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Elementary Reducibility of Automorphisms of a Vector Group

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Abstract

1. Introduction

Throughout this paper, let k be a domain of characteristic $p \geq 0$, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k, where $n \in \mathbf{N}$. We denote by $\operatorname{Aut} \mathbf{A}^n$ the automorphism group of the affine scheme $\mathbf{A}^n := \operatorname{Spec} k[\mathbf{x}]$ over k. Note that $\mathbf{A}^n$ also has a structure of affine algebraic group scheme over k where the coproduct $\mu : k[\mathbf{x}] \to k[\mathbf{x}] \otimes_k k[\mathbf{x}]$, the coidentity $\epsilon : k[\mathbf{x}] \to k$, and the coinverse $\iota : k[\mathbf{x}] \to k[\mathbf{x}]$ are defined by

$$\mu(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon(x_i) = 0 \quad \text{and} \quad \iota(x_i) = -x_i$$

for $i = 1, \ldots, n$, respectively. This algebraic group scheme is called the n-dimensional *vector group* and denoted by $\mathbf{G}_a^n$. The set $\operatorname{Aut} \mathbf{G}_a^n$ of automorphisms of the group $\mathbf{G}_a^n$ becomes a subgroup of $\operatorname{Aut} \mathbf{A}^n$.

Recall that there exists a bijection $\operatorname{Aut} \mathbf{A}^n \ni \phi \mapsto \phi^* \in \operatorname{Aut}_k k[\mathbf{x}]$ such that $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ for each $\phi, \psi \in \operatorname{Aut} \mathbf{A}^n$. Hence, we may identify $\operatorname{Aut} \mathbf{A}^n$ with $\operatorname{Aut}_k k[\mathbf{x}]$. Take any $\phi \in \operatorname{Aut} \mathbf{A}^n$ and set $f_i = \phi^*(x_i)$ for $i = 1, \ldots, n$. Then, ϕ belongs to $\operatorname{Aut} \mathbf{G}_a^n$ if and only if the diagram

$$\begin{tikzcd} k[\mathbf{x}] \arrow[r, "\mu"] \arrow[d, "\phi^*"'] & k[\mathbf{x}] \otimes_k k[\mathbf{x}] \arrow[d, "\phi^* \otimes \phi^*"] \\ k[\mathbf{x}] \arrow[r, "\mu"'] & k[\mathbf{x}] \otimes_k k[\mathbf{x}] \end{tikzcd}$$

commutes, and hence if and only if

$$\begin{aligned} & f_i(x_1 \otimes 1 + 1 \otimes x_1, \ldots, x_n \otimes 1 + 1 \otimes x_n) \\ & \quad = f_i(x_1, \ldots, x_n) \otimes 1 + 1 \otimes f_i(x_1, \ldots, x_n) \end{aligned} \tag{1.1}$$

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holds for $i = 1, \ldots, n$. If $p = 0$, then this implies that ϕ is a *linear automorphism*, i.e.,

$$(f_1, \ldots, f_n) = (x_1, \ldots, x_n)A \quad \text{for some} \quad A \in GL(n, k).$$

Assume that $p > 0$. Then, (1.1) is equivalent to the condition that f_i is a *p-polynomial*, i.e., a linear combination of $x_j^{p^e}$ for $j = 1, \ldots, n$ and $e \in \mathbf{Z}_{\geq 0}$ over k (cf. [2, Sect. 20.3, Lemma A]). For example, Nagata [7] considered $\phi_\lambda \in \operatorname{Aut} \mathbf{A}^2$ defined by

$$\phi_\lambda^*(x_1) = x_1 - 2(\lambda x_1 + x_2^2)x_2 - \lambda(\lambda x_1 + x_2^2)^2, \quad \phi_\lambda^*(x_2) = x_2 + \lambda(\lambda x_1 + x_2^2)$$

for each $\lambda \in k$. If $p = 2$, then ϕ_λ is an element of $\operatorname{Aut} \mathbf{G}_a^2$.

In [10], Tanaka-Kaneta studied the structure of the group $\operatorname{Aut} \mathbf{G}_a^n$ when $p > 0$ and k is an algebraically closed field, and determined a generating set for $\operatorname{Aut} \mathbf{G}_a^n$. The purpose of this paper is to study $\operatorname{Aut} \mathbf{G}_a^n$ from a different point of view.

We say that $\phi \in \operatorname{Aut} \mathbf{A}^n$ is *elementary* if there exists $l \in \{1, \ldots, n\}$ such that $\phi^*(x_i) = x_i$ for all $i \neq l$. If this is the case, then we have

$$\phi^*(x_l) = ax_l + g$$

for some $a \in k^*$ and $g \in k[x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n]$, since k is a domain. The subgroup $\mathrm{T}(n, k)$ generated by all the linear automorphisms of $\mathbf{A}^n$ and all the elementary automorphisms of $\mathbf{A}^n$ is called the *tame subgroup* of $\operatorname{Aut} \mathbf{A}^n$. When k is a field, we know by linear algebra that every linear automorphism of $\mathbf{A}^n$ is obtained by the composition of elementary, linear automorphisms of $\mathbf{A}^n$. Consequently, $\mathrm{T}(n, k)$ is generated only by the elementary automorphisms of $\mathbf{A}^n$.

The following problem is one of the fundamental problems in Affine Algebraic Geometry.

Problem 1.1. *Does it hold that* $\operatorname{Aut} \mathbf{A}^n = \mathrm{T}(n, k)$*?*

It is well known that $\operatorname{Aut} \mathbf{A}^2 = \mathrm{T}(2, k)$ if k is a field by Jung [3] and van der Kulk [4]. On the other hand, we have $\operatorname{Aut} \mathbf{A}^2 \neq \mathrm{T}(2, k)$ whenever the domain k is not a field due to Nagata. Actually, he showed that ϕ_a above does not belong to $\mathrm{T}(2, k)$ if a does not belong to $k^* \cup \{0\}$ (cf. Exercise 1.6 of [7, Part 2]). Note that, if $k = k'[x_3]$ is the polynomial ring in one variable over a field k', then we may regard ϕ_{x_3} as an automorphism of the three-dimensional affine space over k'. Nagata conjectured that this automorphism does not belong to $\mathrm{T}(3, k')$. Recently, Shestakov-Umirbaev [8], [9] settled this conjecture in the affirmative in the case where k' is of characteristic zero. This easily implies that $\operatorname{Aut} \mathbf{A}^3 \neq \mathrm{T}(3, k)$ if $p = 0$. Problem 1.1 is not solved in the other cases.

When $p = 0$, every element of $\operatorname{Aut} \mathbf{G}_a^n$ is linear, and hence belongs to $\mathrm{T}(n, k)$ by definition. In the case where $p > 0$ and k is an algebraically closed field, Tanaka-Kaneta [10, Theorem 1] implies that $\operatorname{Aut} \mathbf{G}_a^n$ is generated by elementary automorphisms belonging to $\mathrm{Aut}\,\mathbf{G}_a^n$, although they defined "elementary automorphism" in a different way. In this paper, we study $\mathrm{Aut}\,\mathbf{G}_a^n$ using a technique of "reductions" of polynomial automorphisms, and give a simpler description of the structure of $\mathrm{Aut}\,\mathbf{G}_a^n$. As a consequence of our main result (Theorem 2.2), it follows that $\mathrm{Aut}\,\mathbf{G}_a^n$ is generated by elementary automorphisms belonging to $\mathrm{Aut}\,\mathbf{G}_a^n$ if $p > 0$ and k is a field (Corollary 2.3). Thus, $\mathrm{Aut}\,\mathbf{G}_a^n$ is contained in $\mathrm{T}(n,k)$ whenever k is a field. On the other hand, ϕ_a belongs to $\mathrm{Aut}\,\mathbf{G}_a^2$ if $p = 2$, while ϕ_a does not belong to $\mathrm{T}(2,k)$ for each $a \in k$ not belonging to $k^* \cup \{0\}$ as mentioned. Hence, $\mathrm{Aut}\,\mathbf{G}_a^2$ is not contained in $\mathrm{T}(2,k)$ if $p = 2$ and k is not a field. In Section 4, we show that the same holds for any $p > 0$ (Theorem 4.1).

We mention that analysis of reductions of polynomial automorphisms is of great importance in the study of Problem 1.1. Actually, Shestakov-Umirbaev [9] constructed a powerful theory of reductions of polynomial automorphisms, from which they derived that $\mathrm{Aut}\,\mathbf{A}^3 \neq \mathrm{T}(3,k)$ (see also [5]). The results of this paper may have some significance in constructing a similar theory in the case of $p > 0$.

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2. Main result

Let Γ be a *totally ordered additive group*, i.e., an additive group equipped with a total ordering such that $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for each $\alpha, \beta, \gamma \in \Gamma$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be an n-tuple of elements of Γ. We define the $\mathbf{w}$-*weighted* Γ-*grading*

$$k[\mathbf{x}] = \bigoplus_{\alpha \in \Gamma} k[\mathbf{x}]_\alpha$$

by setting $k[\mathbf{x}]_\alpha$ to be the k-submodule of $k[\mathbf{x}]$ generated by the monomials $x_1^{i_1} \cdots x_n^{i_n}$ for $i_1, \ldots, i_n \in \mathbf{Z}_{\geq 0}$ with $i_1 w_1 + \cdots + i_n w_n = \alpha$ for each $\alpha \in \Gamma$. Here, $\mathbf{Z}_{\geq 0}$ denotes the set of nonnegative integers. Write $f \in k[\mathbf{x}] \setminus \{0\}$ as $f = \sum_{\gamma \in \Gamma} f_\gamma$, where f_γ is an element of $k[\mathbf{x}]_\gamma$ for each $\gamma \in \Gamma$. Then, we define the $\mathbf{w}$-*degree* of f by

$$\deg_{\mathbf{w}} f = \max\{\gamma \in \Gamma \mid f_\gamma \neq 0\}.$$

We define $f^{\mathbf{w}} = f_\delta$, where $\delta := \deg_{\mathbf{w}} f$. If $\Gamma = \mathbf{Z}$ and $w_i = 1$ for $i = 1, \ldots, n$, then the $\mathbf{w}$-degree of f is the same as the total degree $\deg f$ of f.

For each $\phi \in \mathrm{Aut}\,\mathbf{A}^n$, we define the $\mathbf{w}$-degree of ϕ by

$$\deg_{\mathbf{w}} \phi = \sum_{i=1}^{n} \deg_{\mathbf{w}} \phi^*(x_i).$$

Then, the following lemma holds (cf. [6, Lemma 1.1.1]).

Lemma 2.1. *For any* $\mathbf{w} \in \Gamma^n$ *and* $\phi \in \operatorname{Aut} \mathbf{A}^n$*, we have*

$$\deg_{\mathbf{w}} \phi \geq |\mathbf{w}| := w_1 + \cdots + w_n.$$

Furthermore, it holds that $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$ *if and only if* $\phi^*(x_1)^{\mathbf{w}}, \ldots, \phi^*(x_n)^{\mathbf{w}}$ *are algebraically independent over* k*.*

We say that $\phi \in \operatorname{Aut} \mathbf{A}^n$ admits an *elementary reduction* for the weight $\mathbf{w}$ if there exists an elementary automorphism of $\mathbf{A}^n$ such that

$$\deg_{\mathbf{w}} \epsilon \circ \phi < \deg_{\mathbf{w}} \phi.$$

If $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$, then ϕ admits no elementary reduction for the weight $\mathbf{w}$ by Lemma 2.1.

The following theorem is the main result of this paper.

Theorem 2.2. *Assume that* $n \geq 1$ *and* k *is a field. Let* Γ *be any totally ordered additive group, and let* $\phi \in \operatorname{Aut} \mathbf{G}_a^n$ *and* $\mathbf{w} \in (\Gamma \setminus \{0\})^n$ *be such that* $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$*. Then, there exists* $\epsilon \in \operatorname{Aut} \mathbf{G}_a^n$ *such that* ϵ *is elementary and* $\deg_{\mathbf{w}} \epsilon \circ \phi < \deg_{\mathbf{w}} \phi$*. Hence,* ϕ *admits an elementary reduction for the weight* $\mathbf{w}$*.*

Note that $\phi \in \operatorname{Aut} \mathbf{A}^1$ is defined by $\phi^*(x_1) = ax_1 + b$ for some $a \in k^*$ and $b \in k$. Then, ϕ belongs to $\operatorname{Aut} \mathbf{G}_a^1$ if and only if $b = 0$. If this is the case, we have $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$. Hence, Theorem 2.2 is clear if $n = 1$.

As a consequence of Theorem 2.2, we get the following corollary.

Corollary 2.3. *If* k *is a field, then* $\operatorname{Aut} \mathbf{G}_a^n$ *is generated by elementary automorphisms belonging to* $\operatorname{Aut} \mathbf{G}_a^n$ *for any* $n \in \mathbf{N}$*.*

Proof. Since k is a field, every linear automorphism of $\mathbf{A}^n$ is a composite of elementary linear automorphisms, and hence is a composite of elementary automorphisms belonging to $\operatorname{Aut} \mathbf{G}_a^n$. Thus, the assertion is true if $p = 0$.

Assume that $p > 0$. Let G be the subgroup of $\operatorname{Aut} \mathbf{A}^n$ generated by elementary automorphisms belonging to $\operatorname{Aut} \mathbf{G}_a^n$, and let $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, \ldots, 1)$. We prove that ϕ belongs to G for each $\phi \in \operatorname{Aut} \mathbf{G}_a^n$ by induction on $\deg_{\mathbf{w}} \phi$. By Lemma 2.1, we have $\deg_{\mathbf{w}} \phi \geq |\mathbf{w}| = n$. If $\deg_{\mathbf{w}} \phi = n$, then ϕ is linear. Hence, ϕ belongs to G by the discussion above. Assume that $\deg_{\mathbf{w}} \phi > n$. By Theorem 2.2, there exists $\epsilon \in \operatorname{Aut} \mathbf{G}_a^n$ which is elementary and satisfies $\deg_{\mathbf{w}} \epsilon \circ \phi < \deg_{\mathbf{w}} \phi$. Then, $\epsilon \circ \phi$ belongs to G by induction assumption. Thus, ϕ belongs to G. Therefore, we get $\operatorname{Aut} \mathbf{G}_a^n = G$. □

The following proposition is a key to proving Theorem 2.2.

Proposition 2.4. *Assume that* $p > 0$ *and* k *is a field. Let* $f_1, \ldots, f_r \in k[\mathbf{x}] \setminus \{0\}$ *be* p*-polynomials with* $r \geq 2$*, and* Γ *any totally ordered additive group. If* $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ *are algebraically dependent over* k *for* $\mathbf{w} \in (\Gamma \setminus \{0\})^n$*, then there* *exist* $l \in \{1, \ldots, r\}$, *and* $a_i \in k$ *and* $e_i \in \mathbf{Z}_{\geq 0}$ *for* $i = 1, \ldots, r$ *with* $i \neq l$ *such that*

$$\deg_{\mathbf{w}} \left(f_l + \sum_{i \neq l} a_i f_i^{p^{e_i}} \right) < \deg_{\mathbf{w}} f_l.$$

We prove Proposition 2.4 in the next section. In the rest of this section, we show Theorem 2.2 by assuming this proposition. By the remark after Theorem 2.2, we may assume that $n \geq 2$. Let $\phi \in \operatorname{Aut} \mathbf{G}_a^n$ and $\mathbf{w} \in (\Gamma \setminus \{0\})^n$ be such that $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$. Then, $\phi^*(x_1)^{\mathbf{w}}, \ldots, \phi^*(x_n)^{\mathbf{w}}$ are algebraically dependent over k by Lemma 2.1. First, assume that $p > 0$. Then, $\phi^*(x_1), \ldots, \phi^*(x_n)$ are p-polynomials. Thanks to Proposition 2.4, there exist $l \in \{1, \ldots, n\}$, and $a_i \in k$ and $e_i \in \mathbf{Z}_{\geq 0}$ for $i = 1, \ldots, n$ with $i \neq l$ such that the $\mathbf{w}$-degree of

$$f := \phi^*(x_l) + \sum_{i \neq l} a_i \phi^*(x_i)^{p^{e_i}}$$

is less than $\deg_{\mathbf{w}} \phi^*(x_l)$. Define $\epsilon \in \operatorname{Aut} \mathbf{G}_a^n$ by

$$\epsilon^*(x_l) = x_l + \sum_{i \neq l} a_i x_i^{p^{e_i}}$$

and $\epsilon^*(x_i) = x_i$ for $i \neq l$. Then, ϵ is elementary and

$$\begin{aligned} \deg_{\mathbf{w}} \epsilon \circ \phi &= \sum_{i=1}^{n} \deg_{\mathbf{w}} (\phi^* \circ \epsilon^*)(x_i) = \deg_{\mathbf{w}} f + \sum_{i \neq l} \deg_{\mathbf{w}} \phi^*(x_i) \\ &< \deg_{\mathbf{w}} \phi^*(x_l) + \sum_{i \neq l} \deg_{\mathbf{w}} \phi^*(x_i) = \sum_{i=1}^{n} \deg_{\mathbf{w}} \phi^*(x_i) = \deg_{\mathbf{w}} \phi. \end{aligned}$$

Thus, Theorem 2.2 holds when $p > 0$.

Next, assume that $p = 0$. Then, ϕ is a linear automorphism. Hence, $\phi^*(x_1)^{\mathbf{w}}, \ldots, \phi^*(x_n)^{\mathbf{w}}$ are nonzero linear combinations of $x_1, \ldots, x_n$ over k. Since $\phi^*(x_1)^{\mathbf{w}}, \ldots, \phi^*(x_n)^{\mathbf{w}}$ are algebraically dependent over k, it follows that $\phi^*(x_1)^{\mathbf{w}}, \ldots, \phi^*(x_n)^{\mathbf{w}}$ are linearly dependent over k (cf. Lemma 3.2). Thus, there exist $l \in \{1, \ldots, n\}$, and $a_i \in k$ for $i = 1, \ldots, n$ with $i \neq l$ such that

$$\phi^*(x_l)^{\mathbf{w}} + \sum_{i \neq l} a_i \phi^*(x_i)^{\mathbf{w}} = 0.$$

Then, the $\mathbf{w}$-degree of

$$f := \phi^*(x_l) + \sum_{i \neq l} a_i \phi^*(x_i)$$

is less than $\deg_{\mathbf{w}} \phi^*(x_l)$. Define $\epsilon \in \operatorname{Aut} \mathbf{G}_a^n$ by

$$\epsilon^*(x_l) = x_l + \sum_{i \neq l} a_i x_i$$

and $\epsilon^*(x_i) = x_i$ for $i \neq l$. Then, ϵ is elementary, and $\deg_{\mathbf{w}} \epsilon \circ \phi < \deg_{\mathbf{w}} \phi$ as above. Hence, Theorem 2.2 holds when $p = 0$. This proves Theorem 2.2 by assuming Proposition 2.4.

3. Proof of Proposition 2.4

The goal of this section is to prove Proposition 2.4. Throughout, assume that $p > 0$ and k is a field, and let Γ be a totally ordered additive group, and $\mathbf{w} = (w_1, \ldots, w_n)$ an element of $(\Gamma \setminus \{0\})^n$. We denote by $k[\mathbf{x}]'$ the set of p-polynomials of $k[\mathbf{x}]$. Then, for each $\alpha \in \Gamma$, the k-vector space $k[\mathbf{x}]' \cap k[\mathbf{x}]_\alpha$ is generated by $x_i^{p^e}$ for $e \in \mathbf{Z}_{\geq 0}$ and $i = 1, \ldots, n$ such that $p^e w_i = \alpha$. Since $w_i \neq 0$ by assumption, $p^e w_i = \alpha$ holds for at most one $e \in \mathbf{Z}_{\geq 0}$ for each i. Hence, $x_i^{p^e}$ belongs to $k[\mathbf{x}]_\alpha$ for at most one $e \in \mathbf{Z}_{\geq 0}$ for each i. Therefore, the monomials contained in $k[\mathbf{x}]' \cap k[\mathbf{x}]_\alpha$ are algebraically independent over k.

Lemma 3.1. *For any $\alpha \in \Gamma$ and $f_1, \ldots, f_r \in k[\mathbf{x}]' \cap k[\mathbf{x}]_\alpha$, it holds that $f_1, \ldots, f_r$ are linearly independent over k if and only if $f_1, \ldots, f_r$ are algebraically independent over k.*

The "if" part of Lemma 3.1 is obvious. For the "only if" part, it suffices to prove the following lemma by the discussion above.

Lemma 3.2. *Let $f_1, \ldots, f_r, g_1, \ldots, g_s \in k[\mathbf{x}]$ be such that $f_1, \ldots, f_r$ are algebraically independent over k, and $g_1, \ldots, g_s$ are linear combinations of $f_1, \ldots, f_r$ over k. If $g_1, \ldots, g_s$ are linearly independent over k, then $g_1, \ldots, g_s$ are algebraically independent over k.*

Proof. Let V be the k-vector space generated by $f_1, \ldots, f_r$. Then, $g_1, \ldots, g_s$ belong to V. Since $g_1, \ldots, g_s$ are linearly independent over k, there exist $g_{s+1}, \ldots, g_r \in V$ such that $g_1, \ldots, g_r$ form a basis of V. Then, we have $k(g_1, \ldots, g_r) = k(f_1, \ldots, f_r)$. Since $f_1, \ldots, f_r$ are algebraically independent over k, this implies that $g_1, \ldots, g_r$ are algebraically independent over k. Therefore, $g_1, \ldots, g_s$ are algebraically independent over k. □

For each $f \in k[\mathbf{x}]' \setminus \{0\}$, we define $S(f)$ to be the set of $i \in \{1, \ldots, n\}$ such that $x_i^{p^e}$ appears in $f^{\mathbf{w}}$ with nonzero coefficient for some $e \in \mathbf{Z}_{\geq 0}$. Then, $f^{\mathbf{w}}$ belongs to $k[\{x_i \mid i \in S(f)\}]$.

Lemma 3.3. *Let $f, g \in k[\mathbf{x}]' \setminus \{0\}$ be such that $S(f) \cap S(g) \neq \emptyset$. Then, we have* $\deg_{\mathbf{w}} f = p^e \deg_{\mathbf{w}} g$ *for some $e \in \mathbf{Z}$.*

Proof. Take any $i \in S(f) \cap S(g)$. Then, $x_i^{p^a}$ and $x_i^{p^b}$ appear in $f^{\mathbf{w}}$ and

$g^{\mathbf{w}}$ for some $a, b \in \mathbf{Z}_{\geq 0}$, respectively. This implies that $\deg_{\mathbf{w}} f = p^a w_i$ and $\deg_{\mathbf{w}} g = p^b w_i$. Therefore, we have $\deg_{\mathbf{w}} f = p^{a-b} \deg_{\mathbf{w}} g$. $\square$

Now, let us prove Proposition 2.4. Without loss of generality, we may assume that $\deg_{\mathbf{w}} f_1 \leq \cdots \leq \deg_{\mathbf{w}} f_r$. Since $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ are algebraically dependent over k by assumption, we can find the minimal number l such that $f_1^{\mathbf{w}}, \ldots, f_l^{\mathbf{w}}$ are algebraically dependent over k. Then, we have $l \geq 2$. Actually, since f_1 is a nonzero p-polynomial, $f_1^{\mathbf{w}}$ does not belong to k, and hence is algebraically independent over k. Put $\alpha = \deg_{\mathbf{w}} f_l$, and define I to be the set of $i \in \{1, \ldots, l\}$ such that $p^{e_i} \deg_{\mathbf{w}} f_i = \alpha$ for some $e_i \in \mathbf{Z}_{\geq 0}$. Then, l belongs to I, since $p^{e_l} \deg_{\mathbf{w}} f_l = \alpha$ holds for $e_l = 0$. Hence, we have

$$J := \{1, \ldots, l\} \setminus I = \{1, \ldots, l-1\} \setminus I. \tag{3.1}$$

We show that

$$S := \bigcup_{i \in I} S(f_i) \quad \text{and} \quad T := \bigcup_{j \in J} S(f_j)$$

are disjoint by contradiction. Suppose that $S \cap T \neq \emptyset$. Then, we have $S(f_i) \cap S(f_j) \neq \emptyset$ for some $i \in I$ and $j \in J$. By Lemma 3.3, it follows that $\deg_{\mathbf{w}} f_i = p^e \deg_{\mathbf{w}} f_j$ for some $e \in \mathbf{Z}$. Since i is an element of I, we have $p^{e_i} \deg_{\mathbf{w}} f_i = \alpha$. Hence, we get $p^{e_i + e} \deg_{\mathbf{w}} f_j = \alpha$. Since $j \leq l$, we have $\deg_{\mathbf{w}} f_j \leq \deg_{\mathbf{w}} f_l = \alpha$. This implies that $e_i + e \geq 0$. Thus, j belongs to I, a contradiction. Therefore, we have $S \cap T = \emptyset$. By the definition of S and T, we see that $f_i^{\mathbf{w}}$ belongs to $k[\{x_i \mid i \in S\}]$ for each $i \in I$, and $f_j^{\mathbf{w}}$ belongs to $k[\{x_j \mid j \in T\}]$ for each $j \in J$. Since $S \cap T = \emptyset$ and $f_1^{\mathbf{w}}, \ldots, f_l^{\mathbf{w}}$ are algebraically dependent over k, it follows that $f_i^{\mathbf{w}}$'s for $i \in I$, or $f_j^{\mathbf{w}}$'s for $j \in J$ must be algebraically dependent over k. By the minimality of l, the latter case does not occur due to (3.1). Hence, $f_i^{\mathbf{w}}$'s for $i \in I$ are algebraically dependent over k. Then, $(f_i^{\mathbf{w}})^{p^{e_i}}$'s for $i \in I$ are algebraically dependent over k. Since $(f_i^{\mathbf{w}})^{p^{e_i}}$ belongs to $k[\mathbf{x}]' \cap k[\mathbf{x}]_\alpha$ for each $i \in I$, we know by Lemma 3.1 that $(f_i^{\mathbf{w}})^{p^{e_i}}$'s for $i \in I$ are linearly dependent over k. Hence, there exists $(a_i)_{i \in I} \in k^I \setminus \{0\}$ such that $\sum_{i \in I} a_i (f_i^{\mathbf{w}})^{p^{e_i}} = 0$. On the other hand, $f_i^{\mathbf{w}}$'s for $i \in I' := I \setminus \{l\}$ are algebraically independent over k by the minimality of l. Hence, $(f_i^{\mathbf{w}})^{p^{e_i}}$'s for $i \in I'$ are algebraically independent over k, and so linearly independent over k. Thus, we may assume that $a_l = 1$. Since $e_l = 0$, it follows that

$$f_l^{\mathbf{w}} + \sum_{i \in I'} a_i (f_i^{\mathbf{w}})^{p^{e_i}} = 0.$$

This gives that

$$f_l + \sum_{i \in I'} a_i f_i^{p^{e_i}} = f_l + \sum_{i \in I'} a_i f_i^{p^{e_i}} - \left(f_l^{\mathbf{w}} + \sum_{i \in I'} a_i (f_i^{\mathbf{w}})^{p^{e_i}} \right)$$

$$= (f_l - f_l^{\mathbf{w}}) + \sum_{i \in I'} a_i \Big(f_i^{p^{e_i}} - (f_i^{p^{e_i}})^{\mathbf{w}} \Big).$$

Since $\deg_{\mathbf{w}} f_i^{p^{e_i}} = \deg_{\mathbf{w}} f_l = \alpha$ for each $i \in I$, the $\mathbf{w}$-degree of the right-hand side of this equality is less than $\deg_{\mathbf{w}} f_l$. This completes the proof of Proposition 2.4.

4. The case where k is not a field

In this section, we show that $\operatorname{Aut} \mathbf{G}_a^2$ is not contained in $\mathrm{T}(2, k)$ if $p > 0$ and k is not a field. Take any $\lambda \in k \setminus (k^* \cup \{0\})$. Then, $\phi \in \operatorname{Aut} \mathbf{G}_a^2$ is defined by

$$\phi^*(x_1) = x_1 - \lambda^{p-1}(\lambda x_1 + x_2^p)^p, \quad \phi^*(x_2) = x_2 + \lambda(\lambda x_1 + x_2^p).$$

Actually, noting $\phi^*(\lambda x_1 + x_2^p) = \lambda x_1 + x_2^p$, we can check that ϕ^{-1} is defined by

$$x_1 \mapsto x_1 + \lambda^{p-1}(\lambda x_1 + x_2^p)^p, \quad x_2 \mapsto x_2 - \lambda(\lambda x_1 + x_2^p).$$

If $p = 2$, then ϕ is the same as ϕ_λ defined in Section 1, and hence does not belong to $\mathrm{T}(2, k)$ as mentioned. We show that ϕ does not belong to $\mathrm{T}(2, k)$ in the general case. Let $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, 1)$. Then, we have $\deg_{\mathbf{w}} \phi^*(x_1) = p \deg_{\mathbf{w}} \phi^*(x_2)$. Now, suppose that ϕ belongs to $\mathrm{T}(2, k)$. Then, we have $\phi^*(x_1)^{\mathbf{w}} = \alpha(\phi^*(x_2)^{\mathbf{w}})^p$ for some $\alpha \in k$ (cf. [1, Proposition 1]). Since $\phi^*(x_1)^{\mathbf{w}} = -\lambda^{p-1} x_2^{p^2}$ and $\phi^*(x_2)^{\mathbf{w}} = \lambda x_2^p$, it follows that $-\lambda^{p-1} = \alpha \lambda^p$, and hence $\lambda(-\alpha) = 1$. Thus, λ belongs to k^*, a contradiction. Therefore, ϕ does not belong to $\mathrm{T}(2, k)$.

By the discussion above, we get the following theorem.

Theorem 4.1. *If $p > 0$ and k is not a field, then $\operatorname{Aut} \mathbf{G}_a^2$ is not contained in $\mathrm{T}(2, k)$.*

Because of this theorem, it seems better to define the "tame subgroup" of $\operatorname{Aut} \mathbf{A}^n$ to be the subgroup generated by all the elementary automorphisms of $\mathbf{A}^n$ and the automorphisms of $\mathbf{G}_a^n$. Let us denote this subgroup by $\mathrm{T}'(n, k)$. Then, we have $\mathrm{T}'(n, k) = \mathrm{T}(n, k)$ if $p = 0$ or k is a field by Corollary 2.3, while $\mathrm{T}(2, k)$ is a proper subgroup of $\mathrm{T}'(2, k)$ if $p > 0$ and k is not a field. In the case where $n \geq 3$, $p > 0$ and k is not a field, we do not know whether $\mathrm{T}'(n, k) = \mathrm{T}(n, k)$. We also do not know whether $\operatorname{Aut} \mathbf{A}^n = \mathrm{T}'(n, k)$ if $n \geq 2$ and $p > 0$.

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