$

where $O_\infty(t^{-n/2+N+1})$ is a term whose derivative of arbitrary order with respect to r is estimated as $O(t^{-n/2+N+1})$.

In this paper we are interested in the residues of $\zeta_n(s)$, hence, in the coefficients of the asymptotic expansion of (3.1): refer to Proposition 3.1. For example, in the case where n is odd, (3.5) and (3.8) imply that the trace (3.1) can be expanded as

$$\begin{aligned} \mathrm{Tr}(K_n(t)) &= \mathrm{vol}(S^n)\, e^{-0^2/4t+(n-1)^2t/4}\, t^{-n/2} \sum_{j=0}^{m} t^j A_{m,j}(0) + O_\infty(e^{-1/t}) \\ &= \mathrm{vol}(S^n)\, t^{-n/2} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{(n-1)^2}{4} \right)^l t^l \sum_{j=0}^{m} t^j A_{m,j}(0) + O_\infty(e^{-1/t}). \end{aligned}$$

Thus it will be obvious that, in general, if we set

$$\widetilde{k}_n(t,r) = e^{r^2/4t-(n-1)^2t/4}\, k_n(t,r),$$

then we have only to investigate the asymptotics of

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \widetilde{k}_n(t,r)\, dt$$

to know the residues of $\zeta_n(s)$. Moreover, we may ignore the holomorphic part

$$\frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \widetilde{k}(t,r)\, dt.$$

Namely, it will be enough, for our purpose, to investigate the coefficients of the expansion of

$$\widetilde{\zeta}_n(s,r) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \widetilde{k}_n(t,r)\, dt. \tag{3.10}$$

4. Proof of Theorem 2.1

Theorem 2.1 is an easy consequence of the following two propositions:

Proposition 4.1 *We have*

$$\widetilde{\zeta}_n(s,r) = Z_n(s,r) + \rho(s,r), \tag{4.1}$$

where $\rho(s,r)$ is holomorphic on $\mathbb{C}\,(\ni s)$ and smooth at $r=0$.

Proposition 4.2

(1) *The residues of $\zeta_n(s)$ are expressed as*

$$\begin{aligned}&\operatorname*{Res}_{s=n/2-j} \zeta_n(s) \\ &= \mathrm{vol}(S^n) \sum_{l=0}^{N} \frac{1}{(j-l)!} \left(\frac{(n-1)^2}{4}\right)^{j-l} \frac{\Gamma(n/2-l)}{\Gamma(n/2-j)} \operatorname*{Res}_{s=n/2-l} \widetilde{\zeta}_n(s,0),\end{aligned} \tag{4.2}$$

where

$$N = \begin{cases} \max\{\frac{n-1}{2}, j\} & (n \textit{ is odd}) \\ j & (n \textit{ is even}). \end{cases}$$

(2) *$\widetilde{\zeta}_{2m}(s,r)$ and $\widetilde{\zeta}_{2m+1}(s,r)$ are related to each other as*

$$\begin{aligned}&\widetilde{\zeta}_{2m}(s,r) \\ &= \frac{1}{\Gamma(s)} 2^{1/2} \sum_{j=0}^{m} \Gamma(m+\frac{1}{2}-j) \int_0^1 t^{s-1-m-1/2+j} \\ &\quad \times \int_r^\delta \operatorname*{Res}_{s=m+1/2-j} \widetilde{\zeta}_{2m+1}(s,\sigma) \cdot \frac{\sin\sigma}{(\cos r - \cos\sigma)^{1/2}} e^{-(\sigma^2-r^2)/4t}\, d\sigma\, dt \\ &\quad + \rho(s,r),\end{aligned} \tag{4.3}$$

where $\rho(s,r)$ is holomorphic on $\mathbb{C}\,(\ni s)$ and smooth at $r=0$.

To prove the propositions we need some lemmas.

Lemma 4.3 *We have*

$$\widetilde{\zeta}_1(s,r) = \frac{1}{\Gamma(s)} \frac{2^{-1/2}}{(2\pi)^{1/2}} \frac{1}{s-1/2}, \tag{4.4}$$

and

$$\begin{aligned}\widetilde{\zeta}_2(s,r) &= \frac{1}{\Gamma(s)} \frac{2^{-1}}{(2\pi)^{3/2}} \\ &\quad \times \int_0^1 t^{s-1-3/2} e^{r^2/4t} \left(\int_r^\delta \frac{\sigma e^{-\sigma^2/4t}}{(\cos r - \cos\sigma)^{1/2}}\, d\sigma \right) dt.\end{aligned} \tag{4.5}$$

Proof. The formulas are obtained by straightforward calculation. ■

Lemma 4.4 *We have*

$$\widetilde{k}_{n+2}(t,r) = \frac{1}{2\pi}\frac{-1}{\sin r}\frac{\partial}{\partial r}\widetilde{k}_n(t,r) + \frac{1}{4\pi}\frac{r}{\sin r}\frac{1}{t}\widetilde{k}_n(t,r), \tag{4.6}$$

$$\widetilde{k}_{n+1}(t,r) = 2^{1/2}\int_r^\delta \frac{\widetilde{k}_{n+2}(t,\sigma)\sin\sigma}{(\cos r - \cos\sigma)^{1/2}}\, e^{-(\sigma^2-r^2)/4t}\, d\sigma + O_\infty(e^{-1/t}), \tag{4.7}$$

and $\widetilde{k}_n(t,r)$ *has the following expansion:*

$$\widetilde{k}_n(t,r) = t^{-n/2}\sum_{j=0}^{m} t^j A_{m,j}(r) \quad (n = 2m+1), \tag{4.8}$$

$$\widetilde{k}_n(t,r) = t^{-n/2}\sum_{j=0}^{N} t^j B_{m,j}(r) + O_\infty(t^{-n/2+N+1}) \quad (n = 2m+2), \tag{4.9}$$

where $A_{m,j}(r)$ *are even functions and are smooth at* $r = 0$*, and* $B_{m,j}(r)$ *are smooth at* $r = 0$.

Remark. The formulas (4.8) and (4.9) are essentially equivalent to the formulas (3.8) and (3.9). In the following proof, the parts proving the formulas (4.8) and (4.9) are based on Nagase [5].

Proof. We can check the formulas (4.6) and (4.7) easily by using the formulas (3.6) and (3.7) respectively. Let us prove (4.8) by induction. We have

$$\widetilde{k}_1(t,r) = e^{r^2/4t - 0^2 t/4} k_1(t,r) = \frac{2^{-1/2}}{(2\pi)^{1/2}}\, t^{-1/2}.$$

Thus (4.8) holds if $n = 1$. Next, let us assume that $\widetilde{k}_n(t,r)$ has the expansion (4.8). Then, by using (4.6), we have

$$\begin{aligned}\widetilde{k}_{n+2}(t,r) = \frac{1}{2\pi} t^{-(n+2)/2} &\sum_{j=0}^{m} t^{j+1}\frac{-r}{\sin r}\frac{1}{r}\frac{\partial}{\partial r}A_{m,j}(r)\\ &+ \frac{1}{4\pi} t^{-(n+2)/2}\sum_{j=0}^{m} t^j \frac{r}{\sin r}\, A_{m,j}(r).\end{aligned}$$

In the right hand side of the formula, $\frac{\partial}{\partial r}A_{m,j}(r)$ is an odd function which is smooth at $r = 0$, so that $\frac{1}{r}\frac{\partial}{\partial r}A_{m,j}(r)$ is certainly an even function. Therefore, we have the formula (4.8) for any n. Next, to prove (4.9) first we show

$$\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)^{m+1}\sigma e^{-\sigma^2/4t}=t^{-(m+1)}\sum_{j=0}^{m+1}t^j\beta_{m,j}(\sigma)\sigma e^{-\sigma^2/4t}, \tag{4.10}$$

where $\beta_{m,j}(\sigma)$ are even functions which are smooth at $\sigma=0$. When $m=0$, we have

$$\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\sigma e^{-\sigma^2/4t}=\left(\frac{-1}{\sigma}\frac{\partial}{\partial\sigma}\frac{\sigma}{\sin\sigma}+\frac{\sigma}{\sin\sigma}\frac{1}{2t}\right)\sigma e^{-\sigma^2/4t}.$$

In the right hand side of the formula, $\frac{\partial}{\partial\sigma}\frac{\sigma}{\sin\sigma}$ is an odd function which is smooth at $\sigma=0$, so that $\frac{-1}{\sigma}\frac{\partial}{\partial\sigma}\frac{\sigma}{\sin\sigma}$ is an even function. Thus (4.10) holds if $m=0$. Next, let us assume that (4.10) holds. Then, we have

$$\begin{aligned}\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)^{(m+1)+1}\sigma e^{-\sigma^2/4t}&=\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)t^{-(m+1)}\sum_{j=0}^{m+1}t^j\beta_{m,j}(\sigma)\sigma e^{-\sigma^2/4t}\\
=t^{-(m+2)}\Bigg(\sum_{j=0}^{m+1}t^{j+1}\left(\frac{\partial}{\partial\sigma}\frac{\sigma}{\sin\sigma}\cdot\beta_{m,j}(\sigma)\frac{-1}{\sigma}+\frac{\sigma}{\sin\sigma}\cdot\frac{\partial}{\partial\sigma}\beta_{m,j}(\sigma)\cdot\frac{-1}{\sigma}\right)&\\
+\sum_{j=0}^{m+1}t^j\frac{1}{2}\frac{\sigma}{\sin\sigma}\beta_{m,j}(\sigma)\Bigg)\sigma e^{-\sigma^2/4t}.&\end{aligned}$$

In the last expression, $\frac{\partial}{\partial\sigma}\frac{\sigma}{\sin\sigma}\cdot\beta_{m,j}(\sigma)\frac{-1}{\sigma}$, $\frac{\sigma}{\sin\sigma}\cdot\frac{\partial}{\partial\sigma}\beta_{m,j}(\sigma)\cdot\frac{-1}{\sigma}$, $\frac{1}{2}\frac{\sigma}{\sin\sigma}\beta_{m,j}(\sigma)$ are certainly even functions. Therefore, we have (4.10) for any m. Next, let us prove (4.9). We set $n=2m+2$. By applying integration by parts to the right hand of (3.4), we have

$$\begin{aligned}k_n(t,r)=&\frac{1}{(2\pi)^{(n+1)/2}}t^{-3/2}e^{(n-1)^2t/4}\\
&\times\int_r^\delta(\cos r-\cos\sigma)^{1/2}\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)^{m+1}\sigma e^{-\sigma^2/4t}\,d\sigma\\
&+O_\infty(e^{-1/t}).\end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{k}_n(t,r)=&\frac{1}{(2\pi)^{(n+1)/2}}t^{-3/2}\\
&\underline{\times e^{r^2/4t}\int_r^\delta(\cos r-\cos\sigma)^{1/2}\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)^{m+1}\sigma e^{-\sigma^2/4t}\,d\sigma}\\
&+O_\infty(e^{-1/t}).\end{aligned}$$

Now, by using (4.10), let us calculate the part underlined. We have

$$
\begin{aligned}
&e^{r^2/4t}\int_r^\delta (\cos r-\cos\sigma)^{1/2}\left(\frac{\partial}{\partial\sigma}\frac{-1}{\sin\sigma}\right)^{m+1}\sigma e^{-\sigma^2/4t}\,d\sigma\\
&=t^{-(m+1)}\sum_{j=0}^{m+1}t^j\int_r^\delta (\cos r-\cos\sigma)^{1/2}\beta_{m,j}(\sigma)\sigma e^{-(\sigma^2-r^2)/4t}\,d\sigma\\
&=t^{-(m+1)}\sum_{j=0}^{m+1}t^j\int_r^\delta 2^{-1/2}S(\sigma^2,r^2)(\sigma^2-r^2)^{1/2}\beta_{m,j}(\sigma)\sigma e^{-(\sigma^2-r^2)/4t}\,d\sigma,
\end{aligned}
$$

where we set $S(\sigma^2,r^2)=\left(\frac{\sin\frac{\sigma+r}{2}}{\frac{\sigma+r}{2}}\right)^{1/2}\left(\frac{\sin\frac{\sigma-r}{2}}{\frac{\sigma-r}{2}}\right)^{1/2}$, which is obviously smooth with respect to the variables σ^2 and r^2. Now, by putting $(\sigma^2-r^2)/4t=u$, the part underlined is equal to

$$
2^{1/2}t^{-(m+1)+3/2}\sum_{j=0}^{m+1}t^j\int_0^{(\delta^2-r^2)/4t} S(4tu-r^2,r^2)\beta_{m,j}(4tu-r^2)u^{1/2}e^{-u}\,du.
$$

Since

$$
\int_0^{(\delta^2-r^2)/4t} S(4tu-r^2,r^2)\beta_{m,j}(4tu-r^2)u^{1/2}e^{-u}\,du
$$

is smooth with respect to t, we obtain the expansion (4.9). ■

Lemma 4.5 *We have*

$$
\begin{aligned}
\widetilde{\zeta}_{n+2}(s,r)&=\frac{1}{2\pi}\frac{-1}{\sin r}\frac{\partial}{\partial r}\widetilde{\zeta}_n(s,r)\\
&\quad+\frac{1}{4\pi}\frac{r}{\sin r}\frac{1}{s-1}\widetilde{\zeta}_n(s-1,r)+\rho(s,r),
\end{aligned}
\tag{4.11}
$$

where $\rho(s,r)$ is holomorphic on $\mathbb{C}\,(\ni s)$ and smooth at $r=0$.

Proof. If $n=2m+1$, by referring to (3.10), (4.6) and (4.8), the formula will be obvious. Let us consider the case where $n=2m+2$. The formula (4.9) implies

$$
\begin{aligned}
&\int_0^1 t^{s-1}\frac{\partial}{\partial r}\widetilde{k}_n(t,r)\,dt-\frac{\partial}{\partial r}\int_0^1 t^{s-1}\widetilde{k}_n(t,r)\,dt\\
&=\int_0^1 t^{s-1}\frac{\partial}{\partial r}\left(t^{-n/2}\sum_{j=0}^{N}t^jB_{m,j}(r)+O_\infty(t^{-n/2+N+1})\right)dt\\
&\quad-\frac{\partial}{\partial r}\int_0^1 t^{s-1}\left(t^{-n/2}\sum_{j=0}^{N}t^jB_{m,j}(r)+O_\infty(t^{-n/2+N+1})\right)dt\\
&=\frac{\partial}{\partial r}\int_0^1 t^{s-1}O_\infty(t^{-n/2+N+1})\,dt-\int_0^1\frac{\partial}{\partial r}t^{s-1}O_\infty(t^{-n/2+N+1})\,dt,
\end{aligned}
$$

which is certainly holomorphic on $\{s \in \mathbb{C} | \operatorname{Re}(s) \geq -N + m\}$ and smooth at $r = 0$. Hence, we obtain

$$\int_0^1 t^{s-1} \frac{\partial}{\partial r} \widetilde{k}_n(t,r)\, dt = \frac{\partial}{\partial r} \int_0^1 t^{s-1} \widetilde{k}_n(t,r)\, dt + \rho(s,r). \tag{4.12}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\frac{1}{2\pi} \frac{-1}{\sin r} \frac{\partial}{\partial r} \widetilde{k}_n(t,r) + \frac{1}{4\pi} \frac{r}{\sin r} \frac{1}{t} \widetilde{k}_n(t,r) \right) dt \\
&= \frac{1}{2\pi} \frac{-1}{\sin r} \frac{\partial}{\partial r} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \widetilde{k}_n(t,r)\, dt \\
&\quad + \frac{1}{4\pi} \frac{r}{\sin r} \frac{1}{\Gamma(s)} \int_0^1 t^{s-2} \widetilde{k}_n(t,r)\, dt + \rho(s,r).
\end{aligned}$$

Hence, by using (3.10) and (4.6), we have (4.11). ■

Now, let us prove Proposition 4.1 and Proposition 4.2.

Proof of Proposition 4.1. Using Lemma 4.3 and Lemma 4.5, we obtain (4.1) immediately. ■

Proof of (2) in Proposition 4.2. By using (3.10) and (4.8), we have

$$\begin{aligned}
\widetilde{\zeta}_{2m+1}(s,r) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1-m-1/2} \sum_{j=0}^{m} t^j A_{m,j}(r)\, dt \\
&= \frac{1}{\Gamma(s)} \sum_{j=0}^{m} \frac{A_{m,j}(r)}{s - m - 1/2 + j}.
\end{aligned}$$

Therefore, the coefficients in (4.8) are expressed as

$$A_{m,j}(r) = \Gamma(m + \frac{1}{2} - j) \operatorname*{Res}_{s=m+1/2-j} \widetilde{\zeta}_{2m+1}(s,r).$$

Hence, we have

$$\widetilde{k}_{2m+1}(t,r) = t^{-m-1/2} \sum_{j=0}^{m} t^j \Gamma(m + \frac{1}{2} - j) \operatorname*{Res}_{s=m+1/2-j} \widetilde{\zeta}_{2m+1}(s,r).$$

Now, by using (4.7), we obtain

$$\begin{aligned}
&\widetilde{k}_{2m}(t,r) \\
&= 2^{1/2} \sum_{j=0}^{m} \Gamma(m + \frac{1}{2} - j)\, t^{s-1-m-1/2+j} \\
&\quad \times \int_r^\delta \operatorname*{Res}_{s=m+1/2-j} \widetilde{\zeta}_{2m+1}(s,\sigma) \cdot \frac{\sin\sigma}{(\cos r - \cos\sigma)^{1/2}}\, e^{-(\sigma^2 - r^2)/4t}\, d\sigma \\
&\quad + O_\infty(e^{-1/t}).
\end{aligned}$$

Consequently, by using (3.10), we have (4.3). ∎

Proof of (1) in Proposition 4.2. For $j = 0, 1, \ldots, n/2-1$, the coefficients in the formula (4.9) are expressed as

$$B_{m,j}(r) = \Gamma(\frac{n}{2} - j) \operatorname*{Res}_{s=n/2-j} \widetilde{\zeta}_n(s, r). \tag{4.13}$$

Using this formula, we prove (4.2) only in the case where $n = 2m + 2$. Referring to (4.9), we can calculate $k_n(t, 0)$ as follows:

$$\begin{aligned} k_n(t, 0) &= e^{-0^2/4t+(n-1)^2t/4}\,\widetilde{k}_n(t, 0) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{(n-1)^2}{4}\right)^l \\ &\quad \times \left(t^{-n/2} \sum_{j=0}^{N} t^j \Gamma(\frac{n}{2} - j) \operatorname*{Res}_{s=n/2-j} \widetilde{\zeta}_n(s, 0) + O(t^{-n/2+N+1})\right) \\ &= t^{-n/2} \sum_{j=0}^{N} t^j \sum_{l=0}^{j} \frac{1}{(j-l)!} \left(\frac{(n-1)^2}{4}\right)^{j-l} \\ &\quad \times \Gamma(\frac{n}{2} - j) \operatorname*{Res}_{s=n/2-j} \widetilde{\zeta}_n(s, 0) + O(t^{-n/2+N+1}). \end{aligned}$$

Hence, we have

$$\begin{aligned} \zeta_n(s) &= \operatorname{vol}(S^n) \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} k_n(t, 0)\, dt + \rho(s) \\ &= \operatorname{vol}(S^n) \frac{1}{\Gamma(s)} \sum_{j=0}^{N} \frac{1}{s - n/2 + j} \\ &\quad \times \sum_{l=0}^{j} \frac{1}{(j-l)!} \left(\frac{(n-1)^2}{4}\right)^{j-l} \Gamma(\frac{n}{2} - j) \operatorname*{Res}_{s=n/2-j} \widetilde{\zeta}_n(s, 0)\, dt + \rho(s), \end{aligned}$$

where $\rho(s)$ are holomorphic functions on $\{s \in \mathbb{C} |\, \mathrm{Re}(s) > n/2-N-1\}$. Therefore, we have

$$\begin{aligned} &\operatorname*{Res}_{s=n/2-j} \zeta_n(s) \\ &= \operatorname{vol}(S^n) \sum_{l=0}^{j} \frac{1}{(j-l)!} \left(\frac{(n-1)^2}{4}\right)^{j-l} \frac{\Gamma(n/2 - l)}{\Gamma(n/2 - j)} \operatorname*{Res}_{s=n/2-l} \widetilde{\zeta}_n(s, 0). \end{aligned}$$

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