Saitama Math. J.
Vol. 31 (2017), 161–170

The discriminant of Valentiner reflection group and deformations of a plane curve

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(*Received 28 June, 2016; Revised 12 December, 2016; Accepted 26 December, 2016*)

Abstract

A family of plane curves with three parameters related with the Valentiner reflection group is constructed and the family gives an answer to Arnold's Problem proposed in [1], 1974-5 for the case of the Valentiner reflection group.

1. Introduction

In this paper G always denotes the No.27 group in the list of Shephard-Todd [7]. The group G is called Valentiner reflection group. Since 6,12,30 are the degrees of basic invariants of G, I write such basic invariants as x_1, x_2, x_5 in this introduction and assume that 1,2,5 are their weights. Then the polynomial Δ defined below is regarded as the discriminant of G (cf. [6], [4]):

$$\begin{aligned}\Delta = {} & 65536x_1^{11}x_2^2 - 1765376x_1^9x_2^3 + 17406016x_1^7x_2^4 - 73887360x_1^5x_2^5 \\ & +107371008x_1^3x_2^6 + 34338816x_1x_2^7 - 4096x_1^8x_2x_5 + 96640x_1^6x_2^2x_5 \\ & -707952x_1^4x_2^3x_5 + 1622592x_1^2x_2^4x_5 + 186624x_2^5x_5 + 64x_1^5x_5^2 \\ & -1584x_1^3x_2x_5^2 + 7128x_1x_2^2x_5^2 + 9x_5^3.\end{aligned}$$

It is shown by direct calculation (cf. [6]) that Δ^2 is the discriminant of the polynomial $P_a(u)$ of u:

$$\begin{aligned}P_a(u) = {} & u^6 + x_1u^5 + \tfrac{5(9+c_1)}{16}x_2u^4 + \tfrac{5(11+3c_1)}{64}x_1x_2u^3 + \tfrac{5(37+45c_1)}{512}x_2^2u^2 \\ & + \left\{\tfrac{(-61-5c_1)}{192}x_1^3x_2 + \tfrac{(5407+695c_1)}{3072}x_1x_2^2 + \tfrac{(61+5c_1)}{6144}x_5\right\}u \\ & + \tfrac{179(-279+145c_1)}{221184}x_1^2x_2^2 + \tfrac{5(-45+11c_1)}{1152}x_2^3 + \tfrac{(-279+145c_1)}{221184}x_1x_5 + \tfrac{279-145c_1}{6912}x_1^4x_2\end{aligned}$$

($c_1 = \sqrt{15}i$). It is easy to show that $P_b(v) = P_a\left(v + \frac{3-5c_1}{72}x_1\right)$ is of the form

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2010 Mathematics Subject Classification. 20F55, 14H50, 14B07
Key words and phrases. Complex reflection group, plane curve, deformation
Partially supported by JSPS Grant-in-Aid for Scientific Research (C) (No.26400111)

$$\begin{aligned}P_b(v) = {} & v^6 - \tfrac{5(-3+c_1)}{12}x_1v^5 - \tfrac{5}{288}\left\{(49+25c_1)x_1^2 - 18(9+c_1)x_2\right\}v^4 \\ & + \tfrac{5}{15552}\left\{2(-1377+55c_1)x_1^2 - 81(-101+19c_1)x_2\right\}x_1v^3 \\ & + \tfrac{5}{124416}\left\{(-559+1465c_1)x_1^4 - 54(87+175c_1)x_1^2x_2 + 243(37+45c_1)x_2^2)\right\}v^2 \\ & + \tfrac{1}{4478976}\left\{2(64137+5425c_1)x_1^5 - 54(52007+4095c_1)x_1^3x_2\right. \\ & \qquad \left. +3888(3117+245c_1)x_1x_2^2 + 729(61+5c_1)x_5\right\}v \\ & + \tfrac{5(-45+11c_1)}{11609505792}\left\{(-39+c_1)x_1^2 + 216x_2\right\}^3.\end{aligned}$$

I note here that the constant term of $P_b(v)$ is a cubic power of a polynomial. The purpose of this report is to show the existence of a family of plane curves with three parameters related with the group G. This is an answer to the long-standing problem by Arnol'd [1], p.20 for the group G and it makes clear the reason why the constant term of $P_b(v)$ is a cubic power of a polynomial.

I shall explain the contents of this report briefly. Section 2 is devoted to a survey of the results of Crass [2] which are useful for our considerations. Section 3 is devoted to a construction of a family of plane curves which is defined in a way similar to the case of the versal family of D_6-singularity. After these preparations, I shall construct a family of plane curves which gives an answer to Arnold's Problem [1], 1974-5 in section 4. In section 5, I shall explain an observation on the existence of a G-equivariant correspondence between the set of reflections of the Valentiner reflection group and that of the complex reflection group $G(3,3,6)$.

2. Some consequences of the paper by S. Crass

There are many interesting results in [2] which help our investigation. I will explain some of the results stated in [2] and their consequences which are needed for our purpose.

First of all, I introduce three reflections which generate the group G (cf. [7], p.296):

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ R_2 = \frac{1}{2}\begin{pmatrix} 1 & -\omega\tau & -\omega^2\tau^{-1} \\ -\omega^2\tau & -\tau^{-1} & -\omega \\ -\omega\tau^{-1} & -\omega^2 & \tau \end{pmatrix},\ R_3 = \begin{pmatrix} 0 & -\omega^2 & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tau = \frac{1+\sqrt{5}}{2}$, $\omega = \frac{-1+\sqrt{3}i}{2}$. Let $t = (t_1, t_2, t_3)$ be a linear coordinate of $\mathbf{C}^3$ and each R_i operates on t as follows: $t \to t \cdot {}^tR_i$.

Following [2], I introduce two matrices P, Q by

$$P = \frac{1}{2}\begin{pmatrix} 1 & \tau-1 & -\tau \\ \tau-1 & \tau & 1 \\ \tau & -1 & \tau-1 \end{pmatrix},\quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & -\omega & 0 \end{pmatrix},$$

It is easy to show that

$$P^5 = P \cdot {}^t\!P = I, \quad Q\overline{{}^tQ} = Q^4 = I.$$

I define $C_{\bar{1}}(t) = t_1^2 + t_2^2 + t_3^2$ and

$$C_{\bar{2}}(t) = C_{\bar{1}}(t\overline{Q}), C_{\bar{k}}(t) = C_{\bar{2}}(tP^{2-k})\ (k = 3, 4, 5, 6)$$

as in [2]. Then

$$\begin{aligned}
C_{\bar{2}}(t) &= t_1^2 + \omega^2 t_2^2 + \omega t_3^2, \\
C_{\bar{3}}(t) &= \eta \left\{ \tfrac{\eta}{3}(t_1^2 + \omega t_2^2 + \omega^2 t_3^2) + \omega^2 t_1 t_2 + \omega t_1 t_3 - t_2 t_3 \right\}, \\
C_{\bar{4}}(t) &= \eta \left\{ \tfrac{\eta}{3}(\omega t_1^2 + \omega^2 t_2^2 + t_3^2) - t_1 t_2 + \omega^2 t_1 t_3 + \omega t_2 t_3 \right\}, \\
C_{\bar{5}}(t) &= \eta \left\{ \tfrac{\eta}{3}(\omega t_1^2 + \omega^2 t_2^2 + t_3^2) - t_1 t_2 - \omega^2 t_1 t_3 - \omega t_2 t_3 \right\}, \\
C_{\bar{6}}(t) &= \eta \left\{ \tfrac{\eta}{3}(t_1^2 + \omega t_2^2 + \omega^2 t_3^2) + \omega^2 t_1 t_2 - \omega t_1 t_3 + t_2 t_3 \right\},
\end{aligned}$$

where $\eta = \frac{3+\sqrt{15}i}{4}$.

Each R_i induces a transformation among the six polynomials $C_{\bar{i}}(t)\,(i = 1, 2, \cdots, 6)$. Indeed, by direct computation, we find that

$$\begin{array}{ll}
R_1: & C_{\bar{1}} \to C_{\bar{1}},\ C_{\bar{2}} \to C_{\bar{2}},\ C_{\bar{3}} \leftrightarrow \omega^2 C_{\bar{5}},\ C_{\bar{4}} \leftrightarrow \omega C_{\bar{6}} \\
R_2: & C_{\bar{1}} \leftrightarrow C_{\bar{6}},\ C_{\bar{2}} \to C_{\bar{2}},\ C_{\bar{3}} \leftrightarrow \omega C_{\bar{5}},\ C_{\bar{4}} \to C_{\bar{4}}, \\
R_3: & C_{\bar{1}} \leftrightarrow \omega^2 C_{\bar{2}}, C_{\bar{3}} \to C_{\bar{3}},\ C_{\bar{4}} \leftrightarrow C_{\bar{5}},\ C_{\bar{6}} \to C_{\bar{6}}.
\end{array} \tag{1}$$

As a direct consequence,

$$P_V(u) = \prod_{k=1}^{6} (u - C_{\bar{k}}(t)^3)$$

is a sextic polynomial of u whose coefficients are G-invariant polynomials of t. I introduce G-invariants $X_i(t)\ (i = 1, 2, \cdots, 5)$ and $Y(t) = \prod_{k=1}^{6} C_{\bar{k}}(t)$ by expanding $P_V(u)$:

$$P_V(u) = u^6 + X_1(t)u^5 + X_2(t)u^4 + X_3(t)u^3 + X_4(t)u^2 + X_5(t)u + Y(t)^3.$$

We are going to construct basic G-invariants $I_1(t), I_2(t), I_5(t)$ and express $X_i(t), Y(t)$ as polynomials of $I_1(t), I_2(t), I_5(t)$. Note that the degrees of basic invariants are 6, 12, 30 We first introduce a homogeneous polynomial $I_1(t)$ by

$$\begin{aligned}
I_1(t) = {} & t_1^6 + t_2^6 + t_3^6 + 3(5 - c_1) t_1^2 t_2^2 t_3^2 \\
& - \tfrac{3}{4}(5 + c_1) \left\{ \omega(t_1^4 t_2^2 + t_2^4 t_3^2 + t_3^4 t_1^2) + \omega^2 (t_1^4 t_3^2 + t_2^4 t_1^2 + t_3^4 t_2^2) \right\},
\end{aligned}$$

where $c_1 = \sqrt{15}i$. Note that I_1 is same as the polynomial F in [2], p.221. It is standard to introduce the polynomials $I_2(t)$, $I_5(t)$ (cf. [2]) by

$$I_2(t) = -\frac{1}{20250} \begin{vmatrix} \partial_{t_1}\partial_{t_1} I_1 & \partial_{t_1}\partial_{t_2} I_1 & \partial_{t_1}\partial_{t_3} I_1 \\ \partial_{t_2}\partial_{t_1} I_1 & \partial_{t_2}\partial_{t_2} I_1 & \partial_{t_2}\partial_{t_3} I_1 \\ \partial_{t_3}\partial_{t_1} I_1 & \partial_{t_3}\partial_{t_2} I_1 & \partial_{t_3}\partial_{t_3} I_1 \end{vmatrix},$$

$$I_5(t) = \begin{vmatrix} \partial_{t_1}\partial_{t_1} I_2 & \partial_{t_1}\partial_{t_2} I_2 & \partial_{t_1}\partial_{t_3} I_2 & \partial_{t_1} I_1 \\ \partial_{t_2}\partial_{t_1} I_2 & \partial_{t_2}\partial_{t_2} I_2 & \partial_{t_2}\partial_{t_3} I_2 & \partial_{t_2} I_1 \\ \partial_{t_3}\partial_{t_1} I_2 & \partial_{t_3}\partial_{t_2} I_2 & \partial_{t_3}\partial_{t_3} I_2 & \partial_{t_3} I_1 \\ \partial_{t_1} I_1 & \partial_{t_2} I_1 & \partial_{t_3} I_1 & 0 \end{vmatrix}.$$

Then I_1, I_2, I_5 are homogeneous basic G-invariants and $\deg I_j = 6j$ $(j = 1, 2, 5)$. Since $X_1, X_2, X_3, X_4, X_5, Y$ are G-invariant, they are expressed as polynomials of I_1, I_2, I_5. It is straightforward to show that

$$X_1(t) = \frac{3}{32}(-25 + c_1)I_1,$$

and their concrete forms are given below:

$$\begin{aligned}
X_1 &= \tfrac{3}{32}(-25 + c_1)I_1, \\
X_2 &= -\tfrac{3}{2^{14}}\{45(-237 + 53c_1)I_2 + 176(-61 + 5c_1)I_1^2\}, \\
X_3 &= \tfrac{1}{2^{19}\cdot 5} I_1\{8(-224305 + 27417c_1)I_1^2 + 135(-45615 + 10823c_1)I_2\}, \\
X_4 &= -\tfrac{3}{2^{29}\cdot 5}\left\{\begin{array}{l} 64(-1178831 + 173543c_1)I_1^4 \\ +2160(-300339 + 78571c_1)I_1^2 I_2 \\ +6075(71987 + 43285c_1)I_2^2 \end{array}\right\}, \\
X_5 &= -\tfrac{3}{2^{37}\cdot 5^2\cdot 11^2}\left\{\begin{array}{l} 22528(18558655 + 1150633c_1)I_1^5 \\ -11520(-1015846575 + 227173063c_1)I_1^3 I_2 \\ -17107200(930805 + 686467c_1)I_1 I_2^2 \\ +675(-2003265 + 85673c_1)I_5 \end{array}\right\}, \\
Y &= \omega^2\left\{-\tfrac{9(495-7c_1)}{8192}I_2 + \tfrac{115-43c_1}{10240}I_1^2\right\}.
\end{aligned} \tag{2}$$

Let

$$D = \begin{vmatrix} \partial_{t_1} I_1 & \partial_{t_2} I_1 & \partial_{t_3} I_1 \\ \partial_{t_1} I_2 & \partial_{t_2} I_2 & \partial_{t_3} I_2 \\ \partial_{t_1} I_3 & \partial_{t_2} I_3 & \partial_{t_3} I_3 \end{vmatrix}$$

be the Jacobian of the map $t \to (I_1, I_2, I_5)$. Then D^2 is the discriminant of G and is a polynomial of I_1, I_2, I_5:

$$\begin{aligned}
D^2 = {} & -\tfrac{3970695168(-781+171c_1)}{15625} I_1^{15} - \tfrac{95293145088(223+119c_1)}{3125} I_1^{13} I_2 + \tfrac{6679855890432(-251+13c_1)}{6875} I_1^{11} I_2^2 \\
& + \tfrac{36165234917376(-7+33c_1)}{3025} I_1^9 I_2^3 - \tfrac{2934756458496(61+5c_1)}{55} I_1^7 I_2^4 + \tfrac{19622017622016(-17+7c_1)}{5} I_1^5 I_2^5 \\
& + 18022306553856(11 + 3c_1) I_1^3 I_2^6 + 376648647598080(-7 + c_1) I_1 I_2^7 \\
& + \Big\{ \tfrac{7778304(-7+33c_1)}{625} I_1^{10} + \tfrac{333379584(61+5c_1)}{275} I_1^8 I_2 - \tfrac{5066136576(-17+7c_1)}{121} I_1^6 I_2^2 \\
& \quad - \tfrac{2592705024(11+3c_1)}{11} I_1^4 I_2^3 - 5861859840(-7 + c_1) I_1^2 I_2^4 - 18291018240(1 + c_1) I_2^5 \Big\} I_5 \\
& + \Big\{ -\tfrac{18432(11+3c_1)}{55} I_1^5 + \tfrac{2980800(-7+c_1)}{121} I_1^3 I_2 + \tfrac{1749600(1+c_1)}{11} I_1 I_2^2 \Big\} I_5^2 + \tfrac{900}{121} I_5^3
\end{aligned}$$

To compare the polynomial defined by the right side of the above equation with the polynomial of [2], p.222, (2-1), we define

$$\begin{aligned}
J_1 &= I_1,\\
J_2 &= -\tfrac{15(-1+c_1)}{16} I_2,\\
J_5 &= \tfrac{1}{4460544}\left\{811008 I_1^5 - 8133120(-1+c_1) I_1^3 I_2 - 12355200(7+c_1) I_1 I_2^2\right.\\
&\qquad \left. +125(-11+3c_1) I_5\right\}.
\end{aligned}$$

Then it is straightforward to show

$$\begin{aligned}
\tfrac{78125(781+171c_1)}{9055096730025984} D^2 &= 4J_1J_2(J_1^4+10J_1^2J_2+52J_2^2)(J_1^8+10J_1^6J_2+52J_1^4J_2^2+54J_1^2J_2^3+27J_2^4)\\
&\quad +18J_5(J_1^{10}+11J_1^8J_2+53J_1^6J_2^2-11J_1^4J_2^3-306J_1^2J_2^4-108J_2^5)\\
&\quad -162J_1J_5^2(J_1^4+12J_1^2J_2+9J_2^2)+729J_5^3.
\end{aligned}$$

The polynomial of the right side of the above equality coincides with the polynomial of [2], p.222, (2-1) under the correspondence $J_1 \to F,\ J_2 \to \Phi,\ J_5 \to \Psi$.

3. The versal family of D_6-singularity and its modification

I begin this section with recalling the D_6-singularity. We introduce a polynomial of (u,v) defined by

$$P_{D_6}(u,v) = u^5 - uv^2 + x_1u^4 + x_2u^3 + x_3u^2 + x_4u + x_5 + 2yv,$$

where $x_1, x_2, \cdots, x_5, y$ are parameters. Then $P_{D_6}(u,v) = 0$ defines a plane curve and in the case $x_1 = \cdots = x_5 = y = 0$, $P_{D_6}(u,v) = u^5 - uv^2 = 0$ is known to have D_6-singularity at the origin. There is a relationship between $P_{D_6}(u,v)$ and the Weyl group $W(D_6)$ of type D_6, which we are going to explain. Let V be a 6-dimensional real vector space and let $\xi = (\xi_1, \xi_2, \cdots, \xi_6)$ be its standard coordinate. We naturally regard $\xi_1, \cdots, \xi_6$ as complex variables. The group $W(D_6)$ is the totality of the actions on ξ defined by

$$(\xi_1, \xi_2, \cdots, \xi_6) \to (\varepsilon_1\xi_{\sigma(1)}, \varepsilon_2\xi_{\sigma(2)}, \cdots, \varepsilon_6\xi_{\sigma(6)}),$$

where σ are permutations among the six letters $1, 2, \cdots, 6$ and $\varepsilon_i = \pm 1$ $(i = 1, 2, \cdots, 6)$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_6 = 1$. Let $s_i(\xi_1, \xi_2, \cdots, \xi_6)$ be the i-th fundamental symmetric polynomial of $(\xi_1, \xi_2, \cdots, \xi_6)$. Then putting $x_i = (-1)^i s_i(\xi_1^2, \xi_2^2, \cdots, \xi_6^2)$ and $y = s_6(\xi_1, \xi_2, \cdots, \xi_6)$, we find that

$$P_{D_6}(u,v) = \frac{1}{u}\left\{\prod_{i=1}^{6}(u-\xi_i^2) - (uv - \xi_1\xi_2\cdots\xi_6)^2\right\}.$$

This expression of $P_{D_6}(u,v)$ suggests to introduce the polynomial

$$P_T(u,v) = \frac{1}{u}\left\{\prod_{i=1}^{6}(u-\xi_i^3) - (uv + \xi_1\xi_2\cdots\xi_6)^3\right\}.$$

Then $G(3,3,6)$ is the group which plays the role of $W(D_6)$ for $P_{D_6}(u,v)$ in this case. The complex reflection group denoted by $G(3,3,6)$ is the totality of the actions on $(\xi_1,\xi_2,\cdots,\xi_6)$ of the forms

$$(\xi_1,\xi_2,\cdots,\xi_6)\to(\varepsilon_1\xi_{\sigma(1)},\varepsilon_2\xi_{\sigma(2)},\cdots,\varepsilon_6\xi_{\sigma(6)}),$$

where σ are permutations among the six letters $1,2,\cdots,6$ and $\varepsilon_i^3=1$ $(i=1,2,\cdots,6)$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_6=1$. The basic invariants of $G(3,3,6)$ are $\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_5,\tilde{y}$, where

$$\tilde{x}_i=(-1)^i s_i(\xi_1^3,\xi_2^3,\cdots,\xi_6^3),\quad \tilde{y}=s_6(\xi_1,\xi_2,\cdots,\xi_6).$$

Using $\tilde{x}_1,\cdots,\tilde{x}_5,\tilde{y}$, we obtain

$$P_T(u,v)=u^5-u^2v^3+\tilde{x}_1u^4+\tilde{x}_2u^3+\tilde{x}_3u^2+\tilde{x}_4u+\tilde{x}_5-3\tilde{y}uv^2-3\tilde{y}^2v.$$

It follows from the definition that

$$\partial_vP_T=-3(uv+\tilde{y})^2.$$

Erasing v on the definition of $P_T(u,v)$ by the condition $\partial_vP_T=0$, we obtain

$$uP_T(u,-\tilde{y}/u)=u^6+\tilde{x}_1u^5+\tilde{x}_2u^4+\tilde{x}_3u^3+\tilde{x}_4u^2+\tilde{x}_5u+\tilde{y}^3.$$

4. An answer to one of Arnold's Problems for the group G

Let $e_1,\cdots,e_6$ be a standard basis of the vector space $V_c=V\otimes\mathbf{C}=\mathbf{C}^6$ Let $\tilde{R}_1,\tilde{R}_2,\tilde{R}_3$ be linear transformations of V_c defined by

$$\begin{array}{lcl}\tilde{R}_1 & : & (e_1,e_2,e_3,e_4,e_5,e_6)\longrightarrow(e_1,e_2,\omega^2e_5,\omega e_6,\omega e_3,\omega^2e_4),\\ \tilde{R}_2 & : & (e_1,e_2,e_3,e_4,e_5,e_6)\longrightarrow(e_6,e_2,\omega e_5,e_4,\omega^2e_3,e_1),\\ \tilde{R}_3 & : & (e_1,e_2,e_3,e_4,e_5,e_6)\longrightarrow(\omega^2e_2,\omega e_1,e_3,e_5,e_4,e_6).\end{array}$$

Then clearly $\tilde{R}_1,\tilde{R}_2,\tilde{R}_3\in G(3,3,6)$ and the relation (1) shows that the correspondence

$$\varphi(R_i)=\tilde{R}_i\ (i=1,2,3)$$

induces a group homomorphism of G into $G(3,3,6)$.

It is known that there is a group homomorphism of the real reflection group $W(H_3)$ of type H_3 into the Weyl group $W(D_6)$ of type D_6. This implies a free deformation of the D_6-singularity related to $W(H_3)$ in the sense of T. Yano [8]. I briefly explain the result on the $W(H_3)$ case in [8]. Let $P_{D_6}(u,v)$ be the polynomial introduced before. By definition, $P_{D_6}(u,v)$ has parameters $x_1,\cdots,x_5,y$. Let z_2,z_6,z_{10} be variables and substitute

$$x_1=10z_2,\ x_2=35z_2^2,\ x_3=5z_6+50z_2^3,\ x_4=15z_2z_6+25z_2^4,$$

$$x_5 = 5z_{10} + 5z_2^2 z_6 + 2z_2^5,\ y = \sqrt{5}z_6/2$$

in $P_{D_6}(u,v)$. Then we obtain the polynomial $Q_{H_3}(u,v)$ by

$$\begin{aligned} Q_{H_3}(u,v) = {} & u^5 - uv^2 + 10z_2u^4 + 35z_2^2u^3 + 5(z_6 + 10z_2^3)u^2 + 5(3z_6 + 5z_2^3)z_2u \\ & + 5z_{10} + 5z_2^2z_6 + 2z_2^5 + \sqrt{5}z_6v. \end{aligned}$$

In this case, under the identification of z_2, z_6, z_{10} with basic $W(H_3)$-invariants of homogeneous degrees 2,6,10, we find that the equation

$$C_{H_3}(z_1, z_6, z_{10}) : Q_{H_3}(u,v) = 0$$

defines a curve in (u,v)-plane and the family $\{C_{H_3}(z_2, z_6, z_{10})\})$ parametrized by three variables defines a deformation of type H_3 (cf. [8]).

Applying the above consideration concerning the pair $(W(H_3),\ W(D_6))$ studied by T. Yano to the pair $(G,\ G(3,3,6))$, we construct a family of plane curves with three papameters related with the group G. To explain the result, we recall the polynomial $P_T(u,v)$ introduced in §3. We have a family of plane curves

$$C_{G(3,3,6)}(\tilde{x}_1, \cdots, \tilde{x}_5, \tilde{y}) : P_T(u,v) = 0 \tag{3}$$

on (u,v)-plane with parameters $(\tilde{x}_1, \cdots, \tilde{x}_5, \tilde{y})$. This family is related with the complex reflection group $G(3,3,6)$. By replacing $\tilde{x}_i$ with X_i $(i = 1,2,3,4,5)$ and $\tilde{y}$ with Y in $P_T(u,v)$, where X_i $(i = 1, \cdots, 5)$ and Y are polynomials of I_1, I_2, I_5 by the relation (2), we define a polynomial $Q_G(u,v)$ by

$$Q_G(u,v) = u^5 - u^2v^3 + X_1u^4 + X_2u^3 + X_3u^2 + X_4u + X_5 - 3Yv(uv + Y) \tag{4}$$

depending on I_1, I_2, I_5. Then the equation

$$C_G(I_1, I_2, I_5) : Q_G(u,v) = 0$$

defines a curve in (u,v)-plane and the family $\{C_G(I_1, I_2, I_5)\}$ is parametrized by I_1, I_2, I_5.

We now mention one of Arnold's Problems. In the book [1], p.20, it is written that

"1974-5 Find applications of the (Shephard-Todd) complex reflection groups to singularity theory."

There are many studies concerning this problem (cf. comments by V. V. Goryunov in [1], p.305, [3]). The family of curves $\{C(I_1, I_2, I_5)\}$ defined above is an answer to this problem for the group G. It is underlined here that $C(0,0,0) : u^5 - u^2v^3 = 0$ is not a reduced equation and is not related with a simple curve singularity.

Remark 1. *Consider the case $\partial_v Q_G = 0$. Then*

$$uQ_G(u, -Y/u) = u^6 + X_1u^5 + X_2u^4 + X_3u^3 + X_4u^2 + X_5u + Y^3.$$

The right hand side of this equality is nothing but the polynomial $P_b(u)$ introduced in the Introduction. We explain this briefly. For this purpose, we put

$$(5)\qquad \begin{aligned} I_1 &= -\tfrac{1}{36}(7c_1 + 15)x_1, \\ I_2 &= \tfrac{1}{324}(5c_1 + 61)(6x_2 - x_1^2), \\ I_5 &= -\tfrac{1}{7290}(2339c_1 - 62805)(121x_5 + 27984x_1x_2^2 - 5012x_1^3x_2 + 36x_1^5), \end{aligned}$$

where x_1, x_2, x_5 mean the G-invariants given in Introduction. Therefore if we regard $\{I_1, I_2, I_5\}$ and $\{x_1, x_2, x_5\}$ as two systems of basic G-invariants, (5) gives the relationship between $\{I_1, I_2, I_5\}$ and $\{x_1, x_2, x_5\}$. Since $X_1, \cdots, X_5, Y$ are polynomials of I_1, I_2, I_5, the coefficients of the polynomial $u^6 + X_1u^5 + X_2u^4 + X_3u^3 + X_4u^2 + X_5u + Y^3$ are polynomials of x_1, x_2, x_5 by the relation (5). The polynomial thus obtained coincides with $P_b(u)$ (by replacing c_1 with $-c_1$) of the Introduction. This implies that the constant term of $P_b(u)$ is a cubic power of a polynomial.

Remark 2. *It is easier to show an answer to Arnold's Problem for the group No.24 in the list of [7]. Indeed, in this case, the family of curves is related with that of curves of A_6-singularity via the inclusion of the group No.24 into the alternating group on seven letters using quadratic polynomials constructed by F. Klein. T. Mano studied this case in detail and in particular he clarified a relationship between the family of A_6-singularity for the No.24 group and the flat structure introduced in [4], [5].*

Remark 3. *The purpose of this report is to construct a family of plane curves $\{C_G(I_1, I_2, I_5)\}$ which is a deformation of a plane curve $u^5 - u^2v^3 = 0$. The geometric meaning of the family and its properties are postponed by future studies. As a byproduct, the family of plane curves $\{C_{G(3,3,6)}(\tilde{x}_1, \cdots, \tilde{x}_5, \tilde{y})\}$ is constructed. The idea of the construction of this family is applicable to the case of the series of complex reflection groups denoted by $G(p, p, n)$ ([7]).*

5. Reflections of G and those of $G(3, 3, 6)$

In this section, I explain an observation concerning reflections of G. It is known that there are 45 reflections of G and also there are 45 reflections of $G(3, 3, 6)$. There is a correspondence between these reflections. The result is given in Table I below. In the column A, vectors defining reflections of $G(3, 3, 6)$ are given and in the column B, given are linear forms defining reflections of G corresponding to reflections of $G(3, 3, 6)$ in the column A. Let

$Cx = R_1R_2R_3$ (Coxeter transformation) be the product of generating reflections of G. Then $Cx^5 = \alpha I$, where α is a constant such that $\alpha^6 = 1$. There are 9 orbits of the totality of 45 reflections of G by the action of Cx. The notation $r[i,j]$ ($i = 1,2,\cdots,9, j = 1,2,\cdots,5$) in the column C means reflections of G so that $Cx^{-1} \circ r[i,j] \circ Cx = r[i,j+1]$ ($j = 1,2,\cdots,8$) and $Cx^{-1} \circ r[i,9] \circ Cx = r[i,1]$. We may take $r[3,1]$ as the reflection commutative with R_1, R_3. On the other hand, $e_1 - \omega^2 e_2$ is the unique vector among 45 vectors which is invariant by $\varphi(R_1), \varphi(R_3)$. Then

$$r[3,1] \longleftrightarrow e_1 - \omega^2 e_2$$

induces a map of the totality of the reflections of G to that of $G(3,3,6)$. This map is G-equivariant.

Table I

A	B	C
$e_1 - e_2$	t_1	$r[1,1]$
$e_1 - e_3$	$\frac{1}{2}(-t_1 + t_2 - \tau t_2 + \tau t_3)$	$r[8,3]$
$e_1 - e_4$	$\frac{1}{2}(t_1 - \tau t_1 + \tau t_2 + t_3)$	$r[6,5]$
$e_1 - e_5$	$\frac{1}{2}(-t_1 + \tau t_1 - \tau t_2 + t_3)$	$r[9,4]$
$e_1 - e_6$	$\frac{1}{2}(t_1 - t_2 + \tau t_2 + \tau t_3)$	$r[3,2]$
$e_2 - e_3$	$\frac{1}{2}(\omega t_1 + \tau t_2 + \omega\tau t_2 - t_3 + \tau t_3)$	$r[8,2]$
$e_2 - e_4$	$\frac{1}{2}(\omega t_1 - \omega\tau t_1 - t_2 - \omega t_2 + \tau t_3)$	$r[6,2]$
$e_2 - e_5$	$\frac{1}{2}(-\omega t_1 + \omega\tau t_1 + t_2 + \omega t_2 + \tau t_3)$	$r[3,5]$
$e_2 - e_6$	$\frac{1}{2}(-\omega t_1 - \tau t_2 - \omega\tau t_2 - t_3 + \tau t_3)$	$r[9,5]$
$e_3 - e_4$	$\frac{1}{2}(-\omega t_1 + \tau t_1 + \omega\tau t_1 + \omega t_2 + t_3)$	$r[8,5]$
$e_3 - e_5$	$\frac{1}{2}(t_1 + \omega t_1 + t_2 + \omega\tau t_2 + t_3)$	$r[4,2]$
$e_3 - e_6$	$\frac{1}{2}(-\omega t_1 - \tau t_1 - t_2 + \tau t_2 + \omega\tau t_2)$	$r[5,1]$
$e_4 - e_5$	$\frac{1}{2}(t_1 + \omega t_1 - \omega\tau t_1 + \omega t_2 + \tau t_2)$	$r[2,1]$
$e_4 - e_6$	$\frac{1}{2}(-t_1 - \omega t_1 - t_2 - \omega\tau t_2 + t_3)$	$r[2,5]$
$e_5 - e_6$	$\frac{1}{2}(\omega t_1 - \tau t_1 - \omega\tau t_1 - \omega t_2 + t_3)$	$r[9,2]$
$e_1 - \omega e_2$	$(1+\omega)t_2$	$r[2,3]$
$e_1 - \omega e_3$	$\frac{1}{2}(-t_1 + \tau t_1 + \tau t_2 + t_3)$	$r[4,4]$
$e_1 - \omega e_4$	$\frac{1}{2}(-\tau t_1 - t_2 - t_3 + \tau t_3)$	$r[9,1]$
$e_1 - \omega e_5$	$\frac{1}{2}(\tau t_1 + t_2 - t_3 + \tau t_3)$	$r[6,3]$
$e_1 - \omega e_6$	$\frac{1}{2}(t_1 - \tau t_1 - \tau t_2 + t_3)$	$r[2,4]$
$e_2 - \omega e_3$	$\frac{1}{2}(-\omega t_1 + \omega\tau t_1 - t_2 - \omega t_2 + \tau t_3)$	$r[4,3]$
$e_2 - \omega e_4$	$\frac{1}{2}(\omega\tau t_1 - t_2 - \omega t_2 + \tau t_2 + \omega\tau t_2 + t_3)$	$r[7,5]$
$e_2 - \omega e_5$	$\frac{1}{2}(-\omega\tau t_1 + t_2 + \omega t_2 - \tau t_2 - \omega\tau t_2 + t_3)$	$r[1,5]$
$e_2 - \omega e_6$	$\frac{1}{2}(\omega t_1 - \omega\tau t_1 + t_2 + \omega t_2 + \tau t_3)$	$r[7,2]$
$e_3 - \omega e_4$	$\frac{1}{2}(\omega t_1 - \tau t_1 - \omega\tau t_1 + \omega t_2 + t_3)$	$r[5,2]$
$e_3 - \omega e_5$	$\frac{1}{2}(t_1 + \omega t_1 - t_2 - \omega\tau t_2 + t_3)$	$r[8,1]$
$e_3 - \omega e_6$	$\frac{1}{2}(\omega t_1 - t_2 - \omega t_3 + \tau t_3 + \omega\tau t_3)$	$r[4,5]$
$e_4 - \omega e_5$	$\frac{1}{2}(-\omega t_1 - t_2 - \omega t_3 + \tau t_3 + \omega\tau t_3)$	$r[1,4]$

$e_4 - \omega e_6$	$\frac{1}{2}(\omega t_1 + \tau t_1 + t_3 + \omega t_3 - \omega\tau t_3)$	$r[3,3]$
$e_5 - \omega e_6$	$\frac{1}{2}(-t_2 + \tau t_2 + \omega\tau t_2 + t_3 + \omega t_3 - \omega\tau t_3)$	$r[6,1]$
$e_1 - \omega^2 e_2$	t_3	$r[3,1]$
$e_1 - \omega^2 e_3$	$\frac{1}{2}(\tau t_1 - t_2 - t_3 + \tau t_3)$	$r[5,5]$
$e_1 - \omega^2 e_4$	$\frac{1}{2}(t_1 + t_2 - \tau t_2 + \tau t_3)$	$r[7,3]$
$e_1 - \omega^2 e_5$	$\frac{1}{2}(-t_1 - t_2 + \tau t_2 + \tau t_3)$	$r[7,1]$
$e_1 - \omega^2 e_6$	$\frac{1}{2}(-\tau t_1 + t_2 - t_3 + \tau t_3)$	$r[1,2]$
$e_2 - \omega^2 e_3$	$\frac{1}{2}(-\omega\tau t_1 - t_2 - \omega t_2 + \tau t_2 + \omega\tau t_2 + t_3)$	$r[5,4]$
$e_2 - \omega^2 e_4$	$\frac{1}{2}(-\omega t_1 + \tau t_2 + \omega\tau t_2 - t_3 + \tau t_3)$	$r[9,3]$
$e_2 - \omega^2 e_5$	$\frac{1}{2}(\omega t_1 - \tau t_2 - \omega\tau t_2 - t_3 + \tau t_3)$	$r[2,2]$
$e_2 - \omega^2 e_6$	$\frac{1}{2}(\omega\tau t_1 + t_2 + \omega t_2 - \tau t_2 - \omega\tau t_2 + t_3)$	$r[6,4]$
$e_3 - \omega^2 e_4$	$\frac{1}{2}(t_2 - \tau t_2 - \omega\tau t_2 + t_3 + \omega t_3 - \omega\tau t_3)$	$r[4,1]$
$e_3 - \omega^2 e_5$	$\frac{1}{2}(-\omega t_1 - \tau t_1 + t_3 + \omega t_3 - \omega\tau t_3)$	$r[5,3]$
$e_3 - \omega^2 e_6$	$\frac{1}{2}(-\omega t_1 + t_2 - \omega t_3 + \tau t_3 + \omega\tau t_3)$	$r[8,4]$
$e_4 - \omega^2 e_5$	$\frac{1}{2}(\omega t_1 + t_2 - \omega t_3 + \tau t_3 + \omega\tau t_3)$	$r[3,4]$
$e_4 - \omega^2 e_6$	$\frac{1}{2}(-t_1 - \omega t_1 + t_2 + \omega\tau t_2 + t_3)$	$r[1,3]$
$e_5 - \omega^2 e_6$	$\frac{1}{2}(-\omega t_1 + \tau t_1 + \omega\tau t_1 - \omega t_2 + t_3)$	$r[7,4]$

Acknowledgements

The author thanks to A. G. Aleksandrov for informing him the book [1].

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