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Extremal hyperelliptic fibrations on rational surfaces

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Abstract

The Picard number of a rational surface equipped with a relatively minimal fibration is bounded in terms of the genus g of a general fibre. When the Picard number attains the maximum for $g \geq 2$, we give necessary and sufficient conditions for the Mordell-Weil group of such a fibred surface to be trivial.

1. Introduction

The theory of the Mordell-Weil lattices are sufficiently developed by Oguiso and Shioda in [7] for minimal elliptic rational surfaces. In their work, the even unimodular root lattice E_8 of rank eight played very important role as the predominant frame. For example, it was shown that the Mordell-Weil group is trivial if and only if there exists a singular fibre of type II* in the sense of Kodaira [11] whose dual graph contains E_8 as a subgraph. The lattice E_8 also appears in another application by Shioda [17] to describe a hierarchy of deformations of rational double points.

Let X be a smooth projective rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre is a projective curve of genus $g \geq 2$. We know the Picard number $\rho(X)$ is less than or equal to $4g + 6$ (cf. [15, Theorem 2.8]), and consider the case $\rho(X) = 4g + 6$. Then the maximal Mordell-Weil lattice is isomorphic to the unimodular lattice called D^+_{4g+4} in [2, §7, Ch. 4] of rank $4g + 4$ (cf. [15, Theorem C] and [6, Theorem 2.4]). Furthermore, Saito [14, Remark 4.2] gives an example of $f : X \to \mathbb{P}^1$ whose Mordell-Weil group is trivial and which has an extension of a singular fibre of type II*. Since $D^+_8 = E_8$, we expect an application similar to the elliptic case.

The goal of this paper is to prove Theorem 3.1, which gives necessary and sufficient conditions for the Mordell-Weil group of f to be trivial by singular fibres of f, by defining equations, and so on. For example, the defining equations in (5b) of Theorem 3.1 are of branch divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ of the double cover induced by the relative canonical map when $g \geq 2$. This case is also characterized by the dual graph of a reducible fibre in (2a) of Theorem 3.1, which is the same type as

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in [14, Remark 4.2]. It is a generalization of one of the extremal elliptic cases [12] (see also [13]). The condition (3) of Theorem 3.1 seems like an interesting property as follows. In the case where $g = 2$ and $\rho(X) = 14$, from Theorem 2.2 and (3) of Theorem 3.1, we have that there exist no birational morphisms $X \to \mathbb{P}^2$ if and only if the Mordell-Weil group of f is trivial. Here we note that for any fibred rational surface of genus two whose Picard number is less than fourteen, there exists a birational morphism from the surface to $\mathbb{P}^2$ (cf. [1, § 10.5], [9] and [3]).

2. Preliminaries

We briefly review basic notation and results on fibred rational surfaces and Mordell-Weil lattices. Here, a fibred rational surface means a smooth projective rational surface $X/\mathbb{C}$ together with a relatively minimal fibration $f : X \to \mathbb{P}^1$ whose general fibre F is a smooth projective curve of genus $g \geq 1$. In particular, any fibre of f is connected and contains no (-1)-curves as components. Since X is rational, the first Betti number of X equals zero. The second Betti number of X is equal to the Picard number $\rho(X)$ since the geometric genus of X is zero. Hence, we see that $\rho(X) = 10 - K_X^2 = 4g+6-(K_X+F)^2$ from Noether's formula. The adjoint divisor $(K_X + F)$ is nef when $g \geq 2$ (cf. [10, Lemma 1.1]). Thus we have that $\rho(X) \leq 4g + 6$.

Via f, we can regard X as a smooth projective curve of genus g defined over the rational function field $\mathbb{K} = f^*\mathbb{C}(\mathbb{P}^1)$. We assume that it has a $\mathbb{K}$-rational point O. Let $\mathcal{J}_\mathcal{F}/\mathbb{K}$ be the Jacobian variety of the generic fibre $\mathcal{F}/\mathbb{K}$ of f. The Mordell-Weil group of f is the group of $\mathbb{K}$-rational points $\mathcal{J}_\mathcal{F}(\mathbb{K})$. It is a finitely generated abelian group, since $X/\mathbb{C}$ is a rational surface. The rank $\mathrm{rk}\mathcal{J}_\mathcal{F}(\mathbb{K})$ of the group is called the *Mordell-Weil rank.* From [16] and [18], it is given by

$$\mathrm{rk}\mathcal{J}_\mathcal{F}(\mathbb{K}) = \rho(X) - 2 - \sum_{t\in\mathbb{P}^1}(v_t - 1), \tag{2.1}$$

where v_t denotes the number of irreducible components of the fibre $f^{-1}(t)$. There is a natural one-to-one correspondence between the set of $\mathbb{K}$-rational points $\mathcal{F}(\mathbb{K})$ and the set of sections of f. For $P \in \mathcal{F}(\mathbb{K})$, we denote by (P) the section corresponding to P which is regarded as a horizontal curve on X. In particular, (O) corresponding to the origin O of $\mathcal{J}_\mathcal{F}(\mathbb{K})$ is called the *zero section.* Shioda's main idea in [16] and [18] is to view the free part of $\mathcal{J}_\mathcal{F}(\mathbb{K})$ as a Euclidean lattice with respect to a natural pairing induced by the intersection form on $H^2(X)$. The lattice is called the *Mordell-Weil lattice* of f and is denoted by $\mathrm{MWL}(f)$. In fact, by describing the Néron-Severi group $\mathrm{NS}(X)$, we can explicitly determine the structure of $\mathrm{MWL}(f)$ as follows: Let T be the subgroup of $\mathrm{NS}(X)$ generated by (O) and the irreducible components of the fibres of f. When we equip $\mathrm{NS}(X)$ and T with the bilinear form which is (-1) times of the intersection form, we call them the Néron-Severi lattice $\mathrm{NS}(X)^-$ and the trivial lattice T^- respectively. Since X is a rational surface, $\mathrm{NS}(X)^-$ is a unimodular lattice, that is, the absolute value of the determinant of the Gram matrix equals one. Then the following holds.

THEOREM 2.1 ([16], [18]). *Keep the notation and assumptions as above. Put* $\hat{T} = (T \otimes \mathbb{Q}) \cap \mathrm{NS}(X)$. *Then*

$$\mathcal{J}_{\mathcal{F}}(\mathbb{K}) \simeq \mathrm{NS}(X)/T, \quad \mathcal{J}_{\mathcal{F}}(\mathbb{K})_{\mathrm{tor}} \simeq \hat{T}/T.$$

Let L be the orthogonal complement $(T^-)^{\perp} \subset \mathrm{NS}(X)^-$. *Then the dual lattice*

$$L^* = \{\mathfrak{x} \in L \otimes \mathbb{Q} \mid \langle \mathfrak{x}, \mathfrak{y} \rangle_{L\otimes\mathbb{Q}} \in \mathbb{Z}, \quad {}^{\forall}\mathfrak{y} \in L\}$$

is isomorphic to $\mathrm{MWL}(f)$.

Now, for a non-negative integer d, we put

$$\Sigma_d = \{((X_0 : X_1 : X_2), (Y_0 : Y_1)) | X_1 Y_1^d = X_2 Y_0^d\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

and call it Hirzebruch surface of degree d. The restriction of the second projection to Σ_d gives a structure of $\mathbb{P}^1$-bundle. We also remark that $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Conversely, any $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ is isomorphic to Σ_d for some d. We often consider on the Zariski open subset defined by $X_0 Y_0 \neq 0$ and take $(x, y) = (X_1/X_0, Y_1/Y_0)$ as an affine coordinate. Let $\Delta_{[d]}$ be a minimal section of Σ_d defined by $x = 0$ and $\Gamma_{[d]}$ the fibre defined by $y = 0$. Then we have that $\Delta_{[d]}^2 = -d$, $\Gamma_{[d]}^2 = 0$ and $\Delta_{[d]}.\Gamma_{[d]} = 1$.

To clarify the structure of the Mordell-Weil lattice, we choose a ruling on X and its relatively minimal model Σ_d carefully so that we get a natural $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ which gives us a simple presentation of F. This is done by choosing a birational morphism $\mu : X \to \Sigma_d$ which contracts step by step a (-1)-curve whose intersection number with F is the smallest among all (-1)-curves. When $d = 0$, we may assume that $\Gamma_{[0]}.\mu_* F < \Delta_{[0]}.\mu_* F$ without loss of generality. We call a (-1)-curve E on X a (-1)-*section* of f if $E.F = 1$. When $g = 1$, f has a (-1)-section if and only if f has no multiple fibres (e.g., [12] and [13]).

THEOREM 2.2 (cf. [15, Theorem 4.1] and [6, Theorem 2.4]). *Let X be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus $g \geq 1$. Assume that $\rho(X) = 4g + 6$ and f has no multiple fibres when $g = 1$. Then there exists a birational morphism $\mu : X \to \Sigma_d$ with $d \leq g + 1$ such that the following conditions* (i), (ii) *hold.*

(i) *$\mu_* F$ is linearly equivalent to* $(2\Delta_{[d]} + (g + d + 1)\Gamma_{[d]})$.

(ii) *The pull-back to X of a (-1)-curve contracted by μ intersects with F at just one point.*

In particular, f has at least one (-1)-section and F is a hyperelliptic curve when $g \geq 2$.

Proof. When $g = 1$, the assertion is well-known. In particular, we have that f is the anti-canonical map $\Phi_{|-K_X|}$ from the canonical bundle formula and the assumption, where $|D|$ means the complete linear system of D.

Assume that $g \geq 2$. From $\rho(X) = 4g + 6$ and [10, Remark 1.1], we have $K_X + F \sim (g-1)D$ with $D^2 = 0$, where the symbol $\sim$ means the linear equivalence of divisors. Furthermore, $|D|$ is a base-point-free pencil of rational curves on X. We take a relatively minimal model of X with respect to $\Phi_{|D|}$ and consider the image of F. Then we perform a succession of elementary transformations ([4]) at singular points of the image of F to arrive at a particular relatively minimal model called a #-minimal model in [5]. Let $\mu : (X, F) \to (\Sigma_d, \mu_* F)$ be a birational morphism to a #-minimal model of (X, F). Then we have that μ satisfies the conditions (i), (ii) from $F.D = 2$ and properties of #-minimal model. $\square$

When $g = 1$ and f has no multiple fibres, we have $F.R = -K_X.R = 2$ for all rulings on X with R as general fibres. Therefore, $\left(f, \Phi_{|R|}\right) : X \to \varSigma_0 := \mathbb{P}^1 \times \mathbb{P}^1$ is a generically finite double cover, where $\left(f, \Phi_{|R|}\right)$ is the morphism satisfying $f = \mathrm{pr}_1 \circ \left(f, \Phi_{|R|}\right)$ and $\Phi_{|R|} = \mathrm{pr}_2 \circ \left(f, \Phi_{|R|}\right)$, and the branch divisor is linearly equivalent to $(4\varDelta_{[0]} + 2\varGamma_{[0]})$.

We assume that $g \geq 2$ and $\rho(X) = 4g + 6$. Any fibred rational surface $f : X \to \mathbb{P}^1$ can be considered as a subpencil $\Lambda \subset |2\Delta_{[d]} + (d+g+1)\Gamma_{[d]}|$ through a birational morphism as in Theorem 2.2. Conversely, any fibred rational surface is obtained from a subpencil $\Lambda \subset |2\Delta_{[d]} + (d+g+1)\Gamma_{[d]}|$ by blowing Σ_d up at the $(4g+4)$ base points. Let (O) be a (-1)-section of f and U^- a sublattice generated by (O) and F in $\mathrm{NS}(X)^-$. Then the orthogonal complement $(U^-)^\perp \subset \mathrm{NS}(X)^-$ is isomorphic to the unimodular lattice called D^+_{4g+4} in [2, §7, Ch. 4] of rank $4g + 4$, which contains D_{4g+4} as the maximal root sublattice at index two. Furthermore, all fibres of f are irreducible if and only if $T = U$, and then we have $\mathrm{MWL}(f) \simeq D^+_{4g+4}$ from Theorem 2.1. This case was studied in [15].

We take a birational morphism $\mu : X \to \Sigma_d$ as in Theorem 2.2. We have that $(K_X + F) \sim (g-1)\mu^*\Gamma_{[d]}$ and consider the ruling $\Phi_{|(K_X+F)/(g-1)|} : X \to \mathbb{P}^1$, that is, $\Phi_{|\Gamma_{[d]}|} \circ \mu : X \to \mathbb{P}^1$. All (-1)-sections of f and (-2)-curves contained in fibres of f do not intersect with $(K_X + F)$. Hence they are components of degenerate fibres of the ruling $\Phi_{|(K_X+F)/(g-1)|}$. Conversely, irreducible components of degenerate fibres of $\Phi_{|(K_X+F)/(g-1)|}$ is obtained from base points of a subpencil $\Lambda \subset |2\Delta_{[d]} + (d+g+1)\Gamma_{[d]}|$. Therefore, we have the following.

Corollary 2.3 ([10, Remark 1.1]). *Assume that $g \geq 2$ and $\rho(X) = 4g + 6$. Then any degenerate fibre of the ruling $\Phi_{|(K_X+F)/(g-1)|} : X \to \mathbb{P}^1$ consists of k (-2)-curves contained in a fibre of f and one or two (-1)-sections of f. The dual graph of the configuration has Dynkin diagram of a root lattice A_{k+2} as in*

Figure 1 *or* D_{k+1} *as in Figure* 2. *Here the numbers without the circles denote the multiplicities of components in the degenerate fibre. Conversely,* (-1)*-sections of* f *and* (-2)*-curves contained in fibres of* f *are components of degenerate fibres of the ruling* $\Phi_{|(K_X+F)/(g-1)|} : X \to \mathbb{P}^1$.

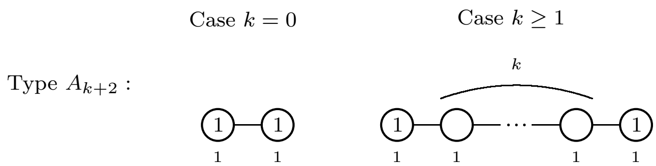

Figure 1.

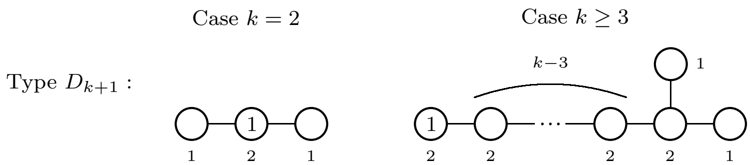

Figure 2.

The hyperelliptic involution of $f : X \to \mathbb{P}^1$ naturally induces a double cover as follows.

Corollary 2.4 ([15, § 4] and [6, § 2]). *If* $g \geq 2$ *and* $\rho(X) = 4g + 6$, *then* $\left(f, \Phi_{|(K_X+F)/(g-1)|}\right) : X \to \Sigma_0$ *is a generically finite double cover and the branch divisor is linearly equivalent to* $((2g + 2)\Delta_{[0]} + 2\Gamma_{[0]})$. *Conversely, the finite double cover of* Σ_0 *branched along a reduced curve which is linearly equivalent to* $((2g + 2)\Delta_{[0]} + 2\Gamma_{[0]})$ *with a minimal resolution of the singularity gives a hyperelliptic fibration of genus* g *on a smooth rational surface whose Picard number is* $(4g + 6)$.

3. Main theorem

Let X be a smooth projective rational surface with $\rho(X) = 4g + 6$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus $g \geq 1$. When $g = 1$, we assume that f has no multiple fibres, or f has a section. Miranda and Persson [12] studied extremal rational elliptic surfaces, where "extremal" means that the Mordell-Weil rank of f is zero (see also [13]). Since f always has a section from Theorem 2.2, we are interested in the Mordell-Weil group and lattice of f. In this section we consider the case where the Mordell-Weil group of f is trivial and prove the following theorem.

THEOREM 3.1. *Let X be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus $g \geq 1$. Assume that $\rho(X) = 4g+6$ and f has no multiple fibres when $g = 1$. Put $\mathbb{K} = f^*(\mathbb{C}(\mathbb{P}^1))$. Then the following conditions are equivalent to each other.*

(1) *The Mordell-Weil group of f is trivial.*

(2a) *f has a reducible fibre whose dual graph is as in Figure 3. Here $\bigcirc$ is a (-2)-curve, $g+1$ is a $(-g-1)$-curve and the numbers without the circles denote the multiplicities of components in the reducible fibre.*

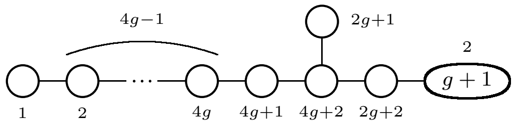

Figure 3.

(2b) *f has a reducible fibre whose dual graph contains, as a subgraph, the extended Dynkin diagram of the unimodular integral lattice D^+_{4g+4} as in Figure 4.*

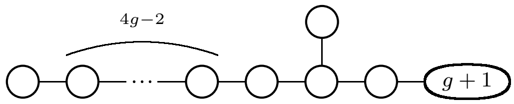

Figure 4.

(3) *X has a unique ruling and possibilities of its relatively minimal models are Hirzebruch surfaces Σ_g of degree g and Σ_{g+1} of degree $g+1$ only.*

(4a) *$f : X \to \mathbb{P}^1$ is obtained from Σ_g by eliminating the base points of the following pencil Λ: Let $\Delta_{[g]}$ be the minimal section and $\Gamma_{[g],0}$ a fibre of Σ_g. Take a curve $H_{[g]}$ which is linearly equivalent to $(2\Delta_{[g]} + (2g+1)\Gamma_{[g],0})$ and which is tangent to $\Gamma_{[g],0}$ at the intersection point of $\Gamma_{[g],0}$ with $\Delta_{[g]}$. Then the pencil Λ is defined by $(2\Delta_{[g]} + (2g+1)\Gamma_{[g],0})$ and $H_{[g]}$.*

(4b) *There exist elements t, x, y in $\mathbb{C}(X)$ and complex numbers $c_{i,j}$, $i = 0, 1, 2$, $j = 0, 1, \ldots, ig+1$ such that they satisfy the following:*

- $\mathbb{C}(X) = \mathbb{C}(x, y)$ *and* $\mathbb{K} = \mathbb{C}(t)$.

- $x^2y^{2g+1} = t\sum c_{i,j}x^iy^j$, $c_{1,0} = c_{0,0} = 0$ *and* $c_{2,0}c_{0,1} \neq 0$.

(5a) $f : X \to \mathbb{P}^1$ *is obtained from* $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ *as follows: Let* $\Delta_{[0],0}$ *be a section of* Σ_0 *with* ${\Delta_{[0],0}}^2 = 0$ *and* $\Gamma_{[0],0}$ *a fibre of* Σ_0. *Take a curve* $B_{[0],0}$ *which is linearly equivalent to* $((2g+1)\Delta_{[0],0} + \Gamma_{[0],0})$ *and which has a contact of order* $(2g+1)$ *with* $\Gamma_{[0],0}$ *at the intersection point of* $\Gamma_{[0],0}$ *with* $\Delta_{[0],0}$. *Let* $\pi^\flat : X^\flat \to \Sigma_0$ *be the finite double cover branched along* $(\Delta_{[0],0}+\Gamma_{[0],0}+B_{[0],0})$. *Then* $X^\flat$ *has only one rational double point of type* D_{4g+4} *as its singularity. Let* $\varsigma : X \to X^\flat$ *be a minimal resolution of the singularity. Define* $f : X \to \mathbb{P}^1$ *as the composite of* $\pi^\flat \circ \varsigma : X \to \Sigma_0$ *and the projection* $\Phi_{|\Gamma_{[0],0}|}$ *of* Σ_0.

(5b) *There exist* $(2g+1)$ *complex numbers* $b_{1,1}, \ldots, b_{1,2g+1}$, *two non-zero complex numbers* $b_{0,2g+1}$, $b_{1,0}$ *and* t *in* $\mathbb{K}$ *such that the followings hold:* $\mathbb{K} = \mathbb{C}(t)$ *and* $\mathbb{C}(X)$ *is isomorphic to the quotient field of* $\mathbb{C}[t, y, z]/(\psi(t, y, z))$, *where*
$$\psi(t, y, z) = z^2 - ty(b_{0,2g+1}y^{2g+1} + b_{1,0}t + ty\textstyle\sum_{j=1}^{2g+1} b_{1,j}y^{j-1}).$$

In order to show Theorem 3.1, we prove some lemmas. As a first step, we show that the conditions (2a), (4a), (4b), (5a) and (5b) are equivalent to each other. As a second step, we show Lemmas 3.4–3.7 below, which induce that (1) $\Leftrightarrow$ (2a) and (2a) $\Leftrightarrow$ (3). As a final step, we deduce (2a) $\Leftrightarrow$ (2b).

Lemma 3.2. *Assume that* f *has a reducible fibre* F_0 *whose dual graph is as in Figure* 3. *Then there exists a birational morphism* $\mu : X \to \Sigma_g$ *such that the images by* μ *of the fibres of* f *forms the pencil* Λ *as in* (4a) *of Theorem* 3.1.

Proof. Let Θ_k, $k = 0, 1, \cdots, 4g+4$ be components of the reducible fibre F_0 such that

$$(\Theta_{i-1}.\Theta_{j-1})_{1\le i,j\le 4g+5} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 & \vdots & \vdots & \vdots \\ \vdots & \ddots & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & -2 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & -g-1 \end{pmatrix}.$$

We know that f has a (-1)-section E_{4g+4} from Theorem 2.2. Since Θ_0 is a unique component whose multiplicity in F_0 is one, E_{4g+4} intersects with Θ_0. Let μ be the birational morphism contracting $E_{4g+4}, \Theta_0, \Theta_1, \ldots, \Theta_{4g+2}$ in turn. Then $(\mu_*\Theta_{4g+3})^2 = 0$ and $(\mu_*\Theta_{4g+4})^2 = -g$. Since $\rho(X) = 4g+6$, the image of X by μ is Σ_g, and we have that $\mu_*\Theta_{4g+3}$ is a fibre of Σ_g and $\mu_*\Theta_{4g+4} = \Delta_{[g]}$. Let

$\Gamma_{[g],0}$ be the fibre $\mu_*\Theta_{4g+3}$. In fact, μ is a birational morphism as in Theorem 2.2, where $d = g$. We take the image by μ of a general fibre of f and denote it by $H_{[g]}$. Then the assertion follows. $\square$

LEMMA 3.3. *The conditions* (2a), (4a), (4b), (5a) *and* (5b) *of Theorem* 3.1 *are equivalent to each other.*

PROOF. (5a) $\Rightarrow$ (2a) comes from a standard calculation for a double cover. Lemma 3.2 states (2a) $\Rightarrow$ (4a). Now, by taking t and y as coordinates of the first component and the second one of $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ respectively, $\psi(t, y, 0) = 0$ in (5b) of Theorem 3.1 defines a branch divisor of a double cover $\varsigma \circ \pi^\flat : X \to \Sigma_0$ in (5a). Furthermore, $\mathbb{C}[t, y, z]/(\psi(t, y, z))$ is an affine coordinate ring of the surface obtained from the finite double cover of Σ_0 whose branch divisor is defined by $\psi(t, y, 0) = 0$. Therefore, (5a) follows from (5b). Conversely, we suppose (5a). We may assume that the intersection point of $\Delta_{[0],0}$ and $\Gamma_{[0],0}$ is $(t, y) = (0, 0)$ by performing projective transformations on Σ_0 if necessary. Let $\psi_B(t, y) = 0$ be a defining equation of the branch divisor $(\Delta_{[0],0} + \Gamma_{[0],0} + B_{[0],0})$ in (5a). Put $\psi(t, y, z) = z^2 - \psi_B(t, y)$. Then we have (5b) from a standard calculation. Similarly, we deduce (4a) $\Leftrightarrow$ (4b).

We show that (4a) and (4b) imply (5a). We consider on the Zariski open subset defined by $X_0Y_0 \neq 0$ in Σ_g. Then the pencil Λ as in (4a) is defined by

$$\begin{aligned}(3.2)\quad \{x^2y^{2g+1} = t(&c_{2,2g+1}x^2y^{2g+1} + c_{2,2g}x^2y^{2g} + \cdots + c_{2,1}x^2y + c_{2,0}x^2 \\ &+ c_{1,g+1}xy^{g+1} + c_{1,g}xy^g + \cdots + c_{1,1}xy + c_{0,1}y)\}_{t\in\mathbb{P}^1},\end{aligned}$$

where $c_{i,j} \in \mathbb{C}$, $i = 0, 1, 2$, $j = 0, 1, \ldots, ig+1$ with $c_{1,0} = c_{0,0} = 0$ and $c_{2,0}c_{0,1} \neq 0$. The meromorphic map defined by $(x, y) \mapsto (t, y)$ induces the generically finite double cover $\left(f, \Phi_{|\Gamma_{[g]}|}\right) : X \to \Sigma_0$ by eliminating the base points of Λ. It is the double cover as in Corollary 2.4 when $g \geq 2$. Therefore, the branch divisor B of $\left(f, \Phi_{|\Gamma_{[g]}|}\right)$ contains the curves defined the discriminant of the equation (3.2) for x. On the other hand, the discriminant

$$ty\left(\begin{aligned}4c_{0,1}y^{2g+1} - 4c_{2,0}c_{0,1}\,t \qquad& \\ -4c_{0,1}t(c_{2,2g+1}y^{2g+1} + c_{2,2g}y^{2g} + \cdots + c_{2,1}y)& \\ +ty(c_{1,g+1}y^g + c_{1,g}y^{g-1} + \cdots + c_{1,2}y + c_{1,1})^2&\end{aligned}\right)$$

defines $\Delta_{[0],0}$, $\Gamma_{[0],0}$ and a section $B_{[0],0}$ of Σ_0 which is linearly equivalent to $((2g + 1)\Delta_{[0]} + \Gamma_{[0]})$. In fact, the discriminant is a defining equation of $B = (\Delta_{[0],0} + \Gamma_{[0],0} + B_{[0],0})$ from Corollary 2.4. Furthermore, we see that $B_{[0],0}$ has a contact of order $(2g + 1)$ with $\Gamma_{[0],0}$ at the intersection point $(0, 0)$ of $\Gamma_{[0],0}$ with $\Delta_{[0],0}$ by considering $c_{2,0}c_{0,1} \neq 0$ and the defining equation. $\square$

LEMMA 3.4. *Assume that f has a reducible fibre F_0 whose dual graph is as in Figure* 3. *Then* (A) *and* (B) *hold*:

(A) *The Mordell-Weil group of f is trivial. In particular, a (-1)-section of f is unique.*

(B) *Let C be an irreducible reduced curve on X. If $C^2 < 0$, then C is a smooth rational curve satisfying one of the following*:

- *C is a (-1)-section of f.*
- *C is an irreducible component of F_0.*

PROOF. (A) : We consider μ as in the proof of Lemma 3.2. Let E_i, $i = 1, 2, \ldots, 4g+4$ be the pull-back to X of $(4g+4)$ (-1)-curves contracted by the birational morphism $\mu : X \to \Sigma_g$. For components Θ_k, $k = 0, 1, \cdots, 4g+4$ of F_0 as in the proof of Lemma 3.2, we have that $\Theta_k = E_{4g+3-k} - E_{4g+4-k}$, $k = 0, 1, \ldots, 4g+2$. For simplicity, we denote the pull-back to X of $\Delta_{[g]}$ and $\Gamma_{[g],0}$ by the same symbols. We remark that $\Theta_{4g+3} = \Gamma_{[g],0} - E_1 - E_2$ and $\Theta_{4g+4} = \Delta_{[g]} - E_1$. Since $\mathrm{NS}(X) = \mathbb{Z}\Delta_{[g]} \oplus \mathbb{Z}\Gamma_{[g],0} \oplus \bigoplus_{i=1}^{4g+4} \mathbb{Z}E_i$, we see that E_{4g+4} and Θ_k, $k = 0, 1, \ldots, 4g+4$ also form $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. Furthermore, we have that $\Theta_1, \Theta_2, \ldots, \Theta_{4g+4}$, F and E_{4g+4} also form $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ by considering

$$\Theta_0 = F_0 - \sum_{k=1}^{4g+1} (k+1)\Theta_k - (2g+2)\Theta_{4g+2} - (2g+1)\Theta_{4g+3} - 2\Theta_{4g+4}.$$

Therefore, the Mordell-Weil group of f is trivial from Theorem 2.1.

(B) : Keep the notation as above. Let C be an irreducible reduced curve on X. We put $C \sim \alpha\Delta_{[g]} + \beta\Gamma_{[g],0} - \sum_{i=1}^{4g+4} m_i E_i$ for some integers α, β and m_i, $i = 1, 2, \ldots, 4g+4$. We assume that C is neither E_{4g+4} nor Θ_k for $k = 0, 1, \ldots, 4g+4$. Since $C.E_{4g+4}$ and $C.\Theta_k$ are non-negative, we have $0 \leq m_{4g+4} \leq m_{4g+3} \leq \cdots \leq m_2 \leq m_1$, $m_1 + m_2 \leq \alpha$ and $\alpha g + m_1 \leq \beta$. Thus

$$\begin{aligned}
C^2 &\geq -\alpha^2 g + 2\alpha\beta - m_1^2 - (4g+3)m_2^2 \\
&\geq \alpha^2 g + 2\alpha m_1 - m_1^2 - (4g+3)m_2^2 \\
&\geq (m_1+m_2)^2 g + 2(m_1+m_2)m_1 - m_1^2 - (4g+3)m_2^2 \\
&= (g+1)(m_1+3m_2)(m_1-m_2) \\
&\geq 0.
\end{aligned}$$

□

LEMMA 3.5. *Assume that f has a reducible fibre F_0 whose dual graph is as in Figure* 3. *Then X has a unique ruling. Furthermore, Σ_g and Σ_{g+1} only can be its relatively minimal model. In particular, there exists no birational morphisms $X \to \mathbb{P}^2$ if $g \geq 2$.*

PROOF. We consider the birational morphism $\mu : X \to \Sigma_g$ as in the proof of Lemma 3.2 and keep the notation. In particular, we remark that $\mu(E_1)$ is on $\Delta_{[g]}$. By performing an elementary transformation $\Sigma_g \dashrightarrow \Sigma_{g+1}$ at $\mu(E_1)$, we have another birational morphism $\mu' : X \to \Sigma_{g+1}$ as the composite of μ and it. Here μ' contracts Θ_{4g+3} in place of Θ_{4g+2}. In fact, μ and μ' give relatively minimal models of the same ruling defined by $|E_{4g+4} + \sum_{k=0}^{4g+3} \Theta_k|$. From (B) of Lemma 3.4, there are no other birational morphisms $X \to \Sigma_d$. Thus the assertion follows. □

LEMMA 3.6. *If the Mordell-Weil group of f is trivial, then f has a unique reducible fibre and its dual graph is as in Figure* 3.

PROOF. Assume that the Mordell-Weil group of f is trivial. From Theorem 2.2, there exists a birational morphism $\mu : X \to \Sigma_d$ in order that $F \sim 2\Delta_{[d]} + (d+g+1)\Gamma_{[d]} - \sum_{i=1}^{4g+4} E_i$. We shall denote the (-1)-section of f by E_{4g+4}. In particular, we remark that a section of f is unique from the assumption. Therefore, in the process of contracting by μ, we may assume that the point corresponding to E_{i+1} is an infinitely near point of that to E_i for $i = 1, 2, \ldots, 4g+3$. Similarly, $\mu(E_2)$ corresponds a tangential direction at $\mu(E_1)$ of a fibre $\Gamma_{[d]}$ of Σ_d. We recall that $\Delta_{[d]}$, $\Gamma_{[d]}$ and E_i, $i = 1, 2, \ldots, 4g+4$ form $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. Since (-2)-curves $\Gamma_{[d]} - E_1 - E_2$ and $E_i - E_{i+1}$, $i = 1, 2, \ldots, 4g+3$ are connected, a reducible singular fibre of f contains all of them. However, they do not generate the reducible fibre of f. From $\rho(X) = 4g+6$ and the equation of the Mordell-Weil rank (2.1), we have that another component of the reducible fibre is unique, where we denote it by Θ, and all other fibres of f are irreducible. Furthermore, the assumption and Theorem 2.1 imply that components $\Gamma_{[d]} - E_1 - E_2$, $E_i - E_{i+1}$, $i = 1, 2, \ldots, 4g+2$, the other component Θ and the unique section E_{4g+4} form $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. Hence we have $\Theta.\Gamma_{[d]} = 1$. We also remark that $\Theta.(\Gamma_{[d]} - E_1 - E_2)$ and $\Theta.(E_i - E_{i+1})$ are non-negative. Then we have that Θ is $\Delta_{[d]} + \beta\Gamma_{[d]}$ or $\Delta_{[d]} + \beta\Gamma_{[d]} - E_1$ for some non-negative integer β in a way similar to the proof of (B) of Lemma 3.4. Here, $\Theta^2 \geq 0$ if $\beta > 0$. It contradicts the fact that Θ is a proper component of the reducible fibre of f. By considering $\Theta.F = 0$, when $\Theta = \Delta_{[d]}$ (resp. $\Theta = \Delta_{[d]} - E_1$), we have $d = g+1$ (resp. $d = g$). Either way, Θ, $\Gamma_{[d]} - E_1 - E_2$ and $E_i - E_{i+1}$, $i = 1, 2, \ldots, 4g+3$ form a singular fibre whose dual graph is as in Figure 3. □

When we show (3) ⇒ (2a), we use a weaker condition than (3) as follows:

LEMMA 3.7. *Assume that $d = g$ or $g+1$ for all birational morphisms $X \to \Sigma_d$ satisfying conditions* (i), (ii) *of Theorem* 2.2. *Then f has a reducible fibre whose dual graph is as in Figure* 3.

PROOF. Let $\mu' : X \to \Sigma_{g+1}$ be a birational morphism satisfying conditions (i), (ii) of Theorem 2.2. Then a base-point-free pencil $|F|$ can be considered as a subpencil $\Lambda' \subset |2(\Delta_{[g+1]} + (g+1)\Gamma_{[g+1]})|$ with $(4g+4)$ simple base points on Σ_{g+1} through μ' (cf. [10, Proof of Lemma 1.2]). Since

$\Delta_{[g+1]}.2(\Delta_{[g+1]} + (g+1)\Gamma_{[g+1]}) = 0$, no base points of Λ' lie on $\Delta_{[g+1]}$. We consider a base point p'_1 on Σ_{g+1} and the corresponding (-1)-curve E'_1. Let $\mu'_1 : X \to X_1$ be the birational morphism which contracts $(4g+3)$ (-1)-curves except E'_1 among $(4g+4)$ (-1)-curves contracted by μ'. The composite of μ' and the elementary transformation $\Sigma_{g+1} \dashrightarrow \Sigma_g$ at p'_1 is a birational morphism $X \to \Sigma_g$ which factors through μ'_1. Here the (-1)-curve contracted by the birational morphism $X_1 \to \Sigma_g$ is the strict transform to X_1 of the fibre of Σ_{g+1} passing through p'_1. In particular, it intersects with $(\mu'_1)_*F$ at just one point. Therefore, there exists a birational morphism $\mu : X \to \Sigma_g$ satisfying conditions (i), (ii) of Theorem 2.2 from the assumption.

The base-point-free pencil $|F|$ can be considered as a subpencil $\Lambda \subset |2\Delta_{[g]} + (2g+1)\Gamma_{[g]}|$ with $(4g+4)$ simple base points on Σ_g through μ. If there exists a base point of Λ on $\Sigma_g \setminus \Delta_{[g]}$, then we obtain a birational morphism $X \to \Sigma_{g-1}$ satisfying conditions (i), (ii) of Theorem 2.2 from the elementary transformation $\Sigma_g \dashrightarrow \Sigma_{g-1}$ at the base point in the same way as in the previous paragraph. It contradicts the assumption. Thus there exists a base point p_1 of Λ on $\Delta_{[g]}$. Since $\Delta_{[g]}.(2\Delta_{[g]} + (2g+1)\Gamma_{[g]}) = 1$, other base points $p_2, p_3, \ldots, p_{4g+4}$ of Λ do not lie on $\Delta_{[g]} \setminus p_1$. Let $\Gamma_{[g],1}$ be the fibre of Σ_g passing through p_1. Then the pull-back of $\Gamma_{[g],1}$ to X is a unique degenerate fibre of the ruling $\Phi_{|\Gamma_{[g]}|} : X \to \mathbb{P}^1$. From Corollary 2.3, the dual graph of the unique degenerate fibre $\mu^*\Gamma_{[g],1}$ of $\Phi_{|\Gamma_{[g]}|}$ is either of type A_{k+2} as in Figure 1 with $k = 4g+3$ or of type D_{k+1} as in Figure 2 with $k = 4g+4$. Hence, we may assume that p_{i+1} is infinitely near point of p_i for simplicity.

We suppose that the dual graph of the unique degenerate fibre $\mu^*\Gamma_{[g],1}$ of $\Phi_{|\Gamma_{[g]}|}$ is of type A_{k+2} as in Figure 1 with $k = 4g+3$. Then the tangential direction of $\Gamma_{[g],1}$ at p_1 is different from the direction corresponding to p_2. Let $X_3 \to \Sigma_g$ be the blowing up at p_1, p_2 and p_3. We denote the (-1)-curves corresponding to p_1, p_2 and p_3 by E_1, E_2 and E_3 respectively. Let $\mu_3 : X \to X_3$ be the birational morphism which contracts $(4g+1)$ (-1)-curves except E_1, E_2 and E_3 among $(4g+4)$ (-1)-curves contracted by μ. We consider the birational morphism $\tau : X_3 \to \Sigma_d$ contracting $(\mu_3)_*(\mu^*\Gamma_{[g],1} - E_1)$, $(\mu_3)_*(E_1 - E_2)$ and $(\mu_3)_*(E_2 - E_3)$ in turn. We have that $(\tau \circ \mu_3)_*(\Delta_{[g]} - E_1)$ is a minimal section of Σ_d and $d = g-1$ since $(\tau \circ \mu_3)_*(\Delta_{[g]} - E_1).(\tau \circ \mu_3)_*\Gamma_{[g]} = 1$ and $(\tau \circ \mu_3)_*(\Delta_{[g]} - E_1)^2 = -g+1 \leq 0$. In fact, $(\mu^*\Gamma_{[g],1} - E_1)$ is a (-1)-section of f. Furthermore, $\tau \circ \mu_3 : X \to \Sigma_{g-1}$ is a birational morphism satisfying conditions (i), (ii) of Theorem 2.2. It contradicts the assumption. Thus, the dual graph of the unique degenerate fibre $\mu^*\Gamma_{[g],1}$ of $\Phi_{|\Gamma_{[g]}|}$ is of type D_{k+1} as in Figure 2 with $k = 4g+4$. In particular, p_2 corresponds to the tangential direction of $\Gamma_{[g],1}$ at p_1. Hence, f has a reducible fibre whose irreducible components are $(\Delta_{[g]} - E_1)$, $(\mu^*\Gamma_{[g],1} - E_1 - E_2)$ and $(E_i - E_{i+1})$, $i = 1, 2, \ldots, 4g+3$. The dual graph of the reducible fibre of f is as in Figure 3. □

Proof of Theorem 3.1. We have (1) $\Leftrightarrow$ (2a) from Lemma 3.6 and (A) in Lemma 3.4. Lemma 3.5 states (2a) $\Rightarrow$ (3). If we assume (3) of Theorem 3.1, then the assumption in Lemma 3.7 is satisfied, which leads to (2a). Therefore, we see that (1), (2a) and (3) of Theorem 3.1 are equivalent. From this and Lemma 3.3, the proof of Theorem 3.1 is completed if we prove the following lemma.

LEMMA 3.8. *The conditions* (2a) *and* (2b) *of Theorem* 3.1 *are equivalent to each other. Furthermore, a reducible fibre of f is unique.*

PROOF. If we assume (2a) of Theorem 3.1, then the equation of the Mordell-Weil rank (2.1) implies the uniqueness of a reducible fibre of f. So (2b) holds. In what follows, we assume (2b). When $g = 1$, the condition (2a) follows from the classification of singular fibres by Kodaira [11]. We consider the case $g \geq 2$ by applying Corollary 2.3.

Let F_0 be a reducible fibre of f whose dual graph contains the extended Dynkin diagram of the unimodular integral lattice D^+_{4g+4} as in Figure 4. We denote the irreducible component of F_0 whose self-intersection number is $(-g-1)$ by Θ_{4g+4}. We remark that the number of the irreducible components of F_0 is at most $(4g+5)$ from the equation of the Mordell-Weil rank (2.1).

At first, we suppose that the number of the irreducible components of F_0 is exactly $(4g+4)$. From Corollary 2.3, there exists a unique (-1)-section E_{4g+4} of f which does not intersect with Θ_{4g+4}. Let $\Gamma_{[d],0}$ be the degenerate fibre of $\Phi_{|(K_X+F)/(g-1)|}$ which consists of E_{4g+4} and components of F_0 except Θ_{4g+4}. We remark that the dual graph of $\Gamma_{[d],0}$ is of type D_{k+1} as in Figure 2 with $k = 4g+3$. From Theorem 2.2, the ruling $\Phi_{|(K_X+F)/(g-1)|}$ has exactly one other degenerate fibre $\Gamma_{[d],\infty}$ whose dual graph is of type A_{k+2} as in Figure 1 with $k = 0$. We remark that $\Gamma_{[d],\infty}$ consists of two (-1)-sections of f which intersect with F_0 on Θ_{4g+4}. We have a birational morphism $\mu : X \to \Sigma_d$ which satisfies conditions (i), (ii) of Theorem 2.2 by contracting E_{4g+4} and $(4g+3)$ of components of $\Gamma_{[d],0}$ and $\Gamma_{[d],\infty}$. Then $\mu_*\Theta_{4g+4}$ intersects with the fibre $\mu_*\Gamma_{[d],0}$ of Σ_d at one point. On the other hand, it intersects with the other fibre $\mu_*\Gamma_{[d],\infty}$ of Σ_d at two point, which is absurd. Therefore, F_0 has exactly one component Θ other than the $(4g+4)$ components corresponding to D^+_{4g+4}.

Next, we suppose $\Theta^2 < -2$. In the quite same argument as in the previous paragraph, the ruling $\Phi_{|(K_X+F)/(g-1)|}$ has exactly two degenerate fibres $\Gamma_{[d],0}$ and $\Gamma_{[d],\infty}$ as above, and we have a birational morphism $\mu : X \to \Sigma_d$. Then $\mu_*\Theta_{4g+4}$ is a minimal section of Σ_d since $\mu_*\Theta_{4g+4}.\mu_*\Gamma_{[d],0} = 1$ and $(\mu_*\Theta_{4g+4})^2 \leq (-g-1)+2 = -g+1 \leq -1$. From Corollary 2.3, Θ is not a component of a degenerate fibre of $\Phi_{|(K_X+F)/(g-1)|}$. Therefore, we have that $\mu_*\Theta$ is a section of Σ_d from (i) of Theorem 2.2. In particular, $(\mu_*\Theta)^2 \geq d = -(\mu_*\Theta_{4g+4})^2 \geq g-1$ and $\mu_*\Theta$ intersects with $\mu_*\Gamma_{[d],0}$ and $\mu_*\Gamma_{[d],\infty}$ transversally. Remark that $\Gamma_{[d],0}$ is a degenerate fibre of $\Phi_{|(K_X+F)/(g-1)|}$ of type D_{k+1} as in Figure 2 with $k = 4g+3$. Then we have $\Theta^2 \geq (g-1)-2 = g-3 \geq -1$, which is absurd. Hence, we get $\Theta^2 = -2$. Thus, we see that the dual graph of F_0 is as in Figure 3 from Corollary 2.3. □

This completes the proof of Theorem 3.1. □

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