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Some remarks on infinitesimal deformations of a conic bundle IV

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Abstract

This paper is the fourth part of successive studies of infinitesimal deformations of a conic bundle of dimension three. We discuss relationship between infinitesimal deformations of a conic bundle and the corresponding infinitesimal displacements of its discriminant locus in case where the discriminant locus has singularity. We shall prove that, if a deformation family of a conic bundle admits no smoothing of the singular points of the discriminant locus, then two kinds of Kodaira-Spencer maps are compatible with each other. We shall also prove that certain conic bundles admit no non-trivial small deformation families fixing their discriminant loci.

1. Introduction and the results

This paper is the fourth part of successive studies of infinitesimal deformations of a conic bundle of dimension three. The first part [2], the second part [3] and the third part [4] shall be referred as Part I, Part II and Part III respectively.

In Part I [2] we have discussed relationship between infinitesimal deformations of a conic bundle $f : X \to Y$ with $\dim X = 3$ and infinitesimal displacements of the discriminant locus $\Delta = \Delta_{X/Y}$ in Y. (For the definitions of a conic bundle and its discriminant locus, we refer to [2, Definition 1.1, Definition 1.2].)

Let $f : X \to Y$ be a conic bundle with $\dim X = 3$. As is known ([1, Proposition 1.2], [7, p.83]), there exist a locally free sheaf $\mathcal{E}$ of rank three on Y, an invertible sheaf $\mathcal{M}$ on Y and a section $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$ such that X is identified with the zero locus of q in $\mathbb{P}_Y(\mathcal{E})$ and that f is the restriction of the natural projection $\pi : \mathbb{P}_Y(\mathcal{E}) \to Y$ to X.

Let $\{f_t : X_t \to Y\}_{t\in M}$ be a deformation family of $f : X \to Y$ with $X_o = X$ and $f_o = f$ (see §2). Horikawa [5] developed deformation theory of holomorphic maps in general. Due to [5] we have the map

$$(1.1) \qquad \tau : T_o(M) \to D_{X/Y} = \mathbb{H}^1(F : \Theta_X \to f^*\Theta_Y)$$

of Kodaira-Spencer type for the family $\{f_t : X_t \to Y\}_{t\in M}$, where $T_o(M)$ denotes the tangent space of the base space M at $o \in M$ and $F : \Theta_X \to f^*\Theta_Y$ is the

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natural homomorphism induced by $f : X \to Y$ (see §2 for precise construction).

On the other hand, the family $\{f_t : X_t \to Y\}_{t\in M}$ naturally induces a family $\{\Delta_t\}_{t\in M}$ of displacements of $\Delta = \Delta_o$. Kodaira [6] studied displacements of submanifolds of a complex manifold . We can apply Kodaira's arguments to our case and we have the map

$$\rho : T_o(M) \to H^0(\Delta, N_{\Delta/Y}), \tag{1.2}$$

which is another type of Kodaira-Spencer map, where $N_{\Delta/Y}$ denotes the normal sheaf of Δ in Y (see §2 for precise construction).

Now we ask the following question.

Question. *How are the maps τ and ρ related to each other?*

Part I [2] has given a partial answer to the question above in case where Δ is smooth. Let us briefly recall our arguments in [2].

First of all, we have the natural homomorphism

$$P : D_{X/Y} \to H^0(X, \mathcal{S}_{X/Y}) \tag{1.3}$$

due to Horikawa [5], where $\mathcal{S}_{X/Y} = \mathrm{Coker}(F : \Theta_X \to f^*\Theta_Y)$ (see §2). On the other hand, we proved that, if Δ is smooth, there exists a natural isomorphism

$$\psi : H^0(\Delta, N_{\Delta/Y}) \to H^0(X, \mathcal{S}_{X/Y}) \tag{1.4}$$

(cf. [2, Corollary 2.6]). Moreover, we proved that

$$\rho = \psi^{-1} \circ P \circ \tau, \tag{1.5}$$

which shows the compatibility of τ and ρ (cf. [2, Theorem 2.12]).

In Part II [3], we discussed a kind of rigidity of a conic bundle. Assume that $Y = \mathbb{P}^2$ and that $f : X \to Y$ is a conic bundle determined by $\mathcal{E}$, $\mathcal{M}$ and q, where $\mathcal{E}$ is a locally free sheaf of rank three on Y, $\mathcal{M}$ is an invertible sheaf on Y and $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$. Assume furthermore that $\mathcal{E}$ is a direct sum of invertible sheaves *and that Δ is smooth.* Then, by using (1.5), we proved that there exists no non-trivial small deformation of $f : X \to Y$ that is again a conic bundle over Y with the same discriminant locus Δ (cf. [3, Corollary 3.14]).

The aim of this paper is to generalize these results [2, Theorem 2.12] and [3, Corollary 3.14] to the case where Δ has singularity.

There is a problem, however, that we do not have the map 'ψ' in general. In Part III [4], we proved that there does not exist such a natural isomorphism as ψ of (1.4), if Δ has singularity.

Let us explain more precisely. Let $f : X \to Y$ be a conic bundle with $\dim X = 3$. Let $B = \{x \in X \mid f \text{ is not smooth at } x\}$. Let $h : B \to X$ and $\iota : \Delta \to Y$ be the natural inclusion maps and we put $g = f \circ h : B \to Y$. The map g factors through $g' : B \to \Delta$ with $g = \iota \circ g'$. Then we have a natural homomorphism

$$\lambda : \Theta_Y \otimes \mathcal{O}_\Delta \to g'_* \mathcal{S}_{X/Y} \tag{1.6}$$

as follows. The canonical homomorphism $f^*\Theta_Y \to \mathcal{S}_{X/Y}$ naturally induces a homomorphism $f^*\Theta_Y \otimes \mathcal{O}_B \to \mathcal{S}_{X/Y}$, since $\mathcal{S}_{X/Y}$ is an $\mathcal{O}_B$-module (cf. [2, Proposition 2.2] and [2, Theorem 2.3]). Noting that $f^*\Theta_Y \otimes \mathcal{O}_B = g^*\Theta_Y = g'^*(\Theta_Y \otimes \mathcal{O}_\Delta)$ and applying g'_* to the homomorphism above, we get a homomorphism $g'_* g'^*(\Theta_Y \otimes \mathcal{O}_\Delta) \to g'_* \mathcal{S}_{X/Y}$. By taking the composition with the standard homomorphism $\Theta_Y \otimes \mathcal{O}_\Delta \to g'_* g'^*(\Theta_Y \otimes \mathcal{O}_\Delta)$, we get the homomorphism λ.

On the other hand, we have the standard homomorphism

$$\mu : \Theta_Y \otimes \mathcal{O}_\Delta \to N_{\Delta/Y}. \tag{1.7}$$

In Part III [4] we proved that $\mathrm{Ker}(\lambda) = \mathrm{Ker}(\mu)$ and that $\mathrm{Coker}(\lambda)$ is isomorphic to $\mathrm{Coker}(\mu)$. We have the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Ker}(\lambda) & \to & \Theta_Y \otimes \mathcal{O}_\Delta & \xrightarrow{\lambda} & g'_* \mathcal{S}_{X/Y} & \to & \mathrm{Coker}(\lambda) \\ =\downarrow & & =\downarrow & & & & \cong\downarrow \\ \mathrm{Ker}(\mu) & \to & \Theta_Y \otimes \mathcal{O}_\Delta & \xrightarrow{\mu} & N_{\Delta/Y} & \to & \mathrm{Coker}(\mu). \end{array} \tag{1.8}$$

We also proved that, if Δ has singularity, there does not exist any isomorphism between $g'_* \mathcal{S}_{X/Y}$ and $N_{\Delta/Y}$ that is compatible with λ and μ (cf. [4, Main Theorem]).

Now we put

$$\mathcal{A} = \mathrm{Im}(\lambda), \quad \mathcal{B} = \mathrm{Im}(\mu). \tag{1.9}$$

Then we have the natural isomorphism

$$\bar{\psi} : \mathcal{A} \to \mathcal{B} \tag{1.10}$$

that is compatible with λ and μ. We also denote by the same symbol $\bar{\psi}$ the isomorphism $H^0(\Delta, \mathcal{A}) \to H^0(\Delta, \mathcal{B})$ induced by (1.10).

Remark 1.1 *We have* $\mathrm{Supp}(\mathrm{Coker}(\lambda)) = \mathrm{Supp}(\mathrm{Coker}(\mu)) = \mathrm{Sing}(\Delta)$ (cf. [4, Corollary 2.3] and [4, Corollary 3.2]). *In particular, if Δ is smooth, λ and μ are surjective. In this case $\bar{\psi}$ naturally induces an isomorphism $H^0(\Delta, g'_* \mathcal{S}_{X/Y}) \to H^0(\Delta, N_{\Delta/Y})$, which coincides with $\psi^{-1} : H^0(X, \mathcal{S}_{X/Y}) \to H^0(\Delta, N_{\Delta/Y})$ via $H^0(X, \mathcal{S}_{X/Y}) = H^0(B, \mathcal{S}_{X/Y}) \cong H^0(\Delta, g'_* \mathcal{S}_{X/Y})$, where ψ is the isomorphism of* (1.4).

Now we have the following results, which are generalization of [2, Theorem 2.12] and [3, Corollary 3.14].

Theorem 1.2 *Let* $f : X \to Y$ *be a conic bundle with* $\dim X = 3$ *and* $\{f_t : X_t \to Y\}_{t \in M}$ *a deformation family of* f. *Assume that the induced family* $\{\Delta_t\}_{t\in M}$ *of discriminant loci does not admit smoothing of any singular point of* Δ.

(1) *We have* $\mathrm{Im}(P \circ \tau) \subset H^0(\Delta, \mathcal{A})$, *where* $\mathrm{Im}(P \circ \tau)$ *is regarded as a subspace of* $H^0(\Delta, g'_*\mathcal{S}_{X/Y})$ *via* $H^0(X, \mathcal{S}_{X/Y}) \cong H^0(\Delta, g'_*\mathcal{S}_{X/Y})$.

(2) *We have* $\mathrm{Im}(\rho) \subset H^0(\Delta, \mathcal{B})$.

(3) *We can define the composition map* $\bar{\psi} \circ P \circ \tau : T_o(M) \to H^0(\Delta, \mathcal{B})$ *and we have* $\rho = \bar{\psi} \circ P \circ \tau$.

Corollary 1.3 *Assume that* $Y = \mathbb{P}^2$ *and that* $f : X \to Y$ *is a conic bundle determined by* $\mathcal{E}$, $\mathcal{M}$ *and* q, *where* $\mathcal{E}$ *is a locally free sheaf of rank three on* Y, $\mathcal{M}$ *is an invertible sheaf on* Y *and* $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$. *Assume furthermore that* $\mathcal{E}$ *is a direct sum of invertible sheaves. Then there exists no non-trivial small deformation of* $f : X \to Y$ *that is again a conic bundle over* Y *with the same discriminant locus* Δ.

2. Notation and preliminaries

Let us recall some definitions and discussions in Part I [2], Part II [3] and Part III [4] in order to fix notation in this paper. Some of the notation in our previous papers shall be slightly changed here.

A. First we shall briefly recall the deformation theory of holomorphic maps due to E. Horikawa [5] (see also Part I [2]).

Let Y be a compact complex manifold of dimension m. By a family of holomorphic maps into Y, we mean a quadruplet $(\mathcal{X}, \Phi, p, M)$ of complex manifolds $\mathcal{X}$, M and holomorhic maps $\Phi : \mathcal{X} \to Y \times M$, $p : \mathcal{X} \to M$ with the following properties:

(i) p is smooth and proper.

(ii) $q \circ \Phi = p$, where $q : Y \times M \to M$ denotes the natural projection.

Putting $X_t = p^{-1}(t)$ and $f_t = \Phi|_{X_t}$ for $t \in M$, we denote the family $(\mathcal{X}, \Phi, p, M)$ by $\{f_t : X_t \to Y\}_{t\in M}$. Let $o \in M$, $X = X_o$ and $f = f_o$. Then the family $\{f_t : X_t \to Y\}_{t\in M}$ is called a deformation family of $f : X \to Y$.

Let $F : \Theta_X \to f^*\Theta_Y$ be the natural homomorphism induced by f. We put $\Theta_{X/Y} = \mathrm{Ker}(f)$, $\mathcal{S}_{X/Y} = \mathrm{Coker}(F)$ and $D_{X/Y} = \mathbb{H}^1(F : \Theta_X \to f^*\Theta_Y)$. Then we have an exact sequence

$$0 \to H^1(X, \Theta_{X/Y}) \to D_{X/Y} \to H^0(X, \mathcal{S}_{X/Y}) \to H^2(X, \Theta_{X/Y}). \tag{2.1}$$

Horikawa showed that the infinitesimal deformations of f are classified by $D_{X/Y}$. He also defined a kind of Kodaira-Spencer map $\tau : T_o(M) \to D_{X/Y}$.

Let $\mathcal{U} = \{U_i\}$ be a finite Stein open covering of X. For a sheaf $\mathcal{F}$ on X, we denote the group of the q-cochains and the q-cocycles with coefficients in $\mathcal{F}$ by $C^q(\mathcal{U},\mathcal{F})$ and $Z^q(\mathcal{U},\mathcal{F})$ respectively. We denote the q-th coboundary map by $\delta : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$. Then we have

$$D_{X/Y} = \frac{\{\,(\tau,\sigma) \in C^0(\mathcal{U}, f^*\Theta_Y) \times Z^1(\mathcal{U},\Theta_X) \,|\, \delta(\tau) = F(\sigma)\,\}}{\{\,(F(g),\delta(g)) \,|\, g \in C^0(\mathcal{U},\Theta_X)\,\}} \tag{2.2}$$

The map τ is determined as follows. Shrinking M, if necessary, we assume the following.

(i) M is an open set in $\mathbb{C}^k$ with coordinates $t = (t_1,\dots,t_k)$ and $o = (0,\dots,0)$.

(ii) $\mathcal{X}$ is covered by a finite number of Stein coordinate open sets $\{\mathcal{U}_i\}$. Each $\mathcal{U}_i$ is covered by a system of coordinates (z_i,t) such that $p(z_i,t) = t$, where $(z_i,t) = (z_i^1,\dots,z_i^n,t_1,\dots,t_k)$.

(iii) $\Phi(\mathcal{U}_i) \subset V_i \times M$, where V_i is an open set of Y covered by a system of coordinates $w_i = (w_i^1,\dots,w_i^m)$.

(iv) Φ is given by $w_i^l = \Phi_i^l(z_i,t)$ for $l = 1,\dots,m$.

(v) $(z_i,t) \in \mathcal{U}_i$ coincides with $(z_j,t) \in \mathcal{U}_j$ if and only if $z_i^l = \phi_{ij}^l(z_j^1,\dots,z_j^n,t)$ for $l = 1,\dots,n$, where ϕ_{ij}^l $(l = 1,\dots,n)$ are holomorphic transition functions.

(vi) $w_i \in V_i$ coincides with $w_j \in V_j$ if and only if $w_i^l = \psi_{ij}^l(w_j^1,\dots,w_j^m)$ for $l = 1,\dots,m$, where ψ_{ij}^l $(l = 1,\dots,m)$ are holomorphic transition functions.

Let $U_i = \mathcal{U}_i \cap X$ and $\mathcal{U}$ denote the covering $\{U_i\}$ of X. For any element $\partial/\partial t \in T_o(M)$, we put

$$\tau_i = \sum_{l=1}^{m} \left.\frac{\partial \Phi_i^l}{\partial t}\right|_{t=0} \cdot \frac{\partial}{\partial w_i^l} \in \Gamma(U_i, f^*\Theta_Y), \tag{2.3}$$

$$\sigma_{ij} = \sum_{l=1}^{n} \left.\frac{\partial \phi_{ij}^l}{\partial t}\right|_{t=0} \cdot \frac{\partial}{\partial z_i^l} \in \Gamma(U_{ij}, \Theta_X). \tag{2.4}$$

Then $\tau = \{\tau_i\} \in C^0(\mathcal{U}, f^*\Theta_Y)$ and $\sigma = \{\sigma_{ij}\} \in Z^1(\mathcal{U},\Theta_X)$ represents an element of $D_{X/Y}$, which we define to be $\tau(\partial/\partial t)$. Thus we can define the map

$$\tau : T_o(M) \to D_{X/Y} \tag{2.5}$$

Let $P : f^*\Theta_Y \to \mathcal{S}_{X/Y}$ be the natural homomorphism. For an element of $D_{X/Y}$, we take a representative $(\tau,\sigma) \in C^0(\mathcal{U}, f^*\Theta_Y) \times Z^1(\mathcal{U},\Theta_X)$ with $\tau = \{\tau_i\}$ and $\sigma = \{\sigma_{ij}\}$. Then the collection $\{P(\tau_i)\}$ patches together to an element of $H^0(X, \mathcal{S}_{X/Y})$. In this way, we can define the map

$$P : D_{X/Y} \to H^0(X, \mathcal{S}_{X/Y}), \tag{2.6}$$

which is nothing but the homomorphism appearing in (2.1), where we use the same symbol P as above.

B. Next we shall briefly discuss displacements of a divisor on a complex manifold.

Let Y be a compact complex manifold of dimension m and $\{V_i\}$ a Stein open covering of Y. Assume that each V_i is a sufficiently small open set with a system of coordinates $w_i = (w_i^1, \dots, w_i^m)$. Let Δ be a reduced divisor on Y. Let

$$\xi_i(w_i) = 0 \tag{2.7}$$

be a defining equation of $\Delta \cap V_i$ in V_i. Let $\mathcal{I}$ be the defining ideal sheaf of Δ in Y. We define the homomorphism $\zeta_i : \Gamma(\Delta \cap V_i, \mathcal{I}/\mathcal{I}^2) \to \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta)$ by $\zeta_i(\xi_i \bmod \mathcal{I}^2) = 1$. Then we have

$$\Gamma(\Delta \cap V_i, N_{\Delta/Y}) = \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta) \cdot \zeta_i. \tag{2.8}$$

Now we define the map ρ. Let $\{\Delta_t\}_{t \in M}$ be a family of displacements of $\Delta = \Delta_o$ defined by local equations

$$\tilde{\xi}_i(w_i, t) = \tilde{\xi}_i(w_i^1, \dots, w_i^m, t_1, \dots, t_k) = 0 \tag{2.9}$$

with $\tilde{\xi}_i(w_i, 0) = \xi_i(w_i)$. For any element $\partial/\partial t \in T_o(M)$, we put

$$\rho_i\left(\frac{\partial}{\partial t}\right) = \left(-\frac{\partial \tilde{\xi}_i}{\partial t}\bigg|_{t=0} \bmod \mathcal{I}\right) \cdot \zeta_i. \tag{2.10}$$

Then the collection $\{\rho_i(\partial/\partial t)\}$ patches together to an element $\rho(\partial/\partial t)$ of $H^0(\Delta, N_{\Delta/Y})$. In this way we can define the map

$$\rho : T_o(M) \to H^0(\Delta, N_{\Delta/Y}). \tag{2.11}$$

Remark 2.1 *K. Kodaira originally studied the case where Δ is smooth. Suppose that $\{\Delta_t\}_{t \in M}$ is defined by local equations $w_i^1 = \tilde{\varphi}_i(w_i^2, \dots, w_i^m, t)$ on V_i. Then he defined $\rho(\partial/\partial t)$ by $((\partial \tilde{\varphi}_i/\partial t)|_{t=0} \bmod \mathcal{I}) \cdot \zeta_i$. In this case we have $\tilde{\xi}_i = w_i^1 - \tilde{\varphi}$. Since we have $\partial \tilde{\varphi}_i/\partial t = -\partial \tilde{\xi}_i/\partial t$,* (2.10) *is a generalization of the original definition.*

C. Here we recall some discussions on conic bundles in Part I [2] and Part III [4]. Let $f : X \to Y$ be a conic bundle with $\dim X = 3$. We put

$$B = \{\, x \in X \mid f \text{ is not smooth at } x \,\}, \quad \Delta = f(B). \tag{2.12}$$

Let $p \in \Delta$. We take a sufficiently small neighbourhood V of p with a system of coordinates (r, s) so that $p = \{\, r = s = 0 \,\}$. In this paper we discuss the following two cases.

Case I: $p \in \Delta \setminus Sing(\Delta)$.

In this case, there exists a unique point $p' \in B$ satisfying $f(p') = p$. We can take such a system (α, β, γ) of local coordinates on X around p' that the map f is given by

$$(\alpha, \beta, \gamma) \mapsto (r, s) = (\beta\gamma, \alpha) \tag{2.13}$$

(cf. [2, Proposition 2.1 (ii) (b)]). Then $F : \Theta_X \to f^*\Theta_Y$ is given by

$$\frac{\partial}{\partial\alpha} \mapsto \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial\beta} \mapsto \gamma\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial\gamma} \mapsto \beta\frac{\partial}{\partial r} \tag{2.14}$$

(cf. [2, Proposition 2.2 (ii) (c)]).

The locus B is defined by $\beta = \gamma = 0$ around p'. We have $\mathcal{S}_{X/Y,\, p'} = \mathcal{O}_{B,\, p'} \cdot \nu$ with $\nu = P(\partial/\partial r)$, where the homomorphism $P : f^*\Theta_Y \to \mathcal{S}_{X/Y}$ is given by

$$\varphi_1(\alpha, \beta, \gamma)\frac{\partial}{\partial r} + \varphi_2(\alpha, \beta, \gamma)\frac{\partial}{\partial s} \mapsto (\varphi_1 \bmod (\beta, \gamma)) \cdot \nu = \varphi_1(\alpha, 0, 0) \cdot \nu \tag{2.15}$$

(see also [2, Proposition 2.2 (ii) (f)]).

The discriminant locus Δ is defined by $r = 0$ around p.

Case II: $p \in Sing(\Delta)$.

In this case, there exist open subsets U_1 and U_2 of X satisfying the following properties, after shrinking V if necessary (cf. [4, §1]).

(i) $X_V := f^{-1}(V) = U_1 \cup U_2$.

(ii) We can choose coordinates $(\alpha_1, \beta_1, \gamma_1)$ on U_1 so that $f|_{U_1}$ is defined by

$$(\alpha_1, \beta_1, \gamma_1) \mapsto (r, s) = (\beta_1^2 - \alpha_1\gamma_1^2, \alpha_1). \tag{2.16}$$

(iii) We can choose coordinates $(\alpha_2, \beta_2, \gamma_2)$ on U_2 so that $f|_{U_2}$ is defined by

$$(\alpha_2, \beta_2, \gamma_2) \mapsto (r, s) = (\alpha_2, \beta_2^2 - \alpha_2\gamma_2^2). \tag{2.17}$$

(iv) On $U_1 \cap U_2$ we have $\alpha_1 = \beta_2^2 - \alpha_2\gamma_2^2$, $\beta_1 = \beta_2\gamma_2^{-1}$ and $\gamma_1 = \gamma_2^{-1}$.

Then $F : \Gamma(U_1, \Theta_X) \to \Gamma(U_1, f^*\Theta_Y)$ is given by

$$\frac{\partial}{\partial\alpha_1} \mapsto -\gamma_1^2\frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial\beta_1} \mapsto 2\beta_1\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial\gamma_1} \mapsto -2\alpha_1\gamma_1\frac{\partial}{\partial r}. \tag{2.18}$$

The locus $B \cap U_1$ is defined by $\beta_1 = \alpha_1\gamma_1 = 0$. We have $\Gamma(B \cap U_1, \mathcal{S}_{X/Y}) = \Gamma(B \cap U_1, \mathcal{O}_B) \cdot \nu_1$ with $\nu_1 = P(\partial/\partial r)$, where $P : \Gamma(U_1, f^*\Theta_Y) \to \Gamma(B \cap U_1, \mathcal{S}_{X/Y})$ is given by

$$\varphi_1(\alpha_1, \beta_1, \gamma_1)\frac{\partial}{\partial r} + \varphi_2(\alpha_1, \beta_1, \gamma_1)\frac{\partial}{\partial s} \mapsto ((\varphi_1 + \gamma_1^2\varphi_2) \bmod (\beta_1, \alpha_1\gamma_1)) \cdot \nu_1. \tag{2.19}$$

On the other hand, the homomorphism $F : \Gamma(U_2, \Theta_X) \to \Gamma(U_2, f^*\Theta_Y)$ is given by

$$\frac{\partial}{\partial\alpha_2} \mapsto \frac{\partial}{\partial r} - \gamma_2^2\frac{\partial}{\partial s}, \quad \frac{\partial}{\partial\beta_2} \mapsto 2\beta_2\frac{\partial}{\partial s}, \quad \frac{\partial}{\partial\gamma_2} \mapsto -2\alpha_2\gamma_2\frac{\partial}{\partial s}. \tag{2.20}$$

The locus $B \cap U_2$ is defined by $\beta_2 = \alpha_2\gamma_2 = 0$. We have $\Gamma(B \cap U_2, \mathcal{S}_{X/Y}) = \Gamma(B \cap U_2, \mathcal{O}_B) \cdot \nu_2$ with $\nu_2 = P(\partial/\partial s)$, where $P : \Gamma(U_2, f^*\Theta_Y) \to \Gamma(B \cap U_2, \mathcal{S}_{X/Y})$ is given by

$$\varphi_1(\alpha_2, \beta_2, \gamma_2)\frac{\partial}{\partial r} + \varphi_2(\alpha_2, \beta_2, \gamma_2)\frac{\partial}{\partial s} \mapsto ((\gamma_2^2\varphi_1 + \varphi_2) \bmod (\beta_2, \alpha_2\gamma_2)) \cdot \nu_2. \tag{2.21}$$

In this case, Δ is defined by $rs = 0$ around p.

Remark 2.2 (1) *All the singular points of B and Δ are ordinary double points.*
(2) *For any singular point q of B, the image $f(q)$ is a singular point of Δ. For each singular point p of Δ, there exists exactly two singular points p'_1 and p'_2 of B satisfying $f(p'_i) = q$ $(i = 1, 2)$.*

3. Proof of Theorem 1.2 (1) and (2)

The aim of this section is to prove Theorem 1.2 (1) and (2).

Let $\{ f_t : X_t \to Y \}_{t \in M}$ be a deformation family of a conic bundle $f : X \to Y$ with $\dim X = 3$. Assume that this family does not admit smoothing of any singular point of Δ. We shall prove that $P \circ \tau(T_o(M)) \subset H^0(\Delta, \mathcal{A})$ and that $\rho(T_o(M)) \subset H^0(\Delta, \mathcal{B})$.

Let us begin with local discussions around a singular point of Δ.

Let $p \in Sing(\Delta)$. We use the notation in §2 C. Case II.

Proposition 3.1 (1) *Any element $\tilde{\phi}$ of $\Gamma(B \cap X_V, \mathcal{S}_{X/Y}) = \Gamma(\Delta \cap V, g'_*\mathcal{S}_{X/Y})$ is written in the following form:*

$$\begin{cases} \tilde{\phi}|_{B\cap U_1} = ((a_0 + a_1\gamma_1 + a_2\gamma_1^2 + \phi_1(\alpha_1)) \bmod (\beta_1, \alpha_1\gamma_1)) \cdot \nu_1, \\ \tilde{\phi}|_{B\cap U_2} = ((a_0\gamma_2^2 + a_1\gamma_2 + a_2 + \phi_2(\alpha_2)) \bmod (\beta_2, \alpha_2\gamma_2)) \cdot \nu_2, \end{cases}$$

where a_i $(i = 0, 1, 2)$ are constants and the functions $\phi_1(\alpha_1)$ and $\phi_2(\alpha_2)$ are holomorphic functions with $\phi_1(0) = 0$ and $\phi_2(0) = 0$ which are defined on U_1 and U_2, respectively.
(2) *The above element $\tilde{\phi}$ belongs to $\Gamma(\Delta \cap V, \mathcal{A})$ if and only if $a_1 = 0$.*

Proof. Straightforward from [4, Proposition 2.1 (1)] and [4, Proposition 2.2 (5)]. (Note that some of the notation in [4] is different from our notation here.) ∎

Now we adjust the notation in §2 **B** to our case in the following way. Since

$\dim Y = 2$ in our case, we put $m = 2$. We also put $V_i = V$, $w_i^1 = r$, $w_i^2 = s$ and $\xi_i(w_i) = \xi(r, s)$.

In this case we have $\xi(r, s) = rs$, since $\Delta \cap V$ is defined by $rs = 0$. Let ζ be the element of $\Gamma(\Delta \cap V, N_{\Delta/Y})$ that satisfies $\zeta : \xi \bmod (rs)^2 \mapsto 1 \bmod (rs)$. Then we have $\Gamma(\Delta \cap V, N_{\Delta/Y}) = \Gamma(\Delta \cap V, \mathcal{O}_\Delta) \cdot \zeta$.

Proposition 3.2 *An element* $\tilde{\varphi} = (\varphi(r, s) \bmod (rs)) \cdot \zeta \in \Gamma(\Delta \cap V, N_{\Delta/Y})$ *belongs to* $\Gamma(\Delta \cap V, \mathcal{B})$ *if and only if* $\varphi(0, 0) = 0$*, that is to say* $\varphi \in (r, s)$*.*

Proof. Straightforward from [4, Proposition 3.1 (3)]. ∎

Let us now discuss the family $\{\Delta_t\}_{t\in M}$ of the discriminant loci.

Let $p \in Sing(\Delta)$. We denote by $\Delta^{(1)}$ the irreducible component of $\Delta \cap V$ determined by $r = 0$. We denote by $\Delta^{(2)}$ the component determined by $s = 0$. We put

$$\xi_1(r, s) = r, \quad \xi_2(r, s) = s. \tag{3.1}$$

Now we assume that the family $\{\Delta_t \cap V\}_{t\in M}$ does not admit smoothing of the singular point p. Then there exist two families $\{\Delta_t^{(1)}\}_{t\in M}$ and $\{\Delta_t^{(2)}\}_{t\in M}$ of displacements of $\Delta^{(1)}$ and $\Delta^{(2)}$, respectively.

Suppose that the family $\{\Delta_t^{(1)}\}_{t\in M}$ is determined by

$$r = \varepsilon_1(s, t) \tag{3.2}$$

with $\varepsilon_1(s, 0) = 0$. We also suppose that $\{\Delta_t^{(2)}\}_{t\in M}$ is determined by

$$s = \varepsilon_2(r, t) \tag{3.3}$$

with $\varepsilon_2(r, 0) = 0$. Let us put

$$\tilde{\xi}_1(r, s, t) = r - \varepsilon_1(s, t), \quad \tilde{\xi}_2(r, s, t) = s - \varepsilon_2(r, t) \quad \text{and} \quad \tilde{\xi} = \tilde{\xi}_1\tilde{\xi}_2. \tag{3.4}$$

Then the family $\{\Delta_t^{(i)}\}_{t\in M}$ (resp. $\{\Delta_t \cap V\}_{t\in M}$) is determined by

$$\tilde{\xi}_i(r, s, t) = 0 \quad (\text{resp. } \tilde{\xi}(r, s, t) = 0) \tag{3.5}$$

with $\tilde{\xi}_i(r, s, 0) = \xi_i(r, s)$ for $i = 1, 2$ (resp. $\tilde{\xi}(r, s, 0) = \xi(r, s) = rs$).

Now we prove Theorem 1.2 (2).

Proof of Theorem 1.2 (2)**.** Let $\partial/\partial t$ be any element of $T_o(M)$. To verify that $\rho(\partial/\partial t) \in H^0(\Delta, \mathcal{B})$, we have only to discuss locally around each point $p \in \Delta$.

If p is a smooth point of Δ, there is nothing to check, since μ is surjective around p.

Let $p \in Sing(\Delta)$. Using the notation above, we have

$$\frac{\partial\tilde{\xi}}{\partial t}\Big|_{t=0} = \tilde{\xi}_1(r,s,0)\frac{\partial\tilde{\xi}_2}{\partial t}\Big|_{t=0} + \tilde{\xi}_2(r,s,0)\frac{\partial\tilde{\xi}_1}{\partial t}\Big|_{t=0} = -r\frac{\partial\varepsilon_2}{\partial t}\Big|_{t=0} - s\frac{\partial\varepsilon_1}{\partial t}\Big|_{t=0}.$$

Thus $\rho(\partial/\partial t)$ is locally expressed as

$$\left(r\frac{\partial\varepsilon_2}{\partial t}\Big|_{t=0} + s\frac{\partial\varepsilon_1}{\partial t}\Big|_{t=0} \bmod (rs)\right)\cdot\zeta \tag{3.6}$$

around p (cf. (2.10)), whence we have $\rho(\partial/\partial t) \in \Gamma(\Delta, \mathcal{B})$ by Proposition 3.2.

Thus Theorem 1.2 (2) is proved. ∎

Next we discuss $\mathrm{Im}(P\circ\tau)$.

Let $p \in Sing(\Delta)$ as before. We adjust the notation in §2 **B** as follows. We have $X_V = U_1 \cup U_2$ as above. We put $n = 3$, $z_i^1 = \alpha_i$, $z_i^2 = \beta_i$, $z_i^3 = \gamma_i$ and $\Phi_i^j(z_i, t) = \Phi_i^j(\alpha_i, \beta_i, \gamma_i, t)$ $(1 \le i \le 2,\ 1 \le j \le 2)$. Thus $\{f_t : X_t \to Y\}_{t\in M}$ is defined by

$$r = \Phi_i^1(\alpha_i,\beta_i,\gamma_i,t), \quad s = \Phi_i^2(\alpha_i,\beta_i,\gamma_i,t) \tag{3.7}$$

on U_i $(i = 1,\ 2)$ with

$$\left\{\begin{array}{l}\Phi_1^1(\alpha_1,\beta_1,\gamma_1,0) = \beta_1^2 - \alpha_1\gamma_1^2,\\ \Phi_1^2(\alpha_1,\beta_1,\gamma_1,0) = \alpha_1,\end{array}\right. \qquad \left\{\begin{array}{l}\Phi_2^1(\alpha_2,\beta_2,\gamma_2,0) = \alpha_2,\\ \Phi_2^2(\alpha_2,\beta_2,\gamma_2,0) = \beta_2^2 - \alpha_2\gamma_2^2.\end{array}\right. \tag{3.8}$$

Let $\partial/\partial t$ be any element of $T_o(M)$. We can describe $P\circ\tau(\partial/\partial t)$ locally around p as follows. Applying (2.3) to this case, we have

$$\tau_i = \frac{\partial\Phi_i^1}{\partial t}\Big|_{t=0}\cdot\frac{\partial}{\partial r} + \frac{\partial\Phi_i^2}{\partial t}\Big|_{t=0}\cdot\frac{\partial}{\partial s} \in \Gamma(U_i, f^*\Theta_Y) \tag{3.9}$$

$(i = 1,\ 2)$. By applying (2.19) and (2.21), we have

$$\left\{\begin{array}{l} P(\tau_1) = \left(\left(\frac{\partial\Phi_1^1}{\partial t}\Big|_{t=0} + \gamma_1^2\frac{\partial\Phi_1^2}{\partial t}\Big|_{t=0}\right) \bmod (\beta_1, \alpha_1\gamma_1)\right)\cdot\nu_1,\\ P(\tau_2) = \left(\left(\gamma_2^2\frac{\partial\Phi_2^1}{\partial t}\Big|_{t=0} + \frac{\partial\Phi_2^2}{\partial t}\Big|_{t=0}\right) \bmod (\beta_2, \alpha_2\gamma_2)\right)\cdot\nu_2.\end{array}\right. \tag{3.10}$$

The sections $P(\tau_1)$ and $P(\tau_2)$ patch together to a section of $\Gamma(\Delta\cap V, g'_*\mathcal{S}_{X/Y})$, which is nothing but $P\circ\tau(\partial/\partial t)|_{\Delta\cap V}$.

On the other hand, $B\cap X_V$ has three irreducible components, which we denote by $B^{(0)}$, $B^{(1)}$ and $B^{(2)}$, satisfying the following properties.

(i) $B^{(0)}\cap U_1$ is defined by $\alpha_1 = \beta_1 = 0$ and $B^{(0)}\cap U_2$ is defined by $\alpha_2 = \beta_2 = 0$. We have $f(B^{(0)}) = \{p\}$.

(ii) $B^{(1)}\cap U_1$ is defined by $\beta_1 = \gamma_1 = 0$ and $B^{(1)}\cap U_2 = \emptyset$. We have $f(B^{(1)}) = \Delta^{(1)}$.

(iii) $B^{(2)}\cap U_1 = \emptyset$ and $B^{(2)}\cap U_2$ is defined by $\beta_2 = \gamma_2 = 0$. We have

$f(B^{(2)}) = \Delta^{(2)}$.

Let us note that any small deformation of a conic bundle is again a conic bundle (cf. [3, Proposition 4.1]), whence we may assume that, for each $t \in M$, $f_t : X_t \to Y$ is a conic bundle, after shrinking M if necessary. Since $\{\Delta_t\}_{t\in M}$ does not admit smoothing of Δ, the corresponding family $\{B_t\}_{t\in M}$ also does not admit smoothing of B (cf. Remark 2.2). Hence each component $B^{(i)}$ has a family $\{B_t^{(i)}\}_{t\in M}$ of its own displacements for $i = 0$, 1, 2.

Now we prove Theorem 1.2 (1).

Proof of Theorem 1.2 (1)**.** Let p be a point of Δ and q a point of B with $g'(q) = p$. Note that $(f^*\Theta_Y)_q \to \mathcal{S}_{X/Y,q}$ is surjective and that $(g'^*(\Theta_Y \otimes \mathcal{O}_\Delta))_q = (f^*\Theta_Y \otimes \mathcal{O}_B)_q \to \mathcal{S}_{X/Y,q}$ is also surjective. If p is a nonsingular point of Δ, then $g' : B \to \Delta$ is a local isomorphism around q (cf. [2, Proposition 1.4 (ii)]). Therefore λ is surjective around a nonsingular point p of Δ, that is to say, $\lambda_p : (\Theta_Y \otimes \mathcal{O}_\Delta)_p \to (g'_*\mathcal{S}_{X/Y})_p$ is surjective. (See also [4, Corollary 2.3].)

Thus we have only to discuss locally around a singular point of Δ. Suppose that p is a singular point of Δ. Let $\partial/\partial t \in T_o(M)$.

The components $\Delta_t^{(1)}$ and $\Delta_t^{(2)}$ of $\Delta \cap V$ intersect at one point, which we denote by p_t. Suppose that the point p_t is defined by

$$r = \eta_1(t), \quad s = \eta_2(t) \tag{3.11}$$

with $\eta_1(0) = \eta_2(0) = 0$. We also suppose that the family $\{B_t^{(0)}\}_{t\in M}$ is defined by

$$\alpha_i = u_i(\gamma_i, t), \quad \beta_i = v_i(\gamma_i, t) \tag{3.12}$$

on U_i ($i = 1$, 2) with $u_i(\gamma_i, 0) = v_i(\gamma_i, 0) = 0$. Noting that $f_t(B_t^{(0)}) = \{p_t\}$, we have

$$\Phi_i^j(u_i(\gamma_i, t), v_i(\gamma_i, t), \gamma_i, t) = \eta_j(t) \tag{3.13}$$

($i = 1$, 2; $j = 1$, 2). Putting $t = 0$ after applying $\partial/\partial t$ to (3.13), we have

$$\begin{aligned} \frac{\partial \Phi_i^j}{\partial \alpha_i}(0,0,\gamma_i,0) \cdot \left.\frac{\partial u_i}{\partial t}\right|_{t=0} &+ \frac{\partial \Phi_i^j}{\partial \beta_i}(0,0,\gamma_i,0) \cdot \left.\frac{\partial v_i}{\partial t}\right|_{t=0} \\ &+ \frac{\partial \Phi_i^j}{\partial t}(0,0,\gamma_i,0) = \left.\frac{\partial \eta_j}{\partial t}\right|_{t=0} \end{aligned} \tag{3.14}$$

for $i = 1$, 2 and $j = 1$, 2. Note that $u_i(\gamma_i, 0) = v_i(\gamma_i, 0) = 0$.

Since we have

$$\frac{\partial \Phi_1^1}{\partial \alpha_1}(0,0,\gamma_1,0) = -\gamma_1^2, \quad \frac{\partial \Phi_1^1}{\partial \beta_1}(0,0,\gamma_1,0) = 0, \tag{3.15}$$

$$\frac{\partial \Phi_2^2}{\partial \alpha_2}(0,0,\gamma_2,0) = -\gamma_2^2, \quad \frac{\partial \Phi_2^2}{\partial \beta_2}(0,0,\gamma_2,0) = 0 \tag{3.16}$$

from (3.8), we obtain the following equalities from (3.14):

$$\frac{\partial \Phi_1^1}{\partial t}(0,0,\gamma_1,0) = \left.\frac{\partial \eta_1}{\partial t}\right|_{t=0} + \gamma_1^2 \left.\frac{\partial u_1}{\partial t}\right|_{t=0}, \tag{3.17}$$

$$\frac{\partial \Phi_2^2}{\partial t}(0,0,\gamma_2,0) = \left.\frac{\partial \eta_2}{\partial t}\right|_{t=0} + \gamma_2^2 \left.\frac{\partial u_2}{\partial t}\right|_{t=0}. \tag{3.18}$$

Since η_1 and η_2 do not depend on γ_1 and γ_2, the above equalities (3.17) and (3.18) imply that $(\partial\Phi_1^1/\partial t)(0,0,\gamma_1,0)$ and $(\partial\Phi_2^2/\partial t)(0,0,\gamma_2,0)$ do not contain terms of γ_1 and γ_2 of degree one, respectively. Then, using Lemma 3.3 below, we can show that $(\partial\Phi_1^1/\partial t)|_{t=0}$, $(\partial\Phi_2^2/\partial t)|_{t=0}$, $(\partial\Phi_1^1/\partial t)|_{t=0} + \gamma_1^2 \cdot (\partial\Phi_1^2/\partial t)|_{t=0}$ and $\gamma_2^2 \cdot (\partial\Phi_2^1/\partial t)|_{t=0} + (\partial\Phi_2^2/\partial t)|_{t=0}$ do not contain terms of γ_1 and γ_2 of degree one, respectively.

Then, applying Proposition 3.1 to (3.10), we have

$$P \circ \tau(\partial/\partial t)|_{\Delta\cap V} \in \Gamma(\Delta \cap V, \mathcal{A}). \tag{3.19}$$

Thus we have $\mathrm{Im}(P\circ\tau) \subset H^0(\Delta, \mathcal{A})$. ∎

Lemma 3.3 *Let $F(\alpha,\beta,\gamma)$ be a holomorphic function with*

$$F(\alpha,\beta,\gamma) \equiv f_0 + f_1\alpha + f_2\alpha^2 + \dots \\ +g_1\gamma + g_2\gamma^2 + \dots \quad \mathrm{mod}\ (\beta, \alpha\gamma).$$

Assume that $F(0,0,\gamma) = f_0 + g_1\gamma + g_2\gamma^2 + \dots$ does not contain a term of γ of degree one (that is to say, $g_1 = 0$), then $F(\alpha,\beta,\gamma)$ does not contain a term of γ of degree one.

Proof. It is obvious. ∎

4. Proof of Theorem 1.2 (3)

In this section we shall prove Theorem 1.2 (3). Assume that the family $\{\Delta_t\}_{t\in M}$ does not admit smoothing of Δ as before. Let $\partial/\partial t$ be any element of $T_o(M)$. We prove Theorem 1.2 (3) by local discussions around $p \in \Delta$.

Case A: $p \in Sing(\Delta)$.

Suppose that the family $\{B_t^{(i)}\}_{t\in M}$ is defined by

$$\beta_i = b_i(\alpha_i, t), \quad \gamma_i = c_i(\alpha_i, t) \tag{4.1}$$

on U_i with $b_i(\alpha_i, 0) = c_i(\alpha_i, 0) = 0$ ($i = 1, 2$). Since we have $f_t(B_t^{(i)}) = \Delta_t^{(i)}$

($i = 1,\ 2$), the following equalities hold.

$$\Phi_1^1(\alpha_1, b_1(\alpha_1,t), c_1(\alpha_1,t), t) = \varepsilon_1(\Phi_1^2(\alpha_1, b_1(\alpha_1,t), c_1(\alpha_1,t), t), t), \tag{4.2}$$

$$\Phi_2^2(\alpha_2, b_2(\alpha_2,t), c_2(\alpha_2,t), t) = \varepsilon_2(\Phi_2^1(\alpha_2, b_2(\alpha_2,t), c_2(\alpha_2,t), t), t). \tag{4.3}$$

Putting $t = 0$ after applying $\partial/\partial t$ to (4.2), we have

$$\begin{aligned} &\frac{\partial \Phi_1^1}{\partial \beta_1}(\alpha_1,0,0,0) \cdot \frac{\partial b_1}{\partial t}\Big|_{t=0} + \frac{\partial \Phi_1^1}{\partial \gamma_1}(\alpha_1,0,0,0) \cdot \frac{\partial c_1}{\partial t}\Big|_{t=0} + \frac{\partial \Phi_1^1}{\partial t}(\alpha_1,0,0,0) \\ = \ &\frac{\partial \varepsilon_1}{\partial s}(\alpha_1,0) \cdot \frac{\partial \Phi_1^2(\alpha_1, b_1(\alpha_1,t), c_1(\alpha_1,t), t)}{\partial t}\Big|_{t=0} + \frac{\partial \varepsilon_1}{\partial t}(\alpha_1, 0). \end{aligned}$$

From (3.8) we have

$$\frac{\partial \Phi_1^1}{\partial \beta_1}(\alpha_1,0,0,0) = \frac{\partial \Phi_1^1}{\partial \gamma_1}(\alpha_1,0,0,0) = 0.$$

Since $\varepsilon_1(s,0) = 0$, we have $(\partial\varepsilon_1/\partial s)(\alpha_1,0) = 0$. Thus we have

$$\frac{\partial \Phi_1^1}{\partial t}(\alpha_1,0,0,0) = \frac{\partial \varepsilon_1}{\partial t}(\alpha_1,0). \tag{4.4}$$

Similarly, putting $t = 0$ after applying $\partial/\partial t$ to (4.3) and noting that

$$\frac{\partial \varepsilon_2}{\partial r}(\alpha_2,0) = \frac{\partial \Phi_2^2}{\partial \beta_2}(\alpha_2,0,0,0) = \frac{\partial \Phi_2^2}{\partial \gamma_2}(\alpha_2,0,0,0) = 0,$$

we obtain

$$\frac{\partial \Phi_2^2}{\partial t}(\alpha_2,0,0,0) = \frac{\partial \varepsilon_2}{\partial t}(\alpha_2,0). \tag{4.5}$$

By Proposition 3.1 (1), $P \circ \tau(\partial/\partial t)|_{\Delta\cap V}$ is determined by $P(\tau_1)$ and $P(\tau_2)$ of the following form:

$$\left\{ \begin{array}{l} P(\tau_1) = ((a_0 + a_1\gamma_1 + a_2\gamma_1^2 + \phi_1(\alpha_1)) \bmod (\beta_1, \alpha_1\gamma_1)) \cdot \nu_1, \\ P(\tau_2) = ((a_0\gamma_2^2 + a_1\gamma_2 + a_2 + \phi_2(\alpha_2)) \bmod (\beta_2, \alpha_2\gamma_2)) \cdot \nu_2, \end{array} \right. \tag{4.6}$$

where a_i ($i = 0,1,2$) are constants and ϕ_i ($i = 1,\ 2$) are holomorphic functions with $\phi_1(0) = \phi_2(0) = 0$. Then, by Theorem 1.2 (1) and Proposition 3.1 (2), we have $a_1 = 0$, whence we have

$$\left\{ \begin{array}{l} P(\tau_1) = ((a_0 + a_2\gamma_1^2 + \phi_1(\alpha_1)) \bmod (\beta_1, \alpha_1\gamma_1)) \cdot \nu_1, \\ P(\tau_2) = ((a_0\gamma_2^2 + a_2 + \phi_2(\alpha_2)) \bmod (\beta_2, \alpha_2\gamma_2)) \cdot \nu_2. \end{array} \right. \tag{4.7}$$

Comparing (4.7) with (3.10), we have

$$a_0 + \phi_1(\alpha_1) = \frac{\partial \Phi_1^1}{\partial t}(\alpha_1,0,0,0) = \frac{\partial \varepsilon_1}{\partial t}(\alpha_1,0),$$

$$a_2 + \phi_2(\alpha_2) = \frac{\partial \Phi_2^2}{\partial t}(\alpha_2, 0, 0, 0) = \frac{\partial \varepsilon_2}{\partial t}(\alpha_2, 0).$$

In particular, we have $a_0 = (\partial\varepsilon_1/\partial t)(0,0)$ and $a_2 = (\partial\varepsilon_2/\partial t)(0,0)$. Thus $P \circ \tau(\partial/\partial t)|_{\Delta\cap V}$ is determined by

$$(4.8) \qquad \begin{cases} P(\tau_1) = \Big(\Big(\gamma_1^2 \dfrac{\partial\varepsilon_2}{\partial t}(0,0) + \dfrac{\partial\varepsilon_1}{\partial t}(\alpha_1, 0)\Big) \bmod (\beta_1, \alpha_1\gamma_1)\Big) \cdot \nu_1, \\ P(\tau_2) = \Big(\Big(\gamma_2^2 \dfrac{\partial\varepsilon_1}{\partial t}(0,0) + \dfrac{\partial\varepsilon_2}{\partial t}(\alpha_2, 0)\Big) \bmod (\beta_2, \alpha_2\gamma_2)\Big) \cdot \nu_2. \end{cases}$$

Next we calculate $\bar\psi \circ P \circ \tau(\partial/\partial t)|_{\Delta\cap V}$. Let us put $\varphi_i(r,s) = (\partial\varepsilon_i/\partial t)|_{t=0}$ ($i = 1$, 2) and

$$(4.9) \qquad \omega = \big(\varphi_1 \bmod (rs)\big)\frac{\partial}{\partial r} + \big(\varphi_2 \bmod (rs)\big)\frac{\partial}{\partial s} \in \Gamma(\Delta\cap V, \Theta_Y \otimes \mathcal{O}_\Delta).$$

Then [4, Proposition 2.2 (1)] implies that $\lambda(\omega) \in \Gamma(\Delta \cap V, g'_*\mathcal{S}_{X/Y})$ is expressed by $\lambda_1(\omega)$ and $\lambda_2(\omega)$ in the following form:

$$(4.10) \qquad \begin{cases} \lambda_1(\omega) = ((\varphi_1(0,\alpha_1) + \gamma_1^2\varphi_2(0,0)) \bmod (\beta_1, \alpha_1\gamma_1)) \cdot \nu_1, \\ \lambda_2(\omega) = ((\gamma_2^2\varphi_1(0,0) + \varphi_2(\alpha_2,0)) \bmod (\beta_2, \alpha_2\gamma_2)) \cdot \nu_2, \end{cases}$$

which is nothing but $P \circ \tau(\partial/\partial t)|_{\Delta\cap V}$. (Compare (4.8) with (4.10). Note that the notation in [4] is different from the notation here. We have to change (s,t) in [4] into (r,s) here.)

On the other hand, [4, Proposition 3.1 (1)] implies that

(4.11)
$$\mu(\omega) = \big((s\varphi_1 + r\varphi_2) \bmod (rs)\big)\cdot\zeta = \Big(\Big(s\frac{\partial\varepsilon_1}{\partial t}\Big|_{t=0} + r\frac{\partial\varepsilon_2}{\partial t}\Big|_{t=0}\Big) \bmod (rs)\Big)\cdot\zeta,$$

which is nothing but $\rho(\partial/\partial t)|_{\Delta\cap V}$. (Compare (3.6) with (4.11).)

These arguments imply that

$$(4.12) \qquad \bar\psi \circ P \circ \tau\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V} = \rho\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V}.$$

Case B: $p \in \Delta \setminus Sing(\Delta)$.

In this case, we use the notation of Case I in §2 C. Suppose that f is determined around p' by (2.13). Suppose furthermore that $\{f_t : X_t \to Y\}_{t\in M}$ is determined by

$$(4.13) \qquad r = \Phi^1(\alpha,\beta,\gamma,t), \quad s = \Phi^2(\alpha,\beta,\gamma,t)$$

with $\Phi^1(\alpha,\beta,\gamma,0) = \beta\gamma$ and $\Phi^2(\alpha,\beta,\gamma,0) = \alpha$. We also suppose that $\{B_t\}_{t\in M}$ is determined around p' by

$$\beta = b(\alpha, t), \quad \gamma = c(\alpha, t) \tag{4.14}$$

with $b(\alpha, 0) = c(\alpha, 0) = 0$ and that $\{\Delta_t\}_{t\in M}$ is determined around p by

$$r = \varepsilon(s, t) \tag{4.15}$$

with $\varepsilon(s, 0) = 0$. Since we have $f_t(B_t) = \Delta_t$, the following equality holds:

$$\Phi^1\big(\alpha, b(\alpha, t), c(\alpha, t), t\big) = \varepsilon\big(\Phi^2(\alpha, b(\alpha, t), c(\alpha, t), t), t\big). \tag{4.16}$$

Putting $t = 0$ after applying $\partial/\partial t$ to (4.16) and noting that

$$\frac{\partial\Phi^1}{\partial\beta}(\alpha, 0, 0, 0) = \frac{\partial\Phi^1}{\partial\gamma}(\alpha, 0, 0, 0) = \frac{\partial\varepsilon}{\partial s}(\alpha, 0) = 0,$$

we have

$$\frac{\partial\Phi^1}{\partial t}(\alpha, 0, 0, 0) = \frac{\partial\varepsilon}{\partial t}(\alpha, 0). \tag{4.17}$$

Then, by (2.15) and (4.17), we obtain

$$P\circ\tau\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V} = P\Big(\frac{\partial\Phi^1}{\partial t}\Big|_{t=0}\cdot\frac{\partial}{\partial r} + \frac{\partial\Phi^2}{\partial t}\Big|_{t=0}\cdot\frac{\partial}{\partial s}\Big) = \frac{\partial\varepsilon}{\partial t}(\alpha, 0)\cdot\nu. \tag{4.18}$$

In this case, $g' : B \to \Delta$ is defined by $\alpha \mapsto s$ and it is locally isomorphic around p'. Then the homomorphism $\lambda : \Gamma(\Delta\cap V, \Theta_Y\otimes\mathcal{O}_\Delta) \to \Gamma(\Delta\cap V, g'_*\mathcal{S}_{X/Y})$ is essentially equal to P via the correspondence $\alpha \mapsto s$ and it is determined by

$$\varphi_1(s)\frac{\partial}{\partial r} + \varphi_2(s)\frac{\partial}{\partial s} \mapsto \varphi_1(s)\cdot\nu \tag{4.19}$$

with $\nu = \lambda(\partial/\partial r)$, whereas $\mu : \Gamma(\Delta\cap V, \Theta_Y\otimes\mathcal{O}_\Delta) \to \Gamma(\Delta\cap V, N_{\Delta/Y})$ is determined by

$$\varphi_1(s)\frac{\partial}{\partial r} + \varphi_2(s)\frac{\partial}{\partial s} \mapsto \varphi_1(s)\cdot\zeta \tag{4.20}$$

with $\zeta = \mu(\partial/\partial r)$, whence we have

$$\bar\psi(\nu) = \zeta. \tag{4.21}$$

Let us now put $\tilde\xi(r, s, t) = r - \varepsilon(s, t)$. Then we have

$$\rho\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V} = -\frac{\partial\tilde\xi}{\partial t}\Big|_{t=0}\cdot\zeta = \frac{\partial\varepsilon}{\partial t}(s, 0)\cdot\zeta \tag{4.22}$$

By (4.18), (4.21) and (4.22), we have

$$\rho\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V} = \bar\psi\circ P\circ\tau\Big(\frac{\partial}{\partial t}\Big)\Big|_{\Delta\cap V}. \tag{4.23}$$

Thus Theorem 1.2 (3) is proved. ∎

5. Proof of Corollary 1.3

Finally we prove Corollary 1.3.

Let $Y = \mathbb{P}^2$ and $f : X \to Y$ a conic bundle determined by $\mathcal{E}$, $\mathcal{M}$ and q. Assume that $\mathcal{E}$ is a direct sum of invertible sheaves.

By [3, Theorem 3.3], we have $f_*\Theta_{X/Y} \cong \mathcal{E} \otimes \det(\mathcal{E})^{-1} \otimes \mathcal{M}^{-1}$, which is also a direct sum of invertible sheaves on $\mathbb{P}^2$. Hence we have $H^1(X, \Theta_{X/Y}) \cong H^1(\mathbb{P}^2, f_*\Theta_{X/Y}) = 0$, since we have $R^i f_*\Theta_{X/Y} = 0$ for $i > 0$ by [3, Lemma 3.1]. Then the exact sequence (2.1) implies that $P : D_{X/Y} \to H^0(X, \mathcal{S}_{X/Y})$ is injective.

If Δ is smooth, we are already done by [3, Corollary 3.14].

Suppose that Δ has singularity. Let $\{f_t : X_t \to Y\}_{t \in M}$ be a deformation family of $f : X \to Y$. Assume that the discriminant locus of f_t coincides with Δ for each $t \in M$. Since the family does not admit smoothing of any singular point of Δ, of course, we have $\mathrm{Im}(P \circ \tau) \subset H^0(\Delta, \mathcal{A})$ and $\mathrm{Im}(\rho) \subset H^0(\Delta, \mathcal{B})$ by Theorem 1.2.

Then Corollary 1.3 is proved, since the map $\bar{\psi} \circ P|_{H^0(\Delta, \mathcal{A})}$ is injective. ∎

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