Saitama Math. J.
Vol. 30 (2013), 39–57

On the trace and the infinitesimally deformed chiral anomaly of Dirac operators on twistor spaces and the change of metrics on the base spaces

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(*Received 19 August, 2013; Accepted 16 December, 2013*)

Abstract

We show that the trace of quotient of two Dirac operators and the infinitesimally deformed chiral anomaly of Dirac operator on a twistor space have adiabatic series expansions. Further their top terms will be explicitly described.

0. Introduction

Let $M = (M, g^M)$ be an even dimensional compact oriented Riemannian manifold equipped with a Spin^q structure introduced in [8]

$$\text{(0.1)} \qquad \Xi^q : P_{\mathrm{Spin}^q(n)}(M) = P_{\mathrm{Spin}(n)}(M) \times_{\mathbb{Z}_2} P_{Sp(1)} \to P_{SO(n)}(M) \times P_{SO(3)},$$

where $P_{SO(n)}(M)$ $(n = \dim M)$ is the reduced structure bundle consisting of $SO(n)$-frames of TM and $P_{SO(3)}$, $P_{\mathrm{Spin}^q(n)}(M)$ are some principal bundles with structure groups $SO(3)$, $\mathrm{Spin}^q(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}_2} Sp(1)$, respectively. Remark that $P_{\mathrm{Spin}(n)}(M)$, $P_{Sp(1)}$ are locally defined bundles and the bundle map Ξ^q is assumed to be equivariant to the canonical Lie group homomorphism $\Xi^q = (\Xi, \mathrm{Ad}) : \mathrm{Spin}^q(n) \to SO(n) \times SO(3)$. Then, using the canonical action of $\mathrm{Spin}^q(n)$ or $Sp(1)$ on $\mathrm{Spin}^q(n)/\mathrm{Spin}^c(n) = Sp(1)/U(1)$ and the identification $Sp(1)/U(1) = \mathbb{C}P^1$ through the representation $r_H : Sp(1) \to GL_{\mathbb{C}}(H) = GL_{\mathbb{C}}(\mathbb{C}^2)$ with $r_H(\alpha + \boldsymbol{j}\beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, we have a $\mathbb{C}P^1$-fibration

$$\text{(0.2)} \qquad \pi : Z = P_{\mathrm{Spin}^q(n)}(M) \times_{\mathrm{can}} \mathbb{C}P^1 = P_{Sp(1)} \times_{\mathrm{can}} \mathbb{C}P^1 \to M.$$

Let us now take an $Sp(1)$-connection A of $P_{Sp(1)}$, so that the **twistor space** Z possesses canonically a Spin structure ([9], [10]). Namely, the connection induces a splitting of TZ into horizontal and vertical components, $TZ = \mathcal{H} \oplus \mathcal{V}$, with natural orientation and with the metric $g^Z = \pi^* g^M + g^{\mathcal{V}}$ $(\pi^* g^M = g^Z|\mathcal{H})$, where $g^{\mathcal{V}}$ is the Riemannian metric on $\mathcal{V}$ induced from the Fubini-Study one of $\mathbb{C}P^1$.

2010 Mathematics Subject Classification. Primary 53C27; Secondary 53C28, 81R25
Key words and phrases. chiral anomaly, Dirac operator

Further we have the locally defined spinor bundle $\$_{g^M}$ associated to $P_{\mathrm{Spin}(n)}(M)$ and a locally defined hermitian vector bundle $\mathbb{H} = P_{Sp(1)} \times_{r_H} H$, which together produce the globally defined vector bundle $\pi^*\$_{g^M} \otimes \pi^*\mathbb{H} = \pi^*\$_{g^M} \otimes \$_{g^{\mathcal{V}}} =: \$_{g^Z}$ on Z, whose rank is certainly equal to $2^{n/2+1}$. Then, the locally defined Clifford action ρ_{g^M} of $\mathbb{C}l(T^*M, g^M)$ on $\$_{g^M}$, together with the action $\rho_{g^{\mathcal{V}}}$ of $\mathbb{C}l(\mathcal{V}^*, g^{\mathcal{V}})$ on $\$_{g^{\mathcal{V}}}$ induced from the fiberwise globally defined canonical Spin structure, gives the globally defined action ρ_{g^Z} of $\mathbb{C}l(T^*Z, g^Z)$ on $\$_{g^Z}$, i.e., $\rho_{g^Z}(\pi^*\xi_b) = \pi^*\rho_{g^M}(\xi_b) \otimes 1$ $(\xi_b \in T^*M)$ and $\rho_{g^Z}(\xi_f) = \pi^*\rho_{g^M}(\tau_{g^M}) \otimes \rho_{g^{\mathcal{V}}}(\xi_f)$ $(\xi_f \in \mathcal{V}^*)$, where τ_{g^M} is the complex volume element of (M, g^M). Thus (Z, g^Z) has a canonical Spin structure, which gives the Dirac operator $\partial\!\!\!/_{g^Z}^{(\pm)} : \Gamma(\$_{g^Z}^{(\pm)}) \to \Gamma(\$_{g^Z}^{(\mp)})$. Note that the canonical splittings $\$_{g^M} = \$_{g^M}^+ \oplus \$_{g^M}^-$, $\$_{g^{\mathcal{V}}} = \pi^*\mathbb{H} = \$_{g^{\mathcal{V}}}^+ \oplus \$_{g^{\mathcal{V}}}^- = \{([v], cv) \in \pi^*\mathbb{H}\} \oplus (\$_{g^{\mathcal{V}}}^+)^\perp$ induce the splitting $\$_{g^Z} = \$_{g^Z}^+ \oplus \$_{g^Z}^-$.

Now, let us take another metric h^M on M and an associated Spin^q structure with the same $P_{SO(3)}$ as in (0.1), whose twistor space is hence equal to the one given at (0.2). We have thus another Spin structure for Z with metric $h^Z = \pi^*h^M + g^{\mathcal{V}}$, which induces another Dirac operator $\partial\!\!\!/_{h^Z}^{(\pm)} : \Gamma(\$_{h^Z}^{(\pm)}) \to \Gamma(\$_{h^Z}^{(\mp)})$. Let us define then the invariants called **the traces of the quotient** $\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}$ by

$$(0.3)\qquad \mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s\, \mathrm{Tr}_\pm\Big(\partial\!\!\!/_{g^Z}\, \partial\!\!\!/_{h^Z}\, e^{-t\partial\!\!\!/_{g^Z}^2}\Big)\, dt,$$
$$\text{with the equalities } \mathrm{Tr}_\pm\Big(\partial\!\!\!/_{g^Z}\, \partial\!\!\!/_{h^Z}\, e^{-t\partial\!\!\!/_{g^Z}^2}\Big) = \mathrm{Tr}_\mp\Big(\partial\!\!\!/_{h^Z}\, \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/_{g^Z}^2}\Big).$$

(The equalities at the second line will be shown at (2.1).) Remark that $e^{-t\partial\!\!\!/_{g^Z}^2}$ is a cross-section of the vector bundle $\$_{g^Z}^{(\pm)} \boxtimes \$_{g^Z}^{(\pm)*}$ over $Z \times Z$, on which the operator $\partial\!\!\!/_{h^Z}$ cannot act in a naive sense. In the paper we will let $\partial\!\!\!/_{h^Z}$ act on it (see (1.6)) by using the method introduced by Bourguignon and Gauduchon ([4], [5]), the explanation for which will be offered at the beginning of the next section. The first purpose is then to study the adiabatic series expansions of (0.3) and the difference $\mathrm{STr}(\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) = \mathrm{Tr}_+(\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) - \mathrm{Tr}_-(\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})$. Namely, by replacing the metrics g^Z etc. by $g^Z_\varepsilon = \varepsilon^{-1}\pi^*g^M + g^{\mathcal{V}} = \pi^*g^M_\varepsilon + g^{\mathcal{V}}$ $(\varepsilon > 0)$ etc., we obtain $\mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})$ etc., and we want to investigate their asymptotic expansions when $\varepsilon \to 0$. Incidentally to express the right hand side of (0.3) by $\mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})$ will be appropriate in the following sense: Using the series of eigenvalues $(0 <)\, \lambda_1^\pm \le \lambda_2^\pm \le \cdots \to \infty$ (see Lemma 2.1) and the corresponding series of orthonormal eigen-cross-sections of the operator $\partial\!\!\!/_{g^Z}^2$ acting on $\Gamma(\$_{g^Z}^\pm)$, let us set $e^{-t\partial\!\!\!/_{g^Z}^2} = \sum e^{-t\lambda_j^\pm}\phi_j^\pm \boxtimes \phi_j^{\pm*}$ and put $\mu_j^\pm = \langle \partial\!\!\!/_{g^Z}\partial\!\!\!/_{h^Z}\phi_j^\pm, \phi_j^\pm\rangle_{L^2}$ where $\langle\,\cdot\,,\cdot\,\rangle_{L^2} = \langle\,\cdot\,,\cdot\,\rangle_{L^2\Gamma(\$_{g^Z}^\pm)}$ is the global inner product which $\Gamma(\$_{g^Z}^\pm)$ has. Then, formally the right hand side of (0.3) is equal to

$$\sum \mu_j^{\pm} \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^s e^{-t\lambda_j^{\pm}} dt = \sum \mu_j^{\pm} \int_0^{\infty} e^{-t\lambda_j^{\pm}} dt = \sum \frac{\mu_j^{\pm}}{\lambda_j^{\pm}}.$$

Second, let us consider some infinitesimal deformation of the so-called chiral anomaly. That is, let us take a symmetric bilinear form X on TM and set $g^M_{(u)} = g^M + uX$ ($0 \le u \le u_0$). The metric induces the Dirac operator $\partial\!\!\!/_{g^Z_{(u)}}$ acting on $\Gamma(\$_{g^Z})$ as above and we have the infinitesimal deformation of $\partial\!\!\!/_{g^Z}$

$$\delta_X \partial\!\!\!/_{g^Z} := \frac{d}{du}\Big|_{u=0} \partial\!\!\!/_{g^Z_{(u)}}. \tag{0.4}$$

We are interested in the associated invariants called **the infinitesimally deformed chiral anomalies of** $\partial\!\!\!/_{g^Z}$ defined by

$$\log\det\,(\delta_X \partial\!\!\!/_{g^Z})^{\pm} = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^s\, \mathrm{Tr}_{\pm}\Big(\partial\!\!\!/_{g^Z}\, \delta_X \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big)\, dt, \tag{0.5}$$

$$\text{with the equalities } \mathrm{Tr}_{\pm}\Big(\partial\!\!\!/_{g^Z}\, \delta_X \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big) = \mathrm{Tr}_{\mp}\Big(\delta_X \partial\!\!\!/_{g^Z}\, \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}}\Big)$$

and we want to investigate the asymptotic expansions of $\log\det\,(\delta_{X_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon})^{\pm}$ and also their difference when $\varepsilon \to 0$. If the operators $\partial\!\!\!/_{g^Z} \partial\!\!\!/_{g^Z_{(u)}}$ acting on $\Gamma(\$^{\pm}_{g^Z})$ happen to have the spectra consisting of eigenvalues $\{\lambda_j(u) = \lambda_j^{\pm}(u)\}$ all of which lie in a positive cone about the positive real axis in $\mathbb{C}$ and have the corresponding orthonormal eigen-cross-sections $\{\phi_j(u) = \phi_j^{\pm}(u)\}$ which are all smooth with respect to the parameter u at $u = 0$, then we have

$$\begin{aligned}
\lambda_j'(0) &:= \frac{d}{du}\Big|_{u=0} \langle \partial\!\!\!/_{g^Z} \partial\!\!\!/_{g^Z_{(u)}} \phi_j(u), \phi_j(u)\rangle_{L^2} \\
&= \langle \partial\!\!\!/_{g^Z} \delta_X \partial\!\!\!/_{g^Z} \phi_j(0), \phi_j(0)\rangle_{L^2} + \langle \partial\!\!\!/^2_{g^Z} \phi_j'(0), \phi_j(0)\rangle_{L^2} + \langle \partial\!\!\!/^2_{g^Z} \phi_j(0), \phi_j'(0)\rangle_{L^2} \\
&= \langle \partial\!\!\!/_{g^Z} \delta_X \partial\!\!\!/_{g^Z} \phi_j(0), \phi_j(0)\rangle_{L^2} + \lambda_j(0) \frac{\partial}{\partial u}\Big|_{u=0} \langle \phi_j(u), \phi_j(u)\rangle_{L^2} \\
&= \langle \partial\!\!\!/_{g^Z} \delta_X \partial\!\!\!/_{g^Z} \phi_j(0), \phi_j(0)\rangle_{L^2} \quad (\text{hence, } \lambda_j(0) > 0 \text{ if } \lambda_j'(0) \neq 0)
\end{aligned}$$

and the right hand side of (0.5) is formally equal to

$$\begin{aligned}
&\sum_{\lambda_j'(0)\neq 0} \lambda_j'(0) \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^s e^{-t\lambda_j(0)} dt = \sum_{\lambda_j(0)>0} \frac{\lambda_j'(0)}{\lambda_j(0)} = \frac{d}{du}\Big|_{u=0} \log \prod_{\lambda_j(0)>0} \lambda_j(u) \\
&= \frac{d}{du}\Big|_{u=0} \Big(-\frac{\partial}{\partial s}\Big|_{s=0} \sum_{(\lambda_j(0)>0)} e^{-s\log\lambda_j(u)}\Big) = \frac{d}{du}\Big|_{u=0} \log\det\,(\partial\!\!\!/_{g^Z} \partial\!\!\!/_{g^Z_{(u)}})^{\pm}.
\end{aligned}$$

Thus, formally (0.5) are the infinitesimal deformations (into the direction X) of **the chiral anomalies** $\log\det\,(\partial\!\!\!/_{g^Z} \partial\!\!\!/_{g^Z_{(u)}})^{\pm}$, which were briefly explained by I.M. Singer [12, Appendix]. (Note that, in general, all but a finite number of eigenvalues

$\lambda_1(u), \cdots, \lambda_k(u)$ lie in a positive cone about the positive real axis and the eigen-cross-sections $\phi_j(u)$ are generalized ones. In the paper he defined the anomalies as $\log\det(\partial\!\!\!/_{g^Z}\partial\!\!\!/_{g^Z_{(u)}})^{\pm} = -\lambda_1(u)\cdots\lambda_k(u)(\partial/\partial s)|_{s=0}\sum_{j>k} e^{-s\log\lambda_j(u)}$.)

Our investigation on the chiral anomaly etc., which will be quite interesting but are not yet widely researched mathematically compared with the other anomalies such as the global gravitational anomaly (the adiabatic limit of η-invariant) ([3]), is an attempt to embody the idea ([3]) that such an operation as replacing g^Z by g^Z_ε and taking the parameter ε up to 0, that is, blowing up the metric g^Z in the base space direction, will extract some intrinsic values from various geometric invariants of Z. We want to emphasize here that it is mainly the general adiabatic expansion theory concerning the kernel $e^{-t\partial\!\!\!/^2_{g^Z}}$ ([11] and Lemma 2.3) that induces our main assertions, i.e., Theorem 1.2 and Corollary 1.3.

1. The operator $\partial\!\!\!/_{h^Z}$ acting on $\Gamma(\$_{g^Z})$ and the Main Assertions

According to the Bourguignon and Gauduchon's method ([4], [5]), first we will make $\partial\!\!\!/_{h^Z}$ act on $\Gamma(\$_{g^Z})$. The projection from the set $F^+(T_pM)$ of positively oriented frames on T_pM to the set $I(T_pM)$ of inner products on T_pM, given by $e \mapsto$ "the inner product $\langle\cdot,\cdot\rangle_e$ which has e as an orthonormal frame", has a structure of principal $SO(n)$-bundle, which is trivial since the base space $I(T_pM)$ is contractible. And the tangent space $T_eF^+(T_pM) \cong \mathfrak{gl}(n)$, $(d/da)|_{a=0}(e\cdot B_a) \leftrightarrow (d/da)|_{a=0}B_a$, has a subspace $\mathcal{H}_e(F^+(T_pM)) \cong \{B \in \mathfrak{gl}(n) \mid B = {}^tB\}$ which is projected onto $T_{\langle\cdot,\cdot\rangle_e}I(T_pM)$ isomorphically. Clearly the distribution $e \mapsto \mathcal{H}_e(F^+(T_pM))$ gives then a connection for the bundle, which induces the parallel displacement $\eta^M : P_{SO(n)}(M)_p \cong P_{SO(n)}(M, h^M)_p$ along the segment from g^M_p to h^M_p. Gathering such displacements we get the bundle isomorphism

$$\begin{aligned} &\eta^M : P_{SO(n)}(M) \cong P_{SO(n)}(M, h^M) \\ &\text{with } \ \eta^M : T^{(*)}M \cong (T^{(*)}M, h^M),\ \eta^M([e^b, v]) = [\eta^M(e^b), v], \end{aligned} \tag{1.1}$$

where we use the canonical expression $TM = P_{SO(n)}(M) \times_{can} \mathbb{R}^n\,(\ni [e^b, v])$, etc. More explicitly, for a g^M-$SO(n)$-frame $e^b = (e^b_1, \cdots, e^b_n)$, set $\eta^b = (\eta^b_{ij}) = (h^M(e^b_i, e^b_j))^{-1/2}$, which is positive and symmetric. Then we have

$$\eta^M(e^b) = e^b\cdot\eta^b\,,\ \eta^M(e^b_i) = \eta^M(e^b)_i = \sum e^b_j\cdot\eta^b_{ji}. \tag{1.2}$$

These come from the fact that, if we take the segment $t \mapsto g^M_p(t) = (1-t)g^M_p + th^M_p$ and for each $g^M_p(t)$ we put $\eta^M_t(e^b) = e^b\cdot(g^M_p(t)(e^b_i, e^b_j))^{-1/2}$, then $\eta^M_t(e^b)$ is a $g^M_p(t)$-$SO(n)$-frame and $(\partial/\partial t)\eta^M_t(e^b)$ is horizontal. We use the common $P_{Sp(1)}$ for the two metrics (see (0.1)), which consequently determines (locally defined) Spin structures $\Xi : P_{\mathrm{Spin}(n)}(M) \to P_{SO(n)}(M)$, Ξ_{h^M} :

$P_{\mathrm{Spin}(n)}(M, h^M) \to P_{SO(n)}(M, h^M)$. Since the above connection for the (trivial) bundle $F^+(T_pM) \to I(T_pM)$ induces a connection for the associated (trivial) Spin(n)-bundle $\tilde{F}^+(T_pM) \to I(T_pM)$, similarly to the above we obtain a bundle isomorphism $\eta^M : P_{\mathrm{Spin}(n)}(M) \cong P_{\mathrm{Spin}(n)}(M, h^M)$ and, further, we have the bundle isometry

$$\begin{aligned} &\eta^M : \$_{g^M} \cong \$_{h^M},\ \eta^M([\psi, s]) = [\eta^M(\psi), s] \\ &\text{with } \ \eta^M \circ \rho_{g^M}(\xi) = \rho_{h^M}(\eta^M(\xi)) \circ \eta^M \quad (\xi \in T^*M). \end{aligned} \tag{1.3}$$

Thus we get the identifications

$$\begin{aligned} &\eta = \eta^M \oplus \mathrm{id} : TZ = \mathcal{H} \oplus \mathcal{V} \cong (TZ, h^Z) = (\mathcal{H}, \pi^* h^M) \oplus \mathcal{V} \\ &\text{given by } \ e_i^b(A) \equiv \pi^* e_i^b,\ e_k^f \ \mapsto \ \pi^* \eta^M(e_i^b),\ e_k^f, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\eta = \eta^M \otimes \mathrm{id} : \$_{g^Z} = \pi^* \$_{g^M} \otimes \$_{g^{\mathcal{V}}} \ \cong \ \$_{h^Z} = \pi^* \$_{h^M} \otimes \$_{g^{\mathcal{V}}} \\ &\text{with } \ \eta \circ \rho_{g^Z}(\xi) = \rho_{h^Z}(\eta(\xi)) \circ \eta \quad (\xi \in T^*Z), \end{aligned} \tag{1.5}$$

where $e^f = (e_1^f, e_2^f)$ is a $g^{\mathcal{V}}$-$SO(2)$-frame of $\mathcal{V}$. Set $e_*(A) = (e_1(A), \cdots) = (e^b(A), e^f)$, which is a g^Z-$SO(n+2)$-frame, and denote its dual by $e^*(A) = (e^1(A), \cdots) = (e_b, e_f(A))$. Then we have the expressions $\partial\!\!\!/_{g^Z} = \sum \rho_{g^Z}(e^i(A)) \, \nabla^{\$_{g^Z}}_{e_i(A)} = \sum \rho_{g^Z}(e^i(A)) \{ e_i(A) + \dfrac{1}{4} \sum g^Z(\nabla^{g^Z}_{e_i(A)} e_{i_1}(A), e_{i_2}(A)) \, \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A)) \}$ etc., where ∇^{g^Z} is the Levi-Civita connection associated to the metric g^Z, and now

$$\begin{aligned} \partial\!\!\!/_{h^Z} &:= \eta^{-1} \circ \partial\!\!\!/_{h^Z} \circ \eta = \sum \rho_{g^Z}(e^i(A)) \, \nabla^{\$_{g^Z}, h^Z}_{\eta(e_i(A))} : \Gamma(\$_{g^Z}) \to \Gamma(\$_{g^Z}) \ \text{ with} \\ \nabla^{\$_{g^Z}, h^Z}_v &= v + \frac{1}{4} \sum g^Z((\eta^{-1} \circ \nabla^{h^Z}_v \circ \eta) \, e_{i_1}(A), e_{i_2}(A)) \, \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A)) \\ &= v + \frac{1}{4} \sum h^Z(\nabla^{h^Z}_v \eta(e_{i_1}(A)), \eta(e_{i_2}(A))) \, \rho_{g^Z}(e^{i_1}(A)) \rho_{g^Z}(e^{i_2}(A)) \end{aligned} \tag{1.6}$$

is the desired one at (0.3). By putting $e_*^\varepsilon(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2} e^b(A), e^f)$ and $e_\varepsilon^*(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2} e_b, e_f(A))$, their adiabatic versions are then expressed as

$$\partial\!\!\!/_{g^Z_\varepsilon} = \sum \rho_{g^Z_\varepsilon}(e^i_\varepsilon(A)) \, \nabla^{\$_{g^Z_\varepsilon}}_{e^\varepsilon_i(A)}, \quad \partial\!\!\!/_{h^Z_\varepsilon} = \sum \rho_{g^Z_\varepsilon}(e^i_\varepsilon(A)) \, \nabla^{\$_{g^Z_\varepsilon}, h^Z_\varepsilon}_{\eta(e^\varepsilon_i(A))}. \tag{1.7}$$

Remark that the map η for g^Z_ε etc. coincides with (1.4) for g^Z etc.

Let us next consider the identity

$$\mathrm{Tr}_\pm\Big(\partial\!\!\!/_{g^Z_\varepsilon} \partial\!\!\!/_{h^Z_\varepsilon} \, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}} \Big) = \mathrm{Tr}_\mp\Big(\partial\!\!\!/^*_{g^Z_\varepsilon, P'} \, \partial\!\!\!/_{h^Z_\varepsilon, P} \, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P, P') \Big), \tag{1.8}$$

where we put $\partial\!\!\!/^*_{g^Z_\varepsilon, P'} \, \partial\!\!\!/_{h^Z_\varepsilon, P} \, \varphi_1(P) \boxtimes \varphi_2(P') = \partial\!\!\!/_{h^Z_\varepsilon, P} \, \varphi_1(P) \boxtimes \partial\!\!\!/_{g^Z_\varepsilon, P'} \, \varphi_2(P')$. The right hand side contains only derivatives up to the first order for each variables P, P'. First we will study $\partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P) = \partial\!\!\!/^*_{g^Z_\varepsilon,P'}\partial\!\!\!/_{h^Z_\varepsilon,P} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,\ P')|_{P=P'}$ (when $\varepsilon \to 0$) regarded as an element of the third side of the identification

$$
\begin{aligned}
(1.9)\qquad & \Gamma(\$_{g^Z_\varepsilon} \otimes \$_{g^Z_\varepsilon}) = \Gamma(\$_{g^Z_\varepsilon} \otimes \$^*_{g^Z_\varepsilon}) = \Gamma(\wedge T^*Z \otimes \mathbb{C}),\\
& s(e^\varepsilon_*(A)) \otimes s(e^\varepsilon_*(A)) \leftrightarrow s(e^\varepsilon_*(A)) \otimes s(e^\varepsilon_*(A))^*,\ \rho_{g^Z_\varepsilon}(e^I_\varepsilon(A)) \leftrightarrow e^I_\varepsilon(A),
\end{aligned}
$$

where $s(e^\varepsilon_*(A))$ is the $SU(2^{n/2+1})$-frame of $\$_{g^Z_\varepsilon}$ induced from $e^\varepsilon_*(A)$, and $I = (I^b, I^f)$ is a multi-index with $I^b = (i^b_1 < \cdots < i^b_{|I^b|})$ and $I^f = (i^f_1 < \cdots < i^f_{|I^f|})$, and we put $e^{I^b}_{b\varepsilon} = e^{i^b_1}_{b\varepsilon} \wedge \cdots \wedge e^{i^b_{|I^b|}}_{b\varepsilon}$, $e^{I^f}_f(A) = e^{i^f_1}_f(A) \wedge \cdots \wedge e^{i^f_{|I^f|}}_f(A)$ and $e^I_\varepsilon(A) = e^{I^b}_{b\varepsilon} \wedge e^{I^f}_f(A)$. Let us take now a (globally defined) tensor field

$$
(1.10)\qquad T_A = \frac{1}{2}\sum \left\{[e^b_i, e^b_j](A) - [e^b_i(A), e^b_j(A)]\right\} \otimes e^i_b \wedge e^j_b =: \sum e^f_k \otimes T^k_A,
$$

where $[e^b_i, e^b_j](A)$ is the $\mathcal{H}$-horizontal lift ($\in \mathcal{H}$) of the bracket $[e^b_i, e^b_j]$. Remark that the difference $[e^b_i, e^b_j](A) - [e^b_i(A), e^b_j(A)]$ is vertical ($\in \mathcal{V}$). Consider then the elliptic operator acting on $\Gamma(\wedge T^*_pM \otimes \$_{g^\mathcal{V}}|Z_p)$ ($Z_p = \pi^{-1}(p)$)

$$
(1.11)\qquad \mathcal{A}^2 = \partial\!\!\!/^2_{g^\mathcal{V}} - \frac{1}{2}\sum T^k_A \wedge\cdot 1\otimes\nabla^{\$_{g^\mathcal{V}}}_{e^f_k} + \frac{1}{16}\Big(\sum T^k_A \wedge\cdot \rho_{g^Z}(e^k_f(A))\Big)^2,
$$

where we put $\partial\!\!\!/_{g^\mathcal{V}} = \sum \rho_{g^Z}(e^k_f(A))\nabla^{\$_{g^\mathcal{V}}}_{e^f_k}$, $\rho_{g^Z}(e^k_f(A)) = (-1)^\ell \otimes \rho_{g^\mathcal{V}}(e^k_f)$ for ℓ-forms in the M-direction and $T^k_A(P) = (1/2)\sum(e^i_b \wedge e^j_b)(p)\cdot T^k_{A,ij}(P)$. This generates a ($C^0$)-semi-group with C^∞-kernel which belongs to $\Gamma(\wedge T^*_pM \otimes (\$_{g^\mathcal{V}}|Z_p \boxtimes \$^*_{g^\mathcal{V}}|Z_p))$. Its value at (P,P) can be canonically regarded as an element of $\wedge(\pi^*T^*M)_P \otimes \wedge\mathcal{V}^*(A)_P \otimes \mathbb{C} = \wedge T^*_PZ \otimes \mathbb{C}$ (see (1.9)), which we denote by $\exp(-t\mathcal{A}^2)(P)$. Then we have

Proposition 1.1. *When $\varepsilon \to 0$, there exists a formal series expansion*

$$
(1.12)\qquad \partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P) = \sum_{m=-2}^{\infty} \varepsilon^{m/2}\, D_{(m/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}),
$$

$$
\begin{aligned}
(1.13)\qquad D_{(-2/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) = & -\theta^\wedge\,\frac{1}{2t}\left\langle e_b \,\middle|\, \eta^b\,\frac{tR^{g^M}}{2}\left\{\coth\frac{tR^{g^M}}{2} - 1\right\} \,\middle|\, e_b\right\rangle(p)\\
& \times \frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}/2}{\sinh(tR^{g^M}/2)}\right)(p)\,\exp\Big(-t\mathcal{A}^2\Big)(P),
\end{aligned}
$$

where we set $p = \pi(P)$, $\theta^\wedge\omega = (-1)^j\omega$ for j-form ω, and $R^{g^M}(p)$ is an anti-symmetric matrix whose (i,j)-entries are equal to $R^{g^M}_{ij}(p) = (1/2)\sum g^M(F(\nabla^{g^M})(e^b_i, e^b_j)e^b_{i_1}, e^b_{i_2})(p)\ (e^{i_1}_b \wedge e^{i_2}_b)(p)$. (See Lemma 2.4 for further *informations for the coefficients.)*

Now let us state the main assertions.

Theorem 1.2. *In the definitions of* $\mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})$ *(see (0.3)), the function* $\frac{1}{\Gamma(s)}\int_0^\infty t^s\,\mathrm{Tr}_\pm\Big(\partial\!\!\!/_{g^Z_\varepsilon}\,\partial\!\!\!/_{h^Z_\varepsilon}\,e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\Big)\,dt$ *is absolutely integrable if* $\mathrm{Re}(s) > n/2+2$ *and has the meromorphic extension to* $\mathbb{C}\,(\ni s)$ *which is analytic at* $s=0$*. When* $\varepsilon\to 0$*, then there exist the asymptotic expansions*

$$
\begin{aligned}
(1.14)\quad & \mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) \\
&= \sum_{m=-2}^{\infty}\varepsilon^{m/2}\,\frac{\mp 2^{n/2}}{(\sqrt{-1})^{n/2+1}} \\
&\qquad\times\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s\int_Z D_{(m/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) \\
&\qquad+\sum_{m=-(n+2)}^{\infty}\varepsilon^{m/2}\,2^{n/2} \\
&\qquad\times\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s\int_Z D_{((m+n)/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})\wedge dg^Z(P),
\end{aligned}
$$

$$
\begin{aligned}
(1.15)\quad & \mathrm{STr}(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) := \mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) - \mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon}) \\
&= -\sum_{m=-2}^{\infty}\varepsilon^{m/2}\,\frac{2^{n/2+1}}{(\sqrt{-1})^{n/2+1}} \\
&\qquad\times\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s\int_Z D_{(m/2)}(t,P\colon \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}),
\end{aligned}
$$

where the functions to be differentiated by s *at* $s=0$ *are also all absolutely integrable if* $\mathrm{Re}(s) > n/2+2$ *and have the meromorphic extensions to* $\mathbb{C}$ *which are analytic at* $s=0$*. In particular, as for (1.15), the coefficients of* $\varepsilon^{m/2}$ *with* $m<0$ *are all pure imaginary.*

As for the infinitesimally deformed chiral anomalies, we have

Corollary 1.3. *In the definitions of* $\log\det(\delta_{X_\varepsilon}\partial\!\!\!/_{g^Z_\varepsilon})^\pm$ *(see (0.5)), the function to be differentiated by* s *is absolutely integrable if* $\mathrm{Re}(s)>n/2+2$ *and has the meromorphic extension to* $\mathbb{C}$ *which is analytic at* $s=0$*. Set*

$$
(1.16)\quad CH_{(m/2)}(t,P\colon \delta_X\partial\!\!\!/_{g^Z}) = \frac{d}{du}\Big|_{u=0} D_{(m/2)}(t,P\colon \partial\!\!\!/_{g^Z_{(u)}}/\partial\!\!\!/_{g^Z}),
$$

$$
\begin{aligned}
(1.17)\quad CH_{(-2/2)}(t,P\colon \delta_X\partial\!\!\!/_{g^Z}) &= \theta^\wedge\frac{1}{4t}\Big\langle e_b\,\Big|\,X\,\frac{tR^{g^M}}{2}\Big\{\coth\frac{tR^{g^M}}{2}-1\Big\}\,\Big|\,e_b\Big\rangle(p) \\
&\quad\times\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}/2}{\sinh(tR^{g^M}/2)}\right)(p)\,\exp\Big(-t\mathcal{A}^2\Big)(P).
\end{aligned}
$$

Then we have the asymptotic expansions when $\varepsilon \to 0$

$$\begin{aligned}
(1.18)\quad & \log\det(\delta_{X_\varepsilon}\not{\partial}_{g^Z_\varepsilon})^{\pm} \\
&= \sum_{m=-2}^{\infty} \varepsilon^{m/2} \frac{\mp 2^{n/2}}{(\sqrt{-1})^{n/2+1}} \\
&\qquad \times \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{(m/2)}(t, P\colon \delta_X \not{\partial}_{g^Z}) \\
&\quad + \sum_{m=-(n+2)}^{\infty} \varepsilon^{m/2}\, 2^{n/2} \\
&\qquad \times \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{((m+n)/2)}(t, P\colon \delta_X \not{\partial}_{g^Z}) \wedge dg^Z(P),
\end{aligned}$$

$$\begin{aligned}
(1.19)\quad & S\text{-}\log\det(\delta_{X_\varepsilon}\not{\partial}_{g^Z_\varepsilon}) := \log\det(\delta_{X_\varepsilon}\not{\partial}_{g^Z_\varepsilon})^{+} - \log\det(\delta_{X_\varepsilon}\not{\partial}_{g^Z_\varepsilon})^{-} \\
&= -\sum_{m=-2}^{\infty} \varepsilon^{m/2} \frac{2^{n/2+1}}{(\sqrt{-1})^{n/2+1}} \\
&\qquad \times \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt \cdot t^s \int_Z CH_{(m/2)}(t, P\colon \delta_X \not{\partial}_{g^Z}).
\end{aligned}$$

So are the functions to be differentiated by s *and the coefficients of* $\varepsilon^{m/2}$ *with* $m < 0$ *at (1.19) as in Theorem 1.2.*

2. Proofs of Theorem 1.2 and Corollary 1.3

First let us show

Lemma 2.1. *There exists a constant* $\lambda_0 > 0$ *satisfying* $\mathrm{Spec}(\not{\partial}^2_{g^Z_\varepsilon}) \geq \lambda_0$ *for every* ε *with* $0 < \varepsilon \leq \varepsilon_0$. *And we have*

$$\begin{aligned}
(2.1)\quad & \mathrm{Tr}_{\pm}\Big(\not{\partial}_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big) = \mathrm{Tr}_{\mp}\Big(\not{\partial}_{h^Z_\varepsilon}\not{\partial}_{g^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big) \\
&= \overline{\mathrm{Tr}_{\mp}\Big(\not{\partial}_{g^Z_\varepsilon}\not{\partial}_{h^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)} + \mathrm{Tr}_{\mp}\Big(\sum_i \frac{\varepsilon^{1/2}\eta(e_i^b)(\det\eta^b)}{\det\eta^b}\, \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\, \not{\partial}_{g^Z_\varepsilon}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}\Big)
\end{aligned}$$

and further there exists a constant $C > 0$ *satisfying*

$$(2.2)\qquad \Big|\,\mathrm{Tr}_{\pm}\Big(\not{\partial}^*_{g^Z_\varepsilon, P'}\, \not{\partial}_{h^Z_\varepsilon, P}\, e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(P, P')\Big)\Big| \leq C\, e^{-t\lambda_0/3}\ \mathrm{Tr}_{\pm}\Big(e^{-(t/6)\not{\partial}^2_{g^Z_\varepsilon}}\Big)$$
$$(\, 0 < \forall\varepsilon \leq \varepsilon_0 \ \text{and}\ 0 < \forall t < \infty\,).$$

Proof. The assertion concerning the spectrum of $\not{\partial}^2_{g^Z_\varepsilon}$ comes from the invertibility of $\not{\partial}_{g^V}$ ([10, (5.15)]) and [3, Proposition 4.41]. Namely, first consider connection $\nabla^{g^{\mathcal{V}}} = P^{\mathcal{V}} \circ \nabla^{g^Z}$ of $\mathcal{V}$, where $P^{\mathcal{V}} : TZ = \mathcal{H} \oplus \mathcal{V} \to \mathcal{V}$ is the projection. This together with the Levi-Civita one $\nabla^{g_\varepsilon^M}$ gives a new connection $\nabla^{g_\varepsilon^Z \oplus} = \pi^* \nabla^{g_\varepsilon^M} \oplus \nabla^{g^{\mathcal{V}}}$ of $TZ = \mathcal{H} \oplus \mathcal{V}$, which is compatible with g_ε^Z and whose torsion is equal to T_A given at (1.10) ([11, Lemma 3.1]). Denote by $\nabla^{\$_{g_\varepsilon^Z} \oplus}$ the associated connection on $\$_{g_\varepsilon^Z}$ and set $T_A^\sharp = \sum e_f^k(A) \otimes T_A^k = \dfrac{1}{2} \sum T_{A,ij}^\sharp \wedge e_b^i \wedge e_b^j$. Then we have

$$
\begin{aligned}
(2.3) \qquad \not{\partial}_{g_\varepsilon^Z} &= \varepsilon^{1/2} \sum \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i) \Big\{ \nabla^{\$_{g_\varepsilon^Z} \oplus}_{e_i^b(A)} + \frac{\varepsilon^{1/2}}{8} \rho_{g_\varepsilon^Z}\big(\sum T_{A,ij}^\sharp \wedge e_{b\varepsilon}^j\big) \Big\} + \not{\partial}_{g^{\mathcal{V}}} \\
&=: \varepsilon^{1/2} \tilde{\not{\partial}}_\varepsilon + \not{\partial}_{g^{\mathcal{V}}}, \\
\not{\partial}_{g_\varepsilon^Z}^2 &= \varepsilon \tilde{\not{\partial}}_\varepsilon^2 + \not{\partial}_{g^{\mathcal{V}}}^2 + \varepsilon^{1/2} \Big\{ \tilde{\not{\partial}}_\varepsilon \circ \not{\partial}_{g^{\mathcal{V}}} + \not{\partial}_{g^{\mathcal{V}}} \circ \tilde{\not{\partial}}_\varepsilon \Big\} \\
&= \varepsilon \tilde{\not{\partial}}_\varepsilon^2 + \not{\partial}_{g^{\mathcal{V}}}^2 + \varepsilon^{1/2} \Big\{ \sum \rho_{g_\varepsilon^Z}(e_{b\varepsilon}^i \wedge e_f^k(A)) \, \nabla^{\$_{g_\varepsilon^Z} \oplus}_{[e_i^b(A), e_k^f]} \\
&\qquad - \frac{\varepsilon^{1/2}}{8} \rho_{g_\varepsilon^Z}\big(\sum T_{A,ij}^\sharp \wedge e_{b\varepsilon}^i \wedge e_{b\varepsilon}^j\big) \circ \not{\partial}_{g^{\mathcal{V}}} \\
&\qquad - \not{\partial}_{g^{\mathcal{V}}} \circ \frac{\varepsilon^{1/2}}{8} \rho_{g_\varepsilon^Z}\big(\sum T_{A,ij}^\sharp \wedge e_{b\varepsilon}^i \wedge e_{b\varepsilon}^j\big) \Big\}.
\end{aligned}
$$

Let $\| \cdot \|_{p,1}$ be the Sobolev H^1-norm of elements of $\Gamma(\$_{g_\varepsilon^Z})$ restricted to Z_p with metric $g_\varepsilon^Z | Z_p$. Then there exist constants $C > 0$, $C' > 0$ such that for any $p \in M$, $\psi \in \Gamma(\$_{g_\varepsilon^Z})$ we have

$$
\Big| \int_{Z_p} \langle \{ \tilde{\not{\partial}}_\varepsilon \circ \not{\partial}_{g^{\mathcal{V}}} + \not{\partial}_{g^{\mathcal{V}}} \circ \tilde{\not{\partial}}_\varepsilon \} \psi, \psi \rangle_{\$_{g_\varepsilon^Z}} dg_\varepsilon^Z | Z_p \Big| \le C \|\psi\|_{p,1}, \; C' \|\not{\partial}_{g^{\mathcal{V}}} \psi\|_{p,1} \ge \|\psi\|_{p,1},
$$

where $\langle \cdot, \cdot \rangle_{\$_{g_\varepsilon^Z}}$ denotes the pointwise inner product which $\Gamma(\$_{g_\varepsilon^Z})$ has. The first estimate comes from the fact that $\tilde{\not{\partial}}_\varepsilon \circ \not{\partial}_{g^{\mathcal{V}}} + \not{\partial}_{g^{\mathcal{V}}} \circ \tilde{\not{\partial}}_\varepsilon$ is a first order differential operator on Z_p and the second comes from the fact that $\not{\partial}_{g^{\mathcal{V}}}$ is invertible. Since further $\tilde{\not{\partial}}_\varepsilon$ is self-adjoint and $\tilde{\not{\partial}}_\varepsilon^2$ is nonnegative, finally we have

$$
\not{\partial}_{g_\varepsilon^Z}^2 \ge \not{\partial}_{g^{\mathcal{V}}}^2 + \varepsilon^{1/2} \Big\{ \tilde{\not{\partial}}_\varepsilon \circ \not{\partial}_{g^{\mathcal{V}}} + \not{\partial}_{g^{\mathcal{V}}} \circ \tilde{\not{\partial}}_\varepsilon \Big\} \ge (1 - \varepsilon^{1/2} C C') \not{\partial}_{g^{\mathcal{V}}}^2.
$$

We have thus shown the assertion concerning the spectrum. As for the equalities (2.1): To simplify the description, let us assume $\varepsilon = 1$. Set $e^{-t\not{\partial}_{g^Z}^2} = \sum e^{-t\lambda_j} \phi_j \boxtimes \phi_j^* \in \Gamma(\$_{g^Z}^+ \boxtimes \$_{g^Z}^{+*})$ as usual (refer to the argument following (0.3)). Because of $\not{\partial}_{g^Z}^2 \not{\partial}_{g^Z} \phi_j / \sqrt{\lambda_j} = \lambda_j \cdot \not{\partial}_{g^Z} \phi_j / \sqrt{\lambda_j} \in \Gamma(\$_{g^Z}^-)$ and $\langle \not{\partial}_{g^Z} \phi_j / \sqrt{\lambda_j}, \not{\partial}_{g^Z} \phi_i / \sqrt{\lambda_i} \rangle_{L^2} = \delta_{ji}$, we have

$$
\mathrm{Tr}_- \Big(\not{\partial}_{g^Z} \not{\partial}_{h^Z} \, e^{-t\not{\partial}_{g^Z}^2} \Big)
$$

$$
\begin{aligned}
&= \sum e^{-t\lambda_j} \int \langle \partial\!\!\!/_{g^Z} \partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z} \phi_j / \sqrt{\lambda_j},\ \partial\!\!\!/_{g^Z} \phi_j / \sqrt{\lambda_j} \rangle_{\$_{g^Z}} dg^Z(P) \\
&= \sum e^{-t\lambda_j} \lambda_j^{-1} \int \langle \partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z} \phi_j,\ \partial\!\!\!/^2_{g^Z} \phi_j \rangle_{\$_{g^Z}} dg^Z(P) \\
&= \sum e^{-t\lambda_j} \int \langle \partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z} \phi_j, \phi_j \rangle_{\$_{g^Z}} dg^Z(P) = \mathrm{Tr}_+\Big(\partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}} \Big).
\end{aligned}
$$

Thus the first equality at (2.1) was proved. Next let us prove the second one. We have

$$
\partial\!\!\!/^*_{h^Z} = \det \eta^b \circ \partial\!\!\!/_{h^Z} \circ \det(\eta^b)^{-1} = \partial\!\!\!/_{h^Z} - \sum \frac{\eta(e_i^b)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z}(e_b^i) \tag{2.4}
$$

because (1.6) implies

$$
\begin{aligned}
&\int \langle \partial\!\!\!/_{h^Z} \psi, \phi \rangle_{\$_{g^Z}} dg^Z = \int \langle \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}, h^Z}_{\eta(e_i(A))} \psi, \det(\eta^b)^{-1} \cdot \phi \rangle_{\$_{g^Z}} dh^Z \\
&= \int \langle \sum \rho_{h^Z}(\eta(e^i(A))) \nabla^{\$_{h^Z}}_{\eta(e_i(A))} \eta(\psi), \det(\eta^b)^{-1} \cdot \eta(\phi) \rangle_{\$_{h^Z}} dh^Z \\
&= \int \langle \eta(\psi), \sum \rho_{h^Z}(\eta(e^i(A))) \nabla^{\$_{h^Z}}_{\eta(e_i(A))} \det(\eta^b)^{-1} \cdot \eta(\phi) \rangle_{\$_{h^Z}} dh^Z \\
&= \int \langle \psi, \det \eta^b \sum \rho_{g^Z}(e^i(A)) \nabla^{\$_{g^Z}, h^Z}_{\eta(e_i(A))} \det(\eta^b)^{-1} \cdot \phi \rangle_{\$_{g^Z}} dg^Z.
\end{aligned}
$$

Hence, using the above expression of $\mathrm{Tr}_+\Big(\partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}} \Big)$, we have

$$
\begin{aligned}
&\mathrm{Tr}_+\Big(\partial\!\!\!/_{h^Z} \partial\!\!\!/_{g^Z}\, e^{-t\partial\!\!\!/^2_{g^Z}} \Big) = \sum e^{-t\lambda_j} \int \langle \partial\!\!\!/_{g^Z} \phi_j(P), \partial\!\!\!/_{h^Z} \phi_j(P) \rangle_{\$_{g^Z}} dg^Z(P) \\
&\quad - \sum e^{-t\lambda_j} \int \langle \partial\!\!\!/_{g^Z} \phi_j(P), \sum \frac{\eta(e_i^b)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z}(e_b^i) \phi_j(P) \rangle_{\$_{g^Z}} dg^Z(P) \\
&= \sum e^{-t\lambda_j} \int \overline{\langle \partial\!\!\!/_{h^Z} \phi_j(P), \partial\!\!\!/_{g^Z} \phi_j(P) \rangle_{\$_{g^Z}}} dg^Z(P) \\
&\quad + \sum e^{-t\lambda_j} \int \langle \sum \frac{\eta(e_i^b)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z}(e_b^i)\, \partial\!\!\!/_{g^Z} \phi_j(P), \phi_j(P) \rangle_{\$_{g^Z}} dg^Z(P).
\end{aligned}
$$

Thus we have proved the second equality. Last, as for the estimate (2.2): Assume $\varepsilon = 1$ and remember the above expression of $e^{-t\partial\!\!\!/^2_{g^Z}}$. We have

$$
\begin{aligned}
&\Big| \mathrm{Tr}_+\Big(\partial\!\!\!/^*_{g^Z} \partial\!\!\!/_{h^Z} e^{-t\partial\!\!\!/^2_{g^Z}} \Big) \Big| \le \sum e^{-t\lambda_j} \Big| \int \langle \partial\!\!\!/_{h^Z} \phi_j(P), \partial\!\!\!/_{g^Z} \phi_j(P) \rangle_{\$_{g^Z}} dg^Z(P) \Big| \\
&\le \sum e^{-t\lambda_j} \|\partial\!\!\!/_{h^Z} \phi_j\|_{L^2} \|\partial\!\!\!/_{g^Z} \phi_j\|_{L^2} \le \sum e^{-t\lambda_j} (C_1 \lambda_j^{1/2} + C_0) \lambda_j^{1/2} \\
&\le C_2 \sum e^{-t\lambda_j/2} \le C_2\, e^{-t\lambda_0/3} \sum e^{-t\lambda_j/6} = C_2\, e^{-t\lambda_0/3}\, \mathrm{Tr}_+(e^{-(t/6)\partial\!\!\!/^2_{g^Z}}).
\end{aligned}
$$

Thus (2.2) with $\varepsilon = 1$ was proved. And, remembering the estimate $\mathrm{Spec}(\partial\!\!\!/^2_{g^Z_\varepsilon}) \geq \lambda_0$ for any ε with $0 < \varepsilon \leq \varepsilon_0$, obviously we know that the above estimation holds also for general ε. ∎

Before we give the proofs of the three assertions stated in the previous section, we will make some preparatory arguments. Take a point $P^0 \in Z$. Though we have taken a g^Z-$SO(n+2)$-frame $e_*(A)$ around P^0 with no specific condition, now it is convenient for the proofs to take such a frame in the following specific way. First fix $e_*(A)(P^0) = (e^b(A)(P^0), e^f(P^0))$. Then let $e_*(A) = (e^b(A), e^f)$ be $\nabla^{g^Z\oplus}$-parallel along the $\nabla^{g^Z\oplus}$-geodesics from P^0 and be equal to the fixed one at P^0, and, further, let $e^*(A) = (e_b, e_f(A))$ be its dual. Remark that $\nabla^{g^Z\oplus}$ ($= \nabla^{g^Z_1\oplus}$) is compatible with the metric g^Z so that $e_*(A)$ is certainly a g^Z-$SO(n+2)$-frame. Note also that $e^b(A)$ coincides with the $\mathcal{H}$-horizontal lift of the g^M-$SO(n)$-frame e^b on a neighborhood U^b ($\subset M$) which is ∇^{g^M}-parallel along the ∇^{g^M}-geodesics from $p^0 = \pi(P^0)$ and is equal to the given $e^b(p^0)$ at p^0. Also take such a $g^{\mathcal{V}}$-$SO(2)$-frame on U^f ($\subset Z_{p^0}$) which coincides with the given $e^f(P^0)$ at P^0 and then spread it on a neighborhood U ($\subset Z$) by the the $\mathcal{H}$-parallel displacement along the ∇^{g^M}-geodesics from p^0. The frame on U thus obtained is certainly equal to the above e^f. Further, let us take the $\nabla^{g^Z\oplus}$-normal coordinate neighborhood $(U = U^b \times U^f, x = (x^b, x^f))$ with $(\partial/\partial x)_{P^0} = e_*(A)(P^0)$. Similarly to the above, $x^b(P)$ are ∇^{g^M}-normal coordinates of $\pi(P)$ and $x^f(P)$ are $\nabla^{\mathcal{V}}$-normal coordinates of the image ($\in Z_{p^0}$) of the point P by the $\mathcal{H}$-parallel displacement. Hence we have

$$
\begin{aligned}
&e^b_i(x^b) = \sum (\partial/\partial x^b_j)_{x^b} \cdot v^b_{ji}(x^b), \quad v^b_{ji}(x^b) = \delta_{ji} + \mathcal{O}(|x^b|^2), \\
&C(\nabla^{g^M})_{i_2 i_1}(e^b_i) := g^M(\nabla^{g^M}_{e^b_i} e^b_{i_1}, e^b_{i_2}) = \mathcal{O}(|x^b|), \quad A(e^b_i) = \mathcal{O}(|x^b|),
\end{aligned} \tag{2.5}
$$

etc. Hereafter we will use the coordinates and the frames thus given and of course the g^Z_ε-$SO(n+2)$-frame $e^\varepsilon_*(A) = (e^{b\varepsilon}(A), e^f) = (\varepsilon^{1/2} e^b(A), e^f)$ and its dual $e^*_\varepsilon(A) = (e_{b\varepsilon}, e_f(A)) = (\varepsilon^{-1/2} e_b, e_f(A))$ (see (1.7)) are assumed to be defined by using such frames. Now first let us show

Lemma 2.2. *On the coordinate neighborhood (U, x), we have*

$$
\begin{aligned}
\partial\!\!\!/_{g^Z_\varepsilon} &= \sum \partial/\partial x^f_k \cdot \rho_{g^Z_\varepsilon}(e^k_f(A)) + \sum \varepsilon^{1/2} \partial/\partial x^b_i \cdot \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}) \\
&\quad - \sum \varepsilon^{2/2} \frac{1}{8} T^k_{A, i_1 i_2}(0) \cdot \rho_{g^Z_\varepsilon}(e^k_f(A)) \rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon}) \rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon}) + \mathcal{O}(|x|),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
\partial\!\!\!/_{h^Z_\varepsilon} &= \sum \partial/\partial x^f_k \cdot \rho_{g^Z_\varepsilon}(e^k_f(A)) + \sum \varepsilon^{1/2} \eta^b_{ji}(0)\, \partial/\partial x^b_j \cdot \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon}) \\
&\quad - \sum \varepsilon^{2/2}\, \eta^b_{j_1 i_1}(0)\, \eta^b_{j_2 i_2}(0)\, \frac{1}{8} T^k_{A, j_1 j_2}(0) \cdot \rho_{g^Z_\varepsilon}(e^k_f(A)) \rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon}) \rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon}) + \mathcal{O}(|x|).
\end{aligned} \tag{2.7}
$$

Proof. Remark that we have $\nabla^{g^M_\varepsilon} = \nabla^{g^M}$ and $\nabla^{h^M_\varepsilon} = \nabla^{h^M}$. Referring to (2.4) we have

$$\begin{aligned}\not{\partial}_{g^Z_\varepsilon} &= \sum \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\varepsilon^{1/2}\Big\{e^b_i(A) + \frac{1}{4}\sum C(\nabla^{g^M})_{i_2i_1}(e^b_i)\,\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\\ &+ \frac{1}{4}\sum C(\nabla^{\mathcal{V}})_{k_2k_1}(e^b_i(A))\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\Big\}\\ &+ \sum \rho_{g^Z_\varepsilon}(e^k_f(A))\Big\{e^f_k + \frac{1}{4}\sum C(\nabla^{\mathcal{V}})_{k_2k_1}(e^f_k)\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\Big\}\\ &- \frac{\varepsilon}{8}\sum T^k_{A,i_1i_2}\,\rho_{g^Z_\varepsilon}(e^k_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon}).\end{aligned}$$

Hence using (2.5) we obtain (2.6). Next, put $C(\nabla^{h^M})_{i_2i_1}(\eta(e^b_i)) = h^M(\nabla^{h^M}_{\eta(e^b_i)}\eta(e^b_{i_1}),\, \eta(e^b_{i_2}))$. Then we have

$$\begin{aligned}\not{\partial}_{h^Z_\varepsilon} &= \sum \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\,\varepsilon^{1/2}\Big\{\eta(e^b_i(A)) + \frac{1}{4}\sum C(\nabla^{h^M})_{i_2i_1}(\eta(e^b_i))\,\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\\ &+ \frac{1}{4}\sum C(\nabla^{\mathcal{V}})_{k_2k_1}(\eta(e^b_i(A)))\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\Big\}\\ &+ \sum \rho_{g^Z_\varepsilon}(e^k_f(A))\Big\{e^f_k + \frac{1}{4}\sum C(\nabla^{\mathcal{V}})_{k_2k_1}(e^f_k)\,\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\rho_{g^Z_\varepsilon}(e^{k_1}_f(A))\Big\}\\ &- \frac{\varepsilon}{8}\sum T^k_A(\eta(e^b_{i_1}),\eta(e^b_{i_2}))\,\rho_{g^Z_\varepsilon}(e^k_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon}).\end{aligned}$$

Hence using (1.2) and (2.5) we obtain (2.7). ∎

Next let us consider the identification

$$\Gamma(\$_{g^Z_\varepsilon}|U \boxtimes \$^*_{g^Z_\varepsilon}|U) = C^\infty(U\times U, \wedge T^*_{p^0}Z) \tag{2.8}$$

given by $s(e^\varepsilon_*(A))(x)\otimes s(e^\varepsilon_*(A))^*(x')\cdot\phi(x,x') \leftrightarrow ((x,x'),\ s(e^\varepsilon_*(A))(0)\otimes s(e^\varepsilon_*(A))^*(0)\cdot\phi(x,x')) \in U\times U\times \$_{g^Z_\varepsilon}|_{P^0}\otimes\$^*_{g^Z_\varepsilon}|_{P^0} \ni ((x,x'),\rho_{g^Z_\varepsilon}(e^I_\varepsilon(A)) \leftrightarrow ((x,x'), e^I_\varepsilon(A)(P^0))$. The Clifford action $\rho_{g^Z_\varepsilon}(e^i_\varepsilon(A))$ acting on the left hand side can be expressed on the right hand side as

$$\rho_{g^Z_\varepsilon}(e^i_\varepsilon(A)) = e^i_\varepsilon(A)\wedge - e^i_\varepsilon(A)\vee \tag{2.9}$$

and the operator $\not{\partial}^*_{g^Z_\varepsilon,P'}$ given at (1.8) can be expressed on the right hand side as

$$\begin{aligned}\not{\partial}^*_{g^Z_\varepsilon} &= \sum \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A))\cdot e^\varepsilon_i(A)(P')\\ &+\frac{1}{4}\sum \rho^*_{g^Z_\varepsilon}(e^{i_2}_\varepsilon(A))\rho^*_{g^Z_\varepsilon}(e^{i_1}_\varepsilon(A))\rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A))\cdot C(\nabla^{g^Z_\varepsilon})(e^\varepsilon_i(A))_{i_2i_1}(P')\\ &\text{with}\ \ \rho^*_{g^Z_\varepsilon}(e^i_\varepsilon(A)) = \theta^\wedge(e^i_\varepsilon(A)\wedge + e^i_\varepsilon(A)\vee).\end{aligned} \tag{2.10}$$

Let us then regard the kernel $e^{-t\not{\partial}^2_{g^Z_\varepsilon}}$ as an element of the right hand side of (2.8) and set $e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(x,x') := \sum e^I(A)(P^0)\cdot\Big(e^{-t\not{\partial}^2_{g^Z_\varepsilon}}(x,x')\Big)_I$, and moreover define its differentiations as

$$\partial_x^{\alpha}\partial_{x'}^{\alpha'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x') := \sum e^I(A)(P^0)\cdot \partial_x^{\alpha}\partial_{x'}^{\alpha'}\Big(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\Big)_I \tag{2.11}$$

$$\text{with}\quad \Big|\partial_x^{\alpha}\partial_{x'}^{\alpha'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\Big|_{g^Z} = \Big\{\sum \Big|\partial_x^{\alpha}\partial_{x'}^{\alpha'}\Big(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(x,x')\Big)_I\Big|^2\Big\}^{1/2}, \tag{2.12}$$

where $\alpha = (\alpha^b, \alpha^f) = (\alpha_1^b, \cdots, \alpha_n^b, \alpha_1^f, \alpha_2^f)$ is a multi-index and we put $\partial_x^\alpha = (\partial/\partial x)^\alpha = (\partial/\partial x^b)^{\alpha^b}(\partial/\partial x^f)^{\alpha^f} = (\partial/\partial x_1^b)^{\alpha_1^b}\cdots(\partial/\partial x_n^b)^{\alpha_n^b}(\partial/\partial x_1^f)^{\alpha_1^f}(\partial/\partial x_2^f)^{\alpha_2^f}$, etc. Then we have

Lemma 2.3 *(the general adiabatic expansion theorem as to $e^{-t\partial\!\!\!/^2_{g^Z}}$: [11, Theorems 1.2, 1.3 and the proof of Proposition 2.2 for $E(t,\varepsilon)$ with t small]).*

(1) *For any integer $m_0 \geq 0$, there exist C^∞-functions $K_{(m/2)}(t, P^0, x, x')$ ($m = 0, 1, \cdots, m_0$), $K_{((m_0+1)/2,\varepsilon^{1/2})}(t, P^0, x, x')$ belonging to the right hand side of (2.8), which are also C^∞ with respect to the variable P^0 (and $\varepsilon^{1/2}$), and satisfying the following condition: For any α and α', (2.11) with $(x, x') = (0, 0)$ has the series expansion*

$$\begin{aligned}\partial_x^{\alpha}\partial_{x'}^{\alpha'} e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0,P^0) = &\sum_{m=0}^{m_0} \varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+m/2}\,\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2)}(t,P^0)\\ &+\varepsilon^{-(|\alpha^b|+|\alpha'^b|)/2+(m_0+1)/2}\,\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0),\end{aligned} \tag{2.13}$$

where we put $|\alpha^b| = \sum \alpha_i^b$ etc. and $\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2)}(t,P^0)$ etc. mean $\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2)}(t, P^0, x, x')|_{x=x'=0}$ etc. Further, there exist constants $\lambda > 0$, $C > 0$ and an integer $N > 0$ satisfying

$$\begin{aligned}&\Big|\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2)}(t,P^0)\Big|_{g^Z} \leq C\,e^{-t\lambda}\,t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+1\Big),\\ &\Big|\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{((m_0+1)/2,\varepsilon^{1/2})}(t,P^0)\Big|_{g^Z} \leq C\,t^{1/2}\Big(\frac{1}{t^{(n+2+|\alpha|+|\alpha'|)/2}}+t^N\Big)\\ &\qquad (\,0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t<\infty,\ \forall P^0\in Z\,).\end{aligned} \tag{2.14}$$

And, if $|\alpha| + |\alpha'| \leq 2$, then, given $T_0 > 0$, we have the series expansion

$$\begin{aligned}&\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2,\cdot)}(t,P^0)\\ &= \frac{1}{(4\pi t)^{(n+2)/2}}\Big\{\sum_{i=-\delta_{0m}}^{i_0} t^i\,\partial_x^{\alpha}\partial_{x'}^{\alpha'} K_{(m/2,\cdot)}(i:P^0)+\mathcal{O}(t^{i_0+1})\Big\}\\ &(\,\forall i_0\geq 0,\ 0\leq\forall m\leq m_0+1,\ 0<\forall\varepsilon^{1/2}\leq\varepsilon_0^{1/2},\ 0<\forall t\leq T_0,\ \forall P^0\in Z\,).\end{aligned} \tag{2.15}$$

(2) *The top term $K_{(0)}(t, P^0, x, x')$ can be written as*

$$K_{(0)}(t,P^0,x,x') = K_M(t,P^0,x^b,x'^b)\,\exp\Big(-t\mathcal{A}^2\Big)(x^f,x'^f)\cdot\det v^b(x'^b) \tag{2.16}$$

where we set

$$K_M(t,P^0,x^b,x'^b)=\frac{1}{(4\pi t)^{n/2}}\det{}^{1/2}\left(\frac{tR^{g^M}(p^0)/2}{\sinh\left(tR^{g^M}(p^0)/2\right)}\right) \tag{2.17}$$
$$\cdot\exp\Big(-\frac{1}{4t}\Big\langle(x^b-x'^b)\Big|\frac{tR^{g^M}(p^0)}{2}\coth\frac{tR^{g^M}(p^0)}{2}\Big|(x^b-x'^b)\Big\rangle$$
$$+\frac{1}{4}\Big\langle x^b\Big|R^{g^M}(p^0)\Big|x'^b\Big\rangle\Big)$$

and $\exp\Big(-t\mathcal{A}^2\Big)$, *see (1.11) around, is here regarded as an element of the right hand side of* $\Gamma(\wedge T^*_{p^0}M\otimes(\$_{g^V}|U^f\boxtimes\$^*_{g^V}|U^f))=C^\infty(U^f\times U^f,\wedge T^*_{P^0}Z)$, *and we have* $\det v^b(x'^b)=\det(g^M(\partial/\partial x^b_i,\partial/\partial x^b_j)(x'^b))^{-1/2}=1+\mathcal{O}(|x'^b|^2)$ *(see (2.5)).*

Proof of Proposition 1.1. (2.10), (2.13) and Lemma 2.2 imply the formal series expansion

$$\partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon}e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0,P^0)\equiv\partial\!\!\!/^*_{g^Z_\varepsilon,P'}\partial\!\!\!/_{h^Z_\varepsilon,P}e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P,P')|_{P=P'=P^0} \tag{2.18}$$
$$=\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i}_{b\varepsilon})\,\eta^b_{ji}(0)\,(\partial/\partial x'^b_{i'})(\partial/\partial x^b_j)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{i}_{b\varepsilon})\,\eta^b_{ji}(0)\,(\partial/\partial x'^f_{k'})(\partial/\partial x^b_j)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{k}_f(A))(\partial/\partial x'^b_{i'})(\partial/\partial x^f_k)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{k}_f(A))(\partial/\partial x'^f_{k'})(\partial/\partial x^f_k)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{2/2+m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{i'}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{k}_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})$$
$$\cdot\eta^b_{j_1i_1}(0)\,\eta^b_{j_2i_2}(0)\Big(-\frac{1}{4}\nu^k(F_{A,j_1j_2})\Big)(0)\,(\partial/\partial x'^b_{i'})K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{2/2+m/2}\sum(\rho_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{i'_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i'_2}_{b\varepsilon}))^*\rho_{g^Z_\varepsilon}(e^{i}_{b\varepsilon})$$
$$\cdot\eta^b_{ji}(0)\,\Big(-\frac{1}{4}\,\nu^{k'}(F_{A,i'_1i'_2})\Big)(0)\,(\partial/\partial x^b_j)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{2/2+m/2}\sum\rho^*_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{k}_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})$$
$$\cdot\eta^b_{j_1i_1}(0)\,\eta^b_{j_2i_2}(0)\Big(-\frac{1}{4}\nu^k(F_{A,j_1j_2})\Big)(0)(\partial/\partial x'^f_{k'})K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{2/2+m/2}\sum(\rho_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{i'_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i'_2}_{b\varepsilon}))^*\rho_{g^Z_\varepsilon}(e^{k}_f(A))$$
$$\cdot\Big(-\frac{1}{4}\,\nu^{k'}(F_{A,i'_1i'_2})\Big)(0)\,(\partial/\partial x^f_k)K_{(m/2)}(t,P^0)$$
$$+\sum\varepsilon^{4/2+m/2}\sum(\rho_{g^Z_\varepsilon}(e^{k'}_f(A))\rho_{g^Z_\varepsilon}(e^{i'_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i'_2}_{b\varepsilon}))^*\rho_{g^Z_\varepsilon}(e^{k}_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})$$
$$\cdot\eta^b_{j_1i_1}(0)\,\eta^b_{j_2i_2}(0)\Big(-\frac{1}{4}\,\nu^{k'}(F_{A,i'_1i'_2})\Big)(0)\Big(-\frac{1}{4}\nu^k(F_{A,j_1j_2})\Big)(0)K_{(m/2)}(t,P^0).$$

Hence, observing (2.10), we know that (2.18) can be expanded as in (1.12). And (2.16) implies further

$$
\begin{aligned}
(2.19)\qquad & D_{(-2/2)}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) \\
&= -\theta^\wedge \sum e_b^{i'} \wedge e_b^i \wedge \eta_{ji}^b(0)\,(\partial/\partial x_{i'}^{\prime b})(\partial/\partial x_j^b) K_{(0)}(t, P^0) \\
&= -\theta^\wedge \frac{1}{2t} \Big\langle e_b \,\Big|\, \eta^b \,\frac{tR^{g^M}}{2}\Big\{\coth \frac{tR^{g^M}}{2} - 1\Big\} \,\Big|\, e_b \Big\rangle (p^0)\, K_{(0)}(t, P^0).
\end{aligned}
$$

Thus we have obtained the formula (1.13). ∎

Further, as for the coefficients in (1.12), Lemmata 2.2 and 2.3 say

Lemma 2.4. *For any integer $m_0 \geq 0$, put $D_{((m_0+1)/2, \varepsilon^{1/2})}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) = \partial\!\!\!/^*_{g^Z_\varepsilon} \partial\!\!\!/_{h^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0, P^0) - \sum_{m=-2}^{m_0} \varepsilon^{m/2}\, D_{(m/2)}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})$. Then there exist constants $\lambda > 0$, $C > 0$ and an integer $N > 0$ satisfying*

$$
\begin{aligned}
(2.20)\quad & \big|D_{(m/2)}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})\big|_{g^Z} \leq C\, e^{-t\lambda}\, t^{(1-\delta_{0m})/2}\Big(\frac{1}{t^{(n+2)/2}} + 1\Big) \quad (m \leq m_0), \\
& \big|D_{((m_0+1)/2, \varepsilon^{1/2})}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})\big|_{g^Z} \leq C\, t^{1/2}\Big(\frac{1}{t^{(n+2)/2}} + t^N\Big) \\
& \qquad\qquad (\, 0 < \forall\varepsilon^{1/2} \leq \varepsilon_0^{1/2},\ 0 < \forall t < \infty,\ \forall P^0 \in Z\,).
\end{aligned}
$$

Further, for given $T_0 > 0$, we have the series expansion

$$
\begin{aligned}
(2.21)\quad & D_{(m/2,\cdot)}(t, P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) \\
&= \frac{1}{(4\pi t)^{(n+2)/2}} \Big\{ \sum_{i=-\delta_{0m}}^{i_0} t^i\, D_{(m/2,\cdot)}(i : P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z}) + \mathcal{O}(t^{i_0+1}) \Big\} \\
& (\,\forall i_0 \geq 0,\ 0 \leq \forall m \leq m_0 + 1,\ 0 < \forall\varepsilon^{1/2} \leq \varepsilon_0^{1/2},\ 0 < \forall t \leq T_0,\ \forall P^0 \in Z\,).
\end{aligned}
$$

As the last preparation, let us investigate the pointwise trace $\mathrm{tr}_\pm(\rho_{g^Z}(e^I(A)))$.

Lemma 2.5. *We have*

$$
(2.22)\quad
\begin{aligned}
& \mathrm{tr}_\pm(\rho_{g^Z}(e^\emptyset(A))) = 2^{n/2}, \quad \mathrm{tr}_\pm(\rho_{g^Z}(e^{(1,\cdots,n+2)}(A))) = \pm \frac{2^{n/2}}{(\sqrt{-1})^{n/2+1}}, \\
& \mathrm{tr}_\pm(\rho_{g^Z}(e^I(A))) = 0 \ (\textit{otherwise}),
\end{aligned}
$$

$$
\begin{aligned}
(2.23)\quad \Omega^\pm(\varepsilon, P) &:= \sum e^I(A)(P) \cdot \varepsilon^{-(n-|I^b|)/2}\, \mathrm{tr}_\pm(\rho_{g^Z}(e^I(A))) \\
&= \varepsilon^{-n/2}\, 2^{n/2} \pm \frac{2^{n/2}}{(\sqrt{-1})^{n/2+1}}\, dg^Z(P).
\end{aligned}
$$

Proof. The first two equalities at (2.22) and the equality $\mathrm{tr}_\pm(\rho_{g^Z}(e^I(A))) = 0$ ($|I|$ is odd), and moreover $\mathrm{tr}_+(\rho_{g^Z}(e^I(A))) = \mathrm{tr}_-(\rho_{g^Z}(e^I(A))) = (1/2)\,\mathrm{tr}(\rho_{g^Z}(e^I(A)))$ ($|I|$ is even and $|I| < 2n$) are all obvious. Hence we have only to prove

$$
(2.24)\qquad \mathrm{tr}(\rho_{g^Z}(e^I(A))) = 0 \quad (0 < |I| = 2m < n + 2).
$$

Take the standard frame $(e^1,\cdots,e^{2r}) = (e_1, Je_1,\cdots,e_r, Je_r)$ of $\mathbb{R}^{2r}$ where J is the standard complex structure, and let us prove (2.24) for the standard Clifford action $\rho:\mathbb{C}l(\mathbb{R}^{2r})\to \mathrm{End}(\wedge^*\mathbb{C}^r)$, i.e., $\rho(e^{2\ell-1}) = e_\ell\wedge - e_\ell\vee$, $\rho(e^{2\ell}) = \sqrt{-1}(e_\ell\wedge + e_\ell\vee)$ and hence $\rho(e^{2\ell-1}\circ e^{2\ell}) = \sqrt{-1}(e_\ell\wedge e_\ell\vee - e_\ell\vee e_\ell\wedge)$. Assume $I = ((2i_1-1, 2i_1),\cdots,(2i_m-1, 2i_m))$ ($\mathbb{I}\equiv(i_1<\cdots<i_m)$, $0<m<r$) and set $(1,2,\cdots,r) = \mathbb{I}\cup\mathbb{J}$ with $\mathbb{J} = (j_1<\cdots<j_{r-m})$. Then we have

$$\begin{aligned}
&\rho(e^I) = (\sqrt{-1})^m\prod_{\ell=1}^m (e_{i_\ell}\wedge e_{i_\ell}\vee - e_{i_\ell}\vee e_{i_\ell}\wedge),\\
&\mathbb{I}\supset\mathbb{K} = (k_1<\cdots<k_{|\mathbb{K}|}),\ \rho(e^I)\, e_{\mathbb{K}}\wedge e_{\mathbb{J}} = (\sqrt{-1})^m(-1)^{m-|\mathbb{K}|}\, e_{\mathbb{K}}\wedge e_{\mathbb{J}},\\
&\mathrm{tr}(\rho(e^I)) = (\sqrt{-1})^m\sum_{\mathbb{K}}(-1)^{m-|\mathbb{K}|} = 0.
\end{aligned}$$

Thus (2.24) for such a type of I was proved. And it will obviously holds if I is not of such a type. ∎

Proof of Theorem 1.2. Let us set $D_{(m/2)}(t,P^0) = D_{(m/2)}(t,P^0 : \partial\!\!\!/_{h^Z}/\partial\!\!\!/_{g^Z})$, etc., to simplify the description, and put $D_{(m/2)}(t,P^0) = \sum e^I(A)(P^0)\cdot D_{(m/2)}(t,P^0)_I$ as in (2.11). Then we have

$$\begin{aligned}
(2.25)\quad \mathrm{Tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t)) &:= \int_Z \mathrm{tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t,P^0))\, dg_\varepsilon^Z(P^0)\\
&= \int_Z\sum \mathrm{tr}_\pm(\rho_{g_\varepsilon^Z}(e_\varepsilon^I(A)))\cdot\varepsilon^{|I^b|/2}\Big(D_{(m/2)}(t,P^0)\Big)_I\, e_\varepsilon^I(A)\wedge\star_{g_\varepsilon^Z} e_\varepsilon^I(A)\\
&= \int_Z D_{(m/2)}(t,P^0)\wedge\star_{g^Z}\Omega^\pm(\varepsilon,P^0),
\end{aligned}$$

where $\mathrm{tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t,P^0))$ mean the pointwise traces of $D_{(m/2)}(t,P^0)$ regarded as an element of $\$_{g_\varepsilon^Z,P^0}\otimes\$^*_{g_\varepsilon^Z,P^0}$ and $\star_{g_\varepsilon^Z}$ is the star operator associated to the metric g_ε^Z. Hence, setting $\Omega^\pm(\varepsilon,P^0) = \sum_{-n\le\ell\le 0}\varepsilon^{\ell/2}\,\Omega^\pm(\ell/2 : P^0)$ (see (2.23)), (1.12) and the above give the formal series expansion

$$\begin{aligned}
(2.26)\qquad \mathrm{Tr}_\pm\Big(\partial\!\!\!/^*_{g_\varepsilon^Z}\partial\!\!\!/_{h_\varepsilon^Z}e^{-t\partial\!\!\!/^2_{g_\varepsilon^Z}}\Big) &= \sum_{m=-2}^\infty \varepsilon^{m/2}\,\mathrm{Tr}_\pm^{g_\varepsilon^Z}(D_{(m/2)}(t))\\
&= \sum_{m=-(n+2)}^\infty\varepsilon^{m/2}\int_Z\sum_{m=m_1+m_2} D_{(m_1/2)}(t,P^0)\wedge\star_{g^Z}\Omega^\pm(m_2/2 : P^0)\\
&=: \sum_{m=-(n+2)}^\infty\varepsilon^{m/2}\int_Z D(m/2 : t, P^0).
\end{aligned}$$

Thus, observing (2.23), we find that $\mathrm{Tr}_\pm(\partial\!\!\!/_{h_\varepsilon^Z}/\partial\!\!\!/_{g_\varepsilon^Z})$ can be expanded into (1.14) (still not asymptotically but) formally. Further the first estimate at (1.17) and the series expansion (1.18) imply that, for given $n_0 > 0$, if $m \le n_0$ then the function (to be differentiated by s)

$$\frac{1}{\Gamma(s)}\int_0^\infty dt\cdot t^s\int_Z D(m/2:t,P^0) \tag{2.27}$$

is absolutely integrable if $\mathrm{Re}(s) > n/2+1$ and has a meromorphic extension to $\mathbb{C}\,(\ni s)$ which is analytic at $s=0$. Hence, to finish the proof of the assertions concerning $\mathrm{Tr}_\pm(\partial\!\!\!/_{h^Z_\varepsilon}/\partial\!\!\!/_{g^Z_\varepsilon})$, we have only to show that so is (2.27) with $D(m/2:t,P^0)$ replaced by the remainder term $D((n_0+1)/2,\varepsilon^{1/2}:t,P^0) = \mathrm{tr}^{g^Z_\varepsilon}_\pm\left(\partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon}e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0,P^0)\right)dg^Z_\varepsilon(P^0) - \sum_{m=-(n+2)}^{n_0}\varepsilon^{m/2}D(m/2:t,P^0)$. To prove it let us investigate the remainder term for t large. That is, fix $T_0>0$ and assume $t \ge T_0$. Then there exists a constant $C=C(T_0)>0$ such that, for any $t\,(\ge T_0)$, we have

$$\left|\mathrm{Tr}_\pm\left(\partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon}e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)\right| \le C\,\varepsilon^{-n/2}\,e^{-t\lambda_0/4}. \tag{2.28}$$

Indeed (2.14) with $\alpha=\alpha'=\emptyset$ implies

$$\left|\mathrm{Tr}_\pm(e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}})\right| = \left|\int_Z e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}(P^0,P^0)\wedge\star_{g^Z}\Omega^\pm(\varepsilon,P^0)\right| \le C'\,\varepsilon^{-n/2}\,t^N,$$

which, combined with (2.2), gives the estimate (2.28). Next let $m_0>0$ be the integer appearing in (2.20). Then (2.28) and the first estimate at (2.20) imply

$$\begin{aligned}&\left|\varepsilon^{(m_0+1)/2}\int_Z D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\right|\\ &=\left|\mathrm{Tr}_\pm\left(\partial\!\!\!/^*_{g^Z_\varepsilon}\partial\!\!\!/_{h^Z_\varepsilon}e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}}\right)-\sum_{m=-(n+2)}^{m_0}\varepsilon^{m/2}\int_Z D(m/2:t,P^0)\right| \le C_1\,\varepsilon^{-(n+2)/2}\,e^{-t\lambda_0/4},\end{aligned}$$

which, combined with the second estimate at (2.20), yields

$$\begin{aligned}&\left|\varepsilon^{(m_0+1)/2}\int_Z D((m_0+1)/2,\varepsilon^{1/2}:t,P^0)\right|\\ &\le C_2\,\varepsilon^{-(n+2)/4}\,e^{-t\lambda_0/8}\varepsilon^{(m_0+1)/4}\,t^{N/2} \le C_3\,\varepsilon^{(m_0-n-1)/4}\,e^{-t\lambda_0/9}.\end{aligned} \tag{2.29}$$

Hence we may take $m_0>0$ so large that we have

$$\left|\int_Z D((n_0+1)/2,\varepsilon^{1/2}:t,P^0)\right| \le C_4\,e^{-t\lambda_0/9}. \tag{2.30}$$

Indeed, since we have

$$\int_Z D((n_0+1)/2, \varepsilon^{1/2} : t, P^0) = \int_Z \Big\{ \sum_{m=n_0+1}^{m_0} \varepsilon^{(m-n_0-1)/2} D(m/2 : t, P^0) \\ + \varepsilon^{(m_0-n_0)/2} D((m_0+1)/2, \varepsilon^{1/2} : t, P^0) \Big\}$$

and (2.29) yields

$$\left| \varepsilon^{(m_0-n_0)/2} \int_Z D((m_0+1)/2, \varepsilon^{1/2} : t, P^0) \right| \le C_3\, \varepsilon^{(m_0-2n_0-n-3)/4}\, e^{-t\lambda_0/9},$$

we have only to take m_0 satisfying $m_0 - 2n_0 - n - 3 \ge 0$. The estimate (2.30) with t large and the series expansion (2.21) with $m = m_0+1$ (and here $m_0 = n_0$) now imply the desired assertion about (2.27) for the remainder term.

Let us show the remained assertions concerning the difference (1.15). (1.15) is obvious. And (1.1) says

$$(2.31)\quad \mathrm{Re}\Big(\mathrm{STr}(\partial\!\!\!/_{h^Z_\varepsilon} / \partial\!\!\!/_{g^Z_\varepsilon}) \Big) \\ = \frac{1}{2}\Big(\mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon} / \partial\!\!\!/_{g^Z_\varepsilon}) - \overline{\mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon} / \partial\!\!\!/_{g^Z_\varepsilon})} \Big) - \frac{1}{2}\Big(\mathrm{Tr}_-(\partial\!\!\!/_{h^Z_\varepsilon} / \partial\!\!\!/_{g^Z_\varepsilon}) - \overline{\mathrm{Tr}_+(\partial\!\!\!/_{h^Z_\varepsilon} / \partial\!\!\!/_{g^Z_\varepsilon})} \Big) \\ = -\frac{d}{ds}\Big|_{s=0} \frac{1}{2\Gamma(s)} \int_0^\infty t^s\, \mathrm{STr}\Big(\sum_i \frac{\varepsilon^{1/2}\eta(e_i^b)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\, \partial\!\!\!/_{g^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}} \Big)\, dt$$

and (2.6) implies

$$\sum \frac{\varepsilon^{1/2}\eta(e_i^b)(\det \eta^b)}{\det \eta^b}\, \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})\, \partial\!\!\!/_{g^Z_\varepsilon}\, e^{-t\partial\!\!\!/^2_{g^Z_\varepsilon}} \\ = \varepsilon^{0/2} \sum \frac{\eta(e_i^b)(\det \eta^b)}{\det \eta^b}(0)\, e^i_b \wedge \Big\{ \sum e^j_b \wedge (\partial/\partial x^b_j) K_{(1/2)}(t, P^0) \\ + \sum \rho_{g^Z}(e^k_f(A))(\partial/\partial x^f_k) K_{(0/2)}(t, P^0) \\ + \sum \Big(-\frac{1}{8}\, T^k_{A,j_1 j_2} \Big)(0)\, \rho_{g^Z}(e^k_f(A))\, e^{j_1}_b \wedge e^{j_2}_b \wedge K_{(0/2)}(t, P^0) \Big\} + \cdots .$$

Thus the series expansion of (2.31) has no term with $\varepsilon^{m/2}$ $(m < 0)$. ∎

Proof of Corollary 1.3. We have $\eta^b_{(u)} := (g^M_{(u)}(e^b_i, e^b_j))^{-1/2} = (E+uX)^{-1/2}$ where E is the unit matrix (compare with (2.1)), which implies

$$(2.32)\qquad \frac{d}{du}\Big|_{u=0} \eta^b_{(u)} = -\frac{1}{2} X.$$

Hence, referring to (2.7) with h^Z replaced by $g^Z_{(u)} = \pi^* g^M_{(u)} + g^{\mathcal{V}}$, we have

$$(2.33)\qquad \delta_{X_\varepsilon} \partial\!\!\!/_{g^Z_\varepsilon} = \frac{d}{du}\Big|_{u=0} \partial\!\!\!/_{g^Z_{(u)\,\varepsilon}} = -\varepsilon^{1/2} \sum \frac{1}{2} X_{ji}(0)\, \partial/\partial x^b_j \cdot \rho_{g^Z_\varepsilon}(e^i_{b\varepsilon})$$

$$
\begin{aligned}
&+\varepsilon^{2/2}\sum\{X_{j_1i_1}\delta_{j_2i_2}+\delta_{j_1i_1}X_{j_2i_2}\}(0)\,\frac{1}{16}T^k_{A,j_1j_2}(0)\\
&\cdot\rho_{g^Z_\varepsilon}(e^k_f(A))\rho_{g^Z_\varepsilon}(e^{i_1}_{b\varepsilon})\rho_{g^Z_\varepsilon}(e^{i_2}_{b\varepsilon})\\
&+\mathcal{O}(|x|).
\end{aligned}
$$

Thus obviously Theorem 1.2 implies the corollary. ∎

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